BRAID COMMUTATORS AND VASSILIEV INVARIANTS

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We establish a relationship between Vassiliev invariants and
the lower central series of the pure braid group, and we use
this to construct infinite families of prime knots or links whose
invariants match those of a given knot or link up to a given
order.

Introduction.

Theorem 1. *Let L and L' be two links which differ by a braid $p \in P_k^n$, the
nth group of the lower central series of P_k, the pure braid group on k strands.
Let v be a link invariant of order less than n. Then $v(L) = v(L')$.*

In Section 1 we will define the order of a link invariant, and also define
what it means for two links to differ by a braid p. As an example, if $\hat{x}$
represents the closure of a braid x, and b and p are any two braids with the
same number of strands, then $\hat{b}$ and $\widehat{pb}$ differ by p. Theorem 1 gives us a
way to modify links without changing their invariants up to some order. We
shall show that these changes can often be guaranteed to give distinct links.

Falk and Randell proved in [**FR**] that the intersection of the lower central
series of P_k is trivial, that is, $\cap_{i=1}^{\infty} P_k^i = \{1\}$, so that Theorem 1 does not give
rise to an easy way of constructing distinct links, *all* of whose finite-order
invariants are equal. We will, however, prove the following in Section 2.

Theorem 2. *For any link L and any positive integer n there exist an infinite
number of prime, nonsplit, alternating links L', (each with the same number
of components as L), such that $v(L') = v(L)$ for any link invariant v of order
less than or equal to n.*

Infinite families of knots whose invariants match up to a given order have
been constructed by Ohyama in [**O**], and also follow from the work of Lin (see
[**B**] and [**L2**]) and Gusarov (see [**Gu2**]). However, all of these examples have
been composite knots. Recall that a link $L \subset S^3$ is *composite* if there exists
a two-sphere R embedded in S^3 which intersects L in exactly two points,
and such that R does not bound a 3-ball B with $B \cap L$ isotopic inside B to
an unknotted arc. L is *prime* if it is not composite. L is *split* if there exists
a 2-sphere R in embedded in S^3 such that R does not intersect L, and R

does not bound a ball B with $B \cap L = \emptyset$. L is *alternating* if it has a diagram D such that as one traverses any component in the diagram, the crossings alternate between over and under.

Knot invariants of finite order are called *Vassiliev invariants*. A *rational Vassiliev invariant* is a Vassiliev invariant taking values in $\mathbb{Q}$, the field of rational numbers. It was proved by Birman and Lin (see [**BL**]) that if the substitution $t = e^x$ is made into the Jones polynomial $J_L(t)$, then the ith coefficient $v_i(L)$ of the resulting power series $J_L(e^x) = \sum_{i=0}^{\infty} v_i(L)x^i$ is a rational Vassiliev invariant of order i. It follows directly from this theorem that there are rational Vassiliev invariants of arbitrarilly high order. However, rational Vassiliev invariants form a graded ring (in fact, a Hopf algebra), and there are many finite-order invariants which are products of invariants of lower orders, and these do not provide any power to distinguish links not already available to their lower order factors. A Vassiliev invariant v is called *primitive* if it is additive under connected sum of knots, that is, $v(K_1 \# K_2) = v(K_1) + v(K_2)$. It turns out that all the rational Vassiliev invariants are polynomials in the primitive ones. See [**B-N**], [**Gu2**].

Corollary 2.1. *There are primitive Vassiliev invariants of arbitrarilly high order.*

This follows from Theorem 2, and from the fact that the Jones polynomial is never trivial (ie, equal to 1, its value on the unknot) on a nontrivial alternating knot (see [**Mu**]). For if there were no primitive Vassiliev invariants of order $\geq n$, then we could construct an alternating knot with all of its finite-order invariants equal to 0, and therefore with trivial Jones polynomial.

Except when otherwise specified, links in this paper can be considered to be either all oriented or all unoriented. We will denote by σ_i the standard generator of the braid group B_k which switches the ith and $(i+1)$st strand of the braid with a positive crossing. A *tangle diagram* will be a link diagram with a single S^1 boundary, which some of the strands in the diagram may intersect transversely. We will sometimes blur the distiction between braids and braid words, and between links and link diagrams when no ambiguity can result.

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1. Invariants of finite order.

A link invariant v taking values in an abelian group A is said to have *finite order* if there exists a positive integer n such that for any link diagram D

with $n + 1$ chosen crossings numbered from 1 to $n + 1$

$$(1.1) \qquad\qquad \sum_{i \subseteq \{1,2,\ldots n+1\}} (-1)^{\#(i)} v(D_i) = 0$$

where $\#(i)$ is the cardinality of the set i, and D_i is obtained from $D = D_\emptyset$ by changing each crossing whose number is in i. The smallest such n is called the *order* of v, and if there is no such n, then v is said to have *infinite order*. In the case of oriented links, one often defines an invariant v of finite order to be such that there exists a positive integer n satisfying 1.1, only now the jth crossing of D_i is taken to be positive or negative according to whether $j \in i$ or $j \notin i$. It is clear that this is the same as our definition, since the quantity on the left side will differ at most by a sign change between the two cases.

Finite-order invariants of oriented links come in abundance from the Jones polynomial and its generalizations. (See [**B**], [**BL**], [**Gu1**], and [**L1**].) The space V_n of rational invariants of order less than or equal to n can be made finite-dimensional either by restricting oneself to links with a fixed number of components or (equivalently, see [**S**]) by requiring v to be invariant under the addition of a disjoint and unknotted component. Finite-order knot invariants were introduced by Vassiliev in [**V**] using singularity theory and algebraic topology, and have since been derived, analyzed, and generalized using several other approaches. (See [**BL**], [**B-N**], [**Gu2**], and [**S**].)

By a *tangle map* $T : B_k \to \{link\ types\}$ we mean for some fixed k a fixed way of putting a braid $b \in B_k$ into a tangle diagram so as to get a link $T(b)$. (This is well-defined because the braid group relations are a subset of Reidemeister moves on link diagrams.) See Figure 1.1. In the case where $T : B_k \to \{oriented\ link\ types\}$ then we do not assume that T preserves any orientation on the strands inherited from the braid group, and in fact the strands of b in $T(1)$ need not have their orientations parallel to each other. Given a tangle map T and a link invariant v, we denote by $f_{v,T}$ the linear extenion of $v \circ T : B_k \to \mathbb{Q}$ to $\mathbb{Q}B_k$.

Two links L and L' are said to *differ by a braid* b if there exists a tangle map T such that $L = T(1)$ and $L' = T(b)$. If L and L' differ by b, then L' and L differ by b^{-1}, and "differing by b" is not necessarilly a reflexive relation on the set of links. However, we are concerned with links which differ by an element of a particular subgoup of the braid group, so this is not a concern. Note that when b is a pure braid then L and L' will have the same number of components.

Consider the ideal $J \in \mathbb{Q}B_k$ generated by $\{(\sigma_i - \sigma_i^{-1}) : 0 < i < k\}$.

Lemma 1.1. *Let v be a link invariant of order n, and let T be a tangle map. Let $x \in J^{n+1} \subset \mathbb{Q}B_k$. Then $f_{v,T}(x) = 0$.*

Proof. J^{n+1} is linearly generated by expressions of the form

$$w_0(\sigma_{m_1} - \sigma_{m_1}^{-1})w_1 \dots w_n(\sigma_{m_{n+1}} - \sigma_{m_{n+1}}^{-1})w_{n+1}$$

where each σ_{m_j} is one of the standard braid generators and each $w_j \in B_k$. Consider a diagram for the braid $b = w_0 \sigma_{m_1} w_1 \sigma_{m_2} \dots \sigma_{m_{n+1}} w_{n+1}$ where each σ_{m_j} is represented by a single crossing (without any cancellation with w_{j-1} or w_j). Let D be a diagram of $f_{v,T}(b)$ which contains the chosen diagram for b, and for $i \subseteq \{1, 2, \dots n+1\}$ let D_i be the diagram obtained from D by changing the crossings corresponding to σ_{m_j} for all $j \in i$. Now it is clear that $f_{v,T}(b)$ is equal to the left-hand side of 1.1. $\square$

$$T(1) \qquad\qquad T(\sigma_2) \qquad\qquad T\left(\sigma_1 \sigma_2^{-1}\right)$$
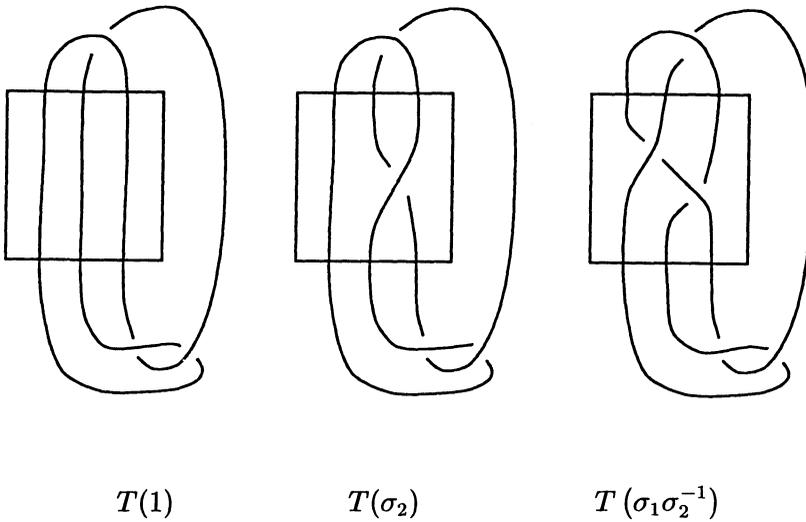
Figure 1.1

Recall that the lower central series $G = G^1, G^2, G^3, \dots$ of a group G is given inductively by $G^{n+1} = [G, G^n]$, where $[H, K] = \langle xyx^{-1}y^{-1} : x \in H \text{ and } y \in K \rangle$.

Lemma 1.2. *Let* $p \in P_k^n$, *the nth group of the lower central series of the pure braid group* P_n. *Then* $p - 1 \in J^n$.

Proof. If p is a pure braid, then p can be "unbraided" by a series of crossing changes from σ_i to σ_i^{-1} or vice-versa. Hence $p - 1 \in J$. If $p \in P_k^n$, then

$$p = [p_1, q_1][p_2, q_2] \dots [p_m, q_m]$$

where $p_i \in P_k$ and $q_i \in P_k^{n-1}$. We have

$$[p_i, q_i] - 1 = (p_i q_i - q_i p_i) p_i^{-1} q_i^{-1} = ((p_i - 1)(q_i - 1) - (q_i - 1)(p_i - 1)) p_i^{-1} q_i^{-1} \in J^n$$

because $p_i - 1 \in J$ and by induction $q_i - 1 \in J^{n-1}$.   □

The ideal $P_k \cap J$ is called the *augmentation ideal* of P_k, and the group $(1 + J^n) \cap P_k$ is called the nth *dimension subgroup* of P_k. Lemma 1.2 corresponds to a well-known fact in group theory, that the nth dimension subgroup of a group G is a subgroup of G^n, the nth group of the lower cenral series. See [**LS**].

Theorem 1. *Let L and L' be two links which differ by a braid $p \in P_k^n$, Let v be a link invariant of order less than n. Then $v(L) = v(L')$.*

Proof. There exists a tangle map T such that $T(1) = L$ and $T(p) = L'$. By Lemmas 1.1 and 1.2, $f_{v,T}(p) - f_{v,T}(1) = f_{v,T}(p - 1) = 0$.   □

2. Constructing alternating links.

Given a subset S of a group G, let $W(S)$ be the set of words $x_1 x_2 \ldots x_n$, for some positive integer n with $x_i \in S$. Given any link L and a positive integer n, we will to make a series of changes to L. Each change will not affect the value $v(L)$ of any invariant v of order less than or equal to n, and when we are finished, L will be an alternating link. We will work with a braid word $w \in W(\{\sigma_1, \sigma_2, \ldots \sigma_{k-1}, \sigma_1^{-1}, \sigma_2^{-1}, \ldots \sigma_{k-1}^{-1}\})$ which represents a braid whose closure $\hat{w}$ is L. Our objective is to modify w until the odd σ_i always occur with positive exponent and the even σ_i always occur with negative exponent, that is, until $w \in W(\{\sigma_i : i \ odd\} \cup \{\sigma_i^{-1} : i \ even\})$. This will ensure that $\hat{w}$ is an alternating link. See Figure 2.1.
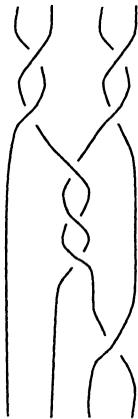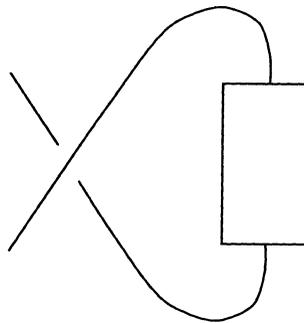
Figure 2.1   **Figure 2.2**

The inspiration for the following lemmas came from [**BM**]. We shall use the following simplified notation in the proofs. In the braid group B_k, let $a = \sigma_i, b = \sigma_{i+1}$. Then $aba = bab$. Let $\bar{x} = (aba)x(aba)^{-1}$. Then $\bar{a} = b$ and $\bar{b} = a$, so if $x \in W(\{a, b^{-1}\})$ then $\bar{x} \in W(\{a^{-1}, b\})$, and if $x \in W(\{a^{-1}, b\})$ then $\bar{x} \in W(\{a, b^{-1}\})$. Also if $x \in W(\{a, b^{-1}\})$ then $x^{-1} \in W(\{a^{-1}, b\})$ and if $x \in W(\{a^{-1}, b\})$ then $x^{-1} \in W(\{a, b^{-1}\})$.

Lemma 2.1. *For any $n > 0$ and $0 < i < k - 1$, there exist $m > 0$, $r \geq 0$, and $x \in W(\{\sigma_i, \sigma_{i+1}^{-1}\})$ such that $t_i = \sigma_i^m x \sigma_i \sigma_{i+1}^{-r} \in P_k^n$.*

Proof. Suppose inductively that $t_i = a^m x a b^{-r} \in W(\{a, b^{-1}\})$ and $t_i \in P_k^n$, with $m > 0$ and $r \geq 0$. Form the commutator

$$
\begin{aligned}
[t_i, b^2] &= a^m x a b^{-r} b^2 b^r a^{-1} x^{-1} a^{-m} b^{-2} \\
&= a^m x (aba) a^{-1} b a^{-1} x^{-1} a^{-m+1} (aba)^{-1} a b^{-1} \\
&= a^m x \overline{a^{-1} b a^{-1} x^{-1} a^{-m+1}} a b^{-1} \in W(\{a, b^{-1}\}).
\end{aligned}
$$

To begin the induction we need only note that there are lots of pure braid words in $W(\{a, b^{-1}\})$ that begin with a (a^2, for instance). □

Remark 2.1. One can also construct words in $W(\{\sigma_i, \sigma_{i+1}^{-1}\}) \cap P_k^n$ that begin with σ_{i+1}^{-1}.

Remark 2.2. Note that t_i is represented by a non-trivial alternating word, but we have not proved that t_i is a non-trivial braid. It is true that $t_i \neq 1$, and this will be seen in the proof of Theorem 2, but we don't need to assume it here.

Lemma 2.2. *For any $n > 0$ and $0 < i < k - 1$, there exists $u_i \in W(\{\sigma_i, \sigma_{i+1}^{-1}\})$ with $u_i(\sigma_i \sigma_{i+1} \sigma_i) \in P_k^n$.*

Proof. By Lemma 2.1, there exists $t_i = a^m x a b^{-r} \in W(\{a, b^{-1}\}) \cap P_k^{n-1}$, with $m > 0$ and $r \geq 0$. Form the commutator

$$
\begin{aligned}
[b^{-2}, t_i] &= b^{-2} a^m x a b^{-r} b^2 b^r a^{-1} x^{-1} a^{-m} \\
&= b^{-2} a^m x (aba) a^{-1} b a^{-1} x^{-1} a^{-m} (aba)^{-1} (aba) \\
&= b^{-2} a^m x \overline{a^{-1} b a^{-1} x^{-1} a^{-m}} (aba).
\end{aligned}
$$

This completes the proof. □

Note that $\bar{u}_i^{-1} \in W(\{\sigma_i, \sigma_{i+1}^{-1}\})$, and $(\sigma_i \sigma_{i+1} \sigma_i)^{-1} \bar{u}_i^{-1} \in P_k^n$ because the subgroups of the lower central series are fully invariant.

Theorem 2. For any link L and any positive integer n there exist an infinite number of prime, nonsplit, alternating links L', such that $v(L') = v(L)$ for any link invariant v of order less than or equal to n.

Remark 2.3. The L and L' constructed will have the same number of components, so that the theorem remains true if "link" is replaced by "knot".

Proof. Let L be represented as the closure $\widehat{w}$ of some braid word w on $k > 2$ strands. To simplify the argument, assume that k is odd. Suppose that $w = z_1 \sigma_i^{-1} z_2$, with i odd. Replace w with

$$z_1 u_i (\sigma_i \sigma_{i+1} \sigma_i) \sigma_i^{-1} (\sigma_i \sigma_{i+1} \sigma_i)^{-1} \overline{u_i}^{-1} z_2 = z_1 u_i \sigma_{i+1}^{-1} \overline{u_i}^{-1} z_2$$

where u_i is as in Lemma 2.2. By Theorem 1, $v(\widehat{w})$ is unchanged by this modification. Repeating the process, we can eliminate all σ_i, i odd, that occur in w with negative exponent. Similarly, whenever $w = z_1 \sigma_{i+1} z_2$, with i odd, replace w with

$$z_1 u_i (\sigma_i \sigma_{i+1} \sigma_i) \sigma_{i+1} (\sigma_i \sigma_{i+1} \sigma_i)^{-1} \overline{u_i}^{-1} z_2 = z_1 u_i \sigma_i \overline{u_i}^{-1} z_2.$$

Call the result of all these modifications L'. L' is now given by an alternating diagram $\widehat{w}$. An alternating diagram D is said to be *reduced* if it contains no crossing as in Figure 2.2. D is said to be *prime* if there does not exist a simple closed planar curve C that intersects D in exactly two points with D non-trivial (as a diagram) on both sides of C. D is *connnected* if it is connected as a projection onto $\mathbb{R}^2$. Menasco proved in [**Me**] that a reduced, connected, prime alternating diagram represents a prime, non-split link. Let α be the braid word $t_1 t_2^{-1} \ldots t_{k-2} t_{k-1}^{-1}$, where t_i is as in Lemma 2.1. Let β be the braid word $\sigma_1 \sigma_2^{-1} \ldots \sigma_{k-2} \sigma_{k-1}^{-1}$. It is easy to verify that $\widehat{\beta^2}$ is a reduced, connnected, prime, alternating diagram, and that none of these four properties will change if we insert some word $x \in W(\{\sigma_i : i \ odd\} \cup \{\sigma_i^{-1} : i \ even\})$ anywhere into β^2. Since $\alpha^2 w$ can be built up from β^2 by adding a sequence of such words, it follows that $\widehat{\alpha^2 w}$ is a reduced, connected, prime, alternating diagram.

Murasugi proved in [**Mu**] that a reduced alternating link diagram has minimal crossing number, so we can produce an infinite number of distinct L' by repeatedly inserting some t_i from the lemma into w, creating a sequence of reduced, connected, prime, alternating diagrams with more and more crossings. $\qquad\square$

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