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# ON BANACH SPACES $Y$ FOR WHICH $B(C(\Omega), Y) = K(C(\Omega), Y)$

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**Let $\Omega$ be a compact Hausdorff space. In this paper we give some necessary conditions and some sufficient conditions on a Banach space $Y$ in order that all continuous linear operators from $C(\Omega)$ into $Y$ are compact. We prove that for a nonscattered compact Hausdorff space $\Omega$, for $Y$ belonging to a large class of Banach spaces all operators from $C(\Omega)$ into $Y$ are compact if and only if all operators from $l^2$ into $Y$ are compact.**

**Introduction.** In this paper by the word "operator" we will mean a "continuous linear operator." E. Dubinsky, A. Pelczynski, and H.P. Rosenthal [**8**] have given a characterization of all Banach spaces $Y$ for which all operators from $\mathcal{L}_\infty$ into $Y$ are absolutely 2-summing. Here, our aim is to characterize all Banach spaces $Y$ for which all operators from a $C(\Omega)$-space into $Y$ are compact. We noticed that such a characterization depends on whether the compact Hausdorff space $\Omega$ is scattered (dispersed) or nonscattered (nondispersed). So we consider two cases separately.

Case 1: $\Omega$ is an infinite scattered compact Hausdorff space. In this case, from some known results we deduce that *all operators from $C(\Omega)$ into a Banach space $Y$ are compact if and only if all operators from a closed subspace of $c_0$ into $Y$ are compact if and only if $Y$ does not contain a copy of $c_0$.*

Case 2: $\Omega$ is a nonscattered compact Hausdorff space. In this case, we present a necessary condition on a Banach space $Y$ for all operators from $C(\Omega)$ into $Y$ to be compact. Specifically, *if each operator from $C(\Omega)$ into $Y$ is compact, then each operator from $l^2$ into $Y$ is compact.* Consequently, *for a Banach space $Y$ for which each operator from $C(\Omega)$ into $Y$ is absolutely 2-summing, each operator from $C(\Omega)$ into $Y$ is compact if and only if each operator from $l^2$ into $Y$ is compact.* Another necessary condition is given by a theorem of T. Terzioglu. Namely, *if each operator from $C(\Omega)$ into $Y$ is compact, then each operator from $C(\Omega)$ into $Y$ factors through a closed subspace of $c_0$*. Next, we see that the above two necessary conditions together are also sufficient. Putting together: *Each operator from $C(\Omega)$ into $Y$ is compact if and only if each operator from $l^2$ into $Y$ is compact and each operator from $C(\Omega)$ into $Y$ factors through a closed subspace of $c_0$.*

In order to prove that another related condition is also sufficient we first generalize a theorem of N.J. Kalton. Then, employing this generalization, and a result of L. Drewnowski we prove: *Each operator from $C(\Omega)$ into $Y$ is compact if and only if each operator from $l^2$ into $Y$ is compact and each operator form $C(\Omega)$ into $Y$ has a weak unconditional compact netted expansion* (Definition 3.5). Consequently, for a Banach space $Y$ with an unconditional basis consisting of finite dimensional subspaces *all operators from $C(\Omega)$ into $Y$ are compact if and only if all operators from $l^2$ into $Y$ are compact.* The conclusion is that the class of all Banach spaces $Y$ for which all operators from $C(\Omega)$ into $Y$ are compact if and only if all operators from $l^2$ into $Y$ are compact is big (see Conclusion 3.12).

In the way we present a necessary and sufficient condition on a Banach space $Y$ for all operators from $l^p$ into $Y$ to be compact for each $p \in [1, \infty)$. We conclude this paper with some results that relate the space of all compact operators on $C(\Omega)$ with the space $\Phi_{c_0}(C(\Omega))$ for all operators factoring through $c_0$.

**1. Notations.** Suppose $X$ and $Y$ are Banach spaces. We will denote the space of all bounded linear operators, compact operators, and absolutely 2-summing operators from $X$ into $Y$ by $B(X,Y)$, $K(X,Y)$, and $\Pi_2(X,Y)$, respectively. By "$X \hookrightarrow Y$" we will mean "$Y$ contains a copy of $X$."

**1.1. Scattered-Compact Spaces.** Recall that a topological space $S$ is said to be **scattered** or **dispersed** if every nonempty closed subset of $S$ has an isolated point in its induced topology (see [**22**]). In this section we will assume that $S$ is a scattered compact Hausdorff space.

PROPOSITION 1.1. *Suppose $X$ is an infinite dimensional closed subspace of $c_0$ and $Y$ is a Banach space. Then, $B(X,Y) = K(X,Y)$*

*if and only if Y does not contain any copy of $c_0$.*

*Proof.* Suppose $Y$ does not contain any copy of $c_0$. Let $T \in B(X,Y)$. Let $\{x_n\}$ be any norm bounded sequence in $E$. We will show that $\{Tx_n\}$ has a norm convergent subsequence. Since $c_0$ does not contain any copy of $l^1$, the space $E$ does not contain any copy of $l^1$. So by the celebrated $l^1$-theorem of H.P. Rosenthal [**20**], a subsequence of $\{x_n\}$ is weakly Cauchy. By passing to the subsequence we can assume that the $\{x_n\}$ itself is weakly Cauchy. Let $y_{m,n} = x_n - x_m$. Then the net $\{y_{m,n}\}$ is weakly null. So is the net $\{Ty_{m,n}\}$. We claim that $\|Ty_{m,n}\| \longrightarrow 0$. To arrive at a contradiction suppose this is not the case. Then there exists an $\epsilon > 0$ and sequences $\{m_k\}$ and $n_k\}$ of natural numbers such that $m_k > m_{k-1} \geq k-1$, $n_k > n_{k-1} \geq k-1$, and $\|Ty_{m_k,n_k}\| > \epsilon$. Now by a theorem of C. Bessaga and A. Pelczynski [**4**] a subsequence of $Ty_{m_k,n_k}$ itself is a basic sequence. Since $y_{m_k,n_k}$ is a weakly null sequence in $c_0$ such that $\inf \|y_{m_k,n_k}\| > 0$, a subsequence of this sequence is a basic sequence and a subsequence of the basic sequence is equivalent to a block basis of the standard basis of $c_0$. Since every normalized block basis of the standard basis is equivalent to the standard basis, it follows that a subsequence of $\{y_{m_k,n_k}\}$ is equivalent to the standard basis. By passing to the subsequence we can assume that $\{y_{m_k,n_k}\}$ itself is such a sequence. That is, $\{y_{m_k,n_k}\}$ is equivalent to the standard basis of $c_0$. Now it is easy to verify that $\sum a_k y_{m_k,n_k}$ converges if and only if $\sum a_k T y_{m_k,n_k}$ does. So, the subspace $[Ty_{m_k,n_k}]$ of $Y$ is isomorphic to $c_0$. This contradicts the hypothesis. The converse is obvious. □

The next result is a corollary of some known results and Proposition 1.1.

COROLLARY 1.2. *For a Banach space Y the following are equivalent*

(a) *For all infinite scattered compact Hausdorff spaces $S$, we have $B(C(S),Y) = K(C(S),Y)$.*
(b) *For some infinite scattered compact Hausdorff space $S$, we have $B(C(S),Y) = K(C(S),Y)$.*
(c) *$Y$ does not contain a copy of $c_0$.*
(d) *For all infinite dimensional subspaces $X$ of $c_0$, we have*

$B(X,Y) = K(X,Y)$.

(e) *For some infinite dimensional subspace $X$ of $c_0$, we have* $B(X,Y) = K(X,Y)$.

*Proof.* (a) $\Rightarrow$ (b) This is obvious.

(b) $\Rightarrow$ (c) By way of contradiction, suppose that $Y$ contains a copy of $c_0$. Since $S$ is an infinite scattered space, there exists a complemented subspace $M$ of $C(S)$ isomorphic to $c_0$ see [**19**, p. 201]). Let $P$ be the projection of $C(S)$ onto $M$ and $T$ be an isomorphism of $M$ onto an isomorphic copy of $c_0$ in $Y$. Then $TP \in B(C(S), Y)$ is a noncompact operator. This contradiction proves *(c)*.

(c) $\Rightarrow$ (a) Let $S$ be an arbitrary infinite scattered compact Hausdorff space. Let $T \in B(C(S), Y)$ be arbitrary. Since $Y$ does not contain any copy of $c_0$, by a result of A. Pelczynski [**17**], the operator $T$ is weakly compact. So, its adjoint $T^* : Y^* \longrightarrow C(S)^*$ is weakly compact. By a well known theorem of W. Rudin [**21**], or (see [**22**, Corollary 19.7.7]), we have $C(S)^* \cong l^1(S)$. By a theorem of Schur (see [**22**, p. 338]), the space $l^1(S)$ has the Schur property. So, $T^*$ is compact. Hence, $T$ is compact.

(c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) This is Proposition 1.1. □

COROLLARY 1.3 (Pitt). *For $1 \leq p < \infty$, we have $B(c_0, l^p) = K(c_0, l^p)$.*

*Proof.* We know that $c_0 \cong C(S)$ for the infinite scattered compact Hausdorff space $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. We also know that $l^p$ does not contain any copy of $c_0$. So, by Corollary 1.2, we have $B(c_0, l^p) = K(c_0, l^p)$. □

**1.2. $l^p_w$-Sequences.** This section gives a complete characterization of all Banach spaces $Y$ (in terms of $l^q_w$-sequences) for which $B(X,Y) = K(X,Y)$ for $X = c_0$ or $l^p$ $(1 \leq p < \infty)$. The results for $X = c_0$ and $l^2$ are already known. We fill in the gap by giving the characterization in the case $X = l^p$ for $1 \leq p < \infty$. This ties the results for $c_0$, $l^2$, and $l^p$ $(p \neq 2)$ together.

Recall that a sequence $\{y_n\}$ of elements in a Banach space $Y$ is said to be a **weak $l^p$-sequence**, or in short an **$l^p_w$-sequence** in $Y$, where $p \in [1, \infty)$, if for every $f \in Y^*$ we have $\sum_{n=1}^{\infty} |f(y_n)|^p < \infty$. The set of all $l^p_w$-sequences of a Banach space $Y$ is denoted by $l^p_w(Y)$

(see [**6**]). For any real number $p > 1$, we denote the number $p/(p-1)$ by $q$. Note that $1/p + 1/q = 1$.

REMARK. (a) If $\{y_n\} \in l_w^p(Y)$, $p \geq 1$, then $\{y_n\} \in l_w^r(Y)$ for any $r \geq p$.

(b) If $\{e_n\}$ is the standard unit vector basis of $l^p$, $1 < p < \infty$, then $\{e_n\} \in l_w^q(l^p)$.

(c) If $\{e_n\}$ is the standard unit vector basis of $c_0$, then $\{e_n\} \in l_w^1(c_0)$.

The next proposition is motivated by [**3**] and [**4**].

PROPOSITION 2.1. *If $\{y_n\}$ is a sequence in a Banach space $Y$ and $1 < p < \infty$, then the following three conditions are equivalent.*

(a) *The sequence $\{y_n\} \in l_w^p(Y)$.*

(b) *The series $\sum_{n=1}^{\infty} a_n y_n$ converges unconditionally for all $\{a_n\} \in l^q$.*

(c) *There exists an operator $T \in B(l^q, Y)$ such that $Te_n = y_n$, where $\{e_n\}$ is the standard unit vector basis of $l^q$.*

*Proof.* (*a*) $\Rightarrow$ (*b*) We suppose that $\{y_n\} \in l_w^p(Y)$, that is, $\{f(y_n)\} \in l^p$ for each $f \in Y^*$. First define a linear operator $S : Y^* \longrightarrow l^p$ by $Sf = \{f(y_n)\}$ for $f \in Y^*$. We will use the closed graph theorem to prove continuity of $S$. So suppose $\{f_n \oplus Sf_n\}$ is a Cauchy sequence in the product space $Y^* \oplus l^p$. Then both $\{f_n\}$ and $\{Sf_n\}$ are Cauchy sequences in $Y^*$ and $l^p$, respectively. Let $f_n \longrightarrow f \in Y^*$. We will show that $Sf_n \longrightarrow Sf$. For every $\epsilon > 0$ there exists a natural number $n_0$ such that $\|Sf_i - Sf_j\|_p < \epsilon$ for all $i, j > n_0$. That is, $\sum_{n=1}^{\infty} |f_i(y_n) - f_j(y_n)|^p < \epsilon^p$ for all $i, j > n_0$. In particular, $\sum_{n=1}^{N} |f_i(y_n) - f_j(y_n)|^p < \epsilon^p$, for all natural numbers $N$ and all natural numbers $i, j > n_0$. By letting $j \longrightarrow \infty$ we get $\sum_{n=1}^{N} |f_i(y_n) - f(y_n)|^p \leq \epsilon^p$. Since this holds for all natural numbers $N$ we get

$$\|Sf_i - Sf\|_p^p = \sum_{n=1}^{\infty} |f_i(y_n) - f(y_n)|^p \leq \epsilon^p$$

for all $i > n_0$. So, $Sf_n \longrightarrow Sf$ in norm. Hence, $S$ is continuous.

Now let $\{a_n\} \in l^q$ be arbitrary, $f \in Y^*$ be such that $\|f\| = 1$, and

$i, j$ be any natural numbers. Then

$$\begin{aligned}\left\| f\left(\sum_{n=1}^{j} a_n y_n\right)\right\| &= \left|\sum_{n=1}^{j} a_n f(y_n)\right| \\ &= \left|\{0, \ldots, 0, a_i, \ldots, a_j, 0, 0, \ldots\} S(f)\right| \\ &\leq \left(\sum_{n=1}^{j} |a_n|^q\right)^{\frac{1}{q}} \|S\|,\end{aligned}$$

where $(0, \ldots, 0, a_i, \ldots, a_j, 0, 0, \ldots)$ is treated as an element of $(l^p)^*$. So,

$$\sup_{\|f\|\leq 1}\left| f\left(\sum_{n=1}^{j} a_n y_n\right)\right| \leq \left(\sum_{n=i}^{j} |a_n|^q\right)^{\frac{1}{q}} \|S\|.$$

Since

$$\sup_{\|f\|\leq 1}\left| f\left(\sum_{n=i}^{j} a_n y_n\right)\right| = \left\|\sum_{n=1}^{j} a_n y_n\right\|,$$

we obtain

$$\left\|\sum_{n=i}^{j} a_n y_n\right\| \leq \left(\sum_{n=i}^{j} |a_n|^q\right)^{\frac{1}{q}} \|S\|, \tag{1}$$

for all natural numbers $i$, $j$. Since $\{a_n\} \in l^q$, $\left(\sum_{n=i}^{j} |a_n|^q\right)^{\frac{1}{q}} \longrightarrow 0$ as $n \longrightarrow \infty$. So, $\left\|\sum_{n=i}^{j} a_n y_n\right\| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, the series $\sum_{n=1}^{\infty} a_n y_n$ converges. Since $\{a_n\} \in l^q$ implies $\{\epsilon_n a_n\} \in l^q$, for any sequence $\{\epsilon_n\}$ of numbers $+1$ and $-1$, we certainly have that the series $\sum_{n=1}^{\infty} \epsilon_n a_n y_n$ converges. That is, the series $\sum_{n=1}^{\infty} a_n y_n$ converges unconditionally in $Y$.

(b) $\Rightarrow$ (c) Define the operator $T : l^q \longrightarrow Y$ by $T(\{a_n\}) = \sum_{n=1}^{\infty} a_n y_n$. Clearly, $T$ is linear and $T(e_n) = y_n$. We will prove that $T$ is bounded. Let $S$ be the bounded linear operator defined above. By letting $i = 1$ and $j \to \infty$ in (1), we obtain $\|\sum_{n=1}^{\infty} a_n y_n\| \leq \|\{a_n\}\| \, \|S\|$. So, $\|T\| \leq \|S\|$.

(c) $\Rightarrow$ (a) Suppose $T \in B(l^q, Y)$ and $T(e_n) = y_n$, for $n = 1, 2, \ldots$. We need to prove that $\{y_n\} \in l^p_w(Y)$. Let $f \in Y^*$ be arbitrary. Then $\sum_{n=1}^{\infty} |f(y_n)|^p = \sum_{n=1}^{\infty} |f \circ T(e_n)|^p < \infty$, because $f \circ T \in (l^q)^*$ and $\{e_n\} \in l^p_w(l^q)$. □

REMARK. On replacing "$l^p_w$" by "$l^1_w$" and "$l^q$" by "$c_0$" in the statement of Proposition 2.1, we obtain a result of C. Bessaga and A. Pelczynski [**4**], whereas on replacing "$l^p_w$" by "$l^2_w$" and $l^q$ by $l^2$ we get a result given in the paper of R. Anantharaman and J. Diestel [**3**].

The next proposition is motivated by a paper of L. Drewnowski [**7**]. Part (c) of the proposition is well known and is included here for the sake of completeness.

PROPOSITION 2.2. *For a Banach space $Y$ and an arbitrary $1 < p < \infty$, the following statements are true.*
(a) *The equality $B(l^p, Y) = K(l^p, Y)$ holds if and only if every $l^q_w$-sequence in $Y$ is a norm null sequence.*
(b) *The equality $B(c_0, Y) = K(c_0, Y)$ holds if and only if every $l^1_w$-sequence in $Y$ is a norm null sequence.*
(c) *The equality $B(l^1, Y) = K(l^1, Y)$ holds if and only if $Y$ is of finite dimension.*

*Proof.* (a) Suppose $B(l^p, Y) = K(l^p, Y)$. Let $\{y_n\}$ be an arbitrary $l^q_w$-sequence in $Y$. By Proposition 2.1, there is an operator $T \in B(l^p, Y)$ such that $T(e_n) = y_n$ for all $n = 1, 2, \ldots$, where $\{e_n\}$ is the standard unit vector basis of $l^p$. By way of contradiction, suppose that $\{y_n\}$ is not norm null. So, there exists a subsequence, say $\{y_{nk}\}$, such that $\|y_{nk}\| > \epsilon$ for some $\epsilon > 0$ and for all $k = 1, 2, \ldots$. Since $\{e_{nk}\}$ is a norm bounded sequence, and $T$ is a compact operator, the sequence $\{Te_{nk}\}$, (i.e., $\{y_{nk}\}$) has a norm convergent subsequence, say $\{y_{nkl}\}$. Suppose $y_{nkl} \xrightarrow{\|\,\|} y \in Y$. Then $y_{nkl} \xrightarrow{w} y\ inY$. Since $\{y_n\}$ is an $l^q_w$-sequence, it is a weakly null sequence. So, $y_{nkl} \xrightarrow{w} 0$. Thus, $y = 0$. Hence, $\|y_{nkl}\| \xrightarrow{\|\,\|} 0$, a contradiction.

For the converse, suppose that every $l^q_w$-sequence of $Y$ is a norm null sequence and take an arbitrary $T \in B(l^p, Y)$. Let $\{x_n\}$ be any norm bounded sequence in $l^p$. We will show that $\{T(x_n)\}$ has a norm convergent subsequence. Since $l^p$ is reflexive, the sequence $\{x_n\}$ has a weakly convergent subsequence. Without loss of generality we can assume that $\{x_n\}$ itself is weakly convergent. Suppose $x_n \xrightarrow{w} x \in l^p$. If $\liminf \|x_n - x\| = 0$, then $\{x_n\}$ has a norm convergent subsequence, and consequently, $\{T(x_n)\}$ has a norm convergent subsequence. So suppose that $\lim \|x_n - x\| > 0$. By the Bessaga-

Pelczynski theorem (see [**6**]), there exists a subsequence of $\{x_n - x\}$ which is a basic sequence. Since $\{x_n - x\}$ is a basic sequence in $l^p$ and $\liminf \|x_n - x\| > 0$, by a theorem of A. Pelczynski [**16**, p. 7], there is a subsequence of $\{x_n - x\}$, which is equivalent to a block basis of the standard basis of $l^p$. Again by passing to a subsequence, we can assume that $\{x_n - x\}$ itself is equivalent to a block basis of the standard basis. Since every block basis of the standard basis of $l^p$ is equivalent to the standard basis (see [**16**]), $\{x_n - x\}$ is equivalent to the standard basis. Since the standard basis is an $l^q_w$-sequence, $\{x_n - x\}$ is an $l^q_w$-sequence. And so $\{T(x_n - x)\}$ is an $l^q_w$-sequence. Consequently, by the hypothesis, $\{T(x_n - x)\}$ is a norm null sequence. That is, $Tx_n \longrightarrow Tx$ in norm. In other words, for every norm bounded sequence $\{x_n\}$ the sequence $\{Tx_n\}$ has a norm convergent subsequence.

(b) Suppose $B(c_0, Y) = K(c_0, Y)$. Let $\{y_n\} \in l^1_w(Y)$ be arbitrary. By Proposition 2.1 there is an operator $T \in B(c_0, Y)$ such that $T(e_n) = y_n$. Note that $\{y_n\}$ converges weakly to zero. So, every subsequence of it converges weakly to zero. Since $T$ is compact, every subsequence of $\{Te_n\}$ (i.e., of $\{y_n\}$) has a subsequence which converges to zero in norm. So, $\{y_n\}$ itself converges to zero in norm.

For the converse, suppose that every $l^1_w$-sequence of $Y$ converges in norm to zero. Notice that the standard unit vector basis $\{e_n\}$ of $c_0$ is an $l^1_w$-sequence, which does not converge to zero in norm. So, $Y$ does not contain any copy of $c_0$. Since $c_0 \cong C(S)$, for some infinite scattered compact Hausdorff space $S$, Corollary 1.2 implies that all operators from $c_0$ into $Y$ are compact.

(c) This follows from the well known fact that every separable Banach space is a quotient of $l^1$. □

NOTE 2.3. For the comparison we mention now the following result that follows from Corollary 3.11. If a Banach space $Y$ has an unconditional basis of finite dimensional subspaces (or more generally, a weak unconditional compact netted expansion of identity), then $B(l_\infty, Y) = K(l_\infty, Y)$ if and only if every $l^2_w$-sequence in $Y$ is a norm null sequence.

COROLLARY 2.4. *Suppose $Y$ is a Banach space and suppose $p \in [1, \infty)$. If $B(l^p, Y) = K(l^p, Y)$, then*
(a) $B(l^r, Y) = K(l^r, Y)$ *for all* $r \in [p, \infty)$ *and*

(b) $B(c_0, Y) = K(c_0, Y)$.

*Proof.* (a) For $p = 1$ the result follows from Proposition 2.2(c). Suppose now that $1 < p \le r < \infty$ and $B(l^p, Y) = K(l^p, Y)$. Then by Proposition 2.2(a) every $l^q_w$-sequence of elements in $Y$ converges to zero in norm. Since $p \le r$ implies that the conjugate number $r'$ satisfies $r' \le q$, we see that every $l^{r'}_w$-sequence of elements in $Y$ is an $l^q_w$-sequence. So, every $l^{r'}_w$-sequence of elements in $Y$ converges to zero in norm. By Proposition 2.2(a), we get $B(l^r, Y) = K(l^r, Y)$.

(b) Since $B(l^p, Y) = K(l^p, Y)$ for some $1 \le p < \infty$, the space $Y$ does not contain any copy of $c_0$. Since $c_0 \cong C(s)$, for some infinite compact scattered Hausdorff space, by Corollary 1.2 we get $B(c_0, Y) = K(c_0, Y)$. □

We conclude this section with the following remark.

REMARK 2.5. For a Banach space $Y$ the following are equivalent.

(a) For all infinite dimensional Hilbert spaces $H$ we have $B(H, Y) = K(H, Y)$.

(b) For some infinite dimensional Hilbert space $H$ we have $B(H, Y) = K(H, Y)$.

(c) We have $B(l^2, Y) = K(l^2, Y)$.

(d) Every $l^2_w$-sequence in $Y$ is a norm null sequence.

**3. Nonscattered-Compact Spaces.** Recall that a topological space $\Omega$ is said to be **nonscattered** or **nondispersed** if $\Omega$ contains a nonempty closed set which has no isolated point in its induced topology. In this section we assume that $\Omega$ is a nonscattered compact Hausdorff space. We begin with a note whose proof is left to the readers.

NOTE 3.1. If $Y$ is a Banach space with the Schur property, then $B(C(\Omega), Y) = K(C(\Omega), Y)$.

THEOREM 3.2. *Let $\Omega$ be a nonscattered compact Hausdorff space, $Y$ be a Banach space. If $B(C(\Omega), Y) = K(C(\Omega), Y)$, then $B(l^2, Y) = K(l^2, Y)$. Furthermore, if $B(C(\Omega), Y) = K(C(\Omega), Y)$, then $B(l^p, Y) = K(l^p, Y)$ for $p \ge 2$.*

*Proof.* By Corollary 2.4 only the case $p = 2$ needs a proof. We proceed by contradiction and assume that $B(l^2, Y) \ne K(l^2, Y)$. Then there is a noncompact operator $T$ in $B(l^2, Y)$. From the proof of Proposition 2.2 it follows that there is a basic sequence $\{u_n\}$ in $l^2$ equivalent to a block basis of the standard basis of $l^2$ such that $\{Tu_n\}$ is an $l^2_w$-sequence with no norm convergent subsequence.

Now we will define a bounded linear operator $\Psi(T) : C(\Omega) \to Y$ which is not compact. Since $\Omega$ is a nonscattered compact Hausdorff space, by a theorem of A. Pelczynski, W. Rudin, and Z. Semedeni (see [**22**, Theorem 19.7.6]) there exists a purely nonatomic Borel probability measure $\mu$ on $\Omega$. Let $\{r_n\}$ be a sequence of Rademacher like functions in $L^2(\mu)$. Then the sequence $\{r_n\}$ is a basic sequence of orthonormal functions. Observe that since $\mu$ is a regular Borel measure, for each function $r_n$ and for each natural number $k$ there exists an $f_{nk} \in C(\Omega)$ such that $\|f_{nk}\| = \sup\{|f_{nk}(\omega)| : w \in \Omega\} = 1$ and $\|f_{nk} - r_n\|_2 < \frac{1}{k}$. Let $M$ be the closed subspace of $L^2(\mu)$ spanned by the sequence $\{r_n\}$ and the sequences $\{f_{nk}\}$ for $n = 1, 2, \ldots$. Let $M_1$ be the closed subspace of $M$ spanned by the sequence $\{r_n\}$ and $M_0$ be the orthogonal complement of $M_1$ in $M$. Then $M$ is the internal direct sum of $M_1$ and $M_0$ (i.e., $M = \{x_1 + x_2 : x_1 \in M_1, x_2 \in M_2\}$ and $\|x_1 + x_2\| = (\|x_1\|^2 + \|x_2\|^2)^{\frac{1}{2}}$). Let $N$ be the closed linear subspace spanned by $\{u_n\}$. We have

$$C(\Omega) \xrightarrow{\Lambda} L^2(\mu) \xrightarrow{P} M \xrightarrow{I} M_1 \oplus M_0 \xrightarrow{J} N \xrightarrow{T|_N} Y,$$

where $\Lambda(f) = f =$ the equivalence class of $f$ in $L^2(\mu)$; the operator $P$ is the orthogonal projection from $L^2(\mu)$ onto $M$; $I$ is the identity map from $M$ onto $M_1 \oplus M_0$; and $J : M_1 \oplus M_0 \to N$ is the operator defined by $J(r_n) = u_n$ for $n = 1, 2, \ldots$ and $J(x) = 0$ for each $x \in M_0$. (Since $\{u_n\}$ is a basic sequence in $l^2$, $J$ is an isomorphism from $M_1$ onto $N$.) Let $\Psi(T) = T|_N JIP\Lambda$. Clearly, $\Psi(T)$ maps $C(\Omega)$ into $Y$. We claim that $\Psi(T)$ is not compact. For this it is enough to show that $\{Tu_n\} \subseteq \overline{\{\Psi(T)(f) : f \in C(\Omega) \text{ and } \|f\| = 1\}}$. To this end, note that

$$\begin{aligned} \|Tu_n - \Psi(T)f_{nk}\| &= \|TJPr_n - TJIP\Lambda f_{nk}\| \\ &\le \|T\|\|JPr_n - JPf_{nk}\| \\ &\le \|T\|\|J\|\|P\|\frac{1}{k} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty. \end{aligned}$$

□

COROLLARY 3.3. *If $Y$ is a Banach space such that $B(C(\Omega),Y) = \Pi_2(C(\Omega),Y)$, then $B(C(\Omega),Y) = K(C(\Omega),Y)$ if and only if $B(l^2,Y) = K(l^2,Y)$.*

*Proof.* In view of Theorem 3.2 we need only to prove that if $B(l^2,Y) = K(l^2,Y)$, then $B(C(\Omega),Y) = K(C(\Omega),Y)$. This follows from Remark 2.5 and the factorization theorem of A. Pietsch [**18**], which states that every absolutely 2-summing operator factors through a Hilbert space. □

COROLLARY 3.4. *For any compact nonscattered Hausdorff space $\Omega$ and any Banach space $Y$, the following are equivalent.*
(a) $B(C(\Omega),Y) = K(C(\Omega),Y)$.
(b) $B(l^2,Y) = K(l^2,Y)$ *and each* $T \in B(C(\Omega),Y)$ *factors through a closed subspace of* $c_0$.

*Proof.* (a)$\Longrightarrow$ (b) This follows from a theorem of T. Terzioglu [**24**] (or see [**1**, Theorem 16.5]) and Theorem 3.2.
(b)$\Longrightarrow$(a) Since $B(l^2,Y) = K(l^2,Y)$, $Y$ does not contain any copy of $c_0$. So, every operator from $c_0$ into $Y$ is compact. Now (a) is clear. □

To present Theorem 3.9 we need some discussion on the spaces of compact operators. Recall [**11**] that an operator $T \in B(X,Y)$ is said to have an **unconditional compact expansion** if there is a sequence $\{T_n\}$ of compact operators from $X$ into $Y$ such that for each $x \in X$ we have $Tx = \sum_{n=1}^{\infty} T_n x$, where the series converges unconditionally in $Y$. Recall also that $T$ is said to have a **finite dimensional expansion** if the operators $T_n$ are of finite rank. We shall now formulate the following definitions.

DEFINITION 3.5. An operator $T \in B(X,Y)$ is said to have a **weak unconditional compact netted expansion** if there is a net $\{T_\mu\}$ of compact operators from $X$ into $Y$ such that for each $x \in X$

$$Tx = \sum_{\mu} T_\mu x,$$

where the series converges weakly unconditionally in $Y$

DEFINITION 3.6. A Banach space $B$ is said to have a **weak unconditional compact netted expansion of identity** if the identity operator $I_B$ on $B$ has a weak unconditional compact netted expansion.

Recall that if $I_B$ in the above definition has an unconditional finite dimensional expansion, then $B$ is said to have an **unconditional finite dimensional expansion of identity**.

REMARKS. Suppose $T$ in $B(X,Y)$ factors through a Banach space $E$.

(a) If E has a weak unconditional compact netted expansion of identity, then T has a weak unconditional compact netted expansion.

(b) If E has an unconditional finite dimensional expansion of identity, then T has an unconditional finite dimensional expansion.

The part (a) of the next proposition is motivated by a result of N.J. Kalton [**13**] and is slightly more general than other known generalizations of the same result.

PROPOSITION 3.7. *Suppose $c_0$ does not embed in $K(X,Y)$ and $T \in B(X,Y)$.*

(a) *If $T$ has a weak unconditional compact netted expansion, then $T$ is compact.*

(b) *If $T$ has a weak unconditional compact netted expansion, then $T$ factors through a closed subspace of $c_0$.*

*Proof.* (a) Let $\{T_\mu\}$ be a weak unconditional compact netted expansion of $T$. We claim that $\{T_\mu\}$ is an unconditional compact netted expansion of $T$. By way of contradiction suppose that for some $x \in B$ the series $\sum_\mu T_\mu x$ does not converge unconditionally. Then there exists an $\epsilon > 0$ and sequences $(F_n)$, $(F'_n)$ of finite subsets of the index set such that for all $m$ and $n$ the sets $F_n$ and $F'_m$ are disjoint and

$$\left\| \sum_{\eta \in F_n} \epsilon_\eta T_\eta x - \sum_{\eta \in F'_n} \epsilon_\eta T_\eta x \right\| > \epsilon.$$

for some choices of signs $\epsilon_\eta$. Set $y_n = \sum_{\eta \in F_n} \epsilon_\eta T_\eta x - \sum_{\eta \in F'_n} \epsilon_\eta T_\eta x$. Then, the series $\sum_n y_n$ converges weakly unconditionally Cauchy in $Y$ and $\inf \|y_n\| \geq \epsilon$. So, by a theorem of Bessaga and Pelczynski [**4**] the space $Y$ contains a copy of $c_0$. This contradicts the hypothesis.

Since the series $\sum_\mu T_\mu x$ converges unconditionally for every $x \in B$, by the uniform boundedness principle

$$\sup \left\| \sum_{\mu \in F} T_\mu \right\| < \infty,$$

where the supremum is taken over all finite subsets $F$ of the index set $M$. Equivalently, the series $\sum_\mu T_\mu$ is weakly unconditionally Cauchy in $K(X,Y)$. Since $K(X,Y)$ does not contain any copy of $c_0$ by a theorem of Bessaga and Pelczynski [**4**], the series converges in norm. Clearly, it converges to $T$.

(b) This is immediate from (a) and a theorem of T. Terzioglu [**24**]. □

This completes the necessary discussion on the spaces of compact operators. The following theorem due to L. Drewnowski [**7**] will also be useful in the proof of Theorem 3.9. Here, the Banach space of all countably additive vector measures from the $\sigma$-algebra $\Sigma$ into the Banach space $Y$ is denoted by $ca(\Sigma, Y)$.

THEOREM 3.8 (Drewnowski). *If a $\sigma$-algebra $\Sigma$ admits an atomless probability measure, then for any Banach space $Y$ the following statements are equivalent.*

(a) $l_\infty \hookrightarrow ca(\Sigma, Y)$.

(b) $c_0 \hookrightarrow ca(\Sigma, Y)$.

(c) $B(l^2, Y) \neq K(l^2, Y)$.

The following theorem gives another necessary and sufficient condition on a Banach space $Y$ for all operators from $C(\Omega)$ into $Y$ to be compact.

THEOREM 3.9. *For any compact nonscattered Hausdorff space $\Omega$ and any Banach space $Y$ the following are equivalent.*

(a) $B(C(\Omega), Y) = K(C(\Omega), Y)$.

(b) $B(l^2, Y) = K(l^2, Y)$ *and each* $T \in B(C(\Omega), Y)$ *has a weak unconditional compact netted expansion.*

*Proof.* (a)$\Longrightarrow$(b) We get the equality $B(l^2, Y) = K(l^2, Y)$ from Theorem 3.2 and that each $T \in B(C(\Omega), Y)$ admits a weak unconditional compact netted expansion is obvious.

(b)$\Longrightarrow$(a) Since $\Omega$ is nonscattered, by a theorem of A. Pelczynski, W. Rudin, and Z. Semadeni (see [**22**, p. 338]), it admits an atomless regular Borel probability measure. Since $B(l^2, Y) = K(l^2, Y)$, by Theorem 3.8, it follows that $c_0 \not\hookrightarrow ca(\Sigma, Y)$, where $\Sigma$ denotes the $\sigma$-algebra of all Borel subsets of $\Omega$. Since $K(C(\Omega), Y)$ is isometrically embeddable in $ca(\Sigma, Y)$ (see [**5**, pp. 152–154]), $c_0 \not\hookrightarrow K(C(\Omega), Y)$. Now the conclusion follows from Proposition 3.7. □

COROLLARY 3.10. *If for some $p$ with $1 \leq p \leq 2$, $B(l^p, Y) = K(l^p, Y)$ and each operator in $B(C(\Omega), Y)$ has a weak unconditional compact netted expansion, then $B(C(\Omega), Y) = K(C(\Omega), Y)$.*

*Proof.* This follows from Corollary 2.4 and Theorem 3.9. □

Recall that a Banach space is said to be **separably universal** if it contains an isometric copy of every separable Banach space. Recall also that for a compact Hausdorff space $\Omega$ the space $C(\Omega)$ is separably universal if and only if $\Omega$ is nonscattered (see [**14**]). Note that if $\mu$ is a regular Borel measure whose support is an infinite compact Hausdorff space, then there exists a nonscattered compact Hausdorff space $\Omega'$ such that $L^\infty(\mu) \cong C(\Omega')$. In particular, $l^\infty \cong C(\Omega')$ for some nonscattered compact Hausdorff space $\Omega'$.

COROLLARY 3.11. *For any nonscattered compact Hausdorff space $\Omega$, any Banach space $Y$ with a weak unconditional compact netted expansion of identity, and any regular Borel measure $\mu$ on a compact Hausdorff space the following statements hold.*

(a) *$B(C(\Omega), Y) = K(C(\Omega), Y)$ if and only if $B(l^2, Y) = K(l^2, Y)$.*

(b) *For any nonscattered compact Hausdorff space $\Omega'$ we have $B(C(\Omega), Y) = K(C(\Omega), Y)$ if and only if $B(C(\Omega'), Y) = K(C(\Omega'), Y)$.*

(c) *$B(C(\Omega), l^p) = K(C(\Omega), l^p)$ for $1 \leq p < 2$.*

(d) *$B(C(\Omega), l^p) \neq K(C(\Omega), l^p)$ for $2 \leq p < \infty$.*

(e) *$B(L^\infty(\mu), l^p) = K(L^\infty(\mu), l^p)$ for $1 \leq p < 2$.*

(f) *$B(L^\infty(\mu), l^p) \neq K(L^\infty(\mu), l^p)$ for $2 \leq p < \infty$.*

*Proof.* (a) This follows from Theorem 3.2 and Theorem 3.9.

(b) This follows from *(a)*.

(c) Since $1 \leq p < 2$, by a result of H.R. Pitt [**16**], we have $B(l^2, l^p) = K(l^2, l^p)$. We know that $l^p$ has a weak unconditional compact netted expansion of identity, so by *(a)* we get $B(C(\Omega), l^p) = K(C(\Omega), l^p)$.

(d) Since $2 \leq p < \infty$, we obviously have $B(l^2, l^p) \neq K(l^2, l^p)$. Now the conclusion follows from Theorem 3.2.

(e) follows from (c) and (f) follows from *(d)*. □

Parts (e) and (f) of Corollary 3.11 follow also from [**19**, Remark 2].

The following conclusion is clear from what we have proved so far.

CONCLUSION 3.12. *Let $\Sigma(\Omega)$ denote the class of all Banach spaces $Y$ for which all operators from $C(\Omega)$ into $Y$ are compact iff all operators from $l^2$ into $Y$ are compact. Then, for a Banach space $Y$ the following statements hold.*

(a) *If $Y$ has an unconditional basis, then $Y \in \Sigma(\Omega)$.*

(b) *If $Y$ has an unconditional basis consisting of finite dimensional subspaces, then $Y \in \Sigma(\Omega)$.*

(c) *If $Y$ has a weak conditional compact netted expansion of identity, then $Y \in \Sigma(\Omega)$.*

(d) *If each operator from $C(\Omega)$ into $Y$ admits a weak unconditional compact netted expansion, then $Y \in \Sigma(\Omega)$.*

(e) *If each operator from $C(\Omega)$ into $Y$ factors through a closed subspace of $c_0$, then $Y \in \Sigma(\Omega)$.*

(f) *If each operator from $C(\Omega)$ into $Y$ is absolutely 2-summing, then $Y \in \Sigma(\Omega)$.*

(g) *If $Y$ has the Schur property, then $Y \in \Sigma(\Omega)$.*

We conclude this section with a remark, whose proof is left to the reader.

REMARK. In Theorem 3.8 the space $l^2$ can not be replaced by an $l^p$-space with $p \neq 2$.

**4. Factorization.** In this section $\Omega$ is any (scattered or nonscattered) compact Hausdorff space. Now we will use some of our earlier theorems to prove some results regarding the space $\Phi_{c_0}(C(\Omega))$ of all operators on $C(\Omega)$ factoring through $c_0$.

PROPOSITION 4.1. *For an infinite compact Hausdorff space $\Omega$, and for a closed subspace $X$ of $c_0$ the following inclusions hold.*
(a) $\Phi_X(C(\Omega)) \subseteq \Phi_{c_0}(C(\Omega))$.
(b) $K(C(\Omega)) \subseteq \Phi_{c_0}(C(\Omega))$, *but* $K(C(\Omega)) \neq \Phi_{c_0}(C(\Omega))$.

*Proof.* (a) Let $T \in \Phi_X(C(\Omega))$ be arbitrary and $T = T_2T_1$ be a factorization of $T$ through $X$. Since $X$ is a closed subspace of $c_0$, by a theorem of J. Lindenstrauss and A. Pelczynski [**15**, Theorem 3.1], $T_2$ extends to a bounded linear operator $\widehat{T}_2$ from $c_0$ into $C(\Omega)$. Clearly, $T = \widehat{T}_2T_1 \in \Phi_{c_0}(C(\Omega))$.

(b) Let $T \in K(C(\Omega))$ be arbitrary. Then by the theorem of T. Terzioglu [**24**], $T$ factors through a closed subspace of $c_0$. Hence, by (a) $T \in \Phi_{c_0}(C(\Omega))$, (i.e., $K(C(\Omega)) \subset \Phi_{c_0}(C(\Omega))$). To prove that $K(C(\Omega)) \neq \Phi_{c_0}(C(\Omega))$ let us first suppose $\Omega$ is scattered. Since $\Omega$ is an infinite set, the space $C(\Omega)$ contains a complemented subspace $M$ isomorphic to $c_0$ (see [**19**, p. 201]). Let $P : C(\Omega) \to M$ be a continuous projection onto $M$, let $M \to C(\Omega)$ be the inclusion map. Clearly, $JP$ factors through $c_0$ and is noncompact. Now suppose $\Omega$ is nonscattered. First note that there is a noncompact operator $T$ in $B(C(\Omega))$. (For, otherwise our Theorem 3.2 would imply that $B(l^2, c_0) = K(l^2, c_0)$. On the other hand, the formal identity map from $l^2$ to $c_0$ is not compact.) Now note that since $\Omega$ is nonscattered there exists an isometry $J$ in $B(c_0, C(\Omega))$. Clearly, $JT \in \Phi_{c_0}(C(\Omega))$ and $JT$ is noncompact. □

THEOREM 4.2. *For a compact Hausdorff space $\Omega$ and for a separable Banach space $X$ the following are equivalent.*
(a) $\Phi_X(C(\Omega)) \subseteq K(C(\Omega))$.
(b) $\Phi_X(C(\Omega)) \subseteq \Phi_{c_0}(C(\Omega))$, *but* $\Phi_X(C(\Omega)) \neq \Phi_{c_0}(C(\Omega))$.

*Proof.* $(a) \Longrightarrow (b)$ This is immediate from Proposition 4.1.

(b)$\Longrightarrow$(a) First observe that $c_0 \not\hookrightarrow X$. For, otherwise since $X$ is separable, a theorem of Sobczyk [**23**], would imply that an isomorphic copy of $c_0$ is complemented in $X$. So, we would get $\Phi_{c_0}(C(\Omega)) \subseteq \Phi_X(C(\Omega))$, contrary to our assumption. To prove that $\Phi_X(C(\Omega)) \subseteq K(C(\Omega))$, it suffices to prove that $B(C(\Omega), X) = K(C(\Omega), X)$. If $\Omega$ is scattered, then $B(C(\Omega), X) = K(C(\Omega), X)$ by Corollary 1.2. If $\Omega$ is nonscattered, then $C(\Omega)$ is separably universal. So, there is an isometry $J : X \to C(\Omega)$. If $T \in B(C(\Omega), X)$, then by our hypothesis $JT \in \Phi_{c_0}(C(\Omega))$. So, suppose $JT = T_2T_1$ is a factorization through $c_0$. Note that $T_2 \in B(c_0, J(X))$ and $c_0 \cong C(S)$ for some scattered compact Hausdorff space $S$. Since $c_0 \not\hookrightarrow J(X)$, by Corollary 1.2 the operator $T_2$ is compact. $\square$

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**Added in proof.** After this paper was accepted for publication we learned that Corollary 1.2 (a)$\Longleftrightarrow$(b)$\Longleftrightarrow$(c) was already known. See Proposition 2 of the following paper.

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