PACIFIC JOURNAL OF MATHEMATICS
Vol. 174, No. 2, 1996

KK-GROUPS OF TWISTED CROSSED PRODUCTS BY GROUPS ACTING ON TREES

KEVIN MCCLANAHAN

An exact sequence of Pimsner for KK-groups of crossed products of C^*-algebras by locally compact groups acting on trees is generalized to the case of crossed products twisted by a circle-valued cocycle. The exact sequence is applied to the case of free products of twisted group C^*-algebras. In particular, the K-groups of the free product of two matrix algebras is computed.

1. Introduction.

The problem motivating this work was that of determining the K-theory of the free product $M_n *_{\mathbb{C}} M_k$ where M_n is the C^*-algebra of n by n complex-entried matrices. Specifically, the desire was to show that the K-groups of this particular free product satisfy the following repeating exact sequence.

$$\longrightarrow K_j(\mathbb{C}) \longrightarrow K_j(M_n) \oplus K_j(M_k) \longrightarrow K_j(M_n *_{\mathbb{C}} M_k) \longrightarrow .$$

This exact sequence is a special case of an exact sequence for amalgamated products conjectured by J. Cuntz in [**Cu2**]. It was proved by Cuntz that if $A *_C B$ is an amalgamated product of C^*-algebras subject to the condition that there are retractions from A and B onto C (homomorphisms which restrict to the identity on C) then the following repeating sequence is exact.

$$\longrightarrow K_j(C) \longrightarrow K_j(A) \oplus K_j(B) \longrightarrow K_j(A *_C B) \longrightarrow .$$

This conjecture that the above sequence holds for arbitrary amalgamated products has been verified for a variety of other cases since. For example, it follows from Pimsner's exact sequence for KK-groups of crossed products by groups acting on trees [**Pi**] that the above sequence is exact if $A = C^*(G_1)$, $B = C^*(G_2)$, and $C = C^*(H)$ where H is a discrete subgroup of the discrete groups G_1 and G_2. The conjecture has also been proved for the case where $A = M_n$, $C = \mathbb{C}$, and B is any unital C^*-algebra which has a retraction onto $\mathbb{C}$ in [**McC2**] (this result is primarily due to J. Cuntz). This conjecture is also true in many cases for reduced free products in the sense of D. Avitzour [**Av**] (see [**McC2**]). In [**McC2**] the author was able to exploit some of the similarities between M_n and $C^*(G)$ for a discrete group G in order to prove the conjecture for some free products of the form $M_n *_{\mathbb{C}} C^*(G)$. The similarity between $C^*(G)$ and M_n is that both C^*-algebras are generated by unitaries such that the product of any two generating unitaries is a scalar multiple of another generating unitary. In the case of $C^*(G)$ the generating unitaries are the elements of G. In the case of M_n, it follows from Takai duality that M_n is isomorphic to a crossed product $C^*(\mathbb{Z}_n) \rtimes \mathbb{Z}_n$ and the generating unitaries can be taken to be the products of elements of the two copies of $\mathbb{Z}_n$. This observation led to the computation of the K-groups of $M_n *_{\mathbb{C}} M_k$ as follows. The Pimsner exact sequence gave a method of computing the K-groups of $C^*(G_1) *_{\mathbb{C}} C^*(G_2) \cong C^*(G_1 * G_2)$ which does not use the fact that there is a retraction from $C^*(G_i)$ onto $\mathbb{C}$ as in the case of Cuntz's result. The similarity between M_n and $C^*(G)$ for a discrete group just mentioned enables one to modify Pimsner's techniques in order to replace each $C^*(G)$ by a matrix algebra.

It was pointed out to the author by several people including A. Paterson, M. Dadarlat, and the referee of the first version of this paper that the similarity between M_n and $C^*(G)$ can be best described in terms of twisted crossed products That is, M_n is isomorphic to the twisted group C^*-algebra $C^*_u(\mathbb{Z}_n)$ for a suitable 2-cocycle u on $\mathbb{Z}_n$. It turns out then that the exact sequence of Pimsner can be generalized directly to the case of twisted crossed products by a circle-valued 2-cocycle. The technique used in the generalization was to write a twisted crossed product as a quotient of a nontwisted crossed product using a well-known technique of G.W. Mackey [**Ma3**, Section 2]. The Pimsner sequence can then be applied to the nontwisted crossed product and the KK-groups exact sequence for the twisted crossed product can be deduced from that for the nontwisted crossed product. This method is somewhat less tedious than generalizing Pimsner's proof step by step.

2. Twisted Crossed Products.

We will review the necessary concepts in the theory of crossed products twisted by a 2-cocycle. The twisted crossed products discussed here are not in the most general form available in the literature [**BS**],[**PR**], [**G2**]. In particular the cocycles will be circle-valued instead of taking values in the unitary group of the multiplier algebra of a C^*-algebra and group actions will be continuous instead of Borel measurable so that Pimsner's results can be applied. For technical reasons, all C^*-algebras will be separable and all groups will be second countable and locally compact.

Recall that a circle-valued 2-cocycle on a group G is a map

$$u : G \times G \to \mathbb{T}$$

such that

$$u(s,t)u(r,st) = u(r,s)u(rs,t) \quad \text{for all } r,s,t \in G.$$

Let A be a separable C^*-algebra, G a second countable locally compact group, α a strongly continuous action of G on A, and u a Borel measurable circle-valued 2-cocycle on G. Then we say that (A, G, α, u) is a twisted C^*-dynamical system. Hereafter when we refer to a twisted C^*-dynamical system we shall mean a dynamical system with the restrictions just given.

Let (A, G, α, u) be a twisted C^*-dynamical system. Define a convolution multiplication and involution on $L^1(G, A)$ as follows:

$$\begin{aligned}(f * h)(x) &= \int_G f(y)\alpha_y[h(y^{-1}x)]u(y, y^{-1}x)dy \\ f^*(x) &= u(x,x^{-1})^*\alpha_x[f(x^{-1})^*]\triangle_G(x^{-1})\end{aligned}$$

where $\triangle_G$ denotes the modular function on G. Then $L^1(G, A)$ is a Banach *-algebra with respect to these operations [**BS**, Theorem 2.2]. The enveloping C^*-algebra of $L^1(G, A)$ will be called the twisted crossed product of A by G and will be denoted $A \rtimes G$.

A covariant representation of (A, G, α, u) is a triple (π, U, H) where H is a Hilbert space, $\pi : A \to B(H)$ is a nondegenerate representation, and $U : G \to \mathcal{U}(H)$ is a Borel measurable map such that

$$\begin{aligned}U_sU_t &= u(s,t)U_{st} \quad \text{for all } s,t \in G \\ \pi(\alpha_s(a)) &= U_s\pi(a)U_s^* \quad \text{for all } s \in G, a \in A.\end{aligned}$$

It is well known that there is a one-to-one correspondence between covariant representations of (A, G, α, u) and nondegenerate representations of $A \rtimes G$ [**BS**, Theorem 3.3]. The correspondence is given by $(\pi, U, H) \leftrightarrow \pi \times U$ where the $\pi \times U : A \rtimes G \to B(H)$ is given by

$$(\pi \times U)(f) = \int_G \pi(f(x))U_x dx$$

for $f \in L^1(G, A)$.

Given a representation $\pi : A \to B(H)$, let $\tilde{\pi} : A \to B(L^2(G, H))$ be defined by

$$(\tilde{\pi}(a)\xi)(s) = \pi(\alpha_{s^{-1}}(a))\xi(s) \quad \xi \in L^2(G,H), \quad a \in A, s \in G.$$

Let $\lambda : G \to B(L^2(G, H))$ be defined by

$$(\lambda_s\xi)(t) = f(s^{-1}t)u(s, s^{-1}t) \quad \xi \in L^2(G,H), s,t \in G. \tag{2.1}$$

It follows that $(\tilde{\pi}, \lambda, H)$ is a covariant representation of (A, G, α, u). The representation $\tilde{\pi} \times \lambda$ will be called the regular representation of $A \rtimes G$ associated with π. The reduced twisted crossed product of A by G is defined to be the completion of $L^1(G, A)$ with respect to the norm given by

$$\|\xi\|_r = \sup\{\|(\tilde{\pi} \times \lambda)(\xi)\| : \pi \text{ is a representation of } A\}$$

for $\xi \in L^2(G, H)$. The reduced twisted crossed product just defined will be denoted $A \rtimes_r G$. It can be shown that if π is a faithful representation of A, then $\|(\tilde{\pi} \times \lambda)(\xi)\| = \|\xi\|_r$ for all $\xi \in L^2(G, H)$ and so $A \rtimes_r G$ can be identified with $(\tilde{\pi} \times \lambda)(A \rtimes G)$. See [**Q**] for more details on reduced twisted crossed products.

We now describe a way in which G can be written as a quotient of a locally compact group G^u in such a way that the twisted crossed products $A \rtimes G$ and $A \rtimes_r G$ are quotients of the nontwisted crossed products $A \rtimes G^u$ and $A \rtimes_r G^u$ respectively. This technique is due to G. Mackey [**Ma3**, Section 2]. First of all we remark that it is possible to replace u by a *normalized* 2-cocycle u' which satisfies $u'(g, g^{-1}) = 1$ for all $g \in G$ in such a way that the twisted crossed products of A by G relative to u and relative to u' are isomorphic. This can be done as follows. Let $\rho : G \to \mathbb{T}$ be defined by $\rho(t) = (u(e, e)u(t, t^{-1}))^{\frac{1}{2}}$ where the square root taken is a Borel measurable function and e denotes the identity element of G. Let $u'(s, t) = \dfrac{\rho(st)}{\rho(s)\rho(t)} u(s, t)$. It follows from [**PR**, Lemma 3.3] that the twisted crossed products of A by G relative to u and u' are isomorphic. We will need the fact later that it follows from the definition of a 2-cocycle that $u'(g, e) = u'(e, g) = 1$ for all $g \in G$. We will assume from now on that the cocycle u is normalized.

Let $G^u = \mathbb{T} \times G$ as a set. Endow G^u with the following multiplication:

$$(z_1, s_1)(z_2, s_2) = (z_1 z_2 u(s_1, s_2), s_1 s_2).$$

It follows that G^u is a group with identity $(1, e)$. It follows from the normalization condition on u that (z^{-1}, s^{-1}) is the inverse of (z, s). It may be the case that the above multiplication is not continuous with respect to the product topology on $\mathbb{T} \times G$. To get around this problem first give G^u the product Borel structure and give G^u the product measure $d\lambda \times dg$ where $d\lambda$ is the normalized Haar measure on $\mathbb{T}$ and dg is a Haar measure on G. It is easy to check that this gives a right-invariant measure on G^u. It follows from [**Ma1**, Theorem 7.1] that there is a unique locally compact topology on G^u under which G^u is a topological group with the Borel structure mentioned above and with $d\lambda \times dg$ as a Haar measure. We remark that if G is a discrete group then the product topology on G^u does make G^u a locally compact group and by the uniqueness part of [**Ma1**, Theorem 7.1] it follows that the product topology is the desired topology in this case.

The action α of G on A can be extended to an action α^u of G^u on A by letting $\alpha^u_{(z,s)} = \alpha_s$. This gives a C^*-dynamical system (A, G^u, α^u) and gives the (nontwisted) crossed products $A \rtimes G^u$ and $A \rtimes_r G^u$. Let

$$\phi : C_c(G^u, A) \to C_c(G, A)$$

be the *-homomorphism defined by

$$\phi(k)(s) = \int_{\mathbb{T}} zk(z,s)dz$$

for $k \in C_c(G^u, A)$, $s \in G$. It follows that this map extends to a surjection:

$$\phi : A \rtimes G^u \to A \rtimes G.$$

This map factors through to the reduced crossed products to give the corresponding surjection:

$$\phi_r : A \rtimes_r G^u \to A \rtimes_r G.$$

To see this let $\pi : A \to B(H)$ be a faithful representation of A and let $\lambda^u : G^u \to B(L^2(G^u, H))$ be the left regular representation. Let $\lambda : G \to B(L^2(G, H))$ be the left regular representation twisted by the cocycle u as in (2.1). For a unit vector $\xi \in L^2(G, H)$, let $\tilde{\xi} \in L^2(G^u, H)$ be the unit vector defined by $\tilde{\xi}(z,t) = \bar{z}\xi(t)$. An elementary computation shows that for $k \in C_c(G^u, A)$,

$$\bar{z}[(\tilde{\pi} \times \lambda)(\phi(k))\xi](t) = \left[(\tilde{\pi} \times \lambda^u)(k)\tilde{\xi}\right](z,t).$$

Hence it follows that

$$\|\phi(k)\|_r = \|(\tilde{\pi} \times \lambda)(\phi(k))\| \leq \|(\tilde{\pi} \times \lambda^u)(k)\| = \|k\|_r$$

and thus ϕ factors through $A \rtimes_r G^u$.

It will be important in what follows that these maps have right inverses. Let

$$\psi : C_c(G, A) \to C_c(G^u, A)$$

be defined by

$$\psi(f)(z,s) = \bar{z}f(s)$$

for $f \in C_c(G, A)$, $(z,s) \in G^u$. This map extends to a map

$$\psi : A \rtimes G \to A \rtimes G^u.$$

This map factors through to a map

$$\psi_r : A \rtimes_r G \to A \rtimes_r G^u$$

of the reduced crossed products. To see this define for each vector $\xi \in L^2(G^u, H)$ a vector $\tilde{\xi} \in L^2(G, H)$ with $\left\|\tilde{\xi}\right\| \leq \|\xi\|$ by

$$\tilde{\xi}(t) = \int_{\mathbb{T}} z\xi(z,t)dz.$$

It follows from a routine computation that for $f \in C_c(G, A)$ the following relation holds:

$$[(\tilde{\pi} \times \lambda^u)(\psi(f))\xi](z,t) = \bar{z}\left[(\tilde{\pi} \times \lambda)(f)\tilde{\xi}\right](t).$$

Hence it follows that

$$\|\psi(f)\|_r = \|(\tilde{\pi} \times \lambda^u)(\psi(f))\| \leq \|(\tilde{\pi} \times \lambda)(f)\| = \|f\|_r$$

and so ψ factors through to the reduced crossed products. It follows easily that ψ (respectively ψ_r) is a right inverse for ϕ (respectively ϕ_r).

We also remark for future reference that if (π, U, H) is a covariant representation of (A, G^u, α^u), then $\pi \times U$ factors through $A \rtimes G$ if and only if $U_{(z,s)} = zU_{(1,s)}$ for all $z \in \mathbb{T}$ and $s \in G$. Also, $\pi \times U$ factors through $A \rtimes_r G$ if and only if $\pi \times U$ factors through $A \rtimes_r G^u$ and $U_{(z,s)} = zU_{(1,s)}$.

The following lemma can be found in [**Pi**, Lemma 4] for nontwisted crossed products. The proof presented there works for the case of crossed products twisted by a cocycle as well. The proof of the exactness of the sequence corresponding to the full crossed product is due to A. Sheu [**Sh**, Theorem 2.6].

Lemma 2.1. *Let G be a locally compact group and let*

$$0 \longrightarrow I \xrightarrow{\ i\ } A \xrightarrow{\ q\ } A/I \longrightarrow 0$$

be an exact sequence of C^*-algebras. Suppose that G acts on I, A, and A/I by partial automorphisms in such a way that (A, G, α, u) is a twisted C^*-dynamical system and that both i and q are G equivariant. Suppose moreover that q has a completely positive cross section ρ with the following properties:

There exists a G equivariant *-representation $\omega : A/I \to \mathcal{M}(A)$ and a G-continuous projection $p \in \mathcal{M}(A)$ satisfying $a(p - g(p)) \in I$ for every $a \in A$, $g \in G$, such that

$$\rho(x) = p\omega(x)p.$$

Then the following commutative diagram has exact rows and the quotient maps admit completely positive cross sections ρ_G and $\rho_{G,r}$ of norm one.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I \rtimes G & \xrightarrow{i_G} & A \rtimes G & \xrightarrow{q_G} & A/I \rtimes G & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \lambda_G^I} & & \Big\downarrow{\scriptstyle \lambda_G^A} & & \Big\downarrow{\scriptstyle \lambda_G^{A/I}} & & \\
0 & \longrightarrow & I \rtimes_r G & \xrightarrow{i_{G,r}} & A \rtimes_r G & \xrightarrow{q_{G,r}} & A/I \rtimes_r G & \longrightarrow & 0
\end{array}$$

3. Actions of Locally Compact Groups on Trees.

In this section we fix some notation concerning the action of locally compact groups on trees. The material in this section is taken from [**Pi**, Section 1].

Let X^0 (resp. X^1) denote the set of vertices (resp. edges) of a tree X. An orientation of X is a map

$$X^1 \to X^0 \times X^0, \quad y \mapsto (o(y), t(y)).$$

The vertices $o(y)$ and $t(y)$ are called the origin and terminus of y.

A locally compact group G is said to act on X if G acts continuously on the discrete spaces X^0 and X^1 and preserves the orientation. If we denote the action of $g \in G$ on $P \in X^0$ (resp. $y \in X^1$) by $P \mapsto gP$ (resp. $y \mapsto gy$) then the orientation preserving property can be stated as $o(gy) = go(y)$ and $t(gy) = gt(y)$ for all $g \in G$ and $y \in X^1$. We denote the orbit of the edge y by X_y^1 and the orbit of the vertex P by X_P^0. We let X_P^1 denote the set of all edges that "point" to P. That is X_P^1 is the set of all edges y for which the unique path from P to $t(y)$ is shorter than the path from P to $o(y)$. We let X_y^0 denote the set of all vertices which are pointed to by y. That is, X_y^0 is the set of all vertices P for which $y \in X_P^1$.

Let $\overline{X}^1$ denote the following two point compactification of X^1. $\overline{X}^1 = X^1 \cup \{-\infty, +\infty\}$. A neighborhood base of $+\infty$ is given by finite intersections of sets of the form $X_P^1 \cup \{+\infty\}$ and a neighborhood base of $-\infty$ is given by complements of finite unions of sets of the form $X_P^1 \cup \{+\infty\}$. For a Banach space E we let $C_+(X^1, E)$ denote the set of continuous functions from X^1 into E which vanish at $-\infty$. The action of G on X^1 extends continuously to $\overline{X}^1$ by letting $g(\pm\infty) = \pm\infty$ [**Pi**, Lemma 2].

Let Σ denote the oriented graph $G\backslash X$ with vertex and edge sets $\Sigma^i = G\backslash X^i$ for $i = 1, 2$ and origin and terminus maps $\hat{o}$ and $\hat{t}$ given by:

$$\hat{o}(\hat{y}) = \widehat{o(y)} \quad \text{and} \quad \hat{t}(\hat{y}) = \widehat{t(y)}$$

where $\hat{y}$ denotes the class of $y \in X^1$ in $G\backslash X^1$. We fix a lifting of Σ. By this we mean the following:

(1) We identify Σ^0 and Σ^1 with subsets of X^0 and respectively X^1. This gives maps $X^0 \ni P \mapsto \hat{P} \in \Sigma^0$ and $X^1 \ni y \mapsto \hat{y} \in \Sigma^1$.

(2) We fix for each $y \in X^1$ an element $g_y \in G$ such that $g_y y = \hat{y}$.

(3) For each $y \in \Sigma^1$ let $y^t, y^o \in X^1$ be the edges for which $\widehat{y^t} = y = \widehat{y^o}$ and $t(y^t), o(y^o) \in \Sigma^0$.

Let G_P (resp. G_y) denote the stabilizer of the vertex P (resp. the edge y). For an edge y we define the group homomorphisms

$$\sigma_y : G_y \to G_{\hat{t}(y)} = G_{t(y^t)}$$
$$\sigma_{\bar{y}} : G_y \to G_{\hat{o}(y)} = G_{o(y^0)}$$

as follows for $g \in G_y$:

$$\sigma_y(g) = g_{y^t}^{-1} g g_{y^t}$$
$$\sigma_{\bar{y}}(g) = g_{y^0}^{-1} g g_{y^0}.$$

Let $\alpha_y, \alpha_{\bar{y}} \in \mathrm{Aut}(A)$ be the automorphisms given by

$$\alpha_y(a) = g_{y^t}^{-1}(a)$$
$$\alpha_{\bar{y}}(a) = g_{y^o}^{-1}(a)$$

for $a \in A$. The pairs (σ_y, α_y) and $(\sigma_{\bar{y}}, \alpha_{\bar{y}})$ are covariant and the maps σ_y and $\sigma_{\bar{y}}$ are homeomorphisms onto their images. Consequently we have the following maps of the crossed products:

$$\sigma_y \times \alpha_y : A \rtimes G_y \to A \rtimes G_{\hat{t}(y)} \quad \sigma_y \times_r \alpha_y : A \rtimes_r G_y \to A \rtimes_r G_{\hat{t}(y)}$$
$$\sigma_{\bar{y}} \times \alpha_{\bar{y}} : A \rtimes G_{\bar{y}} \to A \rtimes G_{\hat{o}(y)} \quad \sigma_{\bar{y}} \times_r \alpha_{\bar{y}} : A \rtimes_r G_{\bar{y}} \to A \rtimes_r G_{\hat{o}(y)}.$$

4. The Toeplitz Extension.

Let (A, G, α, U) be a twisted C^*-dynamical system and consider the following exact sequence.

$$(4.1) \qquad 0 \longrightarrow C_0(X^1, A) \xrightarrow{\ i\ } C_+(X^1, A) \xrightarrow{\ q\ } A \longrightarrow 0\,\cdot$$

Here i denotes the natural inclusion map and q is evaluation at $+\infty$. For $a \in A$ let $\omega(a)$ denote the function on $\overline{X}^1$ constantly equal to a. For $P \in \Sigma_0$ let χ_P denote the characteristic function of the set $X^1_P \cup \{+\infty\}$. The map $\rho : A \to C_+(X^1, A)$ defined by

$$\rho(a) = \chi_P \omega(a) \chi_P$$

is a completely positive cross section of q which has norm one. By [**Pi**, Lemma 2] it follows that $f(\chi_P - g(\chi_P))$ is in $C_0(X^1, A)$ for every $f \in C_+(X^1, A)$. So by Lemma 2.1 we get the following commutative diagram (4.2)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(X^1,A) \rtimes G & \xrightarrow{i_G} & C_+(X^1,A) \rtimes G & \xrightarrow{q_G} & A \rtimes G & \longrightarrow & 0 \\ & & \Big\downarrow \lambda_0 & & \Big\downarrow \lambda_+ & & \Big\downarrow \lambda & & \\ 0 & \longrightarrow & C_0(X^1,A) \rtimes_r G & \xrightarrow{i_{G,r}} & C_+(X^1,A) \rtimes_r G & \xrightarrow{q_{G,r}} & A \rtimes_r G & \longrightarrow & 0 \end{array}$$

with exact rows and completely positive cross sections ρ_G, $\rho_{G,r}$ of q_G,$q_{G,r}$ having norm one. We also get the corresponding commutative diagram with exact rows for the C^*-dynamical system (A, G^u, α^u).

Suppose that (A, G, α, U) is a twisted C^*-dynamical system. Suppose also that G acts continuously on the discrete set Z. Denote the actions of G on A and Z by $(g, a) \mapsto ga \in A$ and $(g, z) \mapsto gz \in Z$. Extend these actions to actions of G^u on A and Z as follows: $(w, g, a) \mapsto ga \in A$ and $(w, g, z) \mapsto gz \in Z$ for $g \in G$, $w \in \mathbb{T}$, $z \in Z$, $a \in A$. Let dz denote the normalized Haar measure on $\mathbb{T}$ and let λ denote the counting measure on Z. The stabilizers G_z of each $z \in Z$ are open subgroups of G and so the restriction of $dz \times d\lambda$ to G_z (still denoted $dz \times d\lambda$) is a left Haar measure on G_z. Let S denote the orbit space $G\backslash Z$ and $z \mapsto \hat{z}$ denote the quotient map. We identify S with a fixed transversal $S\imath Z$. For each $z \in Z$ fix $g_z \in G$ so that $g_z z = \hat{z}$ Let Z_s denote the orbit of $s \in S$. We will need the following result stated in [**Pi**, Proposition 5] for nontwisted crossed products. The result for nontwisted crossed products is due to P. Green [**G1**] and extends to the case of crossed products twisted by a circle-valued 2-cocycle as in the following proposition.

Proposition 4.1. *Let (A, G, α, u) be a twisted C^*-dynamical system.*

(i) *The following isomorphisms hold:*

$$C_0(Z, A) \rtimes G \cong \bigoplus_{s \in S} (A \rtimes G_s) \otimes \mathcal{K}(l^2(Z_s))$$
$$C_0(Z, A) \rtimes_r G \cong \bigoplus_{s \in S} (A \rtimes_r G_s) \otimes \mathcal{K}(l^2(Z_s))$$

where $\mathcal{K}(H)$ denotes the algebra of compact operators on H and where the action of G on $C_0(Z, A)$ is defined by $g(f)(z) = g(f(g^{-1}z))$ for every $f \in C_0(Z, A)$,$g \in G$, and $z \in Z$.

(ii) *If we regard $C_c(G \times Z, A)$ as a subalgebra of $C_c(G, C_0(Z, A))$, then one may describe the above isomorphisms by*

$$\theta(k) = \bigoplus_{s\in S} \sum_{\hat{z}_1=\hat{z}_2=s} k^s_{z_1,z_2} \otimes e_{z_1,z_2}$$

for $k \in C_c(G \times Z, A)$ *and where* e_{z_1,z_2} *is the canonical matrix unit in* $\mathcal{K}(l^2(Z_s))$, *and where* $k^s_{z_1,z_2} \in C_c(G_s, A)$ *is defined by*

$$k^s_{z_1,z_2}(g) = g_{z_1}(k(g_{z_1}^{-1} g g_{z_2}, z_1))\triangle_G(g_{z_2})$$

for every $g \in G_s$.

Suppose that G acts on a tree X and (A, G, α, u) is a twisted C^*-dynamical system. It follows from the preceding proposition that the K-theory of the twisted crossed product $C_0(X^1, A) \rtimes G$ is isomorphic to that of $\oplus_{y\in\Sigma^1}(A \rtimes G_y)$ and similarly for the reduced crossed products.

Now we will show that the K-theory of the crossed product $C_+(X^1, A) \rtimes G$ is isomorphic to that of $C_0(X^0, A) \rtimes G$ and similarly for the reduced crossed products. By Proposition 4.1, $C_0(X^0, A) \rtimes G$ is KK-equivalent to $\oplus_{P\in\Sigma^0}(A \rtimes G_P)$. From these facts we will derive our main result of a an exact sequence of KK-groups. For $Q \in X^0$ let $\chi_Q \in C_+(X^1)$ denote the characteristic function of $X^1_Q \cup \{+\infty\}$. Let $\{e_{PQ}\}_{P,Q\in X^0}$ be a system of matrix units for $l^2(X^0)$. We need the following result of Pimsner [**Pi**, Proposition 14].

Proposition 4.2. *Let* H *be a second countable, locally compact group acting on a countable oriented tree* X *and a separable* C^**-algebra* A. *Let*

$$d : C_0(X^0, A) \rtimes H \to \mathcal{K}(l^2(X^0)) \otimes (C_+(X^1, A) \rtimes H)$$
$$d_r : C_0(X^0, A) \rtimes_r H \to \mathcal{K}(l^2(X^0)) \otimes (C_+(X^1, A) \rtimes_r H)$$

be induced by

$$(dF)(g) = \sum_{Q\in X^0} e_{QQ} \otimes F(g)(Q)\chi_Q,\ \ F \in C_c(H, C_0(X^0, A)),\ \ g \in H.$$

Then the elements

$$\alpha_H \in KK(C_0(X^0, A) \rtimes H, C_+(X^1, A) \rtimes H)$$
$$\alpha_{H,r} \in KK(C_0(X^0, A) \rtimes_r H, C_+(X^1, A) \rtimes_r H)$$

determined by d *and* d_r *are* KK*-equivalences.*

Because of the existence of the cross sections ψ and ψ_r of ϕ and ϕ_r in the following exact sequences

$$\tag{4.3}\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\phi & \xrightarrow{i} & A \rtimes G^u & \xrightarrow{\phi} & A \rtimes G & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle\lambda_{\ker}} & & \big\downarrow{\scriptstyle\lambda_{G^u}} & & \big\downarrow{\scriptstyle\lambda_G} & & \\ 0 & \longrightarrow & \ker\phi_r & \xrightarrow{i_r} & A \rtimes_r G^u & \xrightarrow{\phi_r} & A \rtimes_r G & \longrightarrow & 0 \end{array}$$

we have KK-equivalences

$$A \rtimes G^u \approx_{KK} A \rtimes G \oplus \ker \phi$$
$$A \rtimes_r G^u \approx_{KK} A \rtimes_r G \oplus \ker \phi_r.$$

Specifically,

$$\psi \oplus i \in KK(A \rtimes G \oplus \ker \phi, A \rtimes G^u)$$
$$\psi_r \oplus i_r \in KK(A \rtimes_r G \oplus \ker \phi, A \rtimes_r G^u)$$

are invertible elements. Thus for any C^*-algebra E we have the following commutative diagrams.

$$\tag{4.4} \begin{array}{ccc} KK(E, A \rtimes G^u) & \xleftarrow{\psi_* \oplus i_*} & KK(E, A \rtimes G) \oplus KK(E, \ker \phi) \\ \Big\downarrow \lambda_{G^u,*} & & \Big\downarrow \lambda_{G,*} \oplus \lambda_{\ker,*} \\ KK(E, A \rtimes_r G^u) & \xleftarrow{\psi_{r,*} \oplus i_{r,*}} & KK(E, A \rtimes_r G) \oplus KK(E, \ker \phi_r) \end{array}$$

$$\begin{array}{ccc} KK(A \rtimes G^u, E) & \xrightarrow{\psi^* \oplus i^*} & KK(A \rtimes G, E) \oplus KK(\ker \phi, E) \\ \Big\uparrow \lambda^*_{G^u} & & \Big\uparrow \lambda^*_G \oplus \lambda^*_{\ker} \\ KK(A \rtimes_r G^u, E) & \xrightarrow{\psi^*_r \oplus i^*_r} & KK(A \rtimes_r G, E) \oplus KK(\ker \phi_r, E) \end{array}$$

We now show that the KK-equivalences in Proposition 4.2 factor through to give KK-equivalences in the setting of twisted crossed products. If A is replaced by $C_0(X^0, A)$ (resp. A is replaced by $C_+(X^1, A)$) we let the maps i,ϕ,ψ in (4.3) be denoted by i_0,ϕ_0, ψ_0 (resp. i_+,ϕ_+,ψ_+). In the reduced crossed product setting we let i_r,ϕ_r,ψ_r be denoted by $i_{0,r}$,$\phi_{0,r}$, $\psi_{0,r}$ (resp. $i_{+,r}$,$\phi_{+,r}$,$\psi_{+,r}$). By looking at compactly supported functions it can be shown that the maps d and d_r of Proposition 4.2 with $H = G^u$ can be factored into homomorphisms

$$\tag{4.5} \begin{aligned} d_G &: C_0(X^0, A) \rtimes G \to \mathcal{K}(l^2(X^0)) \otimes (C_+(X^1, A) \rtimes G) \\ d_{G,r} &: C_0(X^0, A) \rtimes_r G \to \mathcal{K}(l^2(X^0)) \otimes (C_+(X^1, A) \rtimes_r G) \end{aligned}$$

$$d_{\ker} : \ker \phi_0 \to \mathcal{K}(l^2(X^0)) \otimes \ker \phi_+$$
$$d_{\ker,r} : \ker \phi_{0,r} \to \mathcal{K}(l^2(X^0)) \otimes \ker \phi_{+,r}.$$

Let

$$\alpha_G \in KK(C_0(X^0, A) \rtimes G, C_+(X^1, A) \rtimes G) \quad \alpha_{\ker} \in KK(\ker \phi_0, \ker \phi_+)$$
$$\alpha_{G,r} \in KK(C_0(X^0, A) \rtimes_r G, C_+(X^1, A) \rtimes_r G) \quad \alpha_{\ker,r} \in KK(\ker \phi_{0,r}, \ker \phi_{+,r})$$

be the elements determined by d_G,$d_{G,r}$,$d_{\ker}$,$d_{\ker,r}$. It follows from Proposition 4.2 that for every C^*-algebra E the following diagrams commute with all maps being isomorphisms.

(4.6)

$$\begin{array}{ccc}
KK(E, C_0(X^0, A) \rtimes G) & \xleftarrow{\psi_{0,*} \oplus i_{0,*}} & KK(E, C_0(X^0, A) \rtimes G) \oplus KK(E, \ker \phi_0) \\
\downarrow \alpha_* & & \downarrow \alpha_{G,*} \oplus \alpha_{\ker,*} \\
KK(E, C_+(X^1, A) \rtimes G) & \xleftarrow{\psi_{+,*} \oplus \imath_{+,*}} & KK(E, C_+(X^1, A) \rtimes G) \oplus KK(E, \ker \phi_+)
\end{array}$$

$$\begin{array}{ccc}
KK(C_0(X^0, A) \rtimes G, E) & \xrightarrow{\psi_0^* \oplus i_0^*} & KK(C_0(X^0, A) \rtimes G, E) \oplus KK(\ker \phi_0, E) \\
\uparrow \alpha^* & & \uparrow \alpha_G^* \oplus \alpha_{\ker}^* \\
KK(C_+(X^1, A) \rtimes G, E) & \xrightarrow{\psi_+^* \oplus \imath_+^*} & KK(C_+(X^1, A) \rtimes G, E) \oplus KK(\ker \phi_+, E)
\end{array}$$

The analog of the above diagram for reduced crossed products exists with the same properties as the above diagram. Since α_*, α^*,α_{r*},α_r^* are isomorphisms for every C^*-algebra E it follows that $\alpha_{G,*}$,α_G^*,$\alpha_{G,r,*}$,$\alpha_{G,r}^*$ are isomorphisms for every C^*-algebra E.

Proposition 4.3. *The elements*

$$\alpha_G \in KK(C_0(X^0, A) \rtimes G, C_+(X^1, A) \rtimes G)$$
$$\alpha_{G,r} \in KK(C_0(X^0, A) \rtimes_r G, C_+(X^1, A) \rtimes_r G)$$

are KK-equivalences.

Proof. Suppose A and B are C^*-algebras and $\phi \in KK(A, B)$ is such that the maps

$$\phi_* \colon KK(E, A) \to KK(E, B)$$
$$\phi^* \colon KK(B, E) \to KK(A, E)$$

are isomorphisms for every C^*-algebra E. Letting $E = B$ in the first isomorphism gives an element $\psi_1 \in KK(B, A)$ so that $\phi_*(\psi_1) = \psi_1 \phi = 1_B \in KK(B, B)$. Letting $E = A$ in the second isomorphism gives an element $\psi_2 \in KK(B, A)$ so that $\phi^*(\psi_2) = \phi \psi_2 = 1_A \in KK(A, A)$. It follows that $\psi_1 = \psi_2 = \phi^{-1} \in KK(B, A)$ and thus ϕ is a KK-equivalence. Since the maps $\alpha_{G,*}$,α_G^*,$\alpha_{G,r,*}$, $\alpha_{G,r}^*$ are isomorphisms for every C^*-algebra E the conclusion follows. □

We will now define the connecting maps of the exact sequence in the main result. For $P \in X^0$ let

$$\tau_P^G : A \rtimes G_P \to A \rtimes G$$
$$\tau_{P,r}^G : A \rtimes_r G_P \to A \rtimes_r G$$

denote the maps induced by the inclusion maps $G_P \hookrightarrow G$. Let

$$\sigma_G^t, \sigma_G^o : \bigoplus_{y\in\Sigma^1} A \rtimes G_y \to \mathcal{K}(l^2(X^1)) \otimes \left[\bigoplus_{P\in\Sigma^0} A \rtimes G_P\right]$$

$$\tau_G : \bigoplus_{P\in\Sigma^0} A \rtimes G_P \to \mathcal{K}(l^2(X^0)) \otimes (A \rtimes G)$$

be the homomorphisms defined as follows. Let $\{e_{yz}\}_{y,z\in X^1}$ (resp. $\{e_{PQ}\}_{P,Q\in X^0}$) denote a system of matrix units for $\mathcal{K}(l^2(X^1))$ (resp. $\mathcal{K}(l^2(X^0))$).

$$\sigma_G^t\left(\bigoplus_{y\in\Sigma^1} x_y\right) = \sum_{y\in\Sigma^1} e_{yy} \otimes (\sigma_y \times \alpha_y)(x_y)$$

$$\sigma_G^o\left(\bigoplus_{y\in\Sigma^1} x_y\right) = \sum_{y\in\Sigma^1} e_{yy} \otimes (\sigma_{\bar{y}} \times \alpha_{\bar{y}})(x_y)$$

$$\tau_G\left(\bigoplus_{P\in\Sigma^0} x_P\right) = \sum_{P\in\Sigma^0} e_{PP} \otimes \tau_P^G(x_P).$$

By making the obvious changes in the above definitions one can define the reduced analogs of the above homomorphisms. The reduced versions of the above homomorphisms will be denoted $\sigma_{G,r}^t, \sigma_{G,r}^o, \tau_{G,r}$. Let

$$j_G : \bigoplus_{y\in\Sigma^1} A \rtimes G_y \to \bigoplus_{y\in\Sigma^1} (A \rtimes G_y) \otimes \mathcal{K}(l^2(X_y^1))$$

be defined by

$$j_G\left(\bigoplus_{y\in\Sigma^1} x_y\right) = \bigoplus_{y\in\Sigma^1} (x_y \otimes e_{yy}).$$

Let the reduced version of j_G which is defined by the same rule be denoted $j_{G,r}$. We now state the main results which are the twisted crossed product versions of Theorems 16-18 of [**Pi**].

Theorem 4.4. *Let (A, G, α, u) be a twisted separable C^*-dynamical system and suppose that G acts on a countable oriented tree X. Then:*

(i) *If B is a separable C^*-algebra, then the diagram*

$$\begin{array}{ccccc}
KK_{n-1}(B, A \rtimes G) & \xrightarrow{\partial} & KK_n\left(B, \bigoplus_{y\in\Sigma^1} A \rtimes G_y\right) & \xrightarrow{\sigma_{G*}^t - \sigma_{G*}^o} & KK_n\left(B, \bigoplus_{P\in\Sigma^0} A \rtimes G_P\right) \\
\downarrow & & \downarrow & & \downarrow \\
KK_{n-1}(B, A \rtimes_r G) & \xrightarrow{\partial} & KK_n\left(B, \bigoplus_{y\in\Sigma^1} A \rtimes_r G_y\right) & \xrightarrow{\sigma_{G,r*}^t - \sigma_{G,r*}^o} & KK_n\left(B, \bigoplus_{P\in\Sigma^0} A \rtimes_r G_P\right)
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\tau_{G*}} & KK_n(B, A \rtimes G) & \xrightarrow{\partial} & KK_{n+1}\left(B, \bigoplus_{y\in\Sigma^1} A \rtimes G_y\right) \\
& \downarrow & & \downarrow \\
\xrightarrow{\tau_{G,r*}} & KK_n(B, A \rtimes_r G) & \xrightarrow{\partial} & KK_{n+1}\left(B, \bigoplus_{y\in\Sigma^1} A \rtimes_r G_y\right)
\end{array}$$

is commutative and has exact rows for every $n \in \mathbb{Z}_2$.

(ii) *If B is an arbitrary C^*-algebra, then the diagram*

$$\begin{array}{ccccc}
KK_{n-1}(A \rtimes G, B) & \xleftarrow{\partial} & KK_n\left(\bigoplus_{y\in\Sigma^1} A \rtimes G_y, B\right) & \xleftarrow{\sigma_G^{t*}-\sigma_G^{o*}} & KK_n\left(\bigoplus_{P\in\Sigma^0} A \rtimes G_P, B\right) \\
\uparrow & & \uparrow & & \uparrow \\
KK_{n-1}(A \rtimes_r G, B) & \xleftarrow{\partial} & KK_n\left(\bigoplus_{y\in\Sigma^1} A \rtimes_r G_y, B\right) & \xleftarrow{\sigma_{G,r}^{t*}-\sigma_{G,r}^{o*}} & KK_n\left(\bigoplus_{P\in\Sigma^0} A \rtimes_r G_P, B\right)
\end{array}$$

$$\begin{array}{ccccc}
\xleftarrow{\tau_G^*} & KK_n(A \rtimes G, B) & \xleftarrow{\partial} & KK_{n+1}\left(\bigoplus_{y\in\Sigma^1} A \rtimes G_y, B\right) \\
& \uparrow & & \uparrow \\
\xleftarrow{\tau_{G,r}^*} & KK_n(A \rtimes_r G, B) & \xleftarrow{\partial} & KK_{n+1}\left(\bigoplus_{y\in\Sigma^1} A \rtimes_r G_y, B\right)
\end{array}$$

is commutative and has exact rows for every $n \in \mathbb{Z}_2$.

The vertical arrows are given by the natural projection from the full crossed product onto the reduced crossed product, while ∂ denotes the boundary maps associated with the Toeplitz extension modulo the isomorphisms induced by

$$\bigoplus_{y\in\Sigma^1} A \rtimes G_y \xrightarrow{j_G} \bigoplus_{y\in\Sigma^1} [A \rtimes G_y \otimes \mathcal{K}(l^2(X^1_y))] \xrightarrow{\theta_G^{-1}} C_0(X^1, A) \rtimes G$$

$$\bigoplus_{y\in\Sigma^1} A \rtimes_r G_y \xrightarrow{j_{G,r}} \bigoplus_{y\in\Sigma^1} [A \rtimes_r G_y \otimes \mathcal{K}(l^2(X^1_y))] \xrightarrow{\theta_{G,r}^{-1}} C_0(X^1, A) \rtimes_r G.$$

The isomorphisms θ_G and $\theta_{G,r}$ are the isomorphisms of Proposition 4.1 (ii) *with $S = \Sigma^1$ and $Z = X^1$.*

Proof. The isomorphisms θ_G and $\theta_{G,r}$ give KK-equivalences between $C_0(X^1, A)$ and $\oplus_{y\in\Sigma^1} A \rtimes G_y$ and between $C_0(X^1, A) \rtimes_r G$ and $\oplus_{y\in\Sigma^1} A \rtimes_r G_y$ respectively. Proposition 4.6 gives KK-equivalences between $C_+(X^1, A) \rtimes G$ and $C_0(X^0, A) \rtimes G$ and between $C_+(X^1, A) \rtimes_r G$ and $C_0(X^0, A) \rtimes_r G$. Applying Proposition 4.1 with $S = \Sigma^0$ and $Z = X^0$ gives KK-equivalences between $C_0(X^0, A) \rtimes G$ and $\oplus_{P\in\Sigma^0} A \rtimes G_P$ and between $C_0(X^0, A) \rtimes_r G$ and $\oplus_{P\in\Sigma^0} A \rtimes_r G_P$. It then follows from the existence of the completely positive cross sections ρ_G and $\rho_{G,r}$ of the full and reduced Toeplitz extensions and the split exactness of KK-theory that there exist commutative diagrams with exact rows as in (i) and (ii) of the theorem statement. It remains to show that the connecting maps are as stated. To see this, extend the action of G on A to an action of G^u on A by letting $\mathbb{T} \times \{1\}$ act trivially. It follows from [**Pi**, Theorem 16] that there are commutative diagrams as in (i) and (ii) with G replaced by G^u and the connecting maps are exactly as described above with the group G^u in place of G. Since the connecting maps with respect to G are just the factorization of the connecting maps with respect to G^u, the theorem follows. □

Theorem 4.5. *In the conditions of the preceding theorem, we get the following commutative diagrams with exact rows.*

(i) *If B is separable and the fundamental domain $G\backslash X$ is finite:*

$$\begin{array}{ccccc}
KK_{n-1}(B, A\rtimes G) & \xrightarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_n(B, A\rtimes G_y) & \xrightarrow{\sigma} & \bigoplus_{P\in\Sigma^0} KK_n(B, A\rtimes G_P) \\
\downarrow & & \downarrow & & \downarrow \\
KK_{n-1}(B, A\rtimes_r G) & \xrightarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_n(B, A\rtimes_r G_y) & \xrightarrow{\sigma_r} & \bigoplus_{P\in\Sigma^0} KK_n(B, A\rtimes_r G_P)
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\tau'} & KK_n(B, A\rtimes G) & \xrightarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_{n+1}(B, A\rtimes G_y) \\
 & \downarrow & & \downarrow \\
\xrightarrow{\tau'_r} & KK_n(B, A\rtimes_r G) & \xrightarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_{n+1}(B, A\rtimes_r G_y)
\end{array}$$

where $\sigma = \sum_{y\in\Sigma^1} ((\sigma_y \times \alpha_y)_* - (\sigma_{\bar y} \times \alpha_{\bar y})_*)$, $\sigma_r = \sum_{y\in\Sigma^1} ((\sigma_y \times_r \alpha_y)_* - (\sigma_{\bar y} \times_r \alpha_{\bar y})_*)$, $\tau' = \sum_{P\in\Sigma^0} \tau^G_{P*}$, $\tau'_r = \sum_{P\in\Sigma^0} \tau^G_{P,r*}$.

(ii) *If B and $G\backslash X$ are arbitrary:*

$$\begin{array}{ccccc}
KK_{n-1}(A\rtimes G, B) & \xleftarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_n(A\rtimes G_y, B) & \xleftarrow{\sigma} & \bigoplus_{P\in\Sigma^0} KK_n(A\rtimes G_P, B) \\
\uparrow & & \uparrow & & \uparrow \\
KK_{n-1}(A\rtimes_r G, B) & \xleftarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_n(A\rtimes_r G_y, B) & \xleftarrow{\sigma_r} & \bigoplus_{P\in\Sigma^0} KK_n(A\rtimes_r G_P, B)
\end{array}$$

$$\begin{array}{ccccc}
\xleftarrow{\tau'} & KK_n(A \rtimes G, B) & \xleftarrow{\partial} & \bigoplus_{y \in \Sigma^1} KK_{n+1}(A \rtimes G_y, B) \\
& \uparrow & & \uparrow \\
\xleftarrow{\tau'_r} & KK_n(A \rtimes_r G, B) & \xleftarrow{\partial} & \bigoplus_{y \in \Sigma^1} KK_{n+1}(A \rtimes_r G_y, B)
\end{array}$$

where

$$\sigma = \sum_{y \in \Sigma^1} ((\sigma_y \times \alpha_y)^* - (\sigma_{\bar{y}} \times \alpha_{\bar{y}})^*),$$

$$\sigma_r = \sum_{y \in \Sigma^1} ((\sigma_y \times_r \alpha_y)^* - (\sigma_{\bar{y}} \times_r \alpha_{\bar{y}})^*),$$

$$\tau' = \sum_{P \in \Sigma^0} \tau_P^{G*}, \qquad \tau'_r = \sum_{P \in \Sigma^0} \tau_{P,r}^{G*}.$$

Proof. This follows from Theorem 4.4, the additivity of the KK-functor in the second variable, and the countable additivity of the KK functor in the first variable [**Bl**, 17.7; 19.7]. □

Theorem 4.6. *Suppose (A, G, α, u) is a separable twisted C^*-dynamical system and that G acts on the countable oriented tree X. Then the following diagram is commutative and has exact rows.*

$$\begin{array}{ccccc}
K_{n-1}(A \rtimes G) & \xrightarrow{\partial} & \bigoplus_{y \in \Sigma^1} K_n(A \rtimes G_y) & \xrightarrow{\sigma} & \bigoplus_{P \in \Sigma^0} K_n(A \rtimes G_P) \\
\downarrow & & \downarrow & & \downarrow \\
K_{n-1}(A \rtimes_r G) & \xrightarrow{\partial} & \bigoplus_{y \in \Sigma^1} K_n(A \rtimes_r G_y) & \xrightarrow{\sigma} & \bigoplus_{P \in \Sigma^0} K_n(A \rtimes_r G_P)
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\tau'} & K_n(A \rtimes G) & \xrightarrow{\partial} & \bigoplus_{y \in \Sigma^1} K_{n+1}(A \rtimes G_y) \\
& \downarrow & & \downarrow \\
\xrightarrow{\tau'_r} & K_n(A \rtimes_r G) & \xrightarrow{\partial} & \bigoplus_{y \in \Sigma^1} K_{n+1}(A \rtimes_r G_y).
\end{array}$$

Proof. If A is separable apply Theorem 4.5 (i) with $B = \mathbb{C}$. The general case is obtained by taking direct limits. □

5. Applications to Free Products.

In this section we will use Theorem 4.6 in order to compute the K-groups of certain free products of C^*-algebras. In our examples, the groups considered will be discrete and K-amenable in the sense of J. Cuntz [**Cu1**]. Recall that a discrete group is K-amenable if the natural surjection $\lambda : C^*(G) \to C^*_r(G)$ induces a KK-equivalence. It was shown in [**Cu1**] that this is equivalent to the natural surjection $\lambda_A : A \rtimes G \to A \rtimes_r G$ being a KK-equivalence for every C^*-dynamical system (A, G, α). The notion of K-amenability has been extended to locally compact abelian groups by Julg and Valette in [**JV**, Definition 1.2]. A locally compact group G is said to be K-amenable if $1_G = (\mathbb{C}, 0, 0)$ in $KK_G(\mathbb{C}, \mathbb{C})$ is homotopic to a Fredholm G-module $(\mathcal{H}_0, \mathcal{H}_1, F)$ where the representation of G on $\mathcal{H}_0$ and $\mathcal{H}_1$ are weakly contained in the left regular representation of G. It was shown in [**JV**, Proposition 3.4] that if a locally compact group G is K-amenable then the natural surjection λ_A is a KK-equivalence for every C^*-dynamical system (A, G, α). In fact, λ_A is a KK-equivalence for every twisted C^*-dynamical system (A, G, α, u). The proof is essentially the same as the proof for nontwisted C^*-dynamical systems given in [**JV**]. We will state the following lemma which generalizes [**JV**, Lemma 3.5] in addition to the KK-equivalence result in order to point out how to modify the proof of Julg and Valette.

Lemma 5.1.

(i) *Let (A, G, α, u) be a twisted C^*-dynamical system. Then there exists a *-homomorphism*

$$\triangle_1 : A \rtimes_r G \to \mathcal{M}(C^*_r(G) \otimes (A \rtimes G))$$

such that the diagram

$$\begin{array}{ccc} A \rtimes G & \xrightarrow{\ \triangle\ } & \mathcal{M}(C^*(G) \otimes (A \rtimes G)) \\ \Big\downarrow{\scriptstyle \lambda_A} & & \Big\downarrow{\scriptstyle \lambda_A \otimes 1} \\ A \rtimes_r G & \xrightarrow{\ \triangle_1\ } & \mathcal{M}(C^*_r(G) \otimes (A \rtimes G)) \end{array}$$

commutes.

(ii) *There exists a *-homomorphism*

$$\triangle_2 : A \rtimes_r G \to \mathcal{M}(C^*(G) \otimes (A \rtimes_r G))$$

such that the diagram

$$\begin{array}{ccc} A \rtimes G & \xrightarrow{\triangle} & \mathcal{M}(C^*(G) \otimes (A \rtimes G)) \\ \Big\downarrow{\scriptstyle \lambda_A} & & \Big\downarrow{\scriptstyle 1\otimes\lambda_A} \\ A \rtimes_r G & \xrightarrow{\triangle_2} & \mathcal{M}(C^*(G) \otimes (A \rtimes_r G)) \end{array}$$

commutes.

Proof. The proof of this lemma proceeds exactly as in the proof of [**JV**, Lemma 3.5] except that the crossed products are twisted by a 2-cocycle and the left regular representation maps must be replaced by the twisted left regular representations as in (2.1). □

Proposition 5.2. *If G is K-amenable, then the element of $KK(A \rtimes G, A \rtimes_r G)$ induced by the natural surjection $\lambda_A : A \rtimes G \to A \rtimes_r G$ is a KK-equivalence.*

Proof. The proposition follows from Lemma 5.1 exactly as Proposition 3.4 follows from Lemma 3.5 in [**JV**]. □

Recall that the unital free product of two unital C^*-algebras A and B is a unital C^*-algebra E with unital injections $j_A : A \to E$, $j_B : B \to E$ such that for every pair of unital *-homomorphisms $\phi_A : A \to C$,$\phi_B : B \to C$ into a C^*-algebra C there is a unique unital *-homomorphism $\phi : E \to C$ such that $\phi \circ j_A = \phi_A$ and $\phi \circ j_B = \phi_B$. It follows easily that E is unique up to isomorphism. E is denoted by $A *_{\mathbb{C}} B$ and ϕ is denoted by $\phi_A * \phi_B$.

Given two groups G_1,G_2 with (normalized) 2-cocycles u_1,u_2 there is a way to define a (normalized) free product 2-cocycle $u = u_1 * u_2$ on the free prod $G_1 * G_2$. If w_1 and w_2 are two reduced words in the free product, then $w_1 w_2$ can be replaced by a unique reduced word after a finite number of cancellations. If w_1 ends in $g_1 \in G_i$ and w_2 begins in $g_2 \in G_j$ where $i \neq j$, then no cancellation occurs and $u(w_1, w_2) = 1$. If $i = j$, then let w_1' and w_2' be the parts of w_1 and w_2 which remain after cancellation. Let g_1' denote the last letter of w_1' and g_2' denote the first letter of w_2'. In this case $u(w_1, w_2) = u_k(g_1', g_2')$ where $g_1', g_2' \in G_k$.

The twisted crossed products $\mathbb{C} \rtimes G$ and $\mathbb{C} \rtimes_r G$ arising from a twisted C^*-dynamical system of the form $(\mathbb{C}, G, \alpha, u)$ are denoted $C_u^*(G)$ and $C_{u,r}^*(G)$ respectively. The following proposition is well-known.

Proposition 5.3. *Let (A, G_1, α_1, u_1) and (A, G_2, α_2, u_2) be twisted C^*-dynamical systems with G_1 and G_2 discrete groups. Let $u = u_1 * u_2$ be the free product cocycle on $G_1 * G_2$. Then*

$$C_u^*(G_1 * G_2) \cong C_{u_1}^*(G_1) *_{\mathbb{C}} C_{u_2}^*(G_2).$$

Let (A_1, ω_1) and (A_2, ω_2) be pairs of unital C^*-algebras and states. We will denote an arbitrary word with letters alternating between A_1 and A_2 by $(a_1)b_1 \cdots a_n(b_n)$ where $a_i \in A_1$, $b_i \in A_2$, and the parentheses around the first and last letters are used to indicate the possibility that these letters may not be present in the word. This allows for words beginning and ending in either A_1 or A_2. Then there is a unique state $\omega_1 * \omega_2$ on $A_1 *_{\mathbb{C}} A_2$ such that

$$(\omega_1 * \omega_2)((a_1)b_1 \cdots a_n(b_n)) = 0$$

whenever $a_i \in \ker(\omega_1)$, $b_i \in \ker(\omega_2)$[**Av**]. Let $(\pi_{\omega_1 * \omega_2}, H_{\omega_1 * \omega_2}, \xi_{\omega_1 * \omega_2})$ be the GNS representation associated with $(A_1 *_{\mathbb{C}} A_2, \omega_1 * \omega_2)$. The *reduced free product* of A_1 and A_2 relative to ω_1 and ω_2 was defined by D. Avitzour in [**Av**] to be $\pi_{\omega_1 * \omega_2}(A_1 *_{\mathbb{C}} A_2)$. When the states ω_i are understood we write $A_1 *_{\mathbb{C}}^{\mathrm{red}} A_2$ for the reduced free product. The following proposition was proved in [**Av**] for nontwisted crossed products.

Proposition 5.4. *Let (A, G_1, α_1, u_1) and (A, G_2, α_2, u_2) be twisted C^*-dynamical systems with G_1 and G_2 discrete groups. Let $u = u_1 * u_2$ be the free product cocycle on $G_1 * G_2$. Then*

$$C^*_{u,r}(G_1 * G_2) \cong C^*_{u_1,r}(G_1) *_{\mathbb{C}}^{\mathrm{red}} C^*_{u_2,r}(G_2),$$

where the reduced free products are taken relative to the states $x \mapsto \langle \lambda_i(x)\delta_e, \delta_e \rangle$ with λ_i being the left regular representation twisted by the cocycle u_i.

Proof. Let λ be the left regular representation on $G_1 * G_2$ twisted by the free product cocycle u as in (2.1). Let λ_i be defined similarly with respect to G_i and u_i. It follows easily that the state on $C^*_{u_1}(G_1) *_{\mathbb{C}} C^*_{u_2}(G_2)$ defined by the composition of the state $x \mapsto \langle \lambda(x)\delta_e, \delta_e \rangle$ on $C^*_u(G_1 * G_2)$ and the isomorphism in Proposition 5.3 is the free product state induced by the states $x \mapsto \langle \lambda_i(x)\delta_e, \delta_e \rangle$ on $C^*_{u_i}(G_i)$. Since the two reduced C^*-algebras under consideration are the images of the GNS representations of $C^*_{u_1}(G_1) *_{\mathbb{C}} C^*_{u_2}(G_2)$ relative to these two states the conclusion follows. $\square$

We now discuss some examples related to the preceding propositions. The C^*-algebra M_n of complex-entried n by n matrices can be written as $C^*_u(\mathbb{Z}_n)$ for a certain 2-cocycle u. To see this first notice that $M_n \cong (\mathbb{C} \rtimes_\alpha \mathbb{Z}_n) \rtimes_{\hat{\alpha}} \mathbb{Z}_n$ where α denotes the trivial action and $\hat{\alpha}$ denotes the dual action of the trivial action given by

$$\hat{\alpha}_\chi\left(\sum_{k \in \mathbb{Z}_n} c_k k\right) = \sum_{k \in \mathbb{Z}_n} c_k \chi(k) k$$

for $\chi \in \hat{\mathbb{Z}}_n \cong \mathbb{Z}_n$. This result is a special case of Takai-Takesaki duality [**T**]. In general if G is a locally compact abelian group acting on a C^*-algebra A by α and $\hat{G}$ acts on $A \rtimes_\alpha G$ by the dual action there is a natural cocycle u which can be defined on $G \times \hat{G}$ in such a way that

$$(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \cong A \rtimes_{\alpha \times \hat{\alpha}, u} (G \times \hat{G}).$$

The cocycle u is defined as follows:

$$u((g_1, \chi_1), (g_2, \chi_2)) = \chi_1(g_2).$$

In light of these observations and Proposition 5.3, $M_n *_{\mathbb{C}} C(\mathbb{T})$ can be realized as a twisted group C^*-algebra since $C(\mathbb{T}) \cong C^*(\mathbb{Z})$. Similarly, $M_n *_{\mathbb{C}} \mathbb{C}^2$ can also be realized as a twisted group C^*-algebra since $\mathbb{C}^2 \cong C^*(\mathbb{Z}_2)$. The relative commutants of M_n in these two C^*-algebras were introduced by L. Brown in [**Br**]. Let U_n^{nc} and G_n^{nc} denote the relative commutants of M_n in $M_n *_{\mathbb{C}} C(\mathbb{T})$ and $M_n *_{\mathbb{C}} \mathbb{C}^2$ respectively. U_n^{nc} is called the noncommutative unitary C^*-algebra. The noncommutative unitary C^*-algebra is the unital C^*-algebra generated by elements u_{ij}, $1 \le i, j \le n$, subject to the relations which make the $n \times n$ matrix $[u_{ij}]$ a unitary matrix in $M_n(U_n^{\mathrm{nc}})$. G_n^{nc} is called the noncommutative Grassmanian C^*-algebra. The noncommutative Grassmanian C^*-algebra is generated by a unit and elements p_{ij}, $1 \le i, j \le n$, subject to the relations which make the $n \times n$ matrix $[p_{ij}]$ a projection. It follows that the map $x \otimes y \mapsto xy$ give the following isomorphisms (see [**McC1**] for details).

$$M_n \otimes U_n^{\mathrm{nc}} \cong M_n *_{\mathbb{C}} C(\mathbb{T})$$
$$M_n \otimes G_n^{\mathrm{nc}} \cong M_n *_{\mathbb{C}} \mathbb{C}^2.$$

Thus U_n^{nc} and G_n^{nc} are KK-equivalent to the free products $M_n *_{\mathbb{C}} C(\mathbb{T})$ and $M_n *_{\mathbb{C}} \mathbb{C}^2$ respectively. The K-theory of U_n^{nc} was computed by N.C. Phillips in [**Ph**] and that of G_n^{nc} was computed by J. Cuntz (see [**McC1**] for a proof). Reduced versions $U_{n,r}^{\mathrm{nc}}$, $G_{n,r}^{\mathrm{nc}}$ of U_n^{nc},G_n^{nc} were defined in [**McC1**]. Namely, they are defined to be the relative commutants of M_n in the reduced free products $M_n *_{\mathbb{C}}^{\mathrm{red}} C(\mathbb{T})$ and $M_n *_{\mathbb{C}}^{\mathrm{red}} \mathbb{C}^2$. The states used in the definitions of these reduced free products are the unique trace on M_n and the trace on $C^*(G)$ defined by $x \mapsto < \lambda(x)\delta_e, \delta_e >$ where λ is the left regular representation of $C^*(G)$ on $l^2(G)$ with the identifications $C(\mathbb{T}) \cong C^*(\mathbb{Z})$ and $\mathbb{C}^2 \cong C^*(\mathbb{Z}_2)$ being used. The K-groups of the reduced versions were shown to be isomorphic to the K-groups of the corresponding full versions in [**McC2**]. Using some of the results in this section we can now show that the full and reduced versions of these C^*-algebras are KK-equivalent. It was also shown there that U_n^{nc} is KK-equivalent to $C(\mathbb{T})$ and G_n^{nc} is KK-equivalent to $\mathbb{C}^2$. It follows from the above observations and Proposition 5.2 that U_n^{nc} is KK- equivalent to $U_{n,r}^{\mathrm{nc}}$ and G_n^{nc} is KK- equivalent to $G_{n,r}^{\mathrm{nc}}$. To see this, notice that the fact that $\mathbb{Z}_n$ and $\mathbb{Z}$ are amenable (and hence K-amenable) implies $\mathbb{Z}_n * \mathbb{Z}$ and $\mathbb{Z}_n * \mathbb{Z}_2$ are K-amenable [**Cu1**, Theorem 2.4]. Letting $\approx_{KK}$ denote KK-equivalence we have the following:

$$\begin{aligned} U_n &\approx_{KK} M_n *_{\mathbb{C}} C(\mathbb{T}) \cong C^*_{u_1}(\mathbb{Z}_n * \mathbb{Z}) \\ U_{n,r} &\approx_{KK} M_n *_{\mathbb{C}}^{\mathrm{red}} C(\mathbb{T}) \cong C^*_{u_1,r}(\mathbb{Z}_n * \mathbb{Z}) \\ G_n &\approx_{KK} M_n *_{\mathbb{C}} \mathbb{C}^2 \cong C^*_{u_2}(\mathbb{Z}_n * \mathbb{Z}_2) \\ G_{n,r} &\approx_{KK} M_n *_{\mathbb{C}}^{\mathrm{red}} \mathbb{C}^2 \cong C^*_{u_2,r}(\mathbb{Z}_n * \mathbb{Z}_2). \end{aligned}$$

Combining this with the known K-groups of these C^*-algebras gives the following proposition.

Proposition 5.5. *The quotient maps*

$$\begin{aligned} \lambda_1 &: U_n^{nc} \to U_{r,\mathrm{red}}^{nc} \\ \lambda_2 &: G_n^{nc} \to G_{r,\mathrm{red}}^{nc} \end{aligned}$$

which are the restrictions of the maps

$$\begin{aligned} \lambda_1 &: M_n *_{\mathbb{C}} C(\mathbb{T}) \to M_n *_{\mathbb{C}}^{\mathrm{red}} C(\mathbb{T}) \\ \lambda_2 &: M_n *_{\mathbb{C}} \mathbb{C}^2 \to M_n *_{\mathbb{C}}^{\mathrm{red}} \mathbb{C}^2 \end{aligned}$$

are KK-equivalences. Moreover U_n^{nc},$U_{n,\mathrm{red}}^{nc}$ are KK-equivalent to $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ and G_n^{nc},$G_{n,\mathrm{red}}^{nc}$ are KK-equivalent to $C^*(\mathbb{Z}_2) \cong \mathbb{C}^2$.

One of the most important applications of Pimsner's exact sequence in the case of nontwisted crossed products is computing the K-groups of group C^*-algebras of amalgamated products of locally compact groups. Results in this direction extend to the twisted group C^*-algebra case as well. The following proposition follows from Theorem 4.7 in exactly the same manner as the corresponding result in the nontwisted case follows from Pimsner's exact sequence. We will include a sketch of the proof for completeness.

Proposition 5.6. *Let G_1,G_2 be countable discrete groups and let u_i be a 2-cocycle on G_i. Let u be the free product cocycle on $G_1 * G_2$.*

(i) *If B is any separable C^*-algebra, then the following diagram commutes and has exact rows.*

$$\begin{array}{ccccccc} \longrightarrow & KK_n(B,\mathbb{C}) & \xrightarrow{\iota_{1*}-\iota_{2*}} & KK_n(B,C^*_{u_1}(G_1)) \oplus KK_n(B,C^*_{u_2}(G_2)) & \xrightarrow{\tau_{1*}+\tau_{2*}} & KK_n(B,C^*_u(G_1 * G_2)) \\ & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \lambda^1_* \oplus \lambda^2_*} & & \downarrow{\scriptstyle \lambda_*} \\ \longrightarrow & KK_n(B,\mathbb{C}) & \xrightarrow{\iota_{1*}-\iota_{2*}} & KK_n(B,C^*_{u_1,r}(G_1)) \oplus KK_n(B,C^*_{u_2,r}(G_2)) & \xrightarrow{\tau_{1*}+\tau_{2*}} & KK_n(B,C^*_{u,r}(G_1 * G_2)) \end{array}$$

(ii) *If B is any C^*-algebra, then the following diagram commutes and has exact rows.*

$$\begin{tikzcd}[column sep=small]
 & KK_n(\mathbb{C},B) \arrow[l] & KK_n(C^*_{u_1}(G_1),B)\oplus KK_n(C^*_{u_2}(G_2),B) \arrow[l, "i_1^*-i_2^*"'] & KK_n(C^*_u(G_1*G_2),B) \arrow[l, "\tau_1^*+\tau_2^*"'] \\
 & KK_n(\mathbb{C},B) \arrow[l] \arrow[u, "Id"'] & KK_n(C^*_{u_1,r}(G_1),B)\oplus KK_n(C^*_{u_2,r}(G_2),B) \arrow[l, "i_1^*-i_2^*"'] \arrow[u, "\lambda^{1*}\oplus\lambda^{2*}"'] & KK_n(C^*_{u,r}(G_1*G_2),B) \arrow[l, "\tau_1^*+\tau_2^*"'] \arrow[u, "\lambda^*"']
\end{tikzcd}$$

In the above diagrams, i_j and τ_j denote the appropriate inclusion maps and λ^j,λ denote the appropriate left regular representations.

Proof. Let $G = G_1 * G_2$. Let X be the tree with edge set $X^1 = G$ and vertex set $X^0 = G/G_1 \amalg G/G_2$ where $\amalg$ denotes the disjoint union. The edge g has the coset gG_1 as its origin and the coset gG_2 as its terminus. The group G acts on X by left translation. It is easy to see that the orbit spaces Σ^0 and Σ^1 consist of two elements and a single element respectively. If we choose the transversals Σ^0 and Σ^1 to consist of the identity element of G and the pair of cosets $\{eG_1, eG_2\}$ respectively the proposition follows from Theorem 4.5. □

We now conclude the computation of the K-groups of the free product of two matrix algebras.

Example 5.7. Let $M_n *_{\mathbb{C}} M_k$ denote the unital free product of M_n and M_k. Let $M_n *_{\mathbb{C}}^{\text{red}} M_k$ denote the reduced free product of M_n and M_k relative to the normalized traces on M_n and M_k. This reduced free product is isomorphic to $C^*_{u,r}(\mathbb{Z}_n * \mathbb{Z}_k)$ for an appropriate cocycle by Proposition 5.3. It follows from the fact that $\mathbb{Z}_n$ is amenable (hence K-amenable) that $\mathbb{Z}_n * \mathbb{Z}_k$ is K-amenable and thus by Proposition 5.2 the full free product $M_n *_{\mathbb{C}} M_k$ and the reduced free product $M_n *_{\mathbb{C}}^{\text{red}} M_k$ are KK- equivalent. Thus it suffices to compute the K-theory for the full free product. The cyclic exact sequence in the top row of Proposition 5.6(i) in the K-group setting ($B = \mathbb{C}$) reduces to the following sequence:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "i"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & K_0(M_n *_{\mathbb{C}} M_k) \arrow[d] \\
K_1(M_n *_{\mathbb{C}} M_k) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where $i(j) = (nj, -kj)$. It then follows that

$$K_0(M_n *_{\mathbb{C}} M_k) \cong K_0(M_n *_{\mathbb{C}}^{\text{red}} M_k) \cong \mathbb{Z}^2/\mathbb{Z}(n,-k) \cong \mathbb{Z}\oplus\mathbb{Z}_{(n,k)}$$
$$K_1(M_n *_{\mathbb{C}} M_k) \cong K_1(M_n *_{\mathbb{C}}^{\text{red}} M_k) = 0$$

where (n,k) denotes the greatest common divisor of n and k. If $\{e_{ij}\}$ and $\{f_{kl}\}$ are the canonical matrix units for M_n and M_k then the generators for $\mathbb{Z}$ and $\mathbb{Z}_{(n,k)}$ can be taken to be x and y where a,b are integers such that $ak+bn=(n,k)$ and

$$x = a[e_{11}]_0 + b[f_{11}]_0$$
$$y = \frac{n}{(n,k)}[e_{11}]_0 - \frac{k}{(n,k)}[f_{11}]_0.$$

The above example can be generalized to the free product of two finite powers of matrix algebras. This follows from the tensor product version of Proposition 5.3. Namely,

$$C_u^*(G_1 \times G_2) \cong C_{u_1}^*(G_1) \otimes_{\max} C_{u_2}^*(G_2)$$

where $\otimes_{\max}$ denotes the maximal tensor product and $u = u_1 \times u_2$ is the product cocycle on $G_1 \times G_2$ given by the formula

$$u((g_1,g_2),(g_1',g_2')) = u_1(g_1,g_1')u_2(g_2,g_2').$$

Thus since

$$\oplus_1^r M_n \cong M_n \otimes \mathbb{C}^r \cong C_u^*(\mathbb{Z}_n \times \mathbb{Z}_r),$$

Proposition 5.6 applies and it follows that

$$K_0(\oplus_1^r M_n *_{\mathbb{C}} \oplus_1^s M_k) \cong \mathbb{Z}^{r+s-1} \oplus \mathbb{Z}_{(n,k)}$$
$$K_1(\oplus_1^r M_n *_{\mathbb{C}} \oplus_1^s M_k) \cong 0.$$

References

[Av] D. Avitzour, *Free products of C^*- algebras*, Trans. Amer. Math. Soc., **271** (1982), 423–465.

[Bl] B. Blackadar, *K-theory for operator algebras*, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1987.

[Br] L. Brown, *Ext of certain free product C^*- algebras*, J. Oper. Theory, **6** (1982), 423–465.

[BS] R.C. Busby and H.A. Smith, *Representations of twisted group C^*-algebras*, Trans. Amer. Math. Soc., **149** (1970), 503–537.

[Ch] W. Ching, *Free products of Von Neumann algebras*, Trans. Amer. Math. Soc., **178** (1973), 147–163.

[Cu1] J. Cuntz, *K-Theoretic amenability for discrete groups*, Crelles Journal, **344** (1983), 181–195.

[Cu2] ———, *The K-groups for free products of C^*-algebras*, Proceedings of Symposia in Pure Mathematics, Amer. Math. Soc., Providence, RI, **38(1)**, (1982), 81-84.

[G1] P. Green, *C^*-algebras of transformation groups with smooth orbit space*, Pac. J. Math., **72** (1977), 71–97.

[G2] ———, *The local structure of twisted covariance algebras*, Acta Math., **140** (1978), 191–250.

[JV] P. Julg and A. Valette, *K-Theoretic amenability for $SL_2(\mathbb{Q}_p)$ and the action on the associated tree*, J. Func. Anal., **58** (1984), 194–215.

[L] E.C. Lance, *K-theory for certain group C^*-algebras*, Acta. Math., **151** (1983), 209–230.

[LP] A.T. Lau and A.L.T. Paterson, *Amenability for twisted covariance algebras and group C^*-algebras*, J. Funct. Anal., **100** (1991), 59–86.

[Ma1] G.W. Mackey, *Borel structure in groups and their duals*, Trans. Amer. Math. Soc., **85** (1957), 134–165.

[Ma2] ———, *The Theory of Unitary Group Representations*, The University of Chicago Press, Chicago/London, 1976.

[Ma3] ———, *Unitary representations of group extensions, I*, Acta Math., **99** (1958), 265–311.

[McC1] K. McClanahan, *C^*-algebras generated by elements of a unitary matrix*, J. Func. Anal., **107** (1992), 439–457.

[McC2] ———, *K-theory for certain reduced free products of C^*-algebras*, preprint.

[N] T. Natsume, *On $K_*(C^*(SL_2(\mathbb{Z})))$ (Appendix to "K-theory for certain group C^*-algebras" by E. C. Lance)*, J. Oper. Theory, (1985), 103–118.

[Pd] G. Pedersen, *C^*-algebras and their Automorphism Groups*, Academic Press, London/New York/San Francisco, 1979.

[Ph] N.C. Phillips, *Classifying algebras for the K-theory of σ-C^*-algebras*, Can. J. Math., **86**, 1021–1089.

[Pi] M. Pimsner, *KK-Groups of crossed products by groups acting on trees*, Invent. Math., **86** (1986), 603–634.

[PR] J.A. Packer and I. Raeburn, *Twisted crossed products of C^*-algebras*, Math. Proc. Camb. Phil. Soc., **106** (1989), 293–311.

[PV] M. Pimsner and D. Voiculescu, *K-groups of reduced crossed products by free groups*, J. Operator Theory, **8** (1982), 131–156.

[Q] J.C. Quigg, *Duality for reduced twisted crossed products of C^*-algebras*, Indiana Univ. Math. J., **35** (1986), 549–571.

[S] J.P. Serre, *Trees*, Springer-Verlag New York/Heidelberg/Berlin.

[Sh] A.J.L. Sheu, *A cancellation theorem for modules over the group C^*-algebras of certain nilpotent Lie groups*, Canad. J. Math., **39** (1987), 365–427.

[T] H. Takai, *On a duality for crossed product C^*- algebras*, J. Func. Anal., **19** (1975), 25–39.

[Tk] M. Takesaki, *Theory of Operator Algebras I*, Springer-Verlag, New York-Heidelberg-Berlin, 1979.

[Voi] D. Voiculescu, *Symmetries of some reduced free product C^*-algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Math., Springer, Berlin, New York, **1132** (1985), 556–588.

[ZM] G. Zeller-Meier, *Produits croisés d'une C^*-algèbre par un group d'automorphismes*, J. Math Pures Appl., **47** (1968), 101–239.

Receive December 26, 1993. This research was supported in part by the National Science Foundation.

UNIVERSITY OF MISSISSIPPI
UNIVERSITY, MS 38677
E-mail address: mmkpm@office.math.olemiss.edu