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ON MODULI OF INSTANTON BUNDLES ON $\mathbb{P}^{2n+1}$

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Let $\mathrm{MI}_{\mathbb{P}^{2n+1}}(k)$ be the moduli space of stable instanton bundles on $\mathbb{P}^{2n+1}$ with $c_2 = k$. We prove that $\mathrm{MI}_{\mathbb{P}^{2n+1}}(2)$ is smooth, irreducible, unirational and has zero Euler-Poincaré characteristic, as it happens for $\mathbb{P}^3$. We find instead that $\mathrm{MI}_{\mathbb{P}^5}(3)$ and $\mathrm{MI}_{\mathbb{P}^5}(4)$ are singular.

1. Definition and preliminaries.

Instanton bundles on a projective space $\mathbb{P}^{2n+1}(\mathbb{C})$ were introduced in [**OS**] and [**ST**]. In [**AO**] we studied their stability, proving in particular that special symplectic instanton bundles on $\mathbb{P}^{2n+1}$ are stable, and that on $\mathbb{P}^5$ every instanton bundle is stable.

In this paper we study some moduli spaces $\mathrm{MI}_{\mathbb{P}^{2n+1}}(k)$ of stable instanton bundles on $\mathbb{P}^{2n+1}$ with $c_2 = k$. For $k = 2$ we prove that $\mathrm{MI}_{\mathbb{P}^{2n+1}}(2)$ is *smooth, irreducible, unirational and has zero Euler-Poincaré characteristic* (Theor. 3.2), just as in the case of $\mathbb{P}^3$ [**Har**].

We find instead that $\mathrm{MI}_{\mathbb{P}^5}(k)$ is *singular* for $k = 3, 4$ (theor. 3.3), which is not analogous with the case of $\mathbb{P}^3$ [**ES**], [**P**]. To be more precise, *all points corresponding to symplectic instanton bundles are singular.* Theor. 3.3 gives, to the best of our knowledge, the first example of a singular moduli space of stable bundles on a projective space. The proof of Theorem 3.3 needs help from a personal computer in order to calculate the dimensions of some cohomology group [**BaS**].

We recall from [**OS**], [**ST**] and [**AO**] the definition of instanton bundle on $\mathbb{P}^{2n+1}(\mathbb{C})$.

Definition 1.1. A vector bundle E of rank $2n$ on $\mathbb{P}^{2n+1}$ is called an instanton bundle of quantum number k if

(i) The Chern polynomial is $c_t(E) = (1 - t^2)^{-k} = 1 + kt^2 + \binom{k+1}{2}t^2 + \ldots$

(ii) $E(q)$ has natural cohomology in the range $-2n - 1 \le q \le 0$ (that is $h^i(E(q)) \ne 0$ for at most one $i = i(q)$)

(iii) $E|_r \simeq \mathcal{O}_r^{2n}$ for a general line r.

Every instanton bundle is simple [**AO**]. There is the following characterization:

Theorem 1.2 ([**ST**], [**AO**]). *A vector bundle E of rank $2n$ on $\mathbb{P}^{2n+1}$ satisfies the properties* (i) *and* (ii) *if and only if E is the cohomology of a monad*

$$\mathcal{O}(-1)^k \xrightarrow{A} \mathcal{O}^{2n+2k} \xrightarrow{B} \mathcal{O}(1)^k. \tag{1.1}$$

With respect to a fixed system of homogeneous coordinates the morphism A (resp. B) of the monad can be identified with a $k \times (2n+2k)$ (resp. $(2n+2k) \times k$) matrix whose entries are homogeneous polynomials of degree 1. Then the conditions that (1.1) is a monad are equivalent to:

$$A, B \text{ have rank } k \text{ at every point } x \in \mathbb{P}^{2n+1}, \ A \cdot B = 0.$$

Definition 1.3. A bundle S appearing in an exact sequence:

$$0 \to S^* \to \mathcal{O}^d \xrightarrow{B} \mathcal{O}(1)^c \to 0 \tag{1.2}$$

is called a Schwarzenberger type bundle (*STB*).

The kernel bundle Ker B in the monad (1.1) is the dual of a STB.

Definition 1.4. An instanton bundle is called special if it arises from a monad (1.1) where the morphism B is defined in some system of homogeneous coordinates $(x_0, \dots, x_n, y_0, \dots, y_n)$ on $\mathbb{P}^{2n+1}$ by the matrix

$$B = \begin{bmatrix} x_0 & & \\ \vdots & \ddots & \\ x_n & & x_0 \\ & \ddots & \vdots \\ & & x_n \\ y_0 & & \\ \vdots & \ddots & \\ y_n & & y_0 \\ & \ddots & \vdots \\ & & y_n \end{bmatrix}$$

Example 1.5. Take

$$A = \begin{bmatrix} & & y_n \cdots y_0 & & -x_n \cdots -x_0 \\ & \cdot\ \ \cdot & & \cdot\ \ \cdot & \\ & \cdot\ \ \cdot & & \cdot\ \ \cdot & \\ & \cdot\ \ \cdot & & \cdot\ \ \cdot & \\ y_n \cdots y_0 & & -x_n \cdots -x_0 & & \end{bmatrix} \qquad B = \begin{bmatrix} x_0 & & \\ \vdots & \ddots & \\ x_n & & x_0 \\ & \ddots & \vdots \\ & & x_n \\ y_0 & & \\ \vdots & \ddots & \\ y_n & & y_0 \\ & \ddots & \vdots \\ & & y_n \end{bmatrix}$$

$E = \operatorname{Ker} B/\operatorname{Im} A$ is a special instanton bundle.

Property (iii) of the definition 1.1 can be checked by the following:

Theorem 1.6 [**OS**]. *Let $E = \operatorname{Ker} B/\operatorname{Im} A$ as in* (1.1). *Let r be the line joining two distinct points $P, Q \in \mathbb{P}^{2n+1}$. Then*

$$E|_r \simeq \mathcal{O}_r^{2n} \Leftrightarrow A(P) \cdot B(Q) \quad \textit{is an invertible matrix.}$$

Example 1.7. Consider the special instanton bundle E of the example 1.5. Let $P = (1, 0, \ldots; 0, \ldots, 0)$, $Q = (0, \ldots, ; 0, \ldots, 1)$. Then

$$A(P) = \begin{bmatrix} & & & -1 \\ & & \cdot & \\ & \cdot & & \\ -1 & & & \end{bmatrix} \qquad B(Q) = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

and $A(P) \cdot B(Q) = \begin{bmatrix} & & & -1 \\ & & \cdot & \\ & \cdot & & \\ -1 & & & \end{bmatrix}$ is invertible. Hence E is trivial on the line $\{x_1 = \ldots = x_n = y_0 = \ldots = y_{n-1} = 0\}$.

Proposition 1.8. *Let E be an instanton bundle as in* (1.1). *Then*

$$H^2(E \otimes E^*) = H^2[(\operatorname{Ker} B) \otimes (\operatorname{Ker} A^t)]$$

Proof. See [**AO**] Theorem 3.13 and Remark 2.22. □

Remark 1.9. If $E \simeq E^*$, then

$$H^2(E \otimes E^*) = H^2[(\operatorname{Ker} A^t) \otimes (\operatorname{Ker} A^t)] = H^2[(\operatorname{Ker} B) \otimes (\operatorname{Ker} B)].$$

Remark 1.10. The single complex associated with the double complex obtained by tensoring the two sequences

$$0 \to \operatorname{Ker} A^t \to \mathcal{O}^{2n+2k} \xrightarrow{A^t} \mathcal{O}(1)^k \to 0$$
$$0 \to \operatorname{Ker} B^t \to \mathcal{O}^{2n+2k} \xrightarrow{B^t} \mathcal{O}(1)^k \to 0$$

gives the resolution

$$\begin{aligned} 0 \to (\operatorname{Ker} A^t) \otimes (\operatorname{Ker} B) &\to \mathcal{O}^{2n+2k} \otimes \mathcal{O}^{2n+2k} \\ &\to \mathcal{O}^{2n+2k} \otimes \mathcal{O}(1)^k \oplus \mathcal{O}(1)^k \otimes \mathcal{O}^{2n+2k} \xrightarrow{\alpha} \mathcal{O}(1)^k \otimes \mathcal{O}(1)^k \to 0 \end{aligned}$$

where $\alpha = (A^t \otimes \mathrm{id}, \mathrm{id} \otimes B)$.

Hence

$$H^2(E \otimes E^*) = \operatorname{Coker} H^0(\alpha)$$

and its dimension can be computed using [**BaS**]. For the convenience of the reader we sketch the steps needed in the computations.

A, B^t are given by $k \times (2n+2k)$ matrices whose entries are linear homogeneous polynomials.

$$A \otimes \mathrm{Id}_k = (a_1, \dots, a_{k(2n+2k)})$$

and

$$\mathrm{Id}_k \otimes B^t = (b_1, \dots, b_{k(2n+2k)})$$

are both $k^2 \times (2n+2k)k$ matrices. Let

$$C = (a_1, \dots, a_{k(2n+2k)}, b_1, \dots, b_{k(2n+2k)}).$$

We will denote by $\operatorname{syz}_m C$ the dimension of the space of the syzygies of C of degree m. Then

$$\begin{aligned} h^2(E \otimes E^*) &= h^0(\mathcal{O}(2)^{k^2}) - (4n+4k)h^0(\mathcal{O}(1)^k) + \operatorname{syz}_1 C \\ &= k(n+1)[k(2n-5) - 8n] + \operatorname{syz}_1 C \\ h^1(E \otimes E^*) &= h^2(E \otimes E^*) + 1 - k^2 + 8n^2k - 4n^2 + 3nk^2 - 2n^2k^2 \\ &= 1 - 6k^2 - 8kn - 4n^2 + \operatorname{syz}_1 C. \end{aligned}$$

Note also that $h^0(E(1)) = \mathrm{syz}_1 B^t - k$ and $h^0(E^*(1)) = \mathrm{syz}_1 A - k$.

Remark 1.11. In the same way we obtain

$$h^1(E \otimes E^*(-1)) = \mathrm{syz}_0 C$$
$$h^2(E \otimes E^*(-1)) = 2k(nk - 2n - k) + \mathrm{syz}_0 C.$$

2. Example on $\mathbb{P}^5$.

Let (a, b, c, d, e, f) be homogeneous coordinates in $\mathbb{P}^5$.

Example 2.1. $(k = 3)$ Let

$$B^t = \begin{bmatrix} a & b & c & & & d & e & f & & \\ & a & b & c & & & d & e & f & \\ & & a & b & c & & & d & e & f \end{bmatrix}$$

$$A = \begin{bmatrix} & & f & e & d & & & -c & -b & -a \\ & f & e & d & & & -c & -b & -a & \\ f & e & d & & & -c & -b & -a & & \end{bmatrix}.$$

The corresponding monad gives a special symplectic instanton bundle on $\mathbb{P}^5$ with $k = 3$. With the notation of remark 1.10, using [**BaS**] we can compute $\mathrm{syz}_0 C = 14, \mathrm{syz}_1 C = 174$. Hence $h^2(E \otimes E^*) = 3$ from the formulas of Remark 1.10. Moreover $h^0(E(1)) = 4$.

Example 2.2. $(k = 3)$ Let B^t as in the Example 2.1 and

$$A = \begin{bmatrix} & & f & e & d & & & -c & -b & -a \\ e & d & & 2f & -b & -a & & & & -2c \\ d & & f & e & -a & & & & -c & -b \end{bmatrix}.$$

We have $\mathrm{syz}_0 C = 10, \mathrm{syz}_1 C = 171$. Hence $h^2(E \otimes E^*) = 0$. We can compute also the syzygies of B^t and A and we get $h^0(E(1)) = 4$, $h^0(E^*(1)) = 3$, hence E is not self-dual.

Example 2.3. $(k = 4)$ Let

$$B^t = \begin{bmatrix} a & b & c & & & & d & e & f & & & \\ & a & b & c & & & & d & e & f & & \\ & & a & b & c & & & & d & e & f & \\ & & & a & b & c & & & & d & e & f \end{bmatrix}$$

$$A = \begin{bmatrix} & & & f & e & d & & & & -c & -b & -a \\ & & f & e & d & & & & -c & -b & -a & \\ & f & e & d & & & & -c & -b & -a & & \\ f & e & d & & & & -c & -b & -a & & & \end{bmatrix}$$

E is a special symplectic instanton bundle with $k = 4$. We compute

$$h^2(E \otimes E^*) = 12.$$

Example 2.4. $(k = 4)$ Let B^t as in the Example 2.3. Let

$$A = \begin{bmatrix} & & & f & e & d & & & & -c & -b & -a \\ e & d & & & & 2f & -b & -a & & & & -2c \\ 3d & & & & f & e & -3a & & & & -c & -b \\ & & f & e & d & & & & -c & -b & -a & \end{bmatrix}.$$

In this case $h^2(E \otimes E^*) = 6$, $h^0(E(1)) = 4$, $h^0(E^*(1)) = 3$.

Example 2.5. $(k = 4)$ Let B^t as in the Example 2.3. Let

$$A = \begin{bmatrix} & & & f & e & d & & & & -c & -b & -a \\ e & d & & & & 2f & -b & -a & & & & -2c \\ 3d & & & & f & e & -3a & & & & -c & -b \\ 5d & & f & e & d+f & e & -5a & & -c & -b & -a-c & -b \end{bmatrix}.$$

Now $H^2(E \otimes E^*) = 0$, $h^0(E(1)) = 4$, $h^0(E^*(1)) = 2$.

3. On the singularities of moduli spaces.

The stable Schwarzenberger type bundles on $\mathbb{P}^m$ (see (1.2)) form a Zariski open subset of the moduli space of stable bundles. Let $N_{\mathbb{P}^m}(k, q)$ be the moduli space of stable STB whose first Chern class is k and whose rank is q. The following proposition is easy and well known:

Proposition 3.1. *The space $N_{\mathbb{P}^m}(k, q)$ is smooth, irreducible of dimension $1 - k^2 - (q + k)^2 + k(q + k)(m + 1)$.*

We denote by $\mathrm{MI}_{\mathbb{P}^{2n+1}}(k)$ the moduli space of stable instanton bundles with quantum number k. It is an open subset of the moduli space of stable $2n$-bundles on $\mathbb{P}^{2n+1}$ with Chern polynomial $(1 - t^2)^{-k}$.

On $\mathbb{P}^5$ (as on $\mathbb{P}^3$) all instanton bundles are stable by [**AO**], Theorem 3.6. $\mathrm{MI}_{\mathbb{P}^{2n+1}}(2)$ is smooth ([**AO**] Theorem 3.14), unirational of dimension $4n^2 + 12n - 3$ and has zero Euler-Poincaré characteristic ([**BE**], [**K**]).

Theorem 3.2. *The space* $\mathrm{MI}_{\mathbb{P}^{2n+1}}(2)$ *is irreducible.*

Proof. The moduli space $N = N_{\mathbb{P}^{2n+1}}(2,\ n + 2)$ of stable STB of rank $2n + 2$ and $c_1 = 2$ is irreducible of dimension $4n^2 + 8n - 3$ by Prop. 3.1

For a given instanton bundle E there is a STB S associated with E, which is stable ([**AO**], Theorem 2.8) and unique (ibid., Prop. 2.17). It is easy to prove that the map $\pi : M \to N$ defined by $\pi([E]) = [S]$ is algebraic, moreover π is dominant by [**ST**]. If $m = [E] \in M$, the fiber $\pi^{-1}(\pi(m))$ is a Zariski open subset of the grassmannian of planes in the vector space $H^0(\mathbb{P}^{2n+1}, S^*(1))$, where $\pi(m) = [S]$; by the Theorem 3.14 of [**AO**], $h^0(\mathbb{P}^{2n+1}, S^*(1)) = 2n+2$, hence $\dim \pi^{-1}(\pi(m)) = 4n$.

In order to prove that M is irreducible, we suppose by contradiction that there are at least two irreducible components M_0 and M_1 of M. Then $M_0 \cap M_1 = \emptyset$ (M is smooth), $\pi(M_0)$ and $\pi(M_1)$ are constructible subset of N by Chevalley's theorem. Looking at the dimensions of M_0, M_1, N and the fibers of π we conclude that both $\pi(M_0)$ and $\pi(M_1)$ must contain an open subset of N, which implies $\pi(M_0) \cap \pi(M_1) \neq \emptyset$ by the irreducibility of N. This is a contradiction because the fibers of π are connected. $\square$

For $n \geq 2$ and $k \geq 3$, it is no longer true that $\mathrm{MI}_{\mathbb{P}^{2n+1}}(k)$ is smooth. In fact on $\mathbb{P}^5$ we have:

Theorem 3.3. *The space* $\mathrm{MI}_{\mathbb{P}^5}(k)$ *is singular for* $k = 3, 4$. *To be more precise, the irreducible component* $M_0(k)$ *of* $\mathrm{MI}_{\mathbb{P}^5}(k)$ *containing the special instanton bundles is generically reduced of dimension* $54(k = 3)$ *or* $65(k = 4)$, *and* $\mathrm{MI}_{\mathbb{P}^5}(k)$ *is singular at the points corresponding to special symplectic instanton bundles.*

Proof. Let E_0 be the special instanton bundle on $\mathbb{P}^5$ of the Example 2.2($k = 3$) or of the Example 2.5($k = 4$). Then $h^2(E_0 \otimes E_0^*) = 0$ and $M_0(k)$ is smooth at the point corresponding to E_0, of dimension $h^1(E_0 \otimes E_0^*) = 54(k = 3)$ or $65(k = 4)$. In particular, $M_0(k)$ is generically reduced. If E_1 is a special symplectic instanton bundle on $\mathbb{P}^5$, the computations in 2.1 and 2.3 show that $h^2(E_1 \otimes E_1^*) = 3(k = 3)$ or $12(k = 4)$, and $h^1(E_1 \otimes E_1^*) = 57$ or 77 respectively. Hence $\mathrm{MI}_{\mathbb{P}^5}(k)$ is singular at E_1 for $k = 3$ and 4. $\square$

Remark 3.4. It is natural to conjecture that $\mathrm{MI}_{\mathbb{P}^{2n+1}}(k)$ is singular for all $n \geq 2$ and $k \geq 3$.

Theorem 3.5. *Let E be an instanton bundle on $\mathbb{P}^{2n+1}$ with $c_2(E) = k$. Then*

$$h^1(E(t)) = 0 \text{ for } t \leq -2 \text{ and } k - 1 \leq t.$$

Proof. The result is obvious for $t \leq -2$. It is sufficient to prove $h^1(S^*(t)) = 0$ for $t \geq k - 1$. We have

$$S^*(t) = \bigwedge^{2n+k-1} S(t-k).$$

Taking wedge products of (1.2) we have the exact sequence

$$0 \to \mathcal{O}(t+1-2n-2k)^{\alpha_0} \to \ldots \to \mathcal{O}(t-k-1)^{\alpha_{2n+k-2}} \\ \to \mathcal{O}(t-k)^{\alpha_{2n+k-1}} \to \bigwedge^{2n+k-1} S(t-k) \to 0$$

for suitable $\alpha_i \in \mathbb{N}$ and from this sequence we can conclude.

Ellia proves Theorem 3.5 in the case of $\mathbb{P}^3$ ([**E**], Prop. IV.1). He also remarks that the given bound is sharp. This holds on $\mathbb{P}^{2n+1}$ as it is shown by the following theorem, which points out that the special symplectic instanton bundles are the "furthest" from having natural cohomology. □

Theorem 3.6. *Let E be a special symplectic instanton bundle on $\mathbb{P}^{2n+1}$ with $c_2 = k$. Then*

$$h^1(E(t)) \neq 0 \textit{ for } -1 \leq t \leq k-2.$$

Proof. For $n = 1$ the thesis is immediate from the exact sequence

$$0 \to \mathcal{O}(t-1) \to E(t) \to \mathcal{J}_C(t+1) \to 0$$

where C is the union of $k+1$ disjoint lines in a smooth quadric surface. Then the result follows by induction on n by considering the sequence

$$0 \to E(t-2) \to E(t-1)^2 \to E(t) \to E(t)|_{\mathbb{P}^{2n-1}} \to 0$$

and the fact that, for a particular choice of the subspace $\mathbb{P}^{2n-1}$, the restriction $E|_{\mathbb{P}^{2n-1}}$ splits as the direct sum of a rank-2 trivial bundle and a special symplectic instanton bundle on $\mathbb{P}^{2n-1}$([**ST**] 5.9). □

Remark 3.7. In [**OT**] it is proved that if E_k is a special symplectic instanton bundle on $\mathbb{P}^5$ with $c_2 = k$ then $h^1(\text{End } E_k) = 20k - 3$.

In the following table we summarize what we know about the component $M_0(k) \subset \text{MI}_{\mathbb{P}^5}(k)$ containing E_k.

Table 3.10

	$h^1(E_k \otimes E_k^*)$	$h^2(E_k \otimes E_k^*)$	$\dim M_0(k)$	$\text{MI}_{\mathbb{P}^5}(k)$
$k=1$	14	0	14	open subset of $\mathbb{P}^{14}$
$k=2$	37	0	37	smooth, irreduc., unirat.
$k=3$	57	3	54	singular
$k=4$	77	12	65	singular
$k \geq 2$	$20k-3$	$3(k-2)^2$	?	?

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Added in proof. After this paper has been written we received a preprint of R. Miró-Roig and J. Orus-Lacort where they prove that the conjecture stated in the Remark 3.4 is true.