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THE QUASI-LINEARITY PROBLEM FOR C^*-ALGEBRAS

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Let $\mathcal{A}$ be a C^*-algebra with no quotient isomorphic to the algebra of all two-by-two matrices. Let μ be a quasi-linear functional on $\mathcal{A}$. Then μ is linear if, and only if, the restriction of μ to the closed unit ball of $\mathcal{A}$ is uniformly weakly continuous.

Introduction.

Throughout this paper, $\mathcal{A}$ will be a C^*-algebra and A will be the real Banach space of self-adjoint elements of $\mathcal{A}$. The unit ball of A is A_1 and the unit ball of $\mathcal{A}$ is $\mathcal{A}_1$. We do not assume the existence of a unit in $\mathcal{A}$.

Definition. A *quasi-linear functional* on A is a function $\mu : A \to \mathbb{R}$ such that, whenever B is an abelian subalgebra of A, the restriction of μ to B is linear. Furthermore μ is required to be bounded on the closed unit ball of A.

Given any quasi-linear functional μ on A we may extend it to $\mathcal{A}$ by defining

$$\tilde{\mu}(x + iy) = \mu(x) + i\mu(y)$$

whenever $x \in A$ and $y \in A$. Then $\tilde{\mu}$ will be linear on each maximal abelian *-subalgebra of $\mathcal{A}$. We shall abuse our notation by writing 'μ' instead of '$\tilde{\mu}$'.

When $\mathcal{A} = \mathrm{M}_2(\mathbb{C})$, the C^* -algebra of all two-by-two matrices over $\mathbb{C}$, there exist examples of quasi-linear functionals on $\mathcal{A}$ which are not linear.

Definition. A local *quasi-linear functional* on A is a function $\mu : A \to \mathbb{R}$ such that, for each x in A, μ is linear on the smallest norm closed subalgebra of A containing x. Furthermore μ is required to be bounded on the closed unit ball of A.

Clearly each quasi-linear functional on A is a local quasi-linear functional. Surprisingly, the converse is false, even when A is abelian (see Aarnes [**2**]). However when A has a rich supply of projections (e.g. when $\mathcal{A}$ is a von Neumann algebra) each local quasi-linear functional is quasi-linear [**3**].

The solution of the Mackey-Gleason Problem shows that every quasi-linear functional on a von Neumann algebra $\mathcal{M}$, where $\mathcal{M}$ has no direct summand of Type I_2, is linear [**4, 5, 6**]. This was first established for positive quasi-linear functionals by the conjunction of the work of Christensen [**7**] and

Yeadon [**11**], and for σ-finite factors by the work of Paschciewicz [**10**]. All build on the fundamental theorem of Gleason [**8**].

Although quasi-linear functionals on general C^* -algebras seem much harder to tackle than the von Neumann algebra problem, we can apply the von Neumann results to make progress. In particular, we prove:

Let $\mathcal{A}$ be a *C^*-algebra with no quotient isomorphic to* $\mathrm{M}_2(\mathbb{C})$. *Let μ be a (local) quasi-linear functional on A. Then μ is linear if, and only if, the restriction of μ to A_1, is uniformly weakly continuous.*

1. Preliminaries: Uniform Continuity.

Let X be a real or complex vector space. Let $\mathcal{F}$ be a locally convex topology for X. Let V be a $\mathcal{F}$ -open neighbourhood of 0. We call V *symmetric* if V is convex and, whenever $x \in V$ then $-x \in V$.

Let B be a subset of X. A scalar valued function on X, μ, is said to be *uniformly continuous* on B, with respect to the $\mathcal{F}$ -topology, if, given any $\epsilon > 0$, there exists an open symmetric neighbourhood of $0, V$, such that whenever $x \in B$, $y \in B$ and $x - y \in V$ then

$$|\mu(x) - \mu(y)| < \epsilon.$$

Lemma 1.1. *Let X be a Banach space and let $\mathcal{F}$ be any locally convex topology for X which is stronger than the weak topology. Let μ be any bounded linear functional on X. Then μ is uniformly $\mathcal{F}$-continuous on X.*

Proof. Choose $\epsilon > 0$. Let

$$\begin{aligned} V &= \{x \in X : |\mu(x)| < \epsilon\} \\ &= \mu^{-1}\{\lambda : |\lambda| < \epsilon\}. \end{aligned}$$

Then V is open in the weak topology of X. Hence V is a symmetric $\mathcal{F}$-open neighbourhood of o such that $x - y \in V$ implies

$$|\mu(x) - \mu(y)| = |\mu(x - y)| < \epsilon.$$

□

Lemma 1.2. *Let X be a subspace of a Banach space Y. Let $\mathcal{G}$ be a locally convex topology for Y which is weaker than the norm topology. Let $\mathcal{F}$ be the relative topology induced on X by $\mathcal{G}$. Let B be a subset of X and let C be the closure of B in Y, with respect to the $\mathcal{G}$ -topology. Let $\mu : B \to \mathbb{C}$ be uniformly continuous on B with respect to the $\mathcal{F}$ -topology. Then there exists* *a function $\overline{\mu} : C \to \mathbb{C}$ which extends μ and which is uniformly $\mathcal{G}$-continuous. Furthermore, if μ is bounded on B then $\overline{\mu}$ is bounded on C.*

Proof. Since $\mathcal{F}$ is the relative topology induced by $\mathcal{G}$, μ is uniformly $\mathcal{G}$-continuous on B. Let K be the closure of $\mu[B]$ in $\mathbb{C}$. Then K is a complete metric space. So, see [**9**, page 125], μ has a unique extension to $\overline{\mu} : C \to K$ where $\overline{\mu}$ is uniformly $\mathcal{G}$-continuous.

If μ is bounded on B then K is bounded and so $\overline{\mu}$ is bounded on C. □

Lemma 1.3. *Let X be a Banach space. Let X_1 be the closed unit ball of X and let X_1^{**} be closed unit ball of X^{**}. Let $\mu : X_1 \to \mathbb{C}$ be a bounded function which is uniformly weakly continuous. Then μ has a unique extension to $\overline{\mu} : X_1^{**} \to \mathbb{C}$ where $\overline{\mu}$ is bounded and uniformly weak*-continuous.*

Proof. Let $\mathcal{G}$ be the weak*-topology on X^{**}. For each $\phi \in X^*$

$$X \cap \{x \in X^{**} : |\phi(x)| < 1\} = \{x \in X : |\phi(x)| < 1\}.$$

So $\mathcal{G}$ induces the weak topology on X. So μ is uniformly $\mathcal{G}$-continuous on X_1. Since X_1 is dense in X_1^{**}, with respect to the $\mathcal{G}$-topology, it follows from Lemma 1.2 that $\overline{\mu}$ exists and has the required properties. □

2. Algebraic Preliminaries.

Lemma 2.1. *Let $\mathcal{B}$ be a non-abelian C^*-subalgebra of a von Neumann algebra $\mathcal{M}$, where $\mathcal{M}$ is of Type I_2. Then $\mathcal{B}$ has a surjective homomorphism onto $\mathrm{M}_2(\mathbb{C})$, the algebra of all two-by-two complex matrices.*

Proof. We have $\mathcal{M} = \mathrm{M}_2(\mathbb{C})\overline{\otimes}C(S)$ where S is hyperstonian. For each $s \in S$ there is a homomorphism π_S from $\mathcal{M}$ onto $\mathrm{M}_2(\mathbb{C})$ defined by

$$\pi_S \begin{Bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{Bmatrix} = \begin{Bmatrix} x_{11}(s) & x_{12}(s) \\ x_{21}(s) & x_{22}(s) \end{Bmatrix}.$$

Clearly, if $\pi_S[\mathcal{B}]$ is abelian for every s then $\mathcal{B}$ is abelian. So, for some s, $\pi_S[\mathcal{B}]$ is a non-abelian*-subalgebra of $\mathrm{M}_2(\mathbb{C})$ and so equals $\mathrm{M}_2(\mathbb{C})$. □

Lemma 2.2. *Let π be a representation of a C^*-algebra $\mathcal{A}$ on a Hilbert space H. Let $\mathcal{M} = \pi[\mathcal{A}]''$ where the von Neumann algebra $\mathcal{M}$ has a direct summand of Type I_2. Then $\mathcal{A}$ has a surjective homomorphism onto $\mathrm{M}_2(\mathbb{C})$.*

Proof. Let e be a central projection of $\mathcal{M}$ such that $e\mathcal{M}$ is of Type I_2. Since $\pi[\mathcal{A}]$ is dense in $\mathcal{M}$ in the strong operator topology, $e\pi[\mathcal{A}]$ is dense in $e\mathcal{M}$. Since $e\mathcal{M}$ is not abelian neither is $e\pi[\mathcal{A}]$. So, by the preceding lemma, $e\pi[\mathcal{A}]$, and hence $\mathcal{A}$, has a surjective homomorphism onto $\mathrm{M}_2(\mathbb{C})$. □

3. Linearity.

We now come to our basic theorem.

Theorem 3.1. *Let $\mathcal{A}$ be a C^*-algebra which has no quotient isomorphic to* $\mathrm{M}_2(\mathbb{C})$. *Let π be a representation of $\mathcal{A}$ on a Hilbert space H. Let $\mathcal{M}$ be the closure of $\mathcal{A}$ in the strong operator-topology of $L(H)$. Let μ be a local quasi-linear functional on $\pi[A]$, which is uniformly continuous on the closed unit ball of $\pi[A]$ with respect to the topology induced on $\pi[A]$ by the strong operator topology of $L(H)$. Then μ is linear.*

Proof. We may suppose, by restricting to a closed subspace of H if necessary, that $\pi[\mathcal{A}]$ has an upward directed net converging, in the strong operator topology to the identity of H. Clearly $\pi[\mathcal{A}]$ has no quotient isomorphic to $\mathrm{M}_2(\mathbb{C})$ for, otherwise, $\mathrm{M}_2(\mathbb{C})$ would be a quotient of $\mathcal{A}$.

So, to simplify our notation we shall suppose that $\mathcal{A} = \pi[\mathcal{A}] \subset \mathrm{L}(H)$.

Let $\mathcal{M}$ be the double commutant of $\mathcal{A}$ in $\mathrm{L}(H)$. Let M_1 be the set of all self-adjoint elements in the unit ball of M. Then, by the Kaplansky Density Theorem, A_1 is dense in M_1 with respect to the strong operator-topology of $\mathrm{L}(H)$.

Then, by Lemma 1.2, there exists $\overline{\mu} : \mathrm{M}_1 \to \mathbb{C}$ such that $\overline{\mu}$ is an extension of $\mu \mid A_1$ and such that $\overline{\mu}$ is continuous with respect to the strong operator topology. Since $\mu[A_1]$ is bounded so, also, is $\overline{\mu}[M_1]$.

We know that for each $a \in A_1$ and each $t \in \mathbb{R}$,

$$\mu(ta) = t\mu(a).$$

We extend the definition of $\overline{\mu}$ to the whole of M by defining

$$\overline{\mu}(x) = \|x\|\overline{\mu}\left(\frac{1}{\|x\|}x\right)$$

whenever $x \in M$ with $\|x\| > 1$. It is then easy to verify that if (a_λ) is a bounded net in A which converges to x in the strong operator topology of $\mathrm{L}(H)$ then

$$\mu(a_\lambda) \to \overline{\mu}(x).$$

Also, whenever $(x_n)(n = 1, 2..)$ is a bounded sequence in M, converging to x in the strong operator topology, then

$$\overline{\mu}(x_n) \to \overline{\mu}(x).$$

Let x be a fixed element of M and let (a_λ) be a bounded net in A which converges to x in the strong operator topology. Then, for each positive whole number $n, a_\lambda^n \to x^n$ in the strong operator topology. So $\mu(a_\lambda^n) \to \overline{\mu}(x^n)$.

Let ϕ_1, ϕ_2 be polynomials with real coefficients and zero constant term. Then, since μ is a local quasi-linear functional,

$$\mu\{\phi_1(a_\lambda)\} + \mu\{\phi_2(a_\lambda)\} = \mu\{(\phi_1 + \phi_2)(a_\lambda)\}.$$

Now

$$\phi_1(a_\lambda) \to \phi_1(x), \phi_2(a_\lambda) \to \phi_2(x).$$

and

$$(\phi_1 + \phi_2)(a_\lambda) \to (\phi_1 + \phi_2)(x)$$

in the strong operator topology. So

$$\overline{\mu}\{\phi_1(x)\} + \overline{\mu}\{\phi_2(x)\} = \overline{\mu}\{\phi_1(x) + \phi_2(x)\}.$$

Let $N(x)$ be the norm-closure of the set of all elements of the form $\phi(x)$, where ϕ is a polynomial with real coefficients and zero constant term. Then, since each norm convergent sequence is bounded and strongly convergent, $\overline{\mu}$ is linear on $N(x)$.

Let $p_1, p_2, ...p_n$ be orthogonal projections in M.

Let

$$x = p_1 + \frac{1}{2}p_2 + ... + \frac{1}{2^{n-1}}p_n + \frac{1}{2^n}\{1 - p_1 - p_2 - ... - p_n\}.$$

Then $(x^k)(k = 1, 2, ...)$ converges in norm to p_1. So p_1 is in $N(x)$. Then

$$\{(2x - 2p_1)^k\}\ (k = 1, 2, ...)$$

converges in norm to p_2. Similarly, $p_3, p_4, ...p_n$ and $1 - p_1 - p_2 - ... - p_n$ are all in $N(x)$.

Let $\nu(p) = \overline{\mu}(p)$ for each projection p in M. Then ν is a bounded finitely additive measure on the projections of M.

Since $\mathcal{A}$ has no quotient isomorphic to $M_2(\mathbb{C})$, it follows from Lemma 2.2 that $\mathcal{M}$ has no direct summand of Type I_2. Hence, by Theorem A of [**4**] or [**6**], ν extends to a bounded linear functional on $\mathcal{M}$, which we again denote by ν. From the argument of the preceding paragraph, $\overline{\mu}$ and ν coincide on finite (real) linear combinations of orthogonal projections. Hence by norm-continuity and spectral theory, $\overline{\mu}(x) = \nu(x)$ for each $x \in M$. Thus μ is linear. □

As an application of the above theorem, we shall see that when a quasi-linear functional μ has a "control functional", it is forced to be linear. We need a definition.

Definition. Let ϕ be a positive linear functional in $\overset{*}{\mathcal{A}}$ and let μ be a quasi-linear functional on $\mathcal{A}$. Then μ is said to be *uniformly absolutely continuous with respect to* ϕ if, given any $\epsilon > 0$ there can be found $\delta > 0$ such that, whenever $b \in A_1$ and $c \in A_1$ and $\phi((b-c)^2) < \delta$, then $|\mu(b) - \mu(c)| < \epsilon$.

Corollary 3.2. *Let $\mathcal{A}$ be a C^*-algebra which has no quotient isomorphic to* $\mathrm{M}_2(\mathbb{C})$. *Let μ be a local quasi-linear functional on $\mathcal{A}$ which is uniformly absolutly continuous with respect to ϕ, where ϕ is a positive linear functional in $\mathcal{A}^*$. Then μ is linear.*

Proof. Let (π, H) be the universal representation of $\mathcal{A}$ on its universal representation space H. We identify $\mathcal{A}$ with its image under π and identify $\pi[\mathcal{A}]''$ with $\mathcal{A}^{**}$.

Let ξ be a vector in H which induces ϕ, that is,

$$\phi(a) = \langle a\xi, \xi\rangle \text{for each } a \in \mathcal{A}.$$

Choose $\epsilon > 0$. Then, by hypothesis, there exists $\delta > 0$ such that, whenever $b \in A_1$ and $c \in A_1$ with

$$\|(b-c)\xi\|^2 < \delta$$

then

$$|\mu(b) - \mu(c)| < \epsilon.$$

So μ is uniformly continuous on A_1, with respect to the strong operator topology of $\mathrm{L}(H)$. Hence, by the preceding theorem μ is linear. □

Theorem 3.3. *Let $\mathcal{A}$ be a C^*-algebra with no quotient isomorphic to* $\mathrm{M}_2(\mathbb{C})$. *Let μ be a (local) quasi-linear functional on A. Then μ is a bounded linear functional if, and only if, μ is uniformly weakly continuous on the unit ball of A.*

Proof. By Lemma 1.1 each bounded linear functional on A is uniformly weakly continuous. We now assume that μ is uniformly weakly continuous on A_1. Let (π, H) be the universal representation of $\mathcal{A}$. Let $\mathcal{M} = \pi[\mathcal{A}]''$. Then A^{**} can be identified with $\mathcal{M}$ and A^{**} with M.

By Lemma 1.3 there exists a function $\overline{\mu} : M_1 \to \mathbb{C}$ which is uniformly continuous with respect to the weak*-topology on M_1 and such that $\overline{\mu}|A_1$ coincides with $\mu|A_1$.

The weak*-topology on M_1 coincides with the weak-operator topology of $\mathrm{L}(H)$, restricted to M_1. This is weaker than the strong operator-topology restricted to M_1. So $\overline{\mu}$ is uniformly continuous on M_1 with respect to the strong operator topology of $\mathrm{L}(H)$. Thus μ is uniformly continuous on A_1

with respect to the strong operator topology of $\mathbb{L}(H)$. Then, by Theorem 3.1, μ is linear. □

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