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THE INTRINSIC MOUNTAIN PASS

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We show how the mountain pass and saddle point theorems can be formulated with out the use of "auxiliary" sets. Moreover, we show that results can still be obtained when some basic hypotheses of these theorems are not satisfied. We then apply our results to semilinear problems for partial differential equations.

1. Introduction.

In the mountain pass and saddle point theorems one is concerned with a C^1 functional G on a Banach space E. One wishes to find a solution of $G'(u) = 0$ or at least a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c,\ G'(u_k) \to 0 \tag{1.1}$$

for some $c \in R$. A general procedure was formulated in Brezis-Nirenberg [**BN**] as follows. One finds a compact metric space K and selects a closed subset K^* of K such that $K^* \neq \phi, K^* \neq K$. One then picks a map $p^* \in C(K^*, E)$ and defines

$$A = \{p \in C(K, E) : \quad p = p^* \text{ on } K^*\}$$

$$a = \inf_{p\in A} \max_{\xi \in K} G(p(\xi)). \tag{1.2}$$

Brezis-Nirenberg assume
(A) For each $p \in A, \max_{\xi\in K} G(p(\xi))$ is attained at a point in $K \setminus K^*$.
They then prove that there is a sequence satisfying

$$G(u_k) \to a, \quad G'(u_k) \to 0. \tag{1.3}$$

In reference to the procedure one can ask three questions

1. Are the sets K, K^* essential to the method, or can they be eliminated?
2. How can one verify (A)?
3. What can be said if (A) fails to hold?

The purpose of the present paper is to address these questions. Concerning the first, we show that indeed a quantity corresponding to (1.2) can be introduced which is "intrinsic" in nature and does not depend on the "auxiliary" sets K, K^*. For this purpose we use a definition given in [**ST**]. We define

$$a = \inf_{\Gamma\in\Phi} \sup_{0\le s\le 1, u\in A} G(\Gamma(s)u) \tag{1.4}$$

where A is a given subset of E and Φ is a family of maps in $C(E\times[0,1],E)$ with certain properties (for a precise definition cf. Section 2). The quantity (1.4) depends only on G and the set A. It is not required that $A = p^*(K^*)$ for some K^* and $p^* \in C(K,E)$. In dealing with (1.4) we replace hypothesis (A) with

(B) For each $\Gamma \in \Phi$,

$$\max_{0\le s\le 1, u\in A} G(\Gamma(s)u)$$

is attained at a point not in A.

We then show that hypothesis (B) implies the existence of a sequence satisfying (1.3) (actually, we use a hypothesis weaker than (B)). From this it follows that

$$a_0 := \sup_A G < a < \infty \tag{1.5}$$

implies the existence of a sequence satisfying (1.3).

Concerning the second question, we show that hypothesis (B) holds if and only if there is a subset B of E such that A links B in the sense of [**ST**] and

$$a_0 := \sup_A G \le b_0 := \inf_B G. \tag{1.6}$$

The definition of linking given in [**ST**] differs from that usually found. Essentially, it says that A and B link if they cannot be "slipped" apart without intersection. In our opinion, this definition is more in keeping with the concept of linking (for a formal definition cf. Section 2). The results of the present paper show that the only practical way of applying hypothesis (B) is to find a subset B of E such that A links B and (1.6) holds (the same is true for hypothesis (A)).

Concerning the third question, we note that the only case not covered is when there is a set B such that A links B and

$$-\infty < b_0 < a_0 = a < \infty. \tag{1.7}$$

Surprisingly, something can be accomplished in this case as well. We have

Theorem 1.1. *Assume* (1.7) *and let*

$$B' = \{v \in B : G(v) < a_0\}.$$

Let α, T *be positive numbers such that*

$$a_0 - b_0 < \alpha T, \quad T < d' = d(A, B'). \tag{1.8}$$

Then for each $\delta > 0$ *sufficiently small there is a* $u \in E$ *satisfying*

$$b_0 - \delta \leq G(u) \leq a_0 + \delta, \quad \|G'(u)\| \leq \alpha \tag{1.9}$$

and either

$$d(u, B') < T \text{ or } d(u, B \setminus B') < \delta/\alpha. \tag{1.10}$$

Corollary 1.2. *Let* $\{B_n\}$ *be a sequence of subsets of* E *such that* A *links* B_n *for each* n *and* $d(A, B'_n) \to \infty$, *where*

$$B'_n = \{v \in B_n : \quad G(v) < a_0\}.$$

Assume that $a < \infty$ *and that*

$$\inf_{B_n} G \geq b_0 > -\infty. \tag{1.11}$$

Then there is a sequence $\{u_k\}$ *satisfying* (1.1) *with* $b_0 \leq c \leq a_0$.

In essence, Corollary 1.2 says that if A links B_n for each n and the part B'_n of B_n on which G is $< a_0$ moves out to infinity and (1.11) holds, then the mountain pass methods still apply. Special cases of this theorem were given in [**Sc1-4**, **Si**]. We present an application in Section 4.

2. A Generalized Mountain Pass.

Before stating our main theorems we recall the definition of linking sets given in [**ST**]. Let E be a Banach space and Let Φ be the set of all continuous maps $\Gamma(t)$ from $E \times [0, 1]$ to E such that

(a) $\Gamma(0) = I$

(b) there is an $x_0 \in E$ such that $\Gamma(1)x = x_0$ for each $x \in E$

(c) $\Gamma(t)x \to x_0$ as $t \to 1$ uniformly on bounded subsets of E

(d) for each $t \in [0, 1), \Gamma(t)$ is a homeomorphism of E onto itself and Γ^{-1} is continuous on $E \times [0, 1)$.

Definition. A subset A of E links a subset B of E if $A \cap B = \phi$ and for each $\Gamma \in \Phi$ there is a $t \in (0,1]$ such that $\Gamma(t)A \cap B \neq \phi$.

Let A, B be subsets of a Banach space E such that A links B, and let G be a C^1 functional on E. Define

$$a := \inf_{\Gamma\in\Phi} \sup_{0\leq s\leq 1, u\in A} G(\Gamma(s)u), b_0 = \inf_B G. \tag{2.1}$$

Since A links B, we have

$$b_0 \leq a. \tag{2.2}$$

Assume that

$$d(A,B) > 0. \tag{2.3}$$

Let

$$B' := \{v \in B : G(v) < a\}. \tag{2.4}$$

We note that

$$B' = \phi \text{ iff } b_0 = a \tag{2.6}$$

(If D is any set, we write $d(D,\phi) = \infty$. Thus $d' = \infty$ when $B' = \phi$.) Let α, T be any positive numbers satisfying

$$a - b_0 < \alpha T, T < d'. \tag{2.7}$$

We have

Theorem 2.1. *Assume in addition that*

$$-\infty < b_0, a < \infty. \tag{2.8}$$

Then for every $\delta > 0$ sufficiently small there is a $u \in E$ such that

$$b_0 - \delta \leq G(u) \leq a + \delta, \quad \|G'(u)\| \leq \alpha \tag{2.9}$$

and either

$$d(u, B') < T \tag{2.10}$$

or

$$d(u, B \setminus B') < \delta/\alpha. \tag{2.11}$$

Corollary 2.2. *If $b_0 = a$, then there is a sequence $\{u_k\} \subset E$ such that*

$$G(u_k) \to a, G'(u_k) \to 0, d(u_k, B) \to 0. \tag{2.12}$$

Corollary 2.3. *Let $\{A_n\}, \{B_n\}$ be sequences of subsets of E, and define*

$$a_n = \inf_{\Gamma\in\Phi} \sup_{0\le s\le 1, u\in A_n} G(\Gamma(s)u), b_{on} = \inf_{B_n} G \tag{2.13}$$

$$a = \liminf a_n, b_0 = \limsup b_{on} \tag{2.14}$$

$$B_n' = \{v \in B_n : G(v) < a_n\} \tag{2.15}$$

$$d_n' = d(A_n, B_n'). \tag{2.16}$$

Assume that A_n links $B_n, d(A_n, B_n) > 0$ and that

$$-\infty < b_0 \le a < \infty \tag{2.17}$$

$$d_n' \to \infty. \tag{2.18}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c, b_0 \le c \le a, G'(u_k) \to 0. \tag{2.19}$$

Similarly we have

Theorem 2.4. *Assume that $A, B \subset E$, B links A, and $G \in C^1(E, R)$. Define*

$$a_0 := \sup_A G, b := \sup_{\Gamma\in\Phi} \inf_{0\le s\le 1, v\in B} G(\Gamma(s)v) \tag{2.20}$$

$$A' = \{u \in A : G(u) > b\}, d'' = d(A', B) \tag{2.21}$$

and let α, T be any positive constants satisfying

$$a_0 - b < \alpha T, T < d''. \tag{2.22}$$

Assume also that

$$-\infty < b, a_0 < \infty \tag{2.23}$$

and that (2.3) holds. Then for every $\delta > 0$ sufficiently small there is a $u \in E$ satisfying

$$b - \delta \leq G(u) \leq a_0 + \delta, \|G'(u)\| \leq \alpha \tag{2.24}$$

and either

$$d(u, A') < T \tag{2.25}$$

or

$$d(u, A \setminus A') < \delta/\alpha. \tag{2.26}$$

Corollary 2.5. *Let $\{A_n\}, \{B_n\}$ be sequences of subsets of E, and define*

$$a_{on} = \sup_{A_n} G, b_n = \sup_{\Gamma \in \Phi} \inf\nolimits_{0 \leq s \leq 1, v \in B_n} G(\Gamma(s)v) \tag{2.27}$$

$$a_0 = \liminf a_{on}, b = \limsup b_n. \tag{2.28}$$

$$A'_n = \{u \in A_n : G(u) > b_n\}, d''_n = d(A'_n, B_n). \tag{2.29}$$

Assume that B_n links $A_n, d(A_n, B_n) > 0$ and that

$$-\infty < b \leq a_0 < \infty, d''_n \to \infty. \tag{2.30}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c, b \leq c \leq a_0, G'(u_k) \to 0. \tag{2.31}$$

Theorem 2.6. *If $a_0 \leq b_0$ and $a < \infty$, then there is a sequence $\{u_k\} \subset E$ such that*

$$G(u_k) \to a, G'(u_k) \to 0. \tag{2.32}$$

We now show that we can remove the reference to the set B. We have

Theorem 2.7. *Assume that $a < \infty$ and that for each $\Gamma \in \Phi$ the set*

$$g_\Gamma := \{v = \Gamma(s)u : s \in (0, 1], u \in A, v \notin A, G(v) \geq a_0\} \tag{2.33}$$

is not empty. Then there is a sequence satisfying (2.32).

Corollary 2.8. *If* $a < \infty$ *and* $a_0 \neq a$*, then there is a sequence satisfying* (2.32).

Corollary 2.9. *If* $a < \infty$ *and for each* $\Gamma \in \Phi$

$$\max_{0 \leq s \leq 1, u \in A} G(\Gamma(s)u) \tag{2.34}$$

is attained at a point not in A*, then there is a sequence satisfying* (2.32).

We can summarize by

Theorem 2.10. *Assume that* A *links* B *and that they satisfy* (2.8). *Then one of the following holds*

(i) $a_0 \neq a$

(ii) $b_0 = a_0 = a$

(iii) $b_0 < a_0 = a$.

In case (i) *there is a sequence satisfying* (2.32). *In case* (ii) *there is a sequence satisfying* (2.12). *In case* (iii) *the conclusions of Theorem* (2.1) *hold.*

Another consequence of Theorem 2.1 is

Theorem 2.11. *Let* M, N *be complementary subspaces of a Banach space* E *with one of them being finite dimensional. Let* G *be a* C^1 *functional on* E*, and define*

$$m_0 := \sup_{v \in N} \inf_{w \in M} G(v + w), m_1 := \inf_{w \in M} \sup_{v \in N} G(v + w). \tag{2.35}$$

Assume that

$$-\infty < m_0 \text{ and } m_1 < \infty. \tag{2.36}$$

Then there is a sequence $\{u_k\} \subset E$ *such that*

$$G(u_k) \to c, m_0 \leq c \leq m_1, G'(u_k) \to 0. \tag{2.37}$$

Corollary 2.2 was proved in [**ST**] as well as Theorem 2.6. Theorem 2.11 generalizes theorems of [**Sc1**, **Si**]. The results of this section will be proved in Section 3.

Finally, we note the following

Theorem 2.12. *There is a $B \subset E$ such that A links B and $a_0 \leq b_0$ if and only if g_Γ defined by* (2.33) *is not empty for each $\Gamma \in \Phi$.*

Remarks. There are basic differences between the approach of [**BN**] and that of [**ST**] and the present paper. The sets K, K^* must be compact; otherwise their arguments fail. In [**ST**] the set A neeed not be compact (or even closed). Also, both K and K^* must be chosen initially. In [**ST**] only the set A is chosen. The sets (2.33) vary with Γ and are determined automatically. For our linking, the set B is only required to intersect the set (2.33) for each Γ. It is not required to intersect $p(K)$ for each continuous map p (a much larger set). There are no counterparts of Theorem 2.1, Corollary 2.3, Theorem 2.4, Corollary 2.5 in the [**BN**] theory.

Assumption (2.3) is not required in Theorems 2.6, 2.7 and Corollary 2.9.

3. The Flow.

In this section we give the proofs of the results of Section 2.

Proof of Theorem 2.1. Assume that the theorem is false. Then there are α, T satisfying (2.7) and a $\delta > 0$ such that

$$\|G'(u)\| > \alpha \tag{3.1}$$

when

$$u \in Q := \{u \in E : b_0 - 3\delta \leq G(u) \leq a + 3\delta \quad \text{and either} \\ d(u, B') \leq T + 3\delta \text{ or } d(u, B \setminus B') \leq 5\delta/\alpha\}.$$

Let

$$Q_0 = \{u \in E : b_0 - 2\delta \leq G(u) \leq a + 2\delta \\ \text{and either } d(u, B') \leq T + 2\delta \text{ or } d(u, B \setminus B') \leq 4\delta/\alpha\}$$
$$Q_1 = \{u \in E : b_0 - \delta \leq G(u) \leq a + \delta \text{ and either } d(u, B') \\ \leq T + \delta \text{ or } d(u, B \setminus B') \leq 3\delta/\alpha\}.$$

Define

$$Q_2 = E \setminus Q_0, \eta(u) = d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)].$$

Note that $\eta = 1$ on $Q_1, \eta = 0$ on $\bar{Q}_2$ and $0 < \eta < 1$ otherwise. Let $d_1 = d(B \setminus B', A)$ and reduce δ, if necessary, so that

$$2\delta < \alpha d_1, \delta < \alpha T - (a_0 - b). \tag{3.3}$$

Let θ satisfy

$$\frac{2}{3} < \theta < 1, 2\delta < \theta\alpha d_1, \delta < \theta\alpha T - (a_0 - b). \tag{3.4}$$

One can show that there is a locally Lipshitz continuous map $Y(u) : \hat{E} \to \hat{E} = \{u \in E : G'(u) \neq 0\}$ satisfying

$$\|Y(u)\| \leq 1, (G'(u), Y(u)) \geq \theta\|G'(u)\|, u \in \hat{E} \tag{3.5}$$

(cf, e.g., [**BN**]). Let $\sigma(t)$ be the flow generated by the vector field $\eta(u)Y(u)$ which is I for $t = 0$. Then

$$\begin{aligned} \|\sigma(t)u - u\| &= \left\| \int_0^t \eta(\sigma(\tau)u)Y(\sigma(\tau)u)d\tau \right\| \\ &\leq \Psi(t, u) := \int_0^t \eta(\sigma(\tau)u)d\tau, u \in E, \end{aligned} \tag{3.6}$$

(3.7)

$$G(\sigma(t)u) - G(u) = \int_0^t \eta(\sigma(\tau)u)(G'(\sigma(\tau)u), Y(\sigma(\tau)u))d\tau \geq \theta\alpha\psi(t, u), u \in E.$$

Thus

$$\|u - v\| \leq \|u - \sigma(t)v\| + \Psi(t, v). \tag{3.8}$$

If $u \in A$ and $v \in B'$, this implies

$$\|u - \sigma(t)v\| \geq d' - T, 0 \leq t \leq T, u \in A, v \in B'. \tag{3.9}$$

On the other hand, if $v \in B\backslash B'$, then $G(v) \geq a$ and $G(\sigma(t)v) \leq \max[G(v), a + 2\delta]$ by the definition of η. Hence by (3.7)

$$\theta\alpha\Phi(t, v) \leq 2\delta, t \geq 0, v \in B \setminus B'. \tag{3.10}$$

Consequently, by (3.8) - (3.10) we have

$$\|u - \sigma(t)v\| \geq \min[d' - T, d_1 - (2\delta/\theta\alpha)] > 0, 0 \leq t \leq T, u \in A, v \in B. \tag{3.11}$$

Let $B_1 = \sigma(T)B$. I claim that A links B_1. By (3.11), $A \cap \sigma(t)B = \phi$ for $0 \leq t \leq T$. Let $\Gamma \in \Phi$, and put $\Gamma_1(s) = \sigma(2sT)^{-1}$ for $0 \leq s \leq \frac{1}{2}, \Gamma_1(s) = \sigma(T)^{-1}\Gamma(2s - 1)$ for $\frac{1}{2} < s \leq 1$. Then $\Gamma_1 \in \Phi$. Since A links B, there is an $s_1 \in [0, 1]$ such that $\Gamma_1(s_1)A \cap B \neq \phi$. But $\sigma(2sT)^{-1}A \cap B = \phi$ for $0 \leq s \leq \frac{1}{2}$ by (3.11). Thus $\frac{1}{2} < s_1 \leq 1$, and $\sigma(T)^{-1}\Gamma(2s_1 - 1)A \cap B \neq \phi$, or equivalently,

$\Gamma(2s_1-1)A\cap B_1 \neq \phi$. Thus A links B_1. Suppose there is a $t_1 \le T$ such that $\sigma(t_1)v \notin Q_1$. Then either

$$G(\sigma(t_1)v) > a+\delta \tag{3.12}$$

or

$$d(\sigma(t_1)v, B') > T+\delta \text{ and } d(\sigma(t_1)v, B\setminus B') > 3\delta/\alpha. \tag{3.13}$$

But if $v \in B'$, then $d(\sigma(t_1)v, B') \le t_1 \le T$ by (3.6), and if $v \in B\setminus B'$, then $d(\sigma(t_1)v, B\setminus B') \le \Phi(t_1, v) \le 2\delta/\theta\alpha < 3\delta/\alpha$ by (3.6) and (3.10). Thus (3.13) is false if $v \in B$. Hence (3.12) must hold. This implies

$$G(\sigma(T)v) > a+\delta. \tag{3.14}$$

On the other hand, if $\sigma(t)v \in Q_1$ for $0 \le t \le T$, then (3.7) gives

$$G(\sigma(T)v) \ge b_0 + \theta\alpha T > a+\delta.$$

Hence (3.14) holds for all $v \in B$. Thus means that

$$\inf_{B_1} G \ge a+\delta. \tag{3.15}$$

But by the definition of a, there is a $\Gamma \in \Phi$ such that

$$\sup_{0\le s\le 1, u\in A} G(\Gamma(s)u) < a + (\delta/2) \tag{3.16}$$

and since A links B_1, there is an $s \in [0,1]$ such that $\Gamma(s)A \cap B_1 \neq \phi$, and consequently (3.16) contradicts (3.15). Thus (3.1) cannot hold for u satisfying (3.2), and the theorem is proved. □

Proof of Corollary 2.2. In this case $B' = \phi$ and $d' = \infty$. For each n we take $T_n = 1, \delta_n = 1/n^2, \alpha_n = 1/n$ and apply Theorem 2.1. □

Proof of Corollary 2.3. For each n we take $\delta_n = 1/n, T_n = d'_n/2, \alpha_n = [(a_n - b_{on})/T_n] + \delta_n$. Then $\alpha_n \to 0$ and there is a u_n such that

$$b_{on} - \delta_n \le G(u_n) \le a_n + \delta_n, \|G'(u_n)\| \le \alpha_n.$$

We take a renamed subsequence such that $a_n \to a, b_{on} \to b_0$ and $G(u_n) \to c$. □

Proof of Theorem 2.4. We interchange A and B and replace G by $-G$ in Theorem 2.1. Then b_0 becomes $-a_0$ and a becomes $-b$. B' becomes A'. We then apply Theorem 2.1. □

The proof of Corollary 2.5 in similar to that of Corollary 2.3.

Proof of Theorem 2.7. Let

$$B = \bigcup_{\Gamma \in \Phi} g_\Gamma. \tag{3.17}$$

Then $A \cap B = \phi$ and for each $\Gamma \in \Phi$ there is a $v \in B$, an $s \in (0, 1]$ and a $u \in A$ such that $\Gamma(s)u = v$. Hence

$$\Gamma(s)A \cap B \neq \phi.$$

Hence A links B. Since $G(v) \geq a_0$ for each $v \in B$, we have $a_0 \leq b_0$. We can now apply Theorem 2.6 to conclude that a sequence satisfying (2.32) exists. □

Proof of Corollary 2.8. If $a_0 < a$, then for each $\Gamma \in \Phi$ there is a $u \in A$ and an $s \in [0, 1]$ such that $G(\Gamma(s)u) > a_0$. Clearly $v = \Gamma(s)u \notin A$. Thus $g_\Gamma \neq \phi$. We can now apply Theorem 2.7. □

Proof of Corollary 2.9. If the maximum (2.34) is attained at a point outside A, then this point is in g_Γ. Hence g_Γ is not empty for all $\Gamma \in \Phi$. Apply Theorem 2.7. □

Proof of Theorem 2.11. Assume, for definiteness, that $\dim N < \infty$. Let $\{\epsilon_k\}$ be a sequence tending to 0. For each k there are $v_k \in N, w_k \in M$ such that

$$\inf\nolimits_{M_k} G > m_0 - \epsilon_k, \sup_{N_k} G < m_1 + \epsilon_k \tag{3.18}$$

where $M_k = v_k \oplus M$ and $N_k = w_k \oplus N$. Let

$$A_k = \{v + w_k : v \in N, \|v - v_k\| = k\}, B_k = M_k.$$

Note that $A_k \subset N_k$ and $A_k \cap B_k = \phi$. It is readily shown that A_k links B_k for each k (cf., Propostion 1.2 of [**ST**]). By (3.18) we have

$$b_{ok} \geq m_0 - \epsilon_k, a_{ok} \leq m_1 + \epsilon_k.$$

Moreover, for each k,

$$\Gamma_k(s)u = s(v_k + w_k) + (1 - s)u$$

is in Φ. Consequently, definition (2.13) gives

$$a_k \leq \sup_{N_k} G < m_1 + \epsilon_k.$$

Moreover, by (2.6)

$$d'_k = d(A_k, B'_k) \geq d(A_k, B_k) = k \to \infty.$$

If a and b_0 are given by (2.14), we see that $m_0 \leq b_0$ and $a \leq m_1$. We can now apply Corollary 2.3 to conclude that a sequence satisfying (2.37) exists. The proof for the case $\dim M < \infty$ is similar. □

Proof of Theorem 2.12. The "if" part was already proved in the proof of Theorem 2.7. Conversely, suppose A links B and $a_0 \leq b_0$. If $g_\Gamma = \phi$ for some $\Gamma \in \Phi$, then $\Gamma(s)A \cap B = \phi$ for each $s \in [0,1]$. This says that A does not link B, a contradiction. □

Remark. The proof of Theorem 2.6 was not given because it was proved in [**ST**]. However, it is a corollary of Theorem 2.1. To see this, note that if $a_0 \leq b_0 = a$, then $B' = \phi, d' = \infty$, and we can take $\alpha = \delta, T = 1$. If $a_0 \leq b_0 < a$, let B_1 be given by (3.17). Then A links B_1 and $a_0 \leq b_1 \equiv \inf_{B_1} G = a$. We can now apply Theorem 2.1 again with B replaced by B_1.

4. An Application.

In this section we show how the theorems of Section 2 can be applied. Let Ω be a smooth, bounded domain in R^n, and let A be a selfadjoint operator on $L^2(\Omega)$ with discrete spectrum $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_j < \cdots$. We assume that $C_0^\infty(\Omega) \subset D := D(A^{1/2}) \subset H^m(\Omega)$ for some $m > 0$(m need not be an integer). Let q be a number satisfying

$$2 \leq q < 2m/(n-2m), 2 \leq q < \infty, 2m < n; \tag{4.1}$$

and let $f(x,t)$ be a Caratheodory function on $\Omega \times R$. Assume

(I) $|f(x,t)| \leq V_0(x)^q|t|^{q-1} + V_0(x)V_1(x)$

where $V_0 \in L^q(\Omega), V_1 \in L^{q'}(\Omega)$ and multiplication by V_0 is a compact operator from D to $L^q(\Omega)$.

(II) For some $\ell > 0$ the function

$$F(x,t) := \int_0^t f(x,s)ds$$

satisfies

$$\lambda_{\ell-1}t^2 - W_0(x) \leq 2F(x,t) \leq \nu t^2 + V(x)p|t|^p + W_1(x) \tag{4.2}$$

and

$$\lambda_\ell t^2 - W_2(x) \leq 2\, F(x,t) \tag{4.3}$$

where $\nu < \lambda_\ell, p > 2$,

$$B_j := \int_\Omega W_j(x)dx < \infty, j = 0, 1, 2 \tag{4.4}$$

and

$$\|Vu\|_p^p \le C\|A^{1/2}u\|^p, u \in D. \tag{4.5}$$

(III) For some $\mu > 2$ the function

$$H_\mu(x,t) := \mu F(x,t) - tf(x,t)$$

satisfies

$$H_\mu(x,t) \le V_3(x)^2 t^2 \sigma(t) + W_3(x) \tag{4.6}$$

where multiplication by V_3 is bounded from D to $L^2(\Omega), W_3 \in L^1(\Omega)$ and $\sigma(t)$ is a continuous function such that $\sigma(t) \to 0$ as $|t| \to \infty$.

(IV) $B_0 + B_1 < (1 - \frac{2}{p})(1 - \frac{\nu}{\lambda_\ell})^{p/p-2}(\frac{2}{pC})^{2/p-2}$.

We have

Theorem 4.1. *Under hypotheses* (I) - (IV) *there is a solution of*

$$Au = f(x,u), \quad u \in D. \tag{4.7}$$

Proof. Under hypothesis (I) the functional

$$G(u) = a(u) - 2\int_\Omega F(x,u)dx \tag{4.8}$$

has a continuous Frechet derivative on D given by

$$(G'(u), v) = 2a(u,v) - 2(f(\cdot, u), v) \tag{4.9}$$

where

$$a(u,v) = (Au, v), a(u) = a(u,u), u, v \in D. \tag{4.10}$$

Let N be the subspace spanned by the eigenfunctions corresponding to the eigenvalues $\lambda_0, \cdots, \lambda_{\ell-1}$, and let $M = N^\perp \cap D$, the orthogonal complement of N on D. On M we have

$$G(w) \ge a(w) - \nu\|w\|^2 - \|Vw\|_p^p - B_1 \ge \left(1 - \frac{\nu}{\lambda_\ell}\right)\|w\|_D^2 - C\|w\|_D^p - B_1$$

by (4.2) and (4.5). If we take $\|w\|_D = \delta$ with $\delta^{p-2} = 2\left(1 - \frac{\nu}{\lambda_\ell}\right)/p$, we have

$$G(w) \geq \left(1 - \frac{2}{p}\right)\left(1 - \frac{\nu}{\lambda_\ell}\right)\delta^2 - B_1 \geq \beta - B_1, \|w\|_D = \delta \tag{4.11}$$

where β is the expression on the right hand side in hypothesis (IV). On the other hand, we have by (4.2)

$$G(v) \leq a(v) - \lambda_{\ell-1}\|v\|^2 + B_0 \leq B_0, v \in N. \tag{4.12}$$

Let w_0 be an eigenfunction corresponding to the eigenvalue λ_ℓ with unit norm, and let N_1 denote the subspace spanned by N and w_0. By (4.3) we have

$$G(v) \leq a(v) - \lambda_\ell\|v\|^2 + B_2 \leq B_2, v \in N_1. \tag{4.13}$$

Let

$$\begin{aligned} B &= \{w \in M : \|w\|_D = \delta\}, B_n = B \\ A_{1n} &= \{v \in N : \|v\|_D \leq n\} \\ A_{2n} &= \{u = v + sw_0 : v \in N, s \geq 0, \|u\|_D = n\}. \end{aligned}$$

One can show that $A_n = A_{1n} \cup A_{2n}$ links $B_n = B$ for each n (cf., [**ST**, Proposition 1.2]). By (4.11)

$$b_{on} = b_0 \geq \beta - B_1, b \geq -B_1. \tag{4.14}$$

If $b_0 \neq b$, then we can find a sequence in D satisfying

$$G(u_k) \to b, G'(u_k) \to 0 \tag{4.15}$$

by Corollary 2.8. If $b_0 = b$, then we have

$$A_n' = \{u \in A_n : G(u) > b\} \subset A_{2n}$$

since

$$G(v) \leq B_0 < \beta - B_1, v \in A_{1n}$$

by (4.12) and hypothesis (IV). Hence

$$d_n'' = d(A_n', B) \geq d(A_{2n}, B) \geq n - \delta \to \infty.$$

By (4.13), $a_{on} \leq B_2$ for each n. Hence we may apply Corollary 2.5 to conclude that there is a sequence in D such that

$$G(u_k) \to c, \beta - B_1 \leq c \leq B_2, G'(u_k) \to 0. \tag{4.16}$$

In either case we claim that the sequence is bounded in D. For suppose $\rho_k = \|u_k\|_D \to \infty$. Let $\tilde{u}_k = u_k/\rho_k$. Then by (4.15) or (4.16) we have

$$\|u_k\|_D^2 - 2\int_\Omega F(x,u_k)dx = 0(1)$$
$$\|u_k\|_D^2 - \int_\Omega f(x,u_k)u_k dx = o(\rho_k).$$

Thus

$$(\mu - 2)\rho_k^2 - 2\int_\Omega H(x,u_k)dx = o(\rho_k). \tag{4.17}$$

Now

$$\int \rho_k^{-2} H(x,u_k)dx \le \int_\Omega V_3(x)^2 \tilde{u}_k^2 \sigma(\rho_k \tilde{u}_k)dx + \rho_k^{-2}\int_\Omega W_3(x)dx. \tag{4.18}$$

There is a renamed subsequence for which $\tilde{u}_k(x)$ converges a.e. By hypothesis $V_3\tilde{u}_k$ is bounded in $L^2(\Omega)$ as well as $V_3\tilde{u}_k\sigma(u_k)$. Moreover, $V_3\tilde{u}_k\sigma(u_k) \to 0$ a.e. Hence the right hand side of (4.18) converges to 0. But this together with (4.17) implies that $\mu \le 2$, contrary to assumption. Hence the sequence $\{u_k\}$ is bounded. It now follows by standard arguments that $\{u_k\}$ has a subsequence which converges in D and that the limit satisfies $G'(u) = 0$ (cf., e.g., [**Ra2**]). It now follows form (4.9) that the limit is a solution of (4.7). □

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