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FROM THE L^1 NORMS OF THE COMPLEX HEAT KERNELS TO A HÖRMANDER MULTIPLIER THEOREM FOR SUB-LAPLACIANS ON NILPOTENT LIE GROUPS

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This paper aims to prove a Hörmander multiplier theorem for sub-Laplacians on nilpotent Lie groups. We investigate the holomorphic functional calculus of the sub-Laplacians, then we link the L^1 norm of the complex time heat kernels with the order of differentiability needed in the Hörmander multiplier theorem. As applications, we show that order $d/2+1$ suffices for homogeneous nilpotent groups of homogeneous dimension d, while for generalised Heisenberg groups with underlying space $\mathbf{R}^{2n+k}$ and homogeneous dimension $2n+2k$, we show that order $n+(k+5)/2$ for k odd and $n+3+k/2$ for k even is enough; this is strictly less than half of the homogeneous dimension when k is sufficiently large.

1. Introduction.

We begin with the classical Laplacian $(-\Delta)$ on the Euclidean space $\mathbf{R}^d$. The multiplier theorem of L. Hörmander [**Ho**] gives a sufficient condition on a function $m : \mathbf{R}^+ \to \mathbf{C}$ for the operator $m(-\Delta)$ to be bounded on $L^p(\mathbf{R}^d)$ whenever $1 < p < \infty$, namely, when m satisfies the condition that

$$\lambda^k \left| m^{(k)}(\lambda) \right| \le c \qquad \forall \lambda \in \mathbf{R}^+ \tag{1}$$

for $0 \le k \le s = [d/2]+1$ where c is a constant and $[d/2]$ is the integral part of $d/2$.

By using fractional differentiation, the value of s in condition (1) can be improved slightly but it is known that for $(-\Delta)$ on $\mathbf{R}^d$, the value cannot be improved beyond $s = d/2$. We call s the order of the Hörmander multiplier theorem.

A lot of work has been done to obtain results of this type for other operators. E.M. Stein [**St**] proved a general result for a large class of operators, but only when the function m is of Laplace transform type, a rather restrictive condition. This was later improved by M. Cowling [**Co**], using the transference method and interpolation. For the sub-Laplacian L on a homogeneous nilpotent Lie group G of homogeneous dimension d, the following results are known. A. Hulanicki and Stein proved a Hörmander multiplier theorem for L with order $3d/2+2$ when G is a stratified group; this was reported by G.B. Folland and Stein [**FS**]. L. De Michele and G. Mauceri [**DM**] improved Hulanicki and Stein's results and obtained order $d/2+1$. Recently, M. Christ [**Ch**] investigated the problem carefully, and proved a Hörmander multiplier theorem with order $d/2$ when G is a homogeneous nilpotent group. His principal result was then reproved and extended by Mauceri and S. Meda [**MM**].

All the above results rely on certain estimates on the heat kernels, L^2 information derived from the spectral theorem, and the Calderón–Zygmund operator theory. However, the factors controlling the order s were to some extent hidden by the complexity of the proofs.

One open question is whether the condition $s \geq d/2$ is necessary as in the Euclidean case [**Ch**]. Another natural question is to decide what factors control the order s. It seemed that $s = d/2$ is the optimal value [**Ch**], but recently D. Müller and Stein (conference announcement) showed that for the Heisenberg group of homogeneous dimension $2n+2$, the order can be lowered to $n+1/2$.

In this paper, we show that the order s is controlled by the behaviour of the $L^1(G)$ norm of the heat kernels for complex time (Theorem 2). As a corollary, we obtain $s = d/2+1$ for homogeneous nilpotent groups (Theorem 3). Although this order is not optimal, our proof is different from and much easier than the previous proofs. Further, if G is the generalised Heisenberg group of homogeneous dimension $2n+2k$, with underlying manifold $\mathbf{R}^{2n+k}$, then we obtain a Hörmander multiplier theorem with order $s = n+k/2+\beta$ where $\beta = 5/2$ for k odd and $\beta = 3$ for k even (Theorem 4). This order is strictly less than half the homogeneous dimension when k is sufficiently large.

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2. H_∞ functional calculus.

The references for this section are the papers of A. McIntosh [**Mc**] and Cowling, I. Doust, McIntosh, and A. Yagi [**CDMY**].

Definition. A closed operator L in a Banach space X is said to be of type ω, $0 \leq \omega < \pi$, if its spectrum is a subset of the closed sector $S_\omega = \{z \in C \mid |\arg z| \leq \omega\} \cup \{0\}$, and the resolvents $(L-\lambda I)^{-1}$ satisfy the inequality

$$\|(L-\lambda I)^{-1}\| \leq c_\mu |\lambda|^{-1}$$

when $|\arg\lambda| \geq \mu > \omega$.

For $\mu > \omega$, let $H_\infty(S^0_\mu)$ be the usual space of bounded holomorphic functions in the open sector S^0_μ, which is just the interior of S_μ. Further, define

$$\Psi(S^0_\mu) = \left\{ m \in H_\infty(S^0_\mu) \mid \exists s > 0, c > 0 \text{ such that } |m(z)| \leq \frac{c|z|^s}{1+|z|^{2s}} \right\}.$$

Suppose that $\omega < \theta < \mu$. Let γ be the contour defined by the function

$$\gamma(t) = \begin{cases} te^{i\theta} & \text{if } 0 \leq t < \infty \\ -te^{-i\theta} & \text{if } -\infty < t \leq 0. \end{cases}$$

We adopt the definitions of H_∞ functional calculus of [**Mc**], as follows. For $m \in \Psi(S^0_\mu)$, then

$$m(L) = \frac{1}{2\pi i} \int_\gamma (L - \lambda I)^{-1} m(\lambda)\, d\lambda.$$

The above integral is absolutely convergent in the norm topology and $m(L)$ is a bounded linear operator which is independent of the choice of θ. For $m \in H_\infty(S^0_\mu)$, we define

$$m(L) = \frac{1}{2\pi i}(I + L)^2 L^{-1} \int_\gamma (L - \lambda I)^{-1} \frac{\lambda m(\lambda)}{(1+\lambda)^2}\, d\lambda$$

when L is a one to one operator of type ω with dense domain and dense range. This definition is consistent with the previous one when $m \in \Psi(S^0_\mu)$.

We now define $\Lambda^\alpha_{\infty,1}(\mathbf{R}^+)$ to be the class of all bounded measurable functions $m : \mathbf{R}^+ \to \mathbf{C}$ such that $\|m\|_{\Lambda^\alpha_{\infty,1}} < \infty$, where

$$\|m\|_{\Lambda^\alpha_{\infty,1}} = \|m\|_\infty + \sum_{n\in\mathbf{Z}} 2^{|n|\alpha} \|(m \circ \exp) * \phi_n^\vee\|_\infty\,;$$

in this definition, for all ξ in $\mathbf{R}$,

$$\begin{aligned} \phi_0(\xi) &= (2 - 2|\xi|)_+ - (1 - 2|\xi|)_+, \\ \phi_1(\xi) &= (1 - 2|\xi - 1|)_+ + \left(\frac{1}{2} - \left|\xi - \frac{3}{2}\right|\right)_+, \end{aligned}$$

and

$$\phi_{n\epsilon}(\xi) = \phi_1(2^{-n}\epsilon\xi),$$

when $n = 1, 2, 3, \ldots$ and $\epsilon = \pm 1$; here $\phi^\vee$ denotes the inverse Fourier transform of ϕ. It is not hard to check, using Fourier analysis, that if condition

(1) holds when $k = 0, 1, 2, ..., s$, then $m \in \Lambda^{\alpha}_{\infty,1}(\mathbf{R}^+)$ when $\alpha < s$. It was observed by Coifman that there were similarities between having functional calculus for bounded analytic functions in all sectors and Hörmander-type theorems. The following theorem is proved in [**CDMY**] (Theorem 4.10).

Theorem 1. *Suppose that L is a one-one operator of type 0 in $L^p(X)$, $1 < p < \infty$. Then the following conditions are equivalent:*

(i) *L admits a bounded $H_\infty(S^0_\mu)$ functional calculus for all positive μ and there exist positive constants C and α such that*

$$\|m(L)\| \leq C\mu^{-\alpha} \|m\|_{H_\infty(S^0_\mu)} \qquad \forall m \in H_\infty(S_\mu) \quad \forall \mu > 0; \tag{2}$$

(ii) *L admits a bounded $\Lambda^{\alpha}_{\infty,1}(\mathbf{R}^+)$ functional calculus.*

In this paper, we prove that the H_∞ functional calculus of the sub-Laplacian on a homogeneous nilpotent group satisfies (2), hence there is a Hörmander type functional calculus. Note that to establish the existence of the H_∞ functional calculus, we just need to prove (2) for m in $\Psi(S_\mu)$, for the extension to m in $H_\infty(S_\mu)$ then follows from the Convergence Lemma in [**CDMY**] (Lemma 2.1).

In the rest of this paper, the constants C and c may vary from line to line.

3. The L^1 norms of the heat kernels and the Hörmander multiplier theorem.

Let $\underline{\mathbf{g}}$ be a finite dimensional nilpotent Lie algebra. Assume that

$$\underline{\mathbf{g}} = \oplus_{i=1}^{m} \underline{\mathbf{g}}_i$$

as a vector space, where $[\underline{\mathbf{g}}_i, \underline{\mathbf{g}}_j] \subseteq \underline{\mathbf{g}}_{i+j}$ for all i, j, and $\underline{\mathbf{g}}_1$ generates $\underline{\mathbf{g}}$ as a Lie algebra.

Let G be the associated connected, simply connected Lie group. Then G has homogeneous dimension d given by the formula

$$d = \sum_{j=1}^{m} j \dim(\underline{\mathbf{g}}_j),$$

where $\dim(\underline{\mathbf{g}}_j)$ denotes the dimension of $\underline{\mathbf{g}}_j$.

Consider any finite subset $\{X_k\}$ of $\underline{\mathbf{g}}_1$ which spans $\underline{\mathbf{g}}_1$. Each X_k can be identified with a unique left invariant vector field on G. Define

$$L = -\sum_k X_k^2;$$

then L is a left invariant second order differential operator. We define $L^p(G)$ with respect to Haar measure (and denote the corresponding norms by $\|\cdot\|_p$), then L is non-negative self-adjoint on $L^2(G)$ and it admits a spectral resolution

$$L = \int_0^\infty \lambda \, dP_\lambda.$$

For any bounded Borel function on $[0, \infty)$, we can define

$$m(L) = \int_0^\infty m(\lambda) \, dP_\lambda$$

which is bounded on $L^2(G)$, and the corresponding operator norm, which we denote by $\|m(L)\|_{2\to 2}$, satisfies $\|m(L)\|_{2\to 2} = \|m\|_\infty$.

Note that the operators $m(L)$ given by the spectral theorem and in Section 2 are identical when both definitions are applicable.

We need the following lemma which gives the upper bounds on the heat kernel and its derivatives.

Lemma . *Let h_z be the kernel of e^{-zL},* Re $z > 0$, *and* $\arg z = \theta$. *Then the following estimates hold:*

$$|h_z(x)| \le C\,(|z|\cos\theta)^{-\frac{d}{2}} \exp\left\{-c\cos\theta \frac{\|x\|^2}{|z|}\right\} \tag{3}$$

$$|X_i h_z(x)| \le C\,(|z|\cos\theta)^{-\frac{d+1}{2}} \exp\left\{-c\cos\theta \frac{\|x\|^2}{|z|}\right\}. \tag{4}$$

Proof. The following estimates on the heat kernel $h_t(x)$ and its derivatives for $t > 0$ are well known (e.g. see Saloff-Coste [**Sa**] and its references):

$$|h_t(x)| \le C\,t^{-\frac{d}{2}} \exp\left\{-c\frac{\|x\|^2}{t}\right\}$$

$$|X_i h_t(x)| \le C\,t^{-\frac{d+1}{2}} \exp\left\{-c\frac{\|x\|^2}{t}\right\}.$$

The required estimates then follow by interpolation as in Theorem 3.4.8 of Davies [**Da**]. □

We now represent the operator $m(L)$, using the semigroup e^{-zL}.

As in Section 2, for $m \in \Psi(S_\delta)$, we choose the contour $\gamma = \gamma_- + \gamma_+$, where

$$\gamma_+(t) = te^{i\mu} \quad \text{if } 0 \le t < \infty$$
$$\gamma_-(t) = -te^{-i\mu} \quad \text{if } -\infty < t \le 0$$

with $\delta > \mu$, and write

$$m(L) = \frac{1}{2\pi i} \int_\gamma (L - \lambda I)^{-1} m(\lambda)\, d\lambda.$$

Assume $\lambda \in \gamma_+$; then we have

$$(L - \lambda I)^{-1} = \int_{\Gamma_+} e^{\lambda z} e^{-zL} dz$$

where the curve Γ_+ is defined by $\Gamma_+(t) = te^{i\theta}$ for $t \geq 0$ and $\theta = (\pi - \mu)/2$. Therefore

$$\begin{aligned} m_+(L) &= \frac{1}{2\pi i} \int_{\gamma_+} \left[\int_{\Gamma_+} e^{\lambda z} e^{-zL}\, dz \right] m(\lambda)\, d\lambda \\ &= \int_{\Gamma_+} \left[\frac{1}{2\pi i} \int_{\gamma_+} e^{\lambda z} m(\lambda)\, d\lambda \right] e^{-zL}\, dz, \end{aligned}$$

by a change in the order of integration. Define Γ_- similarly: $\Gamma_-(t) = te^{-i\theta}$ for $t \geq 0$. A similar argument shows that

$$\begin{aligned} m_-(L) &= \frac{1}{2\pi i} \int_{\gamma_-} \left[\int_{\Gamma_-} e^{\lambda z} e^{-zL}\, dz \right] m(\lambda)\, d\lambda \\ &= \int_{\Gamma_-} \left[\frac{1}{2\pi i} \int_{\gamma_-} e^{\lambda z} m(\lambda)\, d\lambda \right] e^{-zL}\, dz, \end{aligned}$$

and therefore

$$m(L) = \int_{\Gamma_+} e^{-zL} n_+(z)\, dz + \int_{\Gamma_-} e^{-zL} n_-(z)\, dz,$$

where

$$n_\pm(z) = \frac{1}{2\pi i} \int_{\gamma_\pm} e^{\lambda z} m(\lambda)\, d\lambda,$$

which implies the bound

$$|n_\pm(z)| \leq \frac{1}{2\pi} \|m\|_\infty (\cos\theta)^{-1} |z|^{-1}. \tag{5}$$

Consequently, the kernel of $K_m(x)$ of $m(L)$ is given by

$$K_m(x) = \int_{\Gamma_+} h_z(x)\, n_+(z)\, dz + \int_{\Gamma_-} h_z(x)\, n_-(z)\, dz. \tag{6}$$

We now state our main theorem.

Theorem 2. *Let h_z be the kernel of e^{-zL}, $\operatorname{Re} z > 0$, $\arg z = \theta$. Assume that for some $\ell > 0$ the $L^1(G)$ norm of the complex time heat kernel h_z satisfies*

$$\|h_z\|_1 \le C\,(\cos\theta)^{-\ell}.$$

Then the operator $m(L)$ can be extended to a bounded operator on $L^p(G)$ for all $p \in (1,\infty)$ if the function m satisfies the Hörmander condition (1) *of order $s = \ell + 1$.*

Proof. We denote the Haar measure by dx and the control distance associated to the sub-Laplacian L by d. We write $d(e,x) = \|x\|$, where e is the identity element of G.

Our plan of proof is to prove that L has a bounded holomorphic functional calculus as in (i) of Theorem 1 with $\alpha = l+1$. Then Theorem 2 follows from Theorem 1.

Let $m \in \Psi(S_\mu^0)$. To apply Calderón–Zygmund operator theory, we first prove the following estimate

$$I = \int_{\|x\|\ge 2\|y\|} |K_m(x) - K_m(y^{-1}x)|\,dx \le C\,\|m\|_\infty\,(\cos\theta)^{-(l+1+\epsilon)}. \tag{7}$$

Using (5) and (6), and changing the order of integration, we have

$$I \le C\,\|m\|_\infty\,(\cos\theta)^{-1}\int_\Gamma\int_{\|x\|\ge 2\|y\|} |h_z(x) - h_z(y^{-1}x)|\,dx\,|z|^{-1}\,d|z|, \tag{8}$$

where $\int_\Gamma$ is short for $\int_{\Gamma_+} + \int_{\Gamma_-}$. We write

(9)

$$\int_{\|x\|\ge 2\|y\|} |h_z(x) - h_z(y^{-1}x)|\,dx$$
$$= \left(\int_{\|x\|\ge 2\|y\|} |h_z(x) - h_z(y^{-1}x)|\,dx\right)^{\alpha}\left(\int_{\|x\|\ge 2\|y\|} |h_z(x) - h_z(y^{-1}x)|\,dx\right)^{1-\alpha},$$

where α will be specified later. We estimate the second factor:

(10)

$$\left(\int_{\|x\|\ge 2\|y\|} |h_z(x) - h_z(y^{-1}x)|\,dx\right)^{1-\alpha} \le (2\,\|h_z\|_1)^{1-\alpha} \le C\,(\cos\theta)^{-\ell(1-\alpha)}.$$

To estimate the first factor, we use the upper bound on $X_i h_z$ in the lemma to obtain

$$\int_{\|x\|\ge 2\|y\|} |h_z(x) - h_z(y^{-1}x)|\,dx \le C\,\|y\|\int_{\|x\|\ge\|y\|} (|z|\cos\theta)^{-\frac{d+1}{2}} e^{-c\cos\theta\|x\|^2/|z|}\,dx, \tag{11}$$

where the constant c in the right hand side of (11) is half the constant c in the right hand side of (4). We now use polar coordinates in G, and deduce that

$$\begin{aligned}
&\int_{\|x\|\geq 2\|y\|} \left|h_z(x) - h_z(y^{-1}x)\right| dx \\
&\leq C\,\|y\|\,(|z|\cos\theta)^{-\frac{d+1}{2}} \int_{\|y\|}^{\infty} e^{-c\cos\theta r^2/|z|} r^{d-1} dr \\
&= C\,\|y\|\,(|z|\cos\theta)^{-\frac{d+1}{2}} \left(\frac{|z|}{c\cos\theta}\right)^{\frac{d}{2}} \int_{c\|y\|^2\cos\theta/|z|}^{\infty} e^{-s} s^{\frac{d}{2}-1} ds \\
&\leq C\left(\frac{\|y\|^2\cos\theta}{|z|}\right)^{\frac{1}{2}} (\cos\theta)^{-d-1} e^{-c\|y\|^2\cos\theta/|z|} \left[1 + \left(\frac{\|y\|^2\cos\theta}{|z|}\right)^{\frac{d}{2}-1}\right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
(12)\qquad &\int_\Gamma \left(\int_{\|x\|\geq 2\|y\|} \left|h_z(x) - h_z(y^{-1}x)\right| dx\right)^{\alpha} |z|^{-1}\, d|z| \\
&\leq C \int_\Gamma \left(\left(\frac{\|y\|^2\cos\theta}{|z|}\right)^{\frac{1}{2}} (\cos\theta)^{-d-1} e^{-c\|y\|^2\cos\theta/|z|}\right. \\
&\qquad \left.\cdot\left[1 + \left(\frac{\|y\|^2\cos\theta}{|z|}\right)^{\frac{d}{2}-1}\right]\right)^{\alpha} |z|^{-1}\, d|z| \\
&\leq C\int_0^\infty \left(t^{\frac{1}{2}}(\cos\theta)^{-d-1} e^{-ct}\left[1 + t^{\frac{d}{2}-1}\right]\right)^{\alpha} t^{-1} dt \\
&\leq c_\alpha\,(\cos\theta)^{-\alpha(d+1)},
\end{aligned}$$

where c_α becomes large as $\alpha \to 0$. We combine the inequalities (8) to (10) and (12), to get
(13)

$$I = \int_{\|x\|\geq 2\|y\|} \left|K_m(x) - K_m(y^{-1}x)\right| dx \leq c_\alpha\,\|m\|_\infty\,(\cos\theta)^{-1-\ell(1-\alpha)-\alpha(d+1)}.$$

By choosing α in (13) sufficiently small, interpolation shows that for any p, $1 < p \leq 2$, there exists a constant $c_{\epsilon,p}$ for any $\epsilon > 0$ such that

$$\|m(L)\|_{L^p(G)} \leq c_{\epsilon,p}\,\|m\|_\infty\,(\cos\theta)^{-\ell-1-\epsilon}.$$

We now fix p, $1 < p \leq 2$. To get rid of ϵ, we choose $p_1 = \frac{p+1}{2}$ and ϵ sufficiently small in the estimate of $\|m(L)\|_{L^{p_1}(G)}$, then interpolation between p_1 and 2 gives us the desired estimate.

The case $p > 2$ follows from duality. □

4. Hörmander multiplier theorems for sub-Laplacians on Lie groups.

4.1. Nilpotent Lie groups. Theorem 2 reduces the difficult task of controlling the kernel K_m of the operator $m(L)$ as in (7) to finding the $L^1(G)$ norms of the complex heat kernels h_z. The obvious next question is how large the norms $\|h_z\|_1$ are.

To obtain a sharp estimate on $\|h_z\|_1$ in the general setting of nilpotent Lie groups might be difficult but we can get a useful upper bound on $\|h_z\|_1$ without much difficulty. That result is the content of the following theorem.

Theorem 3. *Let L be a sub-Laplacian on a homogeneous nilpotent Lie group G of homogeneous dimension d, as in Section 3. Then for each $\epsilon > 0$, there exists $c_\epsilon > 0$ such that the $L^1(G)$ norms of the complex heat kernels satisfy*

$$\|h_z\|_1 \leq c_\epsilon (\cos \arg z)^{-\frac{d}{2}-\epsilon}.$$

Consequently, the operator $m(L)$ can be extended to a bounded operator on $L^p(G)$ for all $p \in (1, \infty)$ if m satisfies the Hörmander condition (1) up to order $s = \frac{d}{2} + 1$.

Proof. We first estimate the $L^2(G)$ norms of the complex heat kernels as follows. Let $z = t + iv$ and denote the norm of the operator e^{-zL} from $L^2(G)$ to $L^\infty(G)$ by $\|e^{-zL}\|_{2\to\infty}$. We then have

$$\|h_z\|_2 = \|e^{-zL}\|_{2\to\infty}.$$

By spectral theory, e^{-ivL} is an isometry on $L^2(G)$, so

$$\|e^{-zL}\|_{2\to\infty} = \|e^{-tL}\|_{2\to\infty}.$$

We conclude that

$$\|h_z\|_2 = \|h_t\|_2 = Ct^{-\frac{d}{4}} = C(\text{Re } z)^{-\frac{d}{4}}. \tag{14}$$

The middle equality holds by homogeneity.

We observe that by homogeneity, $\|h_z\|_1 = \|h_{z/|z|}\|_1$, hence we can assume $|z| = 1$.

To estimate $\|h_z\|_1$, we denote $\cos \arg z$ by σ, choose $\beta = \frac{1}{2} + v$ and break G into two parts:

$$G_1 = \{x \in G \mid \|x\| < \sigma^{-\beta}\}$$
$$G_2 = \{x \in G \mid \|x\| \geq \sigma^{-\beta}\}.$$

We then have

$$\begin{aligned}(15)\qquad \int_{G_1} |h_z(x)|\,dx &\le (\mathrm{vol}\ G_1)^{\frac{1}{2}} \left(\int_{G_1} |h_z(x)|^2\,dx\right)^{\frac{1}{2}} \\ &\le (\mathrm{vol}\ G_1)^{\frac{1}{2}} \left(\int_{G} |h_z(x)|^2\,dx\right)^{\frac{1}{2}} \\ &\le C\,\sigma^{-\frac{d}{2}-\frac{dv}{2}}.\end{aligned}$$

To estimate $\int_{G_2} |h_z(x)|\,dx$, we use the estimate (3) of the lemma, and then integrate in polar coordinates. It turns out that

$$\begin{aligned}(16)\qquad \int_{G_2} |h_z(x)|\,dx &\le C\int_{\sigma^{-\beta}}^{\infty} \sigma^{-\frac{d}{2}} \exp\{-c\sigma r^2\} r^{d-1} dr \\ &= C\sigma^{-d}\int_{\sigma^{1-2\beta}}^{\infty} \exp\{-cs\} s^{\frac{d}{2}-1} ds \\ &\le c_{d,v}\end{aligned}$$

where $c_{d,v}$ depends only on d and v. It follows from (15) and (16) that by choosing $v = \frac{2\epsilon}{d}$, there exists c_ϵ such that

$$\|h_z\| \le c_\epsilon\,\sigma^{-\frac{d}{2}-\epsilon}.$$

To complete the proof, we apply Theorem 2, and then interpolate to get rid of ϵ (as in the proof of Theorem 2). □

4.2. Generalised Heisenberg groups. In the proof of Theorem 3, estimate (14) shows that the $L^2(G)$ norm of the complex heat kernels is a multiple of $(\cos\arg z)^{-d/4}$. If we use this estimate to obtain an upper bound for the $L^1(G)$ norm of the complex heat kernels, we have the power $d/2$. This is the reason why our Theorem 3 as well as previously known proofs which utilise the $L^2(G)$ estimate only obtain order $s \ge d/2$.

To improve the order beyond half the homogeneous dimension, we need a sharper estimate on the $L^1(G)$ norm of the complex heat kernels. This can be done for the generalised Heisenberg groups (or H-type groups).

We now give a brief definition of generalised Heisenberg groups. For more details, see the thesis of J. Randall [**Ra1**] and its references.

Let $\underline{\mathbf{g}}$ be a 2-step nilpotent Lie algebra with an inner product. Let ζ be the centre of $\underline{\mathbf{g}}$ and ϑ the orthogonal complement of ζ in $\underline{\mathbf{g}}$. For $v \in \vartheta$, let $f_\vartheta = (\ker \mathrm{ad}\ _v) \cap \vartheta$, and denote by ϑ_v the orthogonal complement of f_ϑ in ϑ. Then $\underline{\mathbf{g}}$ is called an H-type algebra or a generalised Heisenberg algebra if $\mathrm{ad}\ _v : \vartheta_v \to \zeta$ is a surjective isometry for every unit vector $v \in \vartheta$.

The connected simply connected Lie group G, associated with $\underline{\mathbf{g}}$ is called an H-type or generalised Heisenberg group.

For the generalised Heisenberg algebra $\underline{\mathbf{g}} = \vartheta \oplus \zeta$, let $\dim(\vartheta) = 2n$ and $\dim(\zeta) = k$; then G is a stratified group with dilations $\gamma_r(v, \xi) = (rv, r^2\xi)$, for $(v, \xi) \in \vartheta \oplus \zeta$, and homogeneous dimension $d = 2n + 2k$.

We can also define the sub-Laplacian L on G. The heat kernel $h_z(x)$ has an explicit representation which can be used to estimate its $L^1(G)$ norm, (see [**Ra1**]). Our next theorem is

Theorem 4. *The $L^1(G)$ norm for h_z satisfies the following estimate:*

$$\|h_z\|_1 \leq \frac{c}{(\cos \arg z)^{n+\ell}} \text{ where } \ell = \begin{cases} \frac{k+3}{2} & \text{for } k \text{ odd} \\ \frac{k}{2} + 2 & \text{for } k \text{ even.} \end{cases}$$

Hence the operator $m(L)$ can be extended to a bounded operator on $L^p(G)$ for all $p \in (1, \infty)$, if m satisfies the Hörmander condition (1) *up to order*

$$s = \begin{cases} n + \frac{k+5}{2} & \text{for } k \text{ odd} \\ n + \frac{k}{2} + 3 & \text{for } k \text{ even.} \end{cases}$$

Proof. The estimate on the $L^1(G)$ norm of the complex time heat kernels, which uses the explicit representation of the heat kernels, is the main result of [**Ra2**].

The second part of this theorem is a consequence of Theorem 2. □

NOTE:

(a) The order s obtained in this theorem is strictly less than half of the homogeneous dimension when k is sufficiently large.

(b) The Hörmander multiplier result in Theorem 4 can be obtained by direct estimate on the kernel K_m of the operator $m(L)$, using the explicit representation of the complex heat kernels [**D2**].

(c) After this paper was written up, it came to the author's knowledge that, by using the real variable method, W. Hebisch was successful in proving that on a product of generalised Heisenberg groups Hörmander type multiplier theorem for the sub-Laplacian is true with the order $s = \frac{D}{2} + \epsilon$, $\epsilon > 0$, where D is the euclidean dimension of the group [**He**].

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