PACIFIC JOURNAL OF MATHEMATICS
Vol. 174, No. 2, 1996

ON THE EXISTENCE OF EXTREMAL METRICS

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We study the well known variational problem proposed by Calabi: Minimize the functional $\int_M s_g^2 dv_g$ among all metrics in a given Kahler class. We are able to establish the existence of the extremal when the closed Riemann surface has genus different from zero. We have also given a different proof of the result originally proved by Calabi that: On a closed Riemann surface, the extremal metric has constant scalar curvature on a closed Riemann surface, the extremal metric has constant scalar curvature, which originally is proved by Calabi.

1. Introduction.

In the early 80's, E. Calabi [**C1**, **C2**] proposed the following variational problem. Let M be a compact, connected, complex $n-$dimensional manifold without boundary and assume that M admits a Kähler metric g locally expressible in the form $ds^2 = 2g_{\alpha\bar{\beta}}dz^\alpha dz^{\bar{\beta}}$. Let us fix the deRham cohomology class Ω of the real valued, closed exterior $(1,1)$ form $\omega = \sqrt{-1}g_{\alpha\bar{\beta}}dz^\alpha \Lambda dz^{\bar{\beta}}$ associated to the metric g, and denote by $\mathcal{C}_\Omega$ the function space of all differentiable Kähler metrics g with the Kähler form $\omega \in \Omega$. On this function space, Calabi introduces the (non negative) real valued functional Φ which assigns to each g the integral

$$\Phi(g) = \int_M s_g^2 dv_g$$

where $dv_g = (\sqrt{-1})^n \det(g_{\alpha\bar{\beta}})\Lambda_{\alpha=1}^n(dz^\alpha \Lambda dz^{\bar{\alpha}})$ denotes the volume element in M associated with the e Kähler metric g, and

$$s_g = -g^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} \log\det(g_{\lambda\bar{\mu}})$$

the scalar curvature.

The variational problem proposed by Calabi is that of minimizing the functional $\Phi(g)$ over all $g \in \mathcal{C}_\Omega$. The motivation for considering this is the fact that, as g varies in $\mathcal{C}_\Omega$, both the volume

$$V = V_g = \int_M dv_g$$

and the total scalar curvature

$$S_g = \int_M s_g dv_g$$

remain constants. Thus by the virtue of the Schwartz inequality, the functional $\Phi(g)$ has a nonnegative lower bound $S_g^2 V$, we wish that the latter can be achieved if and only if there exists a $g \in \mathcal{C}_\Omega$ with constant scalar curvature.

As M. Levine [**L**] points out, if we call the critical metric of Φ the extremal metric, the extremal metric does not necessarily have constant scalar curvature if the dimension $n > 1$. For $n = 1$, E. Calabi is able to show that the extremal metric always has constant scalar curvature if the extremal metric exists (see [**C1**] and also §5 of this paper).

The problem of finding extremal metrics is quite nature but quite difficult. There are severval results about the non-existence ([**C1**],[**C2**],[**L**],[**BB**]). However, in the past decade, there has almost been no progress on the existence of extremal metrics.

The propose of this paper is to show:

Main Theorem. *If $n = 1$ with $\chi(M) \leq 0$, then the extremal metric exists.*

Remember that this is not so surprising at all since there are several methods to reach this conclusion: Poincaré's classical uniformization theorem [**P**]; M. Berger's minimization method [**A**],[**B**]; R. Hamilton's Ricci flow [**CH**], [**H1**],[**H2**]; B. Osgood, R. Phillips and P. Sarnack's minimizing the log determinent of the Laplace operator [**OPS**].

What it is new in this paper is that, we exactly follow the Calabi's original idea, by using the direct method, to show that the minimizer of the Calabi functional can be achieved. Since the Kähler class and the conformal class for $n = 1$ are equivalent, our setting is in the conformal class. The main difficulty is to get H_2^2 norm bound of the conformal factors in terms of the volume bound and the bound on the Calabi functional.

The organization of this paper is as follows: after some preliminaries(§2), we will give the proof of our main theorem for the case $\chi(M) < 0$(§3). §4 will simply indicate the case $\chi(M) = 0$. Since our setting is in the conformal class, we will show, in this setting, that the extremal metrics have constant scalar curvatures (§5). Clearly this is an alternative proof of one of Calabi's theorems [**C1**].

The author would like to thank Professor Paul Yang for letting him know this very interesting problem and Professor William Abikoff for an encouragement on him to write out this proof.

2. Preliminaries.

Consider a compact connected complex $n-$dimensional manifold M without boundary. Assume that M admits a Kähler metric g locally expressible in the form

$$ds^2 = 2g_{\alpha\bar{\beta}}dz^\alpha dz^{\bar{\beta}}$$

where and thereafter the Einstein convention is used. It is well known that there is a real valued, closed exterior $(1,1)$ form $\omega = \sqrt{-1}g_{\alpha\bar{\beta}}dz^\alpha dz^{\bar{\beta}}$ associated to the metric g. This $(1,1)$ form usually is called Kähler form. By the deRham theory, ω determines a deRham cohomology class Ω. Now let us consider the change of the Kähler metric,

$$g'_{\lambda\bar{\mu}} = g_{\lambda\bar{\mu}} + \partial_{\lambda\bar{\mu}}\varphi \tag{2.1}$$

where $\varphi \in C^\infty$ is such that g' is positive definite. Obviously, g' is a Kähler metric. Also it is clear that the Kähler form ω' associated to the metric g' determines the same deRham cohomology class as ω does. Note that under this change of the metric, new metric g' is also a Riemannian metric since it is Kähler. We will denote all such functions φ by $\mathcal{C}_\Omega$.

For a given Kähler metric g, we have the volume element defined in the local coordinate by $dv_g = (\sqrt{-1})^n \det(g_{\alpha\bar{\beta}})\Lambda_{\alpha=1}^n(dz^\alpha \Lambda dz^{\bar{\alpha}})$. And the scalar curvature associated to the metric g is defined by

$$s_g = -g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}}(\log\det(g_{\lambda\bar{\mu}})). \tag{2.2}$$

For a $\varphi \in \mathcal{C}_\Omega$, the Calabi functional can be written as

$$\Phi(\varphi) = \int_M s_{g'}^2 dv_{g'} \tag{2.3}$$

where g' and φ are related by (2.1).

From now on, we only consider M a Riemann surface without boundary, that is, a compact complex $1-$dimensional manifold without boundary. It is known that on a complex $1-$dimensional manifold, any Riemannian metric is conformally flat [**Be**, Theorem 1.169]. Thus any two Riemannian metrics on M are conformally equivalent. If metrics g and g' are related by (2.1), there exists a C^∞ function u such that $g' = e^{2u}g$. In this form, we know that $dv_{g'} = e^{2u}dv_g$ and the scalar curvatures satisfy the relation

$$\Delta u + k_{g'}e^{2u} = k_g \tag{2.4}$$

where $k_{g'}$ and k_g are the Gaussian curvatures of the metrics g' and g respectively, and Δ is the Laplace operator associated to the metric g. The relation between the Gaussian curvature and the scalar curvature is that $s_g = 2k_g$. Thus, up to a constant multiple, the Calabi functional $\Phi(\varphi)$ is equivalent to

$$J(u) = \int_M (k_g - \Delta u)^2 e^{-2u} dv_g \tag{2.5}$$

with the constrain

$$\int_M e^{2u} dv_g = \int_M dv_g. \tag{2.6}$$

Our main propose of this section is going to show that the functional J has the following properties.

Proposition 2.1.
(a) *J is continuously differentiable on H_2^2 where H_2^2 denotes the Hilbert space as usual.*
(b) *The first variation of J at a point u in a direction φ is given by*

$$J'(u)(\varphi) = -2\int_M [e^{2u}(k_g - \Delta u)^2 + \Delta(e^{2u}(k_g - \Delta u))]\varphi dv_g. \tag{2.7}$$

(c) *The Euler equation associated to J under the constrain* (1.6) *is given by*

$$\Delta[e^{-2u}(k_g - \Delta u)] + e^{-2u}(k_g - \Delta u)^2 = \lambda^2 e^{2u} \tag{2.8}$$

for some constant $\lambda \geq 0$.

Proof. (c) can easily follow from (b), Lagrange multiplier and the fact that if we set $G(u) = \int_M e^{2u} dv_g$, $G'(u)(\varphi) = \int_M e^{2u}\varphi dv_g$. (b) will follow from the proof of (a). Before we are going to prove the part (a), let us recall two simple facts.

Fact 1. If the real dimension of a manifold is two, there exists a constant $C > 0$ such that

$$\max |u| \leq C||u||_{H_2^2} \tag{2.9}$$

for all $u \in H_2^2$.

It can be proved as follows. Suppose G is a Green function for Δ. Then it is known that $\int_M G^2 dv_g$ is finite and for any $u \in H_2^2, u(p) = \int_M u dv_g - \int_M G\Delta u dv_g$. Thus the Hölder inequality implies that (2.9) holds with $C = [(\int_M dv_g)^{1/2} + (\int_M G^2 dv_g)^{1/2}]$.

Fact 2. We can choose $\varepsilon_0 > 0$ such that if $0 < t < \varepsilon_0$, then $|e^{-2t} - 1 + 2t| \leq 4t^2$ and $|e^{-2t} - 1| \leq 100t$ where numbers 4 and 100 are not so important.

In order to simplify the notation, from now on, we will denote the norm $||u||_{H_2^2}$ by $||u||$.

Now we are in position to give the proof of the part (a).

Let us now assume that $u \in H_2^2$ such that $J(u) < \infty$. Let φ be any function in H_2^2 such that $\max|\varphi| < \epsilon_0$. Then we have

$$\begin{aligned}
&||\varphi||^{-1}\left|J(u+\varphi) - J(u) + 2\int_M [e^{-2u}(k_g - \Delta u)^2 + \Delta(e^{-2u}(k_g - \Delta u))]\varphi dv_g\right| \\
&= ||\varphi||^{-1}\left|\int_M (k_g - \Delta u)^2 e^{-2u}[e^{-2\varphi} + 2\varphi - 1]dv_g\right. \\
&\quad \left. + \int_M (\Delta\varphi)^2 e^{-2(u+\varphi)}dv_g - 2\int_M \Delta\varphi[e^{-2u}(k_g - \Delta u)](e^{-2\varphi} - 1)dv_g\right| \\
&\le ||\varphi||^{-1}[4C^2||\varphi||^2 J(u) + C_1\exp[2C(||u|| + ||\varphi||)]\cdot||\varphi||^2 \\
&\quad + 200C_2\exp[C||u||]J(u)^{\frac{1}{2}}\cdot||\varphi||^2] \\
&= [4C^2 J(u) + C_1\exp[2C(||u|| + ||\varphi||)] + 200C_2\exp[C||u||]J(u)^{\frac{1}{2}}]\cdot||\varphi||,
\end{aligned}$$

where C is given in (2.9) and C_1 and C_2 are constants. This proves (b) and the half of (a). For the rest of the part (a), we can argue as follows: for a fixed function $u \in H_2^2$, if $v \in H_2^2$ is such that $\max|u - v| < \varepsilon_0$, then we have

$$\begin{aligned}
&|J'(u)(w) - J'(v)(w)| \\
&= 2\left|\int_M [e^{-2u}(k_g - \Delta u)^2 + \Delta(e^{-2u}(k_g - \Delta u))]w dv_g\right. \\
&\quad \left. - \int_M [e^{-2v}(k_g - \Delta v)^2 + \Delta(e^{-2v}(k_g - \Delta v))]w dv_g\right| \\
&= 2\left|\int_M (e^{-2u} - e^{-2v})(k_g - \Delta u)^2 w dv_g\right. \\
&\quad + \int_M e^{-2v}[(k_g - \Delta u)^2 - (k_g - \Delta v)^2]w dv_g \\
&\quad \left. + \int_M (e^{-2u} - e^{-2v})k_g \Delta w dv_g - \int_M (e^{-2u}\Delta u - e^{-2v}\Delta v)\Delta w dv_g\right| \\
&\le 200\max|u - v|J(u)\max|w| + 2e^{2\max|u|}\int_M |\Delta(u - v)||\Delta w|dv_g \\
&\quad + 2e^{2\max|v|}\max|w|\int_M |\Delta(u - v)||2k_g - \Delta(u + v)|dv_g \\
&\quad + 400e^{\max|v|}\int_M |u - v||\Delta w|dv_g \max|k_g| \\
&\quad + 2e^{2\max|u|}\max|u - v|\cdot||v||\cdot||w|| \\
&\le C||u - v||\cdot||w||,
\end{aligned}$$

where $C > 0$ depends on $||u||, ||v||, J(u)$ and $\max|k_g|$. This proves the result. □

Also we note that if we define $H = \{u \in H_2^2 | \int_M e^{2u} dv_g = \int_M dv_g\}$, we have

Proposition 2.2. *H is a weakly closed subset in H_2^2.*

Proof. Weakly convergent subsequences in H_2^2 is strongly convergent sequences in H_1^2. $\int_M e^{2u} dv_g$ is a weakly continuous functional on H_1^2 by Moser's inequality. Thus it is a weakly continuous functional on H_2^2. We are done. □

3. Proof of Main Theorem for $\chi(M) < 0$.

The proof of the main theorem in this case will consist of several lemmas.

Lemma 3.1. *Let (M, g) be a closed Riemann surface with $\chi(M) < 0$. If the Gaussian curvature k_g is positive somewhere, there exists a function $u \in H_2^2$ so that the metric $\tilde{g} = e^{2u} g$ has nonpositive Gaussian curvature $k_{\tilde{g}}$ and $\int_M k_{\tilde{g}}^2 dv_{\tilde{g}} \leq \int_M k_g^2 dv_g$.*

Proof. Choose

$$s = \begin{cases} -k_g & k_g \geq 0 \\ k_g & k_g \leq 0. \end{cases}$$

Then it is clear that $s \leq 0$. Let us consider

$$\Delta u + s e^{2u} = k_g. \tag{3.1}$$

Clearly, if $\varphi = 0$, then

$$\Delta\varphi + s e^{2\varphi} - k_g = \begin{cases} -2k_g & k_g \geq 0 \\ 0 & k_g \leq 0 \end{cases}$$

which is always nonpositive. Let ϕ be a solution of $\Delta\phi = k_g - \frac{\int_M k_g dv_g}{\int_M dv_g}$. By standard elliptic theory, up to add a constant to ϕ, ϕ is unique. We fix a solution by requiring $\int_M \phi dv_g = 0$. Choose $N > 0$ large enough so that $s e^{2\phi - N} - \frac{\int_M k_g dv_g}{\int_M dv_g} > 0$ and $\phi - N/2 < 0$. It is clear that such an N exists. Then set $v = \phi - N/2$, we can get

$$\Delta v + s e^{2v} - k_g = s e^{2\phi - N} - \frac{\int_M k_g dv_g}{\int_M dv_g} > 0$$

and $v < 0$. Therefore $\phi = 0$ is a supper solution and v is a sub-solution of the equation (3.1). By standard elliptic theory, equation (3.1) has a H_2^2 solution w since s and k_g are in $L^2(M)$, [**S**, Theorem 2.4; Chapter 1].

Now choose a constant α such that $\int_M e^{2(w+\alpha)}dv_g = \int_M dv_g$, and set $\tilde{k} = se^{-2\alpha} \leq 0$. Since $w \leq 0$ and $-2\alpha = \log \frac{\int_M e^{2w}dv_g}{\int_M dv_g}$, by the convexity of the exponential function, $\alpha > 0$. Also $\int_M (\tilde{k})^2 e^{2(w+\alpha)}dv_g = \int_M s^2 e^{-4\alpha} e^{2(w+\alpha)}dv_g = \int_M s^2 e^{2(w-\alpha)}dv_g \leq \int_M s^2 dv_g = \int_M k_g^2 dv_g$. Thus if we set $u = w + \alpha$, then $\tilde{g} = e^{2u}g$ will satisfy the requirement. This completes the proof of Lemma 3.1. □

Lemma 3.2. *If $u \in H_2^2, J(u) \leq C_0, k_g - \Delta u \leq 0$ and $\int_M e^{2u}dv_g = \int_M dv_g$, there exist constants C_1, C_2 depending only C_0 and k_g and $\int_M dv_g$ such that the following inequality holds: $C_1 < u < C_2$.*

Proof. Let φ be a solution of the equation $\Delta\varphi = k_g - \frac{\int_M k_g dv_g}{\int_M dv_g}$ with $\int_M \varphi dv_g = 0$. Since k_g is continuous and φ is in C^2, φ is in H_2^2. There exist constants α and β such that $\alpha < \varphi < \beta$ where α and β only depend on k_g.

Now from $J(u) \leq C_0$, we have

$$\begin{aligned}
C_0 \geq J(u) &= \int_M e^{-2u}(k_g - \Delta u)^2 dv_g \\
&\geq \int_M e^{2(\varphi-\beta)}(k_g - \Delta u)^2 e^{-2u} dv_g \\
&= e^{-2\beta} \int_M e^{-2(u-\varphi)}(k_g - \Delta u)^2 dv_g \\
&= e^{-2\beta} \int_M e^{-2(u-\varphi)} \left(k_g - \frac{\int_M k_g dv_g}{\int_M dv_g} + \frac{\int_M k_g dv_g}{\int_M dv_g} - \Delta u \right)^2 \\
&= e^{-2\beta} \int_M e^{-2(u-\varphi)} \left(\frac{\int_M k_g dv_g}{\int_M dv_g} - \Delta(u-\varphi) \right)^2 dv_g \\
&= e^{-2\beta} \left[\left(\frac{\int_M k_g dv_g}{\int_M dv_g} \right)^2 \int_M e^{-2(u-\varphi)} dv_g \right. \\
&\quad - 2\frac{\int_M k_g dv_g}{\int_M dv_g} \int_M e^{-2(u-\varphi)} \Delta(u-\varphi) dv_g \\
&\quad \left. + \int_M e^{-2(u-\varphi)} \left(\Delta(u-\varphi)\right)^2 dv_g \right] \\
&= e^{-2\beta} \left[\left(\frac{\int_M k_g dv_g}{\int_M dv_g} \right)^2 \int_M e^{-2(u-\varphi)} \right.
\end{aligned}$$

$$- 4\frac{\int_M k_g dv_g}{\int_M dv_g}\int_M e^{-2(u-\varphi)}|\nabla(u-\varphi)|^2 dv_g$$
$$+ \int_M e^{-2(u-\varphi)}\left(\Delta(u-\varphi)\right)^2 dv_g\Big] .$$

The above inequality and the fact that $\int_M k_g dv_g = \chi(M)\int_M dv_g < 0$ imply that

$$\int_M e^{-2(u-\varphi)}dv_g \le \frac{[C_0 e^{2\beta}(\int_M dv_g)^2]}{(\int_M k_g dv_g)^2}, \tag{3.2}$$

$$\int_M e^{-2(u-\varphi)}|\nabla(u-\varphi)|^2 dv_g \le \frac{[C_0 e^{2\beta}\int_M dv_g]}{(-4\int_M k_g dv_g)}, \tag{3.3}$$

$$\int_M e^{-2(u-\varphi)}|\Delta(u-\varphi)|^2 \le C_0 e^{2\beta}. \tag{3.4}$$

Simply (3.2) gives us

$$e^{2\alpha}\int_M e^{-2u}dv_g \le \int_M e^{-2u+2\varphi}dv_g \le \frac{[C_0 e^{2\beta}(\int_M dv_g)^2]}{(\int_M k_g dv_g)^2},$$

or equivalently,

$$\int_M e^{-2u}dv_g \le \frac{C_0 e^{2(\beta-\alpha)}(\int_M dv_g)^2}{(\int_M k_g dv_g)^2} := C_3. \tag{3.5}$$

By using the convexity of the exponential function, we can get that $-\int_M 2u dv_g \le \log C_3$ from (3.5). Since $\int_M e^{2u}dv_g = \int_M dv_g$, $2\int_M u dv_g \le \log\int_M dv_g := C_4$.

From now on we will assume that the volume $\int_M dv_g = 1$.

Thus $C_4 = 0$. Anyway, we get $|\int_M u dv_g| \le \frac{\log C_3}{2}$. Now

$$u(p) - \int_M u dv_g = -\int_M G\Delta u dv_g \tag{3.6}$$

where G is Green's function associated to Laplace operator Δ. We can choose G such that it is positive everywhere [**A**, Theorem 4.14]. These give us the estimate on u as follows

$$\begin{aligned} u(p) - \int_M u dv_g &= \int_M G(k_g - \Delta u)dv_g - \int_M k_g G dv_g \\ &\le \left(\int_M (k_g)^2 dv_g\right)^{1/2}\left(\int_M G^2 dv_g\right)^{1/2} \\ &:= C_5. \end{aligned}$$

This implies that

$$u(p) \le \left(\int_M u dv_g\right) + \left(\int_M (k_g)^2 dv_g\right)^{1/2} \left(\int_M G^2 dv_g\right)^{1/2} \le \frac{\log C_3}{2} + C_5 := C_2.$$

As p is arbitrary on M, $u \le C_2$.

Apply (3.6) with u replaced by e^{-u} to get

$$\begin{aligned} e^{-u} - \int_M e^{-u} dv_g &= -\int_M G\Delta e^{-u} dv_g \\ &= \int_M e^{-u} G(\Delta u - |\nabla u|^2) dv_g \\ &\le \int_M e^{-u} G \Delta u dv_g \\ &\le \left(\int_M e^{-2u} (\Delta u)^2 dv_g\right)^{1/2} \left(\int_M G^2 dv_g\right)^{1/2}, \end{aligned} \tag{3.7}$$

where we have used the fact that $\Delta(e^{-u}) = e^{-u}(|\nabla u|^2 - \Delta u)$. But from (3.3), we obtain

$$\begin{aligned} C_0 e^{2\beta} &\ge \int_M e^{-2(u-\varphi)} (\Delta(u-\varphi))^2 dv_g \\ &= \int_M e^{-2(u-\varphi)} [(\Delta u)^2 - 2\Delta u \Delta\varphi + (\Delta\varphi)^2] dv_g \\ &\ge \int_M e^{-(u-\varphi)} (\Delta u)^2 dv_g - 1/2 \int_M (\Delta u)^2 e^{-2(u\varphi)} dv_g \\ &\quad - 2\int_M (\Delta\varphi)^2 e^{-2(u-\varphi)} dv_g + \int_M (\Delta\varphi)^2 e^{-2(u-\varphi)} dv_g \\ &= 1/2 \int_M e^{-2(u-\varphi)} (\Delta u)^2 dv_g - \int_M (\Delta\varphi)^2 e^{-2(u-\varphi)} dv_g. \end{aligned}$$

This simply implies that

$$\begin{aligned} &\frac{1}{2}\int_M e^{-2u} (\Delta u)^2 dv_g e^{2\alpha} \\ &\le \frac{1}{2}\int_M e^{-2(u-\varphi)} (\Delta u)^2 dv_g \\ &\le \int_M e^{-2(u-\varphi)} (\Delta(u-\varphi))^2 dv_g + \int_M e^{-2(u-\varphi)} (\Delta\varphi)^2 dv_g \\ &\le C_0 e^{2\beta} + \int_M \left(\int_M k_g dv_g - k_g\right)^2 \cdot e^{-2u} dv_g \cdot e^{2\beta} \\ &\le \left[C_0 + 2\left(\int_M k_g dv_g\right)^2 \int_M e^{-2u} dv_g + 2\int_M k_g^2 e^{-2u} dv_g\right] e^{2\beta} \end{aligned} \tag{3.8}$$

$$\leq \left\{ C_0 + 2\left[\left(\int_M k_g dv_g \right)^2 + (\max |k_g|)^2 \right] C_3 \right\} e^{2\beta}$$
$$:= C_6$$

by using (3.5) and (3.7). Thus this implies that

$$e^{-u(p)} \leq C_3^{1/2} + \left(\int_M G^2 \right)^{1/2} (2C_6)^{1/2} e^{-\alpha} := e^{-C_1}. \tag{3.9}$$

Hence, by taking log on both sides of (3.9), we have $u(p) \geq C_1$. As before, p is a general point on M, we have what we need to show. □

Lemma 3.3. *If $u \in H_2^2, J(u) \leq C_0, k_g - \Delta u \leq 0$ and $\int_M e^{2u} dv_g = \int_M dv_g$, there exists a constant C_7 depending on C_0, k_g and $\int_M dv_g$ only such that $||u|| \leq C_7$.*

Proof. It is clear from (3.8) and Lemma 3.2 that

$$\frac{1}{2} e^{-2C_2} \cdot e^{2\alpha} \cdot \int_M (\Delta u)^2 dv_g \leq \frac{1}{2} e^{2\alpha} \int_M e^{-2u} (\Delta u)^2 dv_g \leq C_6.$$

If we define C_7 to be $2\{2C_6 e^{2(C_2-\alpha)} + (C_1^2 + C_2^2)\}$, combining with Lemma 3.2, the result follows. □

Lemma 3.4. *The weak solution of the equation* (2.8) *exists.*

Proof. Let $\alpha_0 = \inf_{u \in H} J(u)$. Suppose that $\{u_j\}$ is a minimizing sequence in H_2^2 for the functional J with $\int_M e^{2u_j} dv_g = \int_M dv_g$. Without loss of generality by applying Lemma 3.1, we can choose a minimizing sequence $\{u_j\}$ such that $k_g - \Delta u_j \leq 0$. Then Lemma 3.3 implies that $\{u_j\}$ is bounded in H_2^2. Thus there exists a subsequence of $\{u_j\}$ still denoted by $\{u_j\}$ and a function $u_0 \in H_2^2$ such that u_j is weakly convergent to u_0 in H_2^2 and u_j is pointwise almost everywhere convergent to u_0 because H_2^2 is reflexive and the Proposition 3.43 of [**A**]. Now we have to show that $J(u_0) = \alpha_0$ in order to show that u_0 weakly satisfies the equation (2.8). On the one hand, since the subset H of H_2^2 is weakly closed by Proposition 2.2, u_0 is in the subset H. By definition of α_0, we can easily see that $\alpha_0 \leq J(u_0)$. On the other hand, we have

(3.10)
$$J(u_n) - J(u_0) = \int_M e^{-2u_n} (k_g - \Delta u_n)^2 dv_g - \int_M e^{-2u_0} (k_g - \Delta u_0)^2 dv_g$$

$$\begin{aligned}
&= \int_M (e^{-2u_n} - e^{-2u_0}) k_g^2 dv_g - 2\int_M k_g (e^{-2u_n}\Delta u_n - e^{-2u_0}\Delta u_0) dv_g \\
&\quad + \int_M \left[e^{-2u_n}(\Delta u_n)^2 - e^{-2u_0}(\Delta u_0)^2\right] dv_g \\
&= \int_M (e^{-2u_n} - e^{-2u_0}) k_g^2 dv_g - 2\int_M k_g (e^{-2u_n} - e^{-2u_0})\Delta u_n dv_g \\
&\quad - 2\int_M k_g e^{-2u_0}\Delta(u_n - u_0) dv_g + \int_M e^{-2u_n}(\Delta(u_n - u_0))^2 dv_g \\
&\quad + 2\int_M e^{-2u_n}\Delta u_0 \Delta(u_n - u_0) dv_g + \int_M (e^{-2u_n} - e^{-2u_0})(\Delta u_0)^2 dv_g \\
&\geq \int_M (e^{-2u_n} - e^{-2u_0}) k_g^2 dv_g - 2\int_M k_g (e^{-2u_n} - e^{-2u_0})\Delta u_n dv_g \\
&\quad - 2\int_M k_g e^{-2u_0}\Delta(u_n - u_0) dv_g + 2\int_M (e^{-2u_n} - e^{-2u_0})\Delta u_0 \Delta(u_n - u_0) dv_g \\
&\quad + 2\int_M e^{-2u_0}\Delta u_0 \Delta(u_n - u_0) dv_g + \int_M (e^{-2u_n} - e^{-2u_0})(\Delta u_0)^2 dv_g \\
&:= I + II + III + IV + V + VI.
\end{aligned}$$

As n goes to ∞, I and VI go to zero because e^{-2u_n} goes to e^{-2u_0} pointwise almost everywhere and the Dominated convergence theorem (Theorem 3.32 of [**A**]) can be applied; III and V go to zero by the definition of the weakly convergence in H_2^2. For II,

$$\begin{aligned}
&\left|\int_M k_g (e^{-2u_n} - e^{-2u_0})\Delta u_n dv_g\right| \\
&\leq \max|k_g| \left[\int_M (\Delta u_n)^2 dv_g\right]^{1/2} \left[\int_M (e^{-2u_n} - e^{-2u_0})^2 dv_g\right]^{1/2} \\
&\to 0
\end{aligned}$$

by the Hölder inequality and the Dominated convergence theorem again since $||u_n||$ is uniformly bounded. Similarly we can estimate the term IV, we will leave it to reader.

By letting $n \to \infty$ in (3.10), we have that $\alpha_0 - J(u_0) \geq 0$, i.e., $J(u_0) \leq \alpha_0$. The Lemma is proved. □

Remark. In fact, we can show that $J(u)$ is weakly lower semicontinuous on H_2^2. Since we do not need this, we will not give a proof here.

Lemma 3.5. *The weak solution of the equation* (2.8) *is smooth if* k_g *is smooth.*

Proof. Suppose u_0 satisfies (2.8). Set $s = e^{-2u_0}(k_g - \Delta u_0)$ and $\lambda^2 - s^2 = f$. Then it is clear that equation (2.8) can be written as

$$\Delta s = e^{2u_0} f.$$

Since $u_0 \in H_2^2$, s must be square integrable. If we set $v = s - \int_M s dv_g$, above equation looks like:

$$\Delta v = e^{2u_0} f.$$

By applying Theorem 3.67 of [**A**] on p. 91, we can get that $\int_M |\nabla v|^2 dv_g$ is finite. Then v is in L^p for all p by applying same theorem. Now the above equation tells us that Δv is square integrable. Thus v , so s, is continuous. By using the definition of s, we can see that u_0 is in C^2. Then s is in C^2 which will imply that u_0 is in C^4. Repeating this argument, we can easily see that u_0 is smooth. This finishes the proof. In fact, we will see later that s is a constant. □

4. $\chi(M) = 0$ case.

In this case, for a given metric g on a Riemmann surface with $\int_M k_g dv_g = 0$, we can choose a function φ such that $\Delta\varphi = k$ and $\int_M e^{2\varphi} dv_g = \int_M dv_g$by standard elliptic theory. Now $J(\varphi) = \int_M e^{-2\varphi}(k_g - \Delta\varphi)^2 dv_g = \int_M e^{-2\varphi}(k_g - k_g)^2 dv_g = 0$. Thus φ is a minimizer for J. Of course, φ is smooth.

5. The scalar curvatures of the extremal metrics.

In this section, we will prove the following theroem due to Calabi:

Theorem. *For a Riemann surface, the extremal metrics have constant scalar curvatures.*

The proof will follow easily from the following

Lemma 5.1. *Let (M, g) be a compact Riemann surface. If the Gaussian curvature k of the metric g satisfies*

$$\Delta k + k^2 = \lambda^2 \tag{5.1}$$

for some constant λ, then ∇k is a conformal vector field.

Proof. The ∇k is a conformal vector field if and only if its components satisfy

$$k_{i,j} + k_{j,i} = 2f g_{ij}$$

in local coordinates for some function f. That is to say that we only need to show that the hessian of the Gaussian curvature is proportional to the metric g. The standard Ricci identity shows that

$$\frac{1}{2}\Delta|\nabla k|^2 = \sum k_{ij}^2 + \langle\nabla\Delta k, \nabla k\rangle + R_{ij}k_i k_j.$$

But for a Riemann surface, $R_{ij} = kg_{ij}$. Thus

$$\frac{1}{2}\Delta|\nabla k|^2 = |\operatorname{Hess}(k)|^2 + \langle\nabla\Delta k, \nabla k\rangle + k|\nabla k|^2.$$

By using (5.1), we get

$$\begin{aligned}\frac{1}{2}\Delta|\nabla k|^2 &= |\operatorname{Hess}(k)|^2 - k|\nabla k|^2 \\ &= |\operatorname{Hess}(k)|^2 - \frac{(\Delta k)^2}{2} + \frac{(\Delta k)^2}{2} - k|\nabla k|^2. \qquad (5.2)\end{aligned}$$

Also by using integration by parts and (5.1) we obtain

$$\begin{aligned}\int_M k|\nabla k|^2 dv_g &= \int_M k \cdot \nabla k \cdot \nabla k dv_g \\ &= -\frac{1}{2}\int_M k^2 \Delta k dv_g \\ &= \frac{1}{2}\int_M (\lambda^2 - k^2)\Delta k dv_g \\ &= \frac{1}{2}\int_M (\Delta k)^2 dv_g.\end{aligned}$$

Now, we integrate both sides of (5.2) to get

$$\begin{aligned}0 &= \int_M |\operatorname{Hess} k|^2 dv_g - \int_M \frac{(\Delta k)^2}{2} dv_g \\ &= \int_M \left|\operatorname{Hess}(k) - \frac{\Delta k}{2} g\right|^2 dv_g.\end{aligned}$$

The last equality holds because of a well known identity. The Lemma can be seen easily from this. □

Proof of Theorem. Set $g' = e^{2u_0}g$. Then we have $\Delta_{g'} = e^{-2u_0}\Delta_g$ and $k_{g'} = e^{-2u_0}(k_g - \Delta u_0)$. Thus the equation (2.8) can be written as

$$\Delta_{g'}k_{g'} + k_{g'}^2 = \lambda^2. \qquad (5.3)$$

Thus Lemma 5.1 can be applied to show that $\nabla k_{g'}$ is a conformal vector field with respect to metric g'. But a conformal vector field X on a manifold M satisfies the identity

$$\int_M X \cdot s_g dv_g = 0$$

where s_g is the scalar curvature of the metric g ([**BE**],[**X**]). In our case, $s_g = 2k_{g'}$. Thus we have

$$2\int_M |\nabla k_{g'}|^2 dv_{g'} = 0$$

Therefore, $4|\nabla k_{g'}|^2 = 0$. That is, $k_{g'}$ is a constant. The theorem is proved. □

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Received November 29, 1993 and revised February 22, 1994.

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