DUALITY FOR THE QUANTUM $E(2)$ GROUP

A. VAN DAELE AND S.L. WORONOWICZ

The quantum deformation $E_\mu(2)$ of the group of motions of the plane on the C^*-level was obtained by the second author. He also constructed the Pontryagin dual $\widehat{E}_\mu(2)$ explicitly, together with the unitary bicharacter describing this duality. In this paper we show that the dual of $\widehat{E}_\mu(2)$ is again $E_\mu(2)$.

0. Introduction.

Let μ be any number in $]0,1[$. Consider the $*$-algebra A over $\mathbb{C}$ with an identity, generated by a unitary element v and a normal element n satisfying $vn = \mu nv$. This $*$-algebra can be made into a Hopf $*$-algebra when the comultiplication Φ is defined by

$$\Phi(v) = v \otimes v$$
$$\Phi(n) = v \otimes n + n \otimes v^*.$$

Then, the counit ϵ is given by $\epsilon(v) = 1$ and $\epsilon(n) = 0$. The antipode S is given by $S(v) = v^*$ and $S(n) = -\mu^{-1}n$.

On the other hand, consider the $*$-algebra B over $\mathbb{C}$ with identity, generated by a self-adjoint invertible element a and a normal element b satisfying $ab = \mu ba$. Again B can be made into a Hopf $*$-algebra when Φ is defined by

$$\Phi(a) = a \otimes a$$
$$\Phi(b) = a \otimes b + b \otimes a^{-1}.$$

Then, the counit is given by $\epsilon(a) = 1$ and $\epsilon(b) = 0$ while the antipode is determined by $S(a) = a^{-1}$ and $S(b) = -\mu^{-1}b$.

The pair (A, B) becomes a dual pair of Hopf $*$-algebras when the pairing is defined by

$$\langle v, a \rangle = \mu^{\frac{1}{2}} \qquad \langle n, a \rangle = 0$$
$$\langle v, b \rangle = 0 \qquad \langle n, b \rangle = 1.$$

This pairing is non-degenerate. (For details, see e.g. [5, 6] and [8].)

The algebra A can be considered as a quantized version - on the Hopf *-algebra level - of the group $E(2)$ of motions of the plane. To obtain these two quantum groups on the C*-algebra level (i.e. the quantized version of the continuous functions on the group that tend to zero at infinity), turns out to be non-trivial and certainly more complicated than for the examples of compact quantum groups and the (non-compact) Heisenberg group (see e.g. [9]). The reason is the following. When representing the algebras A and B by means of operators on a Hilbert space $\mathcal{H}$, it turns out that, although the elements $\Phi(n)$ and $\Phi(b)$ are normal in the *-algebras $A \otimes A$ and $B \otimes B$ respectively, these elements are only formally normal as operators on $\mathcal{H} \otimes \mathcal{H}$. In fact, only with certain restrictions on the spectrum, these operators have proper normal extensions. This is necessary to define the comultiplication on the C*-level. (See [1] and [7].)

It was the second author who developed the necessary machinery to deal with these problems. In [1] the theory of unbounded elements affiliated with a C*-algebra is developed and this is used to obtain the quantization $E_\mu(2)$ of the group of motions of the plane on the C*-level. In [3] the results on pairs of normal operators satisfying certain commutation rules are obtained. And in [2] this is used to obtain the dual $\widehat{E}_\mu(2)$. It turns out that this is indeed the C*-version of the second Hopf *-algebra described above.

In this paper, we use the same techniques as in [2] to show that the Pontryagin dual of $\widehat{E}_\mu(2)$ is again $E_\mu(2)$. This is done in section 2 of the paper. In section 1 we recall the definitions of $E_\mu(2)$ and $\widehat{E}_\mu(2)$ and we collect some of the properties that we need on $\widehat{E}_\mu(2)$ and its automorphisms.

1. The quantum groups $E_\mu(2)$ and $\widehat{E}_\mu(2)$.

Fix a number $\mu \in \,]0,1[\,$. The quantum group $E_\mu(2)$ on the C*-algebra level can be described as a C*-crossed product in the following way. Denote

$$\overline{\mathbb{C}}^\mu = \left\{ z \in \mathbb{C} \mid z = 0 \quad \text{or} \quad |z| \in \mu^{\mathbb{Z}} \right\}.$$

Let $C_\infty\left(\overline{\mathbb{C}}^\mu\right)$ be the algebra of continuous complex functions on $\overline{\mathbb{C}}^\mu$, vanishing at infinity. Define an automorphism α by $(\alpha f)(z) = f(\mu z)$. Then the C*-algebra A is defined as the crossed product $C_\infty\left(\overline{\mathbb{C}}^\mu\right) \times_\alpha \mathbb{Z}$.

Given a Hilbert space $\mathcal{H}$, a unitary operator v and a normal operator n with $\mathrm{Sp}(n) \subseteq \overline{\mathbb{C}}^\mu$ such that $vnv^* = \mu n$, there exists a covariant representation of the above covariant system given by $\pi(f) = f(n)$ since $v\pi(f)v^* = f(vnv^*) = f(\mu n) = (\alpha f)(n)$. Hence there is an associated representation of A on $\mathcal{H}$.

If we take $\mathcal{H} = \ell^2(\mathbb{Z} \times \mathbb{Z})$ and we define v and n by

$$(v\xi)(k,\ell) = \xi(k+1,\ell)$$
$$(n\xi)(k,\ell) = \mu^k \xi(k, \ell-1),$$

then the pair satisfies the above requirements. In fact, we have such universal pair in the sense that the corresponding representation of A faithful. This means that A can be identified with the C*-algebra obtained by taking the closed linear span of operators of the form $v^k f(n)$ where $k \in \mathbb{Z}$ and with $f \in C_\infty\left(\overline{\mathbb{C}}^\mu\right)$.

If we take the pair (v, n) and we define

$$\widetilde{v} = v \otimes v$$
$$\widetilde{n} = v \otimes n \dotplus n \otimes v^*$$

(by $\dotplus$ we mean the closure of the sum of the two operators), it follows from the results in [3] that we get again a pair with the same requirements. Hence there is a *-representation Φ of A induced by this pair. This Φ is a morphism of A into $M(A \otimes A)$ and it is coassociative.

Definition 1.1. The quantum group $E_\mu(2)$ is the C*-algebra A together with this comultiplication.

The quantum group $\widehat{E}_\mu(2)$ is a little more difficult to describe.

Denote by $\overline{\Sigma}_\mu$ the closure of the set

$$\Sigma_\mu = \left\{ (s, \mu^r) \mid s \in \mathbb{Z},\ r \in \mathbb{Z} + \frac{s}{2} \right\}$$

and consider again the C*-algebra $C_\infty\left(\overline{\Sigma}_\mu\right)$. Define an automorphism β on this C*-algebra by $(\beta f)(s, \mu^r) = f(s-2, \mu^r)$. Let B_1 be the C*-crossed product $C_\infty\left(\overline{\Sigma}_\mu\right) \times_\beta \mathbb{Z}$. We will consider the C*-subalgebra B of B_1 generated by finitely supported functions $\xi : \mathbb{Z} \to C_\infty\left(\overline{\Sigma}_\mu\right)$ such that $(\xi(k))(r, 0) = 0$ for all r when $k \neq 0$.

Given a Hilbert space $\mathcal{H}$, a self-adjoint operator N and a normal operator b with polar decomposition $b = u|b|$ such that N and $|b|$ strongly commute and have joint spectrum $\mathrm{Sp}(N, |b|) \subseteq \overline{\Sigma}_\mu$ and such that u is a unitary and satisfies $uNu^* = N - 2$, there is a covariant representation of the above covariant system given by $\pi(f) = f(N, |b|)$ since $u\pi(f)u^* = f(N-2, |b|) = (\beta f)(N, |b|)$. Then we also get an associated representation of the C*-algebra B_1. If u is no longer unitary, but only a partial isometry satisfying $uN =$

$(N-2)u$, we still have $u\pi(f) = (\beta f)(N,|b|)u$ and by the choice of B there is still an associated representation of the C*-algebra B.

Take $\mathcal{H} = \ell^2(\Sigma_\mu)$ and define N and b by

$$(N\xi)(s,\mu^r) = s\xi(s,\mu^r)$$
$$(b\xi)(s,\mu^r) = \mu^r\xi(s-2,\mu^r).$$

Then we have operators as above and the pair is universal in the sense that the corresponding representation of B_1 is faithful. We again identify B_1 with the C*-algebra obtained by taking the closed linear span of operators of the form $u^k f(N,|b|)$ where $k \in \mathbb{Z}$ and $f \in C_\infty\left(\overline{\Sigma}_\mu\right)$. Then B is the closed linear span of the operators $u^k f(N,|b|)$ with $f(r,0) = 0$ if $k \neq 0$.

The comultiplication on B is defined as follows. Consider the universal pair (N,b) as above. Let

$$\widetilde{N} = N \otimes 1 \dotplus 1 \otimes N$$
$$\widetilde{b} = b \otimes \mu^{\frac{1}{2}N} \dotplus \mu^{-\frac{1}{2}N} \otimes b$$

where $\mu^{\frac{1}{2}N}$ is $f(N)$ with $f(k) = \mu^{\frac{1}{2}k}$ for $k \in \mathbb{Z}$ and similarly for $\mu^{-\frac{1}{2}N}$. It follows from [3] that this pair again satisfies the right properties and hence defines a representation. This gives the comultiplication $\Phi : B \to M(B \otimes B)$.

Definition 1.2. The quantum group $\widehat{E}_\mu(2)$ is the C*-algebra B together with this comultiplication.

In [2] it is shown that $\widehat{E}_\mu(2)$ is the Pontryagin dual of $E_\mu(2)$. We will show that also the dual of $\widehat{E}_\mu(2)$ is again $E_\mu(2)$. The method that we use is very similar to the one used in [2].

First we want to obtain some more results on the quantum group $\widehat{E}_\mu(2)$.

Define a projection operator Q on $\ell^2(\Sigma_\mu)$ by

$$(Q\xi)(s,\mu^r) = \begin{cases} \xi(s,\mu^r) & \text{if } s \text{ is even} \\ 0 & \text{if } s \text{ is odd.} \end{cases}$$

It is not hard to prove the following result.

Proposition 1.3. *The operator Q is a central projection in the multiplier algebra of B. One has*

$$\Phi(Q) = Q \otimes Q + (1-Q) \otimes (1-Q).$$

Proof. Because Q is a function of N, it is not hard to see that it is an element of the multiplier algebra $M(B)$. Because of the conditions on the joined spectrum of the pair $(N, |b|)$ it is also a function of $|b|$. By the normality of b it follows that Q also commutes with u. Therefore it is a central projection in $M(B)$. We can write $Q = \frac{1}{2}(1 + (-1)^N)$ where $(-1)^N = f(N)$ with $f(k) = (-1)^k$ for $k \in \mathbb{Z}$. We have $\Phi((-1)^N) = (-1)^N \otimes (-1)^N$. Then the above formula for $\Phi(Q)$ follows from an easy calculation. $\qquad\square$

In fact, Q is the spectral projection corresponding to the even part of the spectrum of N. Because of the relation between the spectrum of N and of $|b|$, it is also a spectral projection of $|b|$. The projection Q splits B into a direct sum of parts where the spectra of N and $|b|$ are no longer related. We will see later that these two parts of B are isomorphic.

We now define some *-homomorphisms.

Lemma 1.4. *There is a *-homomorphism $\psi : B \to \mathbb{C}$ defined by*

$$\psi\left(\sum_k u^k f_k(N, |b|)\right) = f_0(1, 0).$$

This is obtained by considering the trivial pair of operators 1 and 0 on the Hilbert space $\mathbb{C}$ and using the universal property of B. Remark that $\psi(Q) = 0$.

Lemma 1.5. *Define $\rho = (\psi \otimes \iota)\Phi$. This is the *-automorphism of B corresponding to replacing N by $N + 1$ and b by $\mu^{-\frac{1}{2}}b$.*

Proof. Formally we have

$$\begin{aligned}
\rho(N) &= (\psi \otimes \iota)\Phi(N) \\
&= (\psi \otimes \iota)(N \otimes 1 \dotplus 1 \otimes N) \\
&= 1 + N
\end{aligned}$$

$$\begin{aligned}
\rho(b) &= (\psi \otimes \iota)\Phi(b) \\
&= (\psi \otimes \iota)\left(b \otimes \mu^{\frac{1}{2}N} \dotplus \mu^{-\frac{1}{2}N} \otimes b\right) \\
&= \mu^{-\frac{1}{2}}b.
\end{aligned}$$

This formal calculation can of course be made precise by considering the right functions of these operators. $\qquad\square$

Remark that $\rho(Q) = 1 - Q$ so that ρ is a *-automorphism that gives an ismomorphism between the two parts of B.

Lemma 1.6. *Define $\theta = (\psi \otimes \iota \otimes \psi)(\Phi \otimes \iota)\Phi$. This is the *-automorphism of B corresponding to replacing N by $N + 2$ and by leaving b invariant.*

Proof.

$$\begin{aligned}
\theta(N) &= (\rho \otimes \psi)\Phi(N) \\
&= (\rho \otimes \psi)(N \otimes 1 \dotplus 1 \otimes N) \\
&= N + 1 + 1 = N + 2
\end{aligned}$$

$$\begin{aligned}
\theta(b) &= (\rho \otimes \psi)\Phi(b) \\
&= (\rho \otimes \psi)\left(b \otimes \mu^{\frac{1}{2}N} \dotplus \mu^{-\frac{1}{2}N} \otimes b\right) \\
&= \mu^{-\frac{1}{2}}b\mu^{\frac{1}{2}} = b.
\end{aligned}$$

$\square$

Remark that $\theta(Q) = Q$.

We will also use the *-endomorphism γ of B obtained by leaving N and replacing b by 0.

2. Unitary representations of $\widehat{E}_\mu(2)$.

A unitary representation of $\widehat{E}_\mu(2)$ on some Hilbert space $\mathcal{H}$ is a unitary element W in the multiplier algebra $M(B \otimes CB(\mathcal{H}))$, where $CB(\mathcal{H})$ denotes the C*-algebra of compact operators on $\mathcal{H}$, satisfying the equation

$$(\Phi \otimes \iota)W = W_{23}W_{13}.$$

Note that this formula differs from the usual one but the difference is not essential. We use it in this form because we work with the dual and we want to obtain the same formula for the bicharacter as in [2].

We will prove the following result.

Theorem 2.1. *If W is as above, then there exist a unitary operator v and a normal operator n on $\mathcal{H}$ such that $\mathrm{Sp}(n) \subseteq \overline{\mathbb{C}}^\mu$ and $vnv^* = \mu n$ and such that*

$$W = F_\mu\left(\mu^{\frac{1}{2}N}b \otimes vn\right)(1 \otimes v)^{N \otimes 1}.$$

Here F_μ is the function on normal operators with spectrum in $\overline{\mathbb{C}}^\mu$ as defined in [3]. By $(1 \otimes v)^{N \otimes 1}$ we mean the operator $\sum_{k \in \mathbb{Z}} e_k \otimes v^k$ when $N = \sum_{k \in \mathbb{Z}} k\, e_k$ is the spectral resolution of N.

So, in what follows, fix a Hilbert space $\mathcal{H}$ and a unitary W as above. We will first obtain the unitary v.

Definition 2.2. Define $v = (\psi \otimes \iota)(W)$.

This gives a unitary on $\mathcal{H}$. We will need the following formula.

Lemma 2.3. *When γ is the endomorphism of B obtained by leaving N invariant and sending b to 0, we get $(\gamma \otimes \iota)(W) = (1 \otimes v)^{N \otimes 1}$.*

Proof. For any integer s we consider the *-homomorphism ψ_s from B to $\mathbb{C}$ defined by the representation $(s, 0)$ of the pair (N, b). Recall that $\psi = \psi_1$. Then, it is easy to see that $\psi_s \psi_t = \psi_{s+t}$ for all $s, t \in \mathbb{Z}$. So we find that $(\psi_s \otimes \iota)(W) = v^s$. Because γ is induced by the pair $(N, 0)$, it is clear that $(\gamma \otimes \iota)(W) = (1 \otimes v)^{N \otimes 1}$. $\square$

Because $\rho = (\psi \otimes \iota)\Phi$ we get

$$
\begin{aligned}
(\rho \otimes \iota)(W) &= (\psi \otimes \iota \otimes \iota)(\Phi \otimes \iota)W \\
&= (\psi \otimes \iota \otimes \iota)(W_{23} W_{13}) \\
&= W(1 \otimes v).
\end{aligned}
$$

And since $\rho(Q) = 1 - Q$ it follows that

$$
W((1 - Q) \otimes 1) = (\rho \otimes \iota)(W(Q \otimes 1))(1 \otimes v^*).
$$

So we see that it will be sufficient to determine the part $W(Q \otimes 1)$. Therefore, we will work now mainly in the subspace $Q\ell^2(\Sigma_\mu)$. We will denote restrictions of operators, *-automorphisms, ... to this subspace with an index 0. We will use e.g. W_0 for $W(Q \otimes 1)$, θ_0 for the map $xQ \to \theta(xQ)$, ... We will use Φ_0 for the map $xQ \to \Phi(xQ)(Q \otimes Q)$. Notice that $\operatorname{Sp} N_0 \subseteq 2\mathbb{Z}$ where $N_0 = NQ$.

Lemma 2.4. *Define $U_0 = (1 \otimes v)^{-\frac{1}{2} N_0 \otimes 1} W_0 (1 \otimes v)^{-\frac{1}{2} N_0 \otimes 1}$. Then $(\theta_0 \otimes \iota)(U_0) = U_0$.*

Proof. We have

$$
\begin{aligned}
(\theta \otimes \iota)(W) &= (\psi \otimes \iota \otimes \psi \otimes \iota)(\Phi \otimes \iota \otimes \iota)(\Phi \otimes \iota)(W) \\
&= (\psi \otimes \iota \otimes \psi \otimes \iota)(W_{34} W_{24} W_{14}) \\
&= (1 \otimes v)W(1 \otimes v).
\end{aligned}
$$

And because $\theta(N) = N + 2$ we get

$$
\begin{aligned}
(\theta_0 \otimes \iota)(U_0) &= (1 \otimes v)^{-\frac{1}{2}(N_0+2)\otimes 1}(1 \otimes v)W_0(1 \otimes v)(1 \otimes v)^{-\frac{1}{2}(N_0+2)\otimes 1} \\
&= (1 \otimes v)^{-\frac{1}{2}N_0\otimes 1}W_0(1 \otimes v)^{-\frac{1}{2}N_0\otimes 1} \\
&= U_0.
\end{aligned}
$$

$\square$

Lemma 2.5. *There is a continuous function* $f : \mathrm{Sp}(b_0) \to \mathcal{B}(\mathcal{H})$ *such that* $f(z)$ *is a unitary operator for all* $z \in \mathrm{Sp}(b_0)$ *and* $U_0 = f(b_0)$.

Proof. Let $B \otimes C\mathcal{B}(\mathcal{H})$ act on $\ell^2(\Sigma_\mu) \otimes \mathcal{H}$. Then W is a unitary operator on this space. Hence W_0 is a unitary operator on $Q\ell^2(\Sigma_\mu) \otimes \mathcal{H}$. Since the *-automorphism θ_0 is implemented by the unitary u_0, defined by

$$
(u_0\xi)(s, \mu^r) = \xi(s - 2, \mu^r)
$$

we must have that U_0 commutes with this operator $u_0 \otimes 1$. Recall that $u_0|b_0|$ is the polar decomposition of b_0. Because $|b|$ commutes with all of B, also U_0 commutes with $|b_0| \otimes 1$. Hence U_0 commutes with all functions of b_0. But these generate a maximal abelian subalgebra $\mathcal{M}$ on $Q\ell^2(\Sigma_\mu)$. So $U_0 \in \mathcal{M} \otimes \mathcal{B}(\mathcal{H})$, the von Neumann tensor product of $\mathcal{M}$ with $\mathcal{B}(\mathcal{H})$. This means that U_0 has the form $f(b_0)$ where $f : \mathrm{Sp}(b_0) \to \mathcal{B}(\mathcal{H})$ is a measurable function with values in the unitary operators. However, U_0 is in the multiplier algebra of $QB \otimes C\mathcal{B}(\mathcal{H})$. And the fixed points in QB for the automorphism θ_0 are clearly of the form $g(b_0)$ with $g \in C_\infty(\mathrm{Sp}(b_0))$. Therefore, f must be a continuous function. $\square$

Lemma 2.6. *We have*

$$
f(R \dotplus S) = f(R)f(S)
$$

when $R = \mu^{-\frac{1}{2}N_0} \otimes b_0$ *and* $S = b_0 \otimes \mu^{\frac{1}{2}N_0}$.

Proof. Consider again the formula

$$
(\Phi \otimes \iota)W = W_{23}W_{13}
$$

and cut down by $Q \otimes Q \otimes 1$. Recall that $W_0 = (1 \otimes v)^{\frac{1}{2}N_0\otimes 1}f(b_0)(1 \otimes v)^{\frac{1}{2}N_0\otimes 1}$ So we obtain for the left hand side :

$$
(1 \otimes 1 \otimes v)^{\frac{1}{2}(N_0\otimes 1\otimes 1 \dotplus 1\otimes N_0\otimes 1)}
$$
$$
\cdot f\left(b_0 \otimes \mu^{\frac{1}{2}N_0} \dotplus \mu^{-\frac{1}{2}N_0} \otimes b_0\right)(1 \otimes 1 \otimes v)^{\frac{1}{2}(N_0\otimes 1\otimes 1 \dotplus 1\otimes N_0\otimes 1)}
$$

Now consider the endomorphism γ that takes N to N and b to 0. We showed in Lemma 2.3 that $(\gamma \otimes \iota)(W) = (1 \otimes v)^{N \otimes 1}$. If we apply this to the first factor in the tensor product we obtain

$$(1 \otimes 1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1 \otimes 1 + 1 \otimes N_0 \otimes 1)} f\left(\mu^{-\frac{1}{2}N_0} \otimes b_0\right) (1 \otimes 1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1 \otimes 1 + 1 \otimes N_0 \otimes 1)}$$
$$= (W_0)_{23}(1 \otimes 1 \otimes v)^{N_0 \otimes 1 \otimes 1}.$$

If we apply it to the second factor we get

$$(1 \otimes 1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1 \otimes 1 + 1 \otimes N_0 \otimes 1)} f\left(b_0 \otimes \mu^{\frac{1}{2}N_0}\right) (1 \otimes 1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1 \otimes 1 + 1 \otimes N_0 \otimes 1)}$$
$$= (1 \otimes 1 \otimes v)^{1 \otimes N_0 \otimes 1}(W_0)_{13}.$$

If we multiply these two formulas we get

$$(W_0)_{23}(W_0)_{13} = (1 \otimes 1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1 \otimes 1 + 1 \otimes N_0 \otimes 1)} f\left(\mu^{-\frac{1}{2}N_0} \otimes b_0\right)$$
$$(1 \otimes 1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1 \otimes 1 + 1 \otimes N_0 \otimes 1) - (N_0 \otimes 1 \otimes 1)}$$
$$(1 \otimes 1 \otimes v)^{-(1 \otimes N_0 \otimes 1) + \frac{1}{2}(N_0 \otimes 1 \otimes 1 + 1 \otimes N_0 \otimes 1)}$$
$$f\left(b_0 \otimes \mu^{\frac{1}{2}N_0}\right) (1 \otimes 1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1 \otimes 1 + 1 \otimes N_0 \otimes 1)}.$$

If we combine this with the formula in the beginning, we precisely get the desired result. $\qquad\square$

These operators R and S precisely satisfy the conditions of the pairs considered in [3]. Indeed

$$R = \mu^{-\frac{1}{2}N_0} \otimes b_0 = (1 \otimes u_0)\left(\mu^{-\frac{1}{2}N_0} \otimes |b_0|\right)$$
$$S = b_0 \otimes \mu^{\frac{1}{2}N_0} = (u_0 \otimes 1)\left(|b_0| \otimes \mu^{\frac{1}{2}N_0}\right)$$

and $|b_0|$ and N_0 strongly commute, $\operatorname{Sp} N_0 = 2\mathbb{Z}$, $\operatorname{Sp}|b_0| = \overline{\mathbb{C}}_\mu$, $u_0^* N_0 u_0 = N_0 + 2Q$. Therefore we can apply Theorem 4.2 of [3].

Lemma 2.7. *There exist a normal operator n on $\mathcal{H}$ with $\operatorname{Sp}(n) \subseteq \overline{\mathbb{C}}^\mu$ such that*

$$f(b_0) = F_\mu(b_0 \otimes n)$$

where F_μ is as in [2].

Lemma 2.8. *We have $vnv^* = \mu n$.*

Proof. Apply the *-automorphism ρ^2 to the formula

$$W_0 = (1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1)} F_\mu(b_0 \otimes n)(1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1)}.$$

We know that $\rho^2(N_0) = N_0 + 2$, $\rho^2(b_0) = \mu^{-1}b_0$ and $\rho^2(W_0) = W_0(1 \otimes v^2)$. So we get

$$W_0(1 \otimes v^2) = (1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1)}(1 \otimes v)F_\mu(\mu^{-1}b_0 \otimes n)(1 \otimes v)(1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1)}$$

$$W_0 = (1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1)}F_\mu(\mu^{-1}b_0 \otimes vnv^*)(1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1)}.$$

By uniqueness we get $vnv^* = \mu n$. $\square$

Lemma 2.9.   *We have $W_0 = F_\mu\left(\mu^{\frac{1}{2}N_0}b_0 \otimes nv\right)(1 \otimes v)^{N_0 \otimes 1}$.*

Proof.

$$(1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1)}(b_0 \otimes n)(1 \otimes v)^{-\frac{1}{2}(N_0 \otimes 1)}$$

$$= (1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1)}(u_0|b_0| \otimes n)(1 \otimes v)^{-\frac{1}{2}(N_0 \otimes 1)}$$

$$= (u_0 \otimes 1)(1 \otimes v)^{\frac{1}{2}(N_0 \otimes 1)}(1 \otimes v)(|b_0| \otimes n)(1 \otimes v)^{-\frac{1}{2}(N_0 \otimes 1)}$$

$$= (u_0 \otimes 1)\left(\mu^{\frac{1}{2}N_0}|b_0| \otimes vn\right)$$

$$= \mu^{\frac{1}{2}N_0}b_0 \otimes \mu^{-1}vn.$$

$\square$

From this we see that

$$W(Q \otimes 1) = F_\mu\left(\mu^{\frac{1}{2}N}bQ \otimes nv\right)(1 \otimes v)^{N \otimes 1}(Q \otimes 1).$$

If we apply ρ we get

$$W(1 \otimes v)((1 - Q) \otimes 1)$$
$$= F_\mu\left(\mu^{\frac{1}{2}(N+1)}\mu^{-\frac{1}{2}}b(1 - Q) \otimes nv\right)(1 \otimes v)^{(N+1) \otimes 1}(1 - Q) \otimes 1$$

and so

$$W((1 - Q) \otimes 1) = F_\mu\left(\mu^{\frac{1}{2}N}b(1 - Q) \otimes nv\right)(1 \otimes v)^{N \otimes 1}(1 - Q) \otimes 1.$$

Therefore we get precisely the same formula on Q and $(1 - Q)$. Hence, if we replace n by $vnv^* = \mu n$, we get the formula in our theorem.

References

[1]   S.L. Woronowicz, *Unbounded elements affiliated with C^*-algebras and non-compact quantum groups*, Commun. Math. Phys., **136** (1991), 399-432.

[2]   ———, *Quantum $E(2)$ group and its Pontryagin dual*, Lett. in Math. Phys., **23** (1991), 251-263.

[3] _____, *Operator equalities related to the quantum $E(2)$ group*, Commun. Math. Phys., **144** (1992), 417-428.

[4] S.L. Woronowicz and K. Napiórkowski, *Operator theory in the C^*-algebra framework*, Rep. Math. Phys., **31** (1992), 353-373.

[5] A. Van Daele, *Dual Pairs of Hopf *-algebras*, Lecture notes (first version). K.U. Leuven, (1991).

[6] _____, *Dual Pairs of Hopf *-algebras*, Lecture notes (second version). Bull. of the Lond. Math. Soc., **25** (1993), 209-230.

[7] _____, *The operator $a \otimes b + b \otimes a^{-1}$ when $ab = \lambda ba$*, Preprint K.U. Leuven, (1989).

[8] H.T. Koelink, *On quantum groups and q-special functions*, Ph.D. Thesis Leiden, (1991).

[9] A. Van Daele, *A quantum deformation of the Heisenberg group*, Preprint K.U. Leuven, (1989).

KATHOLEIKE UNIVERSITEIT LEUVEN
CELESTIJNENLAAN 200B
3001 LEUVEN (HEVERLEE)
BELGIUM
E-mail address: alfons.vandaele@wis.kuleuven.ac.be