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THE SPHERICAL MEAN VALUE OPERATOR FOR COMPACT SYMMETRIC SPACES

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When M is a compact symmetric space, the spherical mean value operator L_r (for a fixed $r > 0$) acting on $L^2(M)$ is considered. The eigenvalues λ for $L_r f = \lambda f$ are explicitly determined in terms of the elementary spherical functions associated with the symmetric space. Alternative proofs are also provided for some results of T. Sunada regarding the special eigenvalues $+1$ and -1 using a purely harmonic analytic point of view.

1. Introduction. In a series of papers ([**Su1**, **Su2**, **Su3**]) T. Sunada has considered (among other things) the "spherical mean operator" of a fixed radius r on a compact Riemannian manifold Y and has examined its connections with the so-called 'Geodesic Random Walk' problem. If $r > 0$, the spherical mean operator L_r is defined on $L^2(Y)$ by:

$$(L_r f)(x) = \int_{\{X \in T_x(Y): \|X\|=1\}} f(\mathrm{Exp}_x rX) d\sigma(X).$$

(Here $T_x(Y)$ is the tangent space at $x \in Y$ equipped with the inner product arising from the Riemannian structure, Exp_x the exponential map from $T_x(Y)$ into Y and $d\sigma$ the normalized measure on the surface of the unit sphere in $T_x(Y)$.) Roughly speaking $(L_r f)(x)$ is the mean value of f at a geodesic distance r from x. This note grew out of an attempt to understand the results of Sunada from a group theoretic/harmonic analytic point of view. In fact we show that for symmetric spaces of the compact type the ergodicity and eigenvalue problems considered in [**Su1**] are consequences of simple and elementary arguments (Propositions 2.4 and 2.5). This point of view also sheds some light on the difference between spheres, symmetric spaces of rank 1 and higher rank spaces.

The outline of the paper is as follows: L_r is given by a convolution operator with a K-biinvariant measure and consequently the eigenvalues are $\psi_\pi(r)$ (see Section 2). Sunada's results about ergodicity of L_r follow from a simple group theoretic fact (Proposition 2.4) and Proposition 2.5. Furthermore, these arguments also imply that in most cases -1 is not an eigenvalue.

2. The main results. Let Y be a symmetric space of the compact type and $G = I_0(Y)$ the connected component of the group of isometries of Y. Let $q_0 \in Y$ and $K = \{k \in G : k \cdot q_0 = q_0\}$. Then G is semi-simple and compact, (G, K) is a symmetric pair (of the compact type) and Y can be identified with G/K - see **[H1]** for the definition and details. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the Lie algebra of K and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated "Cartan decomposition" (see **[H1]** for details). Then the geodesics through q_0 are precisely $\gamma_X(t) = (\exp tX)q_0$, $X \in \mathfrak{p}$ (where exp is the exponential map on the Lie algebra $\mathfrak{g}$) and $\mathfrak{p}$ can be naturally identified with the tangent space $T_{q_0}(Y)$ of Y at q_0. ($\mathfrak{p}$ can be equipped with the inner product arising from the Killing form restricted to $\mathfrak{p}$ and this in turn gives the Riemannian structure on $Y = G/K$.) Let $q = g \cdot q_0$, $g \in G$ and Exp_q the 'Exponential map' at q of the Riemannian manifold Y. If $r > 0$, it follows from the identification made above that for any $f \in C(Y)$

$$(L_r f)(q) \stackrel{\text{def}}{=} \int_{\{X:\|X\|_{T_q(M)}=1\}} f(\operatorname{Exp}_q rX)\,d\sigma(X) =$$

$$\int_{\{X\in\mathfrak{p}:\|X\|_{\mathfrak{p}}=1\}} f(g \exp rX \cdot q_0)\,d\sigma(X).$$

In the first integral $d\sigma(X)$ denotes the normalized surface measure on the unit sphere in $T_q(Y)$ and in the second integral it is the normalized surface measure on the unit sphere in $\mathfrak{p}$. L_r is thus a linear map of $C(Y)$ into $C(Y)$ and will be called the spherical mean value operator corresponding to r. Actually, L_r extends to a linear bounded self adjoint operator from $L^2(Y)$ to $L^2(Y)$. This will be clear from the discussion to follow later. (Actually, generically speaking, L_r will even be a compact operator - see **[Su1]** - but we will not need this fact in this note.)

We will now describe some facts from elementary harmonic analysis on compact symmetric spaces. A good source for this is **[H2]**.

Equip Y with the canonical Riemannian measure which in this case is just the canonical G-invariant measure on G/K. By $L^p(Y)$ we mean the set of L^p-functions with respect to this measure. A function f (or a measure μ) on $Y = G/K$ will sometimes be viewed as a function (or measure) on G which is right invariant under K. Thus we will sometimes view $C(Y)$, $L^p(Y)$ etc as subspaces of $C(G)$, $L^p(G)$ etc.

Fix $r > 0$. We will now associate with r a certain specific probability measure ν on Y as follows:

$$\nu(f) = \int_{\{X \in \mathfrak{p}: \|X\|=1\}} f(\exp rX \cdot q_0) d\sigma(X), \; f \in C(Y).$$

Thus ν can be viewed as a right K-invariant probability measure on G. Due to the properties of the Cartan decomposition, one can show that the above probability measure is also left K-invariant. Further one can also show that, considered as a measure on G, ν is invariant under the map $g \mapsto g^{-1}$ of G onto G. Using all this one can easily show that if $f \in C(Y)$, then:

$$L_r f = f * \nu.$$

(On the right hand side the convolution is on the group G and since f is a continuous right K-invariant function and ν is K-biinvariant, $f*\nu$ is right K-invariant and hence can be viewed as a continuous function on Y. Recall that if h is a function on G and μ is a measure, $(h*\mu)(g) = \int h(gx^{-1}) d\mu(x)$.)

We record the above discussion in the form of a lemma:

LEMMA 2.1. *With the identifications described above, $L_r f = f*\nu$, for $f \in C(G/K) = C(Y)$. Thus L_r extends to a bounded self-adjoint operator on $L^2(Y)$.*

(The boundedness is clear because L_r is realized as convolution against a probability measure. That it is self-adjoint follows from the fact ν is real and invariant under $g \mapsto g^{-1}$.)

Notice that if f is a constant function, then f is an eigenfunction for L_r with eigenvalue $+1$. Adopting the terminology in **[Su1]** we have:

DEFINITION 2.2. L_r is said to be *ergodic* iff $L_r = f$, $f \in L^2(Y)$ implies f=constant (a.e). In this case we say $+1$ is a simple eigenvalue of L_r.

In order to state the main consequence of Lemma 2.1, we need to introduce some terminology from harmonic analysis. Let π be an irreducible unitary representation of G on a (finite dimensional) Hilbert space H. Then π is said to be class-1 if $\exists 0 \neq v_0 \in H$ such that $\pi(k)v_0 = v_0$, $\forall k \in K$. It is known that if π is a class-1 representation, then $\dim H_0 = 1$, where
$H_0 = \{v : \pi(k)v = v, \forall k \in K\}$. $\widehat{G}_1$ will denote the collection of pairwise inequivalent irreducible unitary (finite dimensional) class-1 representations of G. Let $\pi \in \widehat{G}_1$ and H be a Hilbert space on which π acts. Let $v_0 \in H_0$, $\|v_0\| = 1$ and let ϕ_π be defined by: $\phi_\pi(g) = \langle v_0, \pi(g)v_0\rangle$. Then we will call ϕ_π *the* elementary spherical function associated with π. (Notice if $v_0' \in H_0$, $\|v_0'\| = 1$, then $\langle \pi(g)v_0, v_0\rangle = \langle \pi(g)v_0', v_0'\rangle$, since $\dim H_0 = 1$.) Further ϕ_π is a K-biinvariant continuous (in fact real analytic) function and $\phi_\pi(e) = 1$. If $\pi \in \widehat{G}_1$ and μ is a K-biinvariant measure on G, define $C_{\mu,\pi}$ by

$$C_{\mu,\pi} = \int_G \phi_\pi(x)d\mu(x).$$

The function $\pi \to C_{\mu,\pi}$ defined on $\widehat{G}_1$ is called the "spherical Fourier transform" of the measure μ.

We now record a fact from the harmonic analysis on G/K: Let T_μ be the bounded linear operator on $L^2(G/K)$ defined by $T_\mu(f) = f*\mu$. Then by Frobenius reciprocity it follows that each $\pi \in \widehat{G}_1$ "occurs" exactly once in the decomposition of $L^2(G/K)$ under the left regular action of G. If we denote the subspace of $L^2(G/K)$ corresponding to π^c by $L^2(G/K)_\pi$, then T_μ acts as the scalar $C_{\mu,\pi}$ on this space. From this it follows that the eigenvalues of T_μ are precisely $C_{\mu,\pi}|_{\pi\in\widehat{G}_1}$. (Here π^c is the irreducible representation contragredient to π.)

Now let ν be the specific K-biinvariant measure associated with L_r described earlier. Then:

$$C_{\nu,\pi} = \int_{\{X\in\mathfrak{p}:\|X\|=1\}} \phi_\pi(\exp rX \cdot q_0)d\sigma(X).$$

(Note that here we are viewing ϕ_π as a function on $Y = G/K$.)

For $t \geq 0$, let ψ_π be the function defined by

$$\psi_\pi(t) = \int_{\{X\in\mathfrak{p}:\|X\|=1\}} \phi_\pi(\exp tX \cdot q_0)d\sigma(X).$$

Thus $\psi_\pi(t)$ is the average value of ϕ_π on $\{\exp tX \cdot q_0\}$. Note that if Y is a rank-1 symmetric space then $\psi_\pi(t) = \phi_\pi(\exp tX_0)$ for *any* $X_0 \in \mathfrak{p}$ with $\|X_0\| = 1$. This is because in this case, K acts transitively on the unit sphere of $\mathfrak{p}$. (Also in this case a complete list of rank-1 compact symmetric spaces is available and for these spaces one can get explicit expressions for ϕ_π and ψ_π in terms of well-known special functions.)

We now return to the main question discussed in the introduction: Fix $r > 0$ and consider the eigenvalue problem $L_r f = \lambda f$, $f \in L^2(G/K)$. Let

$$\operatorname{Eig}(L_r) = \{\lambda : \exists 0 \neq f \in L^2(G/K) \text{ such that } L_r f = \lambda f\}.$$

Lemma 2.1 and the preceding discussion immediately yield:

PROPOSITION 2.3. $\operatorname{Eig}(L_r) = \{\psi_\pi(r) : \pi \in \hat{G}_1\}$.

REMARK. To the harmonic analysts among the readers this Proposition should not come as a surprise at all.

Next we would like to take up the question of the special eigenvalues $+1$ and -1. Before that we need some preliminary results. The first proposition is a simple group theoretic lemma and is well-known in the 'folklore'. We therefore omit the proof.

PROPOSITION 2.4. *Let L be a* compact *group and μ a probability measure on L. Assume that the group generated by* $\operatorname{supp}\mu$ *is dense in L. (Here* $\operatorname{supp}\mu$ *denotes the (closed) support of μ.) Then if $f \in L^1(L)$ and $f*\mu = f$ or $\mu*f = f$, then $f =$* const*(a.e).*

We now come to one of the main results of this section. Fix $r > 0$ and let μ be the specific probability measure on Y introduced earlier associated with L_r. We again think of ν as a right K-invariant probability measure on G. Then we have the following crucial observation:

PROPOSITION 2.5. *Let the rank of Y considered as a compact symmetric space be greater than 1. Then for any $r > 0$, the group generated by* $\operatorname{supp}\nu$ *is dense in G.*

Proof. As usual let $\mathfrak{g}$=Lie algebra of G, $\mathfrak{k}$=Lie algebra of K, $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form and $\mathfrak{g} =$

$\mathfrak{k} \oplus \mathfrak{p}$ the 'Cartan decomposition' (see [**H1**]). Let $\pi : G \to G/K = Y$ be the canonical map. Then $d\pi$ is an isometry from $\mathfrak{p}$ to $T_{q_0}(Y)$ and further $\pi(\exp X) = \mathrm{Exp}_{q_0}(d\pi(X))$ for $X \in \mathfrak{p}$. Thus it follows that if $'S_r$ denotes the sphere of radius r in $T_{q_0}(Y)$, then $\mathrm{Exp}_{q_0}[S_r]) = \pi\{\exp X : \|X\| = r\}$. Hence $\pi^{-1}(\mathrm{Exp}_{q_0}(S_r)) = \exp[S_r^{\mathfrak{p}}]K$ (where $S_r^{\mathfrak{p}}$ denotes the sphere of radius r in $\mathfrak{p}$). This is clearly $\mathrm{supp}\,\nu$ in G. Thus it suffices to prove that the group generated by $\exp(S_r^{\mathfrak{p}})K$, denoted by $\mathrm{gp}([\exp S_r^{\mathfrak{p}}]K)$ is dense in G. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Then $\dim \mathfrak{a} = \mathrm{rank}\, Y > 1$. Let A be the analytic (abelian) subgroup of G corresponding to $\mathfrak{a}$. Let $A_1 = \overline{A}$. (Remark: in the case of *non*-compact symmetric space A obtained as above will be closed but in the compact situation, this may not be the case - that is why we are forced to take the closure.) Clearly $A = \exp \mathfrak{a}$ and $A_1 = \overline{\exp \mathfrak{a}}$. Clearly A_1 is a torus in G, $\dim A_1 \geq \dim A \geq 2$. Let $\mathfrak{a}_1$ be the Lie algebra of A_1. Then $\mathfrak{a}_1$ is abelian and $\mathfrak{a}_1 \supset \mathfrak{a}$. Let $\mathfrak{l}$ be the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{a}_1$. Then one can show using the properties of the Cartan decomposition that $\mathfrak{l} \cap \mathfrak{p} = 0$ and in fact $\mathfrak{l}$ is orthogonal to $\mathfrak{p}$ and hence $\mathfrak{l} \subseteq \mathfrak{k}$. Now since A_1 is a torus of dimension $m \geq 2$, by Kronecker's theorem, for any $X \subset \mathfrak{a}_1$ whose coordinates (with respect to an *o.n.* basis) $(x_1, ..., x_m)$ are such that $1, x_1, ..., x_m$ are rationally independent, $\exp X$ generates a dense subgroup of A_1 (see [**CFS**]). Now consider the subset $S = \{X_1 + X_2 : X_1 \in S_r^{\mathfrak{a}}, X_2 \in \mathfrak{l}\}$, i.e. the cylinder on $S_r^{\mathfrak{a}}$ in $\mathfrak{a}_1$. Then by an elementary measure theoretic argument one can show that there exists $X \in S$ whose coordinates $(x_1, ..., x_m)$ are such that $1, x_1, ..., x_m$ are rationally independent. (Note that $\dim S_r^{\mathfrak{a}} \geq 1$.) Hence $A_1 = \overline{\mathrm{gp}[\exp(S)]}$ and $kA_1k^{-1} = \overline{\mathrm{gp}(k[\exp(S)]k^{-1})}$, $\forall k \in K$. Now one may write any $X \in S$ as $X = X_1 + X_2$, $X_1 \in S_r^{\mathfrak{a}}$, $X_2 \in \mathfrak{l}$ and since $[X_1, X_2] = 0$ we have $(\exp X)^m = (\exp X_1)^m (\exp X_2)^m$.

So $\mathrm{gp}(\exp X) \subset \mathrm{gp}(\exp X_1)K$, $\forall X \in S$. Thus

$$k\,\mathrm{gp}(\exp X)k^{-1} \subset \mathrm{gp}(k[\exp(X_1)]k^{-1})K, \ \forall X \in S, \ k \in K,$$

i.e. $k\,\mathrm{gp}(\exp S)k^{-1} \subset \mathrm{gp}[k(\exp S_r^{\mathfrak{a}})k^{-1}]K$.
But $k(\exp S_r^{\mathfrak{a}})k^{-1} = \exp(\mathrm{Ad}\,k \cdot S_r^{\mathfrak{a}}) \subset \exp S_r^{\mathfrak{p}})$ since $S_r^{\mathfrak{a}} \subset S_r^{\mathfrak{p}}$ and $\mathrm{Ad}\,k$ preserves $S_r^{\mathfrak{p}}$, $\forall k \in K$ i.e.

$$k\,\mathrm{gp}[\exp S]k^{-1}K \subset \mathrm{gp}[\exp(S_r^{\mathfrak{p}})]K, \ \forall k \in K.$$

But since $\overline{CK} = \overline{C}K$, $C \subset G$ (G is compact, as is K), one has

$$\overline{k\,\mathrm{gp}[\exp S]k^{-1}K} \subset \overline{\mathrm{gp}[\exp S_r^{\mathfrak{p}}]K}.$$

However

$$\overline{k\,\mathrm{gp}[\exp S]k^{-1}} = k\overline{\mathrm{gp}[\exp(S)]}k^{-1} = kA_1k^{-1}.$$

So

$$kA_1k^{-1}k \subset \overline{\mathrm{gp}[\exp S_r^{\mathfrak{p}}]K}, \ \forall k \in K.$$

Now $G = \bigcup_k (kAk^{-1})K$ by Theorem 6.7 Ch.V in **[H1]** and $A \subseteq A_1 \implies G = \bigcup_k (kA_1k^{-1})K$. So $G = \overline{\mathrm{gp}[\exp S_r^{\mathfrak{p}}]K}$. But note that $\mathrm{gp}[\exp S_r^{\mathfrak{p}}]K \subseteq \mathrm{gp}[\exp S_r^{\mathfrak{p}}K]$ and so $G = \overline{\mathrm{gp}[\exp S_r^{\mathfrak{p}}K]}$, which is what we wanted and the proof of the proposition is complete. □

REMARK. It is instructive to see what the above proposition says in the group situation i.e. let L be a compact, connected Lie group such that the dimension of the maximal torus of L is strictly greater than 1. Fix a biinvariant Riemannian structure on L and let $\mathfrak{l}$ be the Lie algebra of L. Fix $r > 0$ and let $S_r = \{\exp rX : X \in \mathfrak{l}, \|X\| = 1\}$. Then the group generated by S_r is dense in L. Of course this follows from Proposition 2.5, thinking of L as the symmetric space $(L \times L)/\Delta$ where Δ is the diagonal $\{(g,g) : g \in L\}$. However the proof of this fact is much simpler and more direct though one uses the same basic idea as in the proof of Proposition 2.5.

An immediate consequence of Proposition 2.4 and Proposition 2.5 is a theorem of Sunada:

THEOREM 2.6 **[Su1]**. *Let Y be a compact symmetric space of rank greater than 1. Then for any $r > 0$, L_r is ergodic (i.e. $+1$ is a simple eigenvalue).*

Proof. For $f \in L^2(Y)$, $L_r f = f{*}\nu$ by lemma 2.1. Now use propositions 2.4 and 2.5. □

From this we immediately have another result of Sunada:

THEOREM 2.7 **[Su1]**. *Let r and Y be as above. Then -1 is not an eigenvalue of L_r i.e. $f \in L^2(Y)$ and $L_r f = -f$ implies $f = 0$ (a.e.).*

Proof. $L_r f = -f \implies f{*}\nu = -f$. Thus $f{*}(\nu{*}\nu) = -f{*}\nu = -(-f) = f$. Now since ν is a probability measure, it is easy to show $\operatorname{supp}(\nu*\nu) = (\operatorname{supp}\nu)\cdot(\operatorname{supp}\nu)$. G is a compact, connected Lie group and it follows easily that since the closure of the group generated by $\operatorname{supp}\nu$ in G (Prop. 2.5), the closure of the group generated by $(\operatorname{supp}\nu)\cdot(\operatorname{supp}\nu)$ is also G. Hence by Proposition 2.4, $f = \text{const}(a.e.)$. But now $L_r f = -f$ implies $f = 0$ (a.e.). $\square$

We now take up the case of *compact symmetric space of rank-one.*

We first need to make a few observations. Let $\mathfrak{k}$ be a Lie algebra of K. Then since $\operatorname{rank}(G/K) = 1$, it follows that $\mathfrak{k}$ is a maximal Lie sub-algebra of $\mathfrak{g}$. (This is true for any irreducible symmetric pair $(\mathfrak{g}, \mathfrak{k})$ and so in particular for rank-1 pairs.) We use this to see that if S is any closed submanifold of G/K such that $\dim S \geq 1$ and $\pi : G \to G/K$ is the canonical map, then the group generated by $\pi^{-1}(S)$ is dense in G. For if L is the closure of this group then $\dim L > \dim K$, because L will contain a subset homeomorphic to something of the form $U_1 \times U_2$, U_1 a neighborhood in K and U_2 a neighborhood in $S \subset G/K$. Now L clearly contains K and so if $\mathfrak{l}$ is the Lie algebra of L, we would have

$\dim \mathfrak{l} > \dim \mathfrak{k}$. Therefore $\mathfrak{l} = \mathfrak{g}$ by the maximality of $\mathfrak{k}$.

Next, in the case of rank-1 compact symmetric spaces all geodesics are closed and have the same length $= 2L$, say. Now consider $S_r = \{\exp rX \cdot q_0 : X \in \mathfrak{p}, \|X\| = 1\}$. If $r < L$, then one knows from [**H1**] that S_r is a diffeomorphic copy of $\{rX \in \mathfrak{p} : \|X\| = 1\}$. If $r = L$, S_L is the so-called antipodal manifold to q_0 and we have to consider two cases: (a) Y =Sphere - In this case S_L is a single point; (b) Y is *not* a sphere. In this case S_L is a proper submanifold of $Y = G/K$. (These facts follow from Theorem 10.3, Ch.VII in [**H1**].) Putting all the above discussion together we have: (i) if Y is a sphere the group generated by $\pi^{-1}(S_r)$ is dense in G if $r \notin \{L, 2L, 3L, ...\}$. Thus, since clearly $\operatorname{supp}\nu = \pi^{-1}(S_r)$, in this case the group generated by *suppν* is dense in G *(ii)* If Y is not a sphere, the group generated by $\pi^{-1}S_r$ is dense in G if $r \notin \{2L, 4L, 6L, ...\}$. Again in this case the group generated by $\operatorname{supp}\nu$ is dense in G.

We have thus established.

THEOREM 2.8 [**Su1**]. *Let $Y = G/K$ be a compact symmetric space of rank - 1. (a) If Y is the k-sphere (in $\mathbb{R}^{k+1}$, $k = 2, 3, ...$), then for $r \notin \{L, 2L, 3L, ...\}$, $+1$ is a simple eigenvalue of L_r and -1 is not an eigenvalue of L_r. (b) If Y is not a sphere, then for*

$r \notin \{2L, 4L, ...\}$, $+1$ is a simple eigenvalue of L_r and -1 is not an eigenvalue of L_r.

Our approach to the proofs of Theorems 2.6 and 2.8 "explains" why there is a difference between spheres and other rank-1 spaces and also why there is a difference between rank-1 spaces and spaces of rank greater than 1.

Finally, in conclusion, we would like to point out that the question of whether 0 is an eigenvalue is also of independent interest and is related to the so-called Pompeiu problem (see for instance [**BeZ**]). In fact, Badertscher ([**Ba**]) also views L_r as a convolution operator in order to analyze the Pompeiu problem on locally symmetric spaces. The spherical mean value operator has also been considered for the Heisenberg group (see for instance [**T**]).

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