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ON METRICS DEFINED BY MODULES

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Jacqueline Ferrand gave a very general definition of a conformal invariant $\lambda_G(x,y)$ for a domain G by the use of modules of curve families and showed that, in dimension n, $\lambda_G(x,y)^{-1/n}$ is a metric. The question as to whether $\lambda_G(x,y)^{-1/(n-1)}$ is itself a metric was raised by Vuorinen and studied by him and Jacqueline Ferrand. In particular he asked whether this held for $n = 2$ and G the punctured plane. In this paper it is shown that the answer is affirmative for any domain of finite connectivity on the sphere.

1. Jacqueline Ferrand [**4**] gave a very general method for defining metrics by the use of modules of curve families. Let G be a domain in R^n, $x, y \in G$, C_x, C_y connected closed subsets of G with $x \in C_x$, $y \in C_y$, $\operatorname{Cl} C_x \cap \partial G \neq 0$, $\operatorname{Cl} C_y \cap \partial G \neq 0$. Let $\Delta(C_x, C_y, G)$ denote the family of all curves in G joining C_x and C_y, $M()$ denote the module of a curve family. Let

$$\lambda_G(x,y) = \text{g.l.b.} M(\Delta(C_x, C_y, G))$$

taken over the above configurations. She proved by a standard extremal metric argument that $\lambda_G(x,y)^{-1/n}$ is a metric on G. The question has been raised by her and especially by Vuorinen (see [**8**], p.193) under what circumstances $\lambda_G(x,y)^{-1/(n-1)}$ is itself a metric. In [**2**] it is shown that when G is the n-ball $B^n = \{x \in R^n;\ ||x|| < 1\}$, $\lambda_G(x,y)^{-p}$ is a metric if and only if $p \in [0, 1/(n-1)]$. In the summer of 1987 Vuorinen raised the above question to me, asking whether it held even in the case of the punctured plane. Sometime after that I gave him a very simple proof answering that question in the affirmative. In the present paper it will be shown that the same is true for any domain of finite connectivity on the sphere.

2. We begin by treating the special case mentioned above in which the essence of the proof is revealed without involving some of the technical matters which occur in the general case.

THEOREM 1. *Let G be the domain obtained by deleting from the sphere two points which we may take to be 0 and ∞. Let $x, y \in G$. Let C_x be a continuum containing x and 0 or ∞ and C_y a continuum containing y and the other. Let $M(C_x, C_y)$ denote the module of the family of curves joining C_x, C_y. Let $\lambda_G(x, y)$ be the* g.l.b. *of this quantity for all such configurations. Then $\lambda_G(x, y)^{-1}$ is a metric on G.*

Proof. Evidently we may assume C_x, C_y disjoint. Then they determine a doubly-connected domain $D(x, y)$ and $M(C_x, C_y)$ is the reciprocal of the module $M(x, y)$ of $D(x, y)$. The level curves of this domain determine a free (unsensed) homotopy class $\mathcal{H}$ in $G - \{x, y\}$. The module of this homotopy class is equal to the maximal module of a domain such as $D(x, y)$. This follows from the present author's fundamental theorem [**5**]. There is a homotopy class $\mathcal{H}^*$ (unique except in certain special cases) for which the module is maximal. We call this value $m(x, y)$; it is evidently equal to $\lambda_G(x, y)^{-1}$. For definiteness we will assume that it arises from continua joining x and $0, y$ and ∞.

Let now $z \in G$ be distinct from x and y. $\mathcal{H}^*$ will determine homotopy classes Γ_1, Γ_2 of Jordan curves separating 0 and x from z and ∞ and separating 0 and z from y and ∞. Let $\rho_1(w)|dw|$, $\rho_2(w)|dw|$ be the extremal metrics for these classes. Let $\rho = \max(\rho_1, \rho_2)$. Then $\rho(w)|dw|$ is an admissible metric for the family

$$\mathcal{H}^* - \{\text{curves through } z\}.$$

Since the latter set of curves has module zero we have

$$m(x, y) \leq M(\Gamma_1) + M(\Gamma_2) \leq m(x, z) + m(z, y).$$

This completes the proof of Theorem 1. □

3. THEOREM 2. *Let G be a domain of finite connectivity on the sphere. Let $x, y \in G$. Let C_x be a (relatively) closed subset of G with $x \in C_x$ and $ClC_x \cap \partial G \neq 0$, C_y a similar subset of G with $y \in C_y$ and $ClC_y \cap \partial G \neq 0$. Let $\Delta(C_x, C_y, G)$ denote the family of all curves in G joining C_x and C_y, $M(\Delta(C_x, C_y, G))$ its module. Set*

$$\lambda_G(x, y) = \text{g.l.b.} M(\Delta(C_x, C_y, G))$$

taken over the above configurations. Then $\lambda_G(x,y)^{-1}$ *is a metric on* G.

Proof. Evidently we may assume C_x, C_y disjoint. Then the extremal metric for the module problem defining $M(\Delta(C_x, C_y, G))$ is given by $|\mathrm{grad} u||dz|$ where u is a bounded harmonic function on G with boundary values 0 on C_x, 1 on C_y and vanishing normal derivative on the remaining hyperbolic boundary of G in terms of border uniformizers (see [**1**], p.65, where a highly detailed discussion is given or [**3**], p.367). Each level set of u is made up of a Jordan curve or a finite number of arcs joining boundary components of G. The module of the family of level sets is $M(\Delta(C_x, C_y, G))^{-1}$. The level sets determine a family of homotopy classes of corresponding Jordan curves or arcs (for the latter this being understood in the sense indicated in [**5**]). Let $\mathrm{Cl}\, C_x$ meet the boundary component A_x of G, $\mathrm{Cl}\, C_y$ the boundary component A_y. If either is a point boundary they must be disjoint, otherwise they can coincide. Let $\Gamma(x,y)$ denote the family of elements each represented by a selection from homotopy classes possessing the same separating properties as above. It is clear that

$$M(\Delta(C_x, C_y, G))^{-1} \leq M(\Gamma(x,y)).$$

The extremal metric for the module problem determining $M(\Gamma(x,y))$ is given by $|Q(z)|^{1/2}|dz|$ where $Q(z)dz^2$ is a quadratic differential on G negative on hyperbolic border components of G. If A_x or A_y is a point boundary, $Q(z)dz^2$ will have a simple pole there and it will have a trajectory arc with end points at x and A_x or at y and A_y as the case may be. If A_x or A_y is non-degenerate there will be a trajectory arc of $Q(z)dz^2$ tending from x or y to a border element P_x or P_y of A_x or A_y. These play the role of C_x or C_y. These results require a slight extension of the results of [**5**] with quadrangles possibly being replaced by families of quadrangles using the definition of module found in [**6**]. They are readily established by the methods given in [**7**]. Thus $\lambda_G(x,y)^{-1}$ is seen to be the maximum of $M(\Gamma(x,y))$ attained for a family of homotopy classes with elements Γ separating x and a point on a suitable A_x from y and a point on a suitable A_y. If z is a point of G distinct from x and y and Γ_1, Γ_2 are the subsets of Γ separating x from z and y or x and z from y as in the proof of Theorem 1 we have

$$\lambda_G(x,y)^{-1} = m(\Gamma) \leq m(\Gamma_1) + m(\Gamma_2) \leq \lambda_G(x,z)^{-1} + \lambda_G(z,y)^{-1}.$$

This completes the proof of Theorem 2. $\square$

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