PREPOLAR DEFORMATIONS AND A NEW LÊ-IOMDINE FORMULA

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In this short paper, we wish to describe a class of deformations of affine hypersurface singularities. These deformations, which we call *prepolar deformations*, are the ones which we found to be important in our work on the Lê numbers of a hypersurface singularity. However, the results of this paper are not directly related to our work on the Lê numbers.

Introduction.

The main result of this paper is to use prepolar deformations to put lower bounds on the top Betti number of the Milnor fibre of a non-isolated hypersurface singularity. These lower bounds are a corollary of a generalization of the formula of Lê and Iomdine.

Let $\mathcal{U}$ be an open neighborhood of the origin in $\mathbb{C}^{n+1}$, let $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be an analytic function, and let Σf denote the critical locus of f. Let $L : \mathbb{C}^{n+1} \to \mathbb{C}$ be a generic linear form (we need for L to be *prepolar* with respect to f; we shall define this in section 2).

The formula of Lê and Iomdine says that, if $\dim_{\mathbf{0}} \Sigma f = 1$, then, for all large j, $f + L^j$ has an isolated singularity at the origin and

$$b_n(f + L^j) \ = \ \mu(f + L^j) \ = \ b_n(f) - b_{n-1}(f) + j \sum_{\nu} m_\nu \delta_\nu(f),$$

where $b_i()$ denotes the i-th Betti number of the Milnor fibre of a function at the origin, μ denotes the Milnor number of the isolated singularity at the origin, the summation is over all components, ν, of Σf, m_ν is the local degree of L restricted to ν at the origin, and $\delta_\nu(f)$ is the Milnor number of a generic hyperplane slice of f at a point $\mathbf{p} \in \nu - \mathbf{0}$ sufficiently close to the origin.

This formula has, at least, two possible generalizations. One generalization is in terms of Lê numbers; this is the formula which appears in [**Mas1**], [**Mas3**], and [**M-S**]. But, while there are Morse inequalities between the Lê numbers and the Betti numbers of the Milnor fibre, the Lê numbers are not themselves (generally) Betti numbers of the Milnor fibre. So, one might ask

for a generalization of the formula of Lê and Iomdine which generalizes the Betti number information.

In this paper, we prove that, if $\dim_0 \Sigma f = s \geqslant 1$, then, for all large j, $\dim_0 \Sigma(f + L^j) = s - 1$ and

$$b_n(f + L^j) \ = \ b_n(f) - b_{n-1}(f) + j\left(b_{n-1}(f_{|V(L)}) - \left(\Gamma^1_{f,L} \cdot V(L)\right)_0\right),$$

where $\Gamma^1_{f,L}$ denotes the relative polar curve ([L-T], [Te1], [Te2]) of f with respect to L and $(\cdot)$ denotes the scheme-theoretic intersection number. In the case where f has a one-dimensional critical locus at the origin, it is easy to show that this new formula reduces to that of Lê and Iomdine.

We consider this new Lê-Iomdine formula interesting because it implies that

$$b_{n-1}(f_{|V(L)}) \geqslant \left(\Gamma^1_{f,L} \cdot V(L)\right)_0.$$

In terms of deformations, this says that the top possible non-zero Betti number of the Milnor fibre of $f_{|V(L)}$ is greater than or equal to $\left(\Gamma^1_{f,L} \cdot V(L)\right)_0$ for any f which has $V(L)$ as a prepolar slice. Thus, if we define f to be a prepolar deformation of $f_{|V(L)}$ precisely when $V(L)$ is a prepolar slice of f, we obtain a class of deformations of $f_{|V(L)}$ which give lower bounds on the top Betti number of the Milnor fibre. This also suggests that it might be useful to study prepolar deformations for which $\left(\Gamma^1_{f,L} \cdot V(L)\right)_0$ obtains its maximum value.

1. The Relative Polar Curve.

Throughout this section, $\mathcal{U}$ will denote an open subset of $\mathbb{C}^{n+1}$ containing the origin. We denote the coordinates of $\mathbb{C}^{n+1}$ by $(z_0, \ldots, z_n)$. We use vertical lines, $||$, to denote the underlying set of an analytic scheme.

Let W be analytic subset of $\mathcal{U}$ and let α be a coherent sheaf of ideals in $\mathcal{O}_\mathcal{U}$. At each point, $\mathbf{x}$, of $V(\alpha)$, we wish to consider scheme-theoretically those components of $V(\alpha)$ which are not contained in the set $|W|$.

Definition 1.1. Let A denote $\mathcal{O}_{\mathcal{U},\mathbf{x}}$; we write $\alpha_\mathbf{x}$ for the stalk of α in A. Let S be the multiplicatively closed set $A - \bigcup p$ where the union is over all prime ideals $p \in Ass(A/\alpha_\mathbf{x})$ with $|V(p)| \nsubseteq |W|$. Then, we define $\alpha_\mathbf{x}/W$ to equal $S^{-1}\alpha_\mathbf{x} \cap A$.

Thus, $\alpha_\mathbf{x}/W$ is the ideal in A consisting of the intersection of those primary components, q, (possibly embedded), of $\alpha_\mathbf{x}$ such that $|V(q)| \nsubseteq |W|$.

Now, we have defined $\alpha_\mathbf{x}/W$ in each stalk. By [Si-Tr], if we perform this operation simultaneously at all points of $V(\alpha)$, then we obtain a coherent sheaf of ideals called a *gap sheaf*; we write this sheaf as α/W. If $V = V(\alpha)$, we let V/W denote the scheme $V(\alpha/W)$.

It is important to note that the scheme V/W does not depend on the structure of W as a scheme, but only as an analytic set.

The following lemma is very useful for calculating V/W, and is merely an exercise in localization (see [**Mas2**] and [**Mas3**]).

Lemma 1.2. *Let $(X, \mathcal{O}_x)$ be an analytic space, let α, β, and γ be coherent sheaves of ideals in $\mathcal{O}_x$, let $f, g \in \mathcal{O}_x$, and let W be an analytic subset of X. If $V(\alpha + \gamma) \subseteq W$, then $((\alpha \cap \beta) + \gamma)/W = (\beta + \gamma)/W$, and thus*

$$((V(\alpha) \cup V(\beta)) \cap V(\gamma))/W = (V(\beta) \cap V(\gamma))/W$$

as schemes.

We will now use these gap sheaves to define the relative polar curve of a map [**L-T2**], [**Te2**], [**Te3**] with respect to a linear form. Let

$$f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$$

be an analytic map, let Σf denote its critical locus, and let $L : \mathbb{C}^{n+1} \to \mathbb{C}$ be a linear form.

Definition 1.3. After a linear change of coordinates, we may use $(L, z_1, \ldots, z_n)$ as coordinates on $\mathcal{U}$. Using these coordinates, we define the *relative polar curve* of f with respect to L to be the subscheme of $\mathcal{O}_{\mathcal{U}}$ defined by

$$\Gamma^1_{f,L} := V\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)\bigg/ \Sigma f$$

(again, see [**Mas1**], [**Mas2**], [**Mas3**]). Thus, on the level of defining ideals, $\Gamma^1_{f,L}$ consists of those components of $V\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$ which are not contained in $|\Sigma f|$. The key point of this definition is that the dimension of the singular set of f is allowed to be arbitrary.

Of course, for arbitrary L, $\Gamma^1_{f,L}$ need not actually be a curve, even as a germ at the origin. However, as we shall see in the next section, if L is prepolar with respect to f at the origin – a genericly satisfied condition – then $\Gamma^1_{f,L}$ is, in fact, one-dimensional (or empty) at the origin. Moreover, it follows immediately from the definition that whenever $\Gamma^1_{f,L}$ is one-dimensional at the origin, $\Gamma^1_{f,L} \cap V\left(\frac{\partial f}{\partial L}\right)$ is zero-dimensional or empty at the origin.

When we take intersection numbers of schemes in this paper, we are of course actually taking the intersection numbers of the associated cycles. Thus, it will be clear from the context what structure we are using for the polar curve, and we need not introduce any extra notation to distinguish the cycle from the scheme.

2. Good Stratifications and Prepolar Deformations.

As in the last section, we continue with $\mathcal{U}$ being an open subset of $\mathbb{C}^{n+1}$ containing the origin, and $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ being an analytic map.

In [**H-L**], a good stratification for f at $\mathbf{0}$ is an analytic stratification, $\mathcal{S}$, of $V(f)$ in a neighborhood of the origin such that: if $\mathbf{p} \in S \in \mathcal{S}$, and we have a sequence of $\mathbf{p}_i \in \mathcal{U} - \Sigma f$ with $\mathbf{p}_i \to \mathbf{p}$ and $T_{\mathbf{p}_i} V(f - f(\mathbf{p}_i)) \to \mathcal{T}$, then $T_{\mathbf{p}} S \subseteq \mathcal{T}$. We always assume that the smooth part of $V(f)$ has been chosen as a good stratum. Hamm and Lê prove in [**H-L**] that good stratifications always exist.

Definition 2.1. Let $\mathcal{S}$ be a good stratification for f at $\mathbf{0}$. A *prepolar slice of f at $\mathbf{0}$ with respect to $\mathcal{S}$* is a hyperplane H such that H transversely intersects each stratum of $\mathcal{S}$ in a neighborhood of the origin except, perhaps, at the origin itself.

We say simply that H is a *prepolar slice of f at $\mathbf{0}$* if there exists a good stratification for f at the origin with respect to which H is prepolar. For a fixed f, prepolar slices are generic (see [**H-L**], [**Mas3**]).

We define prepolar deformations in the obvious manner. If $\mathcal{W}$ is an open neighborhood of the origin in $\mathbb{C}^n$ and $f_0 : (\mathcal{W}, \mathbf{0}) \to (\mathbb{C}, 0)$ is an analytic function, then a *prepolar deformation* of f_0 consists of an open neighborhood, $\mathcal{U}$, of the origin in $\mathbb{C}^{n+1}$ such that $(0 \times \mathbb{C}^n) \cap \mathcal{U} = 0 \times \mathcal{W}$ and an analytic function $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ such that $0 \times \mathbb{C}^n$ is a prepolar slice of f at $\mathbf{0}$ and $f_{|0 \times \mathcal{W}} = f_0$. That is, a prepolar deformation is a one-parameter deformation which has the original function as a prepolar slice.

Our primary interest in prepolar slices is due to the following result of Lê:

Theorem 2.2 [Lê1], [Mas1]. *If $L : \mathbb{C}^{n+1} \to \mathbb{C}$ is a linear form such that $V(L)$ is a prepolar slice for f at the origin, then $\Gamma^1_{f,L}$ is one-dimensional or empty at the origin, $\Gamma^1_{f,L} \cap V(f)$ is zero-dimensional or empty at the origin, and*

$$b_n(f) - b_{n-1}(f) + b_{n-1}(f_{|V(L)}) = \left(\Gamma^1_{f,L} \cdot V(f) \right)_0,$$

where $b_i()$ denotes the i-th Betti number of the Milnor fibre at the origin and $(\cdot)$ denotes the scheme-theoretic intersection number.

3. A New General Lê-Iomdine Formula.

As before, we let $\mathcal{U}$ be an open subset of $\mathbb{C}^{n+1}$ containing the origin, and let $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ be an analytic map. Also, we use the coordinates $(z_0, \ldots, z_n)$ on $\mathcal{U}$.

We now take a new variable, u, and would like to apply 2.2 to the function $f + u^j : (\mathbb{C} \times \mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ and the hyperplane slice $V(z_0 - u)$. This is a desirable thing to do because the Milnor fibre of $f + u^j$ has the homotopy-type of the one-point union of $j - 1$ copies of the suspension of the Milnor fibre of f [**Mas3**], [**Ok**], [**Sak**], [**Se-Th**]. We first need to show that $V(z_0)$ being a prepolar slice for f implies that $V(u - z_0)$ is a prepolar slice for $f + u^j$ – at least for all large j.

Proposition 3.1. *If $j \geqslant 2$ and $\mathcal{S}$ is a good stratification for f at the origin, then*

$$\{V(f + u^j) - \Sigma(f + u^j)\} \ \cup \ \{0 \times S \mid S \text{ is a singular stratum of } \mathcal{S}\}$$

is a good stratification for $f + u^j$ at the origin.

Proof. As $j \geqslant 2$, $\Sigma(f + u^j) = \{0\} \times \Sigma f$.

Let $\mathbf{p} = (0, \mathbf{q}) \in \{0\} \times \Sigma f$, where $S \in \mathcal{S}$, and let $\mathbf{p}_i = (u_i, \mathbf{q}_i)$ be a sequence of points in $\mathbb{C} \times \mathcal{U} - \{0\} \times \Sigma f$ such that $\mathbf{p}_i \to \mathbf{p}$ and $T_{\mathbf{p}_i} V(f + u^j - (f + u^j)_{|_{\mathbf{p}_i}}) \to \mathcal{T}$. We wish to show that $T_{\mathbf{p}}(\{0\} \times S) = \{0\} \times T_{\mathbf{q}} S \subseteq \mathcal{T}$.

If $\mathcal{T} = T_{\mathbf{p}} V(u) = \{0\} \times \mathbb{C}^{n+1}$, then we are finished. So, suppose otherwise. Then, by taking a subsequence, we may assume that $\mathbf{q}_i \notin \Sigma f$ and that $T_{\mathbf{q}_i} V(f - f(\mathbf{q}_i)) \to \eta$.

As $\mathcal{T}$ transversely intersects $T_{\mathbf{p}} V(u)$, $T_{\mathbf{p}_i} V(f + u^j - (f + u^j)_{|_{\mathbf{p}_i}})$ transversely intersects $T_{\mathbf{p}_i} V(u - u_i)$ for all $\mathbf{p}_i$ close to $\mathbf{p}$. Thus,

$$\mathcal{T} \cap (\{0\} \times \mathbb{C}^{n+1}) = \lim T_{\mathbf{p}_i} V(f + u^j - (f + u^j)_{|_{\mathbf{p}_i}}) \cap T_{\mathbf{p}_i} V(u - u_i) =$$

$$\lim T_{\mathbf{p}_i} V(f + u^j - (f + u^j)_{|_{\mathbf{p}_i}}, u - u_i) = \lim T_{\mathbf{p}_i} V(f - f(\mathbf{q}_i), u - u_i) = \{0\} \times \eta.$$

Now, as S is a good stratum for f, $T_{\mathbf{q}}(S) \subseteq \eta$ and the desired conclusion follows. $\qquad\square$

Corollary 3.2. *If $V(z_0)$ is a prepolar slice for f at $\mathbf{0}$ then, for all large j, $V(z_0 - u)$ is a prepolar slice for $f + u^j$ at $\mathbf{0}$.*

Proof. In light of the proposition, all that we must show is that, for all large j, $\Sigma(f + u^j) \cap V(z_0 - u) = \Sigma(f + u^j_{|_{V(z_0 - u)}})$.

Now,

$$\Sigma(f + u^j) \cap V(z_0 - u) = (\{0\} \times \Sigma f) \cap V(z_0 - u) = \{0\} \times (\Sigma f \cap V(z_0)).$$

On the other hand,

$$\Sigma(f + u^j_{|_{V(z_0 - u)}}) = (\mathbb{C} \times \Sigma(f + z_0^j)) \cap V(z_0 - u).$$

But, near the origin and for large j, $\Sigma(f + z_0^j) = \Sigma f \cap V(z_0)$ by Lemma 3.1 of [**Mas2**]. The conclusion follows. $\square$

We are now in a position to apply 2.2 to $f + u^j$ and the hyperplane slice $V(z_0 - u)$. First, however, we will calculate the intersection number $\left(\Gamma^1_{f+u^j, z_0-u} \cdot V(f + u^j)\right)_0$.

Proposition 3.3. *If $V(z_0)$ is a prepolar slice for f at $\mathbf{0}$ then, for all large j,*

i) $\Gamma^1_{f+u^j, z_0-u} = \left(\mathbb{C} \times \Gamma^1_{f, z_0}\right) \cdot V\left(\frac{\partial f}{\partial z_0} + ju^{j-1}\right)$ *as 1-cycles;*

ii) $\left(\Gamma^1_{f+u^j, z_0-u} \cdot V(z_0 - u)\right)_0 = \left(\Gamma^1_{f, z_0} \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_0;$

iii) $\left(\Gamma^1_{f+u^j, z_0-u} \cdot V\left(\frac{\partial(f+u^j)}{\partial(z_0-u)}\right)\right)_0 = (j-1)\left(\Gamma^1_{f, z_0} \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_0;$

iv) $\left(\Gamma^1_{f+u^j, z_0-u} \cdot V(f + u^j)\right)_0 = j\left(\Gamma^1_{f, z_0} \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_0;$

v)   *if $s := \dim_0 \Sigma f \geqslant 1$, then $\dim_0 \Sigma(f + z_0^j) = s - 1$.*

Proof.

i)   Using the coordinate system $(z_0 - u, z_0, \ldots, z_n)$, we see that

$$\Gamma^1_{f+u^j, z_0-u} = V\left(\frac{\partial f}{\partial z_0} + ju^{j-1}, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right) \bigg/ \Sigma(f + u^j) =$$

$$\left[V\left(\frac{\partial f}{\partial z_0} + ju^{j-1}\right) \cap \mathbb{C} \times (\Gamma^1_{f, z_0} \cup R)\right] \bigg/ \{0\} \times \Sigma f,$$

where R consists of the union of the primary components of $V\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$ which are contained in Σf $\left(\text{for some primary decomposition of } \left\langle \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right\rangle\right)$. Regardless of the decomposition, $R \subseteq \Sigma f$ and so $(\mathbb{C} \times R) \cap V\left(\frac{\partial f}{\partial z_0} + ju^{j-1}\right) \subseteq \{0\} \times \Sigma f$. Therefore, by 1.2,

$$\Gamma^1_{f+u^j, z_0-u} = \left[V\left(\frac{\partial f}{\partial z_0} + ju^{j-1}\right) \cap (\mathbb{C} \times \Gamma^1_{f, z_0})\right] \bigg/ \{0\} \times \Sigma f.$$

But, if C were a component of $V\left(\frac{\partial f}{\partial z_0} + ju^{j-1}\right) \cap (\mathbb{C} \times \Gamma^1_{f,z_0})$ contained in $\{0\} \times \Sigma f$, then

$$C \subseteq V\left(\frac{\partial f}{\partial z_0} + ju^{j-1}\right) \cap (\mathbb{C} \times \Gamma^1_{f,z_0}) \cap (\{0\} \times \Sigma f) =$$

$$\left(\{0\} \times \Gamma^1_{f,z_0}\right) \cap \left(\{0\} \times V\left(\frac{\partial f}{\partial z_0}\right)\right),$$

which is impossible as $\Gamma^1_{f,z_0} \cap V\left(\frac{\partial f}{\partial z_0}\right)$ is 0-dimensional.

Thus, up to embedded subvariety,

$$\Gamma^1_{f+u^j, z_0-u} = V\left(\frac{\partial f}{\partial z_0} + ju^{j-1}\right) \cap \left(\mathbb{C} \times \Gamma^1_{f,z_0}\right)$$

and so these schemes define the same 1-cycle (we know that $\Gamma^1_{f+u^j, z_0-u}$ is 1-dimensional since $V(z_0 - u)$ is a prepolar slice of $f + u^j$). That this 1-cycle equals $V\left(\frac{\partial f}{\partial z_0} + ju^{j-1}\right) \cdot \left(\mathbb{C} \times \Gamma^1_{f,z_0}\right)$ follows from [**Fu**, Prop. 7.1.b], since $\mathbb{C} \times \Gamma^1_{f,z_0}$ has no embedded subvarieties and $V\left(\frac{\partial f}{\partial z_0} + ju^{j-1}\right)$ contains no components of $\mathbb{C} \times \Gamma^1_{f,z_0}$. This proves i).

ii)   By i),

$$\left(\Gamma^1_{f+u^j, z_0-u} \cdot V(z_0 - u)\right)_0$$

$$= \left(V\left(\frac{\partial f}{\partial z_0} + ju^{j-1}\right) \cdot \left(\mathbb{C} \times \Gamma^1_{f,z_0}\right) \cdot V(z_0 - u)\right)_0$$

$$= \left(\left(\mathbb{C} \times V\left(\frac{\partial f}{\partial z_0} + jz_0^{j-1}\right)\right) \cdot \left(\mathbb{C} \times \Gamma^1_{f,z_0}\right) \cdot V(z_0 - u)\right)_0$$

$$= \left(\mathbb{C} \times \left(V\left(\frac{\partial f}{\partial z_0} + jz_0^{j-1}\right) \cdot \Gamma^1_{f,z_0}\right) \cdot V(z_0 - u)\right)_0$$

$$= \left(V\left(\frac{\partial f}{\partial z_0} + jz_0^{j-1}\right) \cdot \Gamma^1_{f,z_0}\right)_0 \cdot$$

But, for all sufficiently large j, this last intersection number equals

$$\left(\Gamma^1_{f,z_0} \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_0 \cdot$$

iii)   Again using i) and the coordinate system $(z_0 - u, z_0, \ldots, z_n)$,

$$
\left(\Gamma^1_{f+u^j, z_0-u} \cdot V\left(\frac{\partial(f+u^j)}{\partial(z_0-u)} \right) \right)_0
$$

$$
= \left(\Gamma^1_{f+u^j, z_0-u} \cdot V(ju^{j-1}) \right)_0
$$

$$
= \left(V\left(\frac{\partial f}{\partial z_0} + ju^{j-1} \right) \cdot \left(\mathbb{C} \times \Gamma^1_{f,z_0} \right) \cdot V(ju^{j-1}) \right)_0
$$

$$
= (j-1) \left(V\left(\frac{\partial f}{\partial z_0} + ju^{j-1} \right) \cdot \left(\mathbb{C} \times \Gamma^1_{f,z_0} \right) \cdot V(u) \right)_0
$$

$$
= (j-1) \left(\Gamma^1_{f,z_0} \cdot V\left(\frac{\partial f}{\partial z_0} \right) \right)_0.
$$

iv)   By [**Lê1**] or [**Mas3**], if $\Gamma^1_{h,L}$ is a curve, then

$$
\left(\Gamma^1_{h,L} \cdot V(h) \right)_0 = \left(\Gamma^1_{h,L} \cdot V\left(L\frac{\partial h}{\partial L} \right) \right)_0
$$

$$
= \left(\Gamma^1_{h,L} \cdot V(L) \right)_0 + \left(\Gamma^1_{h,L} \cdot V\left(\frac{\partial h}{\partial L} \right) \right)_0.
$$

Therefore, iv) is a result of ii) and iii).

v)   This follows at once by combining the prepolarity of $V(z_0)$ with Lemma 3.1 of [**Mas2**], which says that, for all large j, $\Sigma(f+z_0^j) = \Sigma f \cap V(z_0)$ near the origin. $\square$

Theorem 3.4. *If $V(z_0)$ is a prepolar slice of f at $\mathbf{0}$ then, for all large j,*

$$
b_n(f + z_0^j) = b_n(f) - b_{n-1}(f) + j\left(b_{n-1}(f_{|V(z_0)}) - \left(\Gamma^1_{f,z_0} \cdot V(z_0) \right)_0 \right).
$$

In particular, $b_{n-1}(f_{|V(z_0)}) \geqslant \left(\Gamma^1_{f,z_0} \cdot V(z_0) \right)_0$.

Proof. After applying 2.2 to $f + u^j$ and the slice $V(z_0 - u)$, we have

$$
b_{n+1}(f + u^j) - b_n(f + u^j) + b_n(f + z_0^j) = \left(\Gamma^1_{f+u^j, z_0-u} \cdot V(f+u^j) \right)_0
$$

which, by 3.3, equals $j\left(\Gamma^1_{f,z_0} \cdot V\left(\frac{\partial f}{\partial z_0} \right) \right)_0$.

Now, as the Milnor fibre of $f + u^j$ has the homotopy-type of the one-point union of $j-1$ copies of the suspension of the Milnor fibre of f [**Mas3**], [**Ok**],

[Sak], **[Se-Th]**, we obtain

$$(j-1)b_n(f) - (j-1)b_{n-1}(f) + b_n(f+z_0^j) = j\left(\Gamma^1_{f,z_0} \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_0.$$

Using 2.2 on f itself and rearranging, we get

$$b_n(f+z_0^j) = b_n(f) - b_{n-1}(f)$$
$$+ j\left[\left(\Gamma^1_{f,z_0} \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_0 - \left((\Gamma^1_{f,z_0} \cdot V(f))_0 - b_{n-1}(f_{|V(z_0)})\right)\right].$$

Finally, using once again (as in 3.3.iv) that

$$\left(\Gamma^1_{f,z_0} \cdot V(f)\right)_0 = \left(\Gamma^1_{f,z_0} \cdot V(z_0)\right)_0 + \left(\Gamma^1_{f,z_0} \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_0,$$

we obtain the desired result. $\square$

4. Why Look at Prepolar Deformations?

If h is a complex analytic function, the Betti numbers of the Milnor fibre of h at the origin are invariants of the ambient topological type of $V(h)$ at the origin and, hence, these Betti numbers are important data in the study of the topology of complex analytic singularities. Despite their importance, there is no known general method for calculating the Betti numbers of the Milnor fibre for a non-isolated hypersurface singularity. However, Theorem 3.4 provides some useful information: every prepolar deformation, f, of a fixed f_0 yields a lower bound on the top Betti number of the Milnor fibre of f_0.

In fact, dividing the equation of 3.4 by j and taking limits, one obtains the interesting asymptotic formula

$$b_{n-1}(f_0) = \left(\Gamma^1_{f,z_0} \cdot V(z_0)\right)_0 + \lim_{j\to\infty} \frac{b_n(f+z_0^j)}{j}.$$

One might hope that, by considering a prepolar deformation, f, for which $\left(\Gamma^1_{f,z_0} \cdot V(z_0)\right)_0$ obtains its maximal value, one would actually obtain the top Betti number of the Milnor fibre of f_0. This seems unlikely however; certain singularities seem to be "rigid" with respect to prepolar deformations – in the weak sense that any prepolar deformation, f, has no polar curve at the origin.

Nonetheless, the lower bounds provided by prepolar deformations give new data which helps describe the Milnor fibre of a completely general affine

hypersurface singularity; this data does not appear to follow from our Morse inequalities between the Betti numbers of the Milnor fibre and the Lê numbers of the hypersurface, as described in [**Mas1**], [**Mas2**], and [**Mas3**].

As part of these Morse inequalities, we show that the 0-th Lê number of f_0, $\lambda_{f_0}^0(\mathbf{0})$, provides an upper-bound on the top Betti number of the Milnor fibre of f_0. Also, we show in [**Mas3**] that if f is a prepolar deformation of f_0, then

$$\lambda_{f_0}^0(\mathbf{0}) = \lambda_f^1(\mathbf{0}) + \left(\Gamma_{f,z_0}^1 \cdot V(z_0)\right)_{\mathbf{0}},$$

where $\lambda_f^1(\mathbf{0})$ is a Lê number of f itself. Hence, for a fixed f_0, a prepolar deformation, f, with maximal $\left(\Gamma_{f,z_0}^1 \cdot V(z_0)\right)_{\mathbf{0}}$ will have minimal $\lambda_f^1(\mathbf{0})$. We prefer to call such a deformation a *minimal prepolar deformation*, instead of a maximal one.

It follows that, given a prepolar deformation, f, of f_0, the top Betti number of the Milnor fibre of f_0 is bounded by:

$$\left(\Gamma_{f,z_0}^1 \cdot V(z_0)\right)_{\mathbf{0}} \leqslant b_{n-1}(f_0) \leqslant \lambda_f^1(\mathbf{0}) + \left(\Gamma_{f,z_0}^1 \cdot V(z_0)\right)_{\mathbf{0}},$$

and a minimal prepolar deformation will not only provide the maximal possible lower bound on the top Betti number of the Milnor fibre, it will also provide the smallest difference between our general upper and lower bounds.

One might hope that it is always possible to find a prepolar deformation, f, for which $\lambda_f^1(\mathbf{0}) = 0$, for then we would have $b_{n-1}(f_0) = \left(\Gamma_{f,z_0}^1 \cdot V(z_0)\right)_{\mathbf{0}}$; unfortunately, Proposition 1.28 of [**Mas1**] implies that it is usually impossible to find such a deformation.

One final comment: the inequality $b_{n-1}(f_{|V(z_0)}) \geqslant \left(\Gamma_{f,z_0}^1 \cdot V(z_0)\right)_{\mathbf{0}}$ appearing in Theorem 3.4 also follows from our Proposition 3.1 in [**Mas1**], where we describe a cancellation of handles in the Milnor fibre of f. However, this handle cancellation argument requires far more technical machinery than this paper does, and Proposition 3.1 of [**Mas1**] still does not yield the generalized Lê-Iomdine formula of Theorem 3.4.

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