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EXPLICIT SOLUTIONS FOR THE CORONA PROBLEM WITH LIPSCHITZ DATA IN THE POLYDISC

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This paper contains considerations of various versions of the classical corona problem on domains in complex n-dimensional space. Although we do not solve the H^∞ corona problem, we do obtain positive results in other topologies. We also provide explicit constructions for solutions.

1. Introduction.

Let Δ be the unit disc in the complex plane, and let Δ^n be the unit polydisc in $\mathbb{C}^n$. We let $\mathcal{H}(\Delta^n)$ denote the space of all holomorphic functions on the polydisc, and $\mathcal{H}^p(\Delta^n)$ the holomorphic Hardy space on Δ^n (see [**Rud**]).For each $0 < \alpha < \infty$, we let $\Lambda_\alpha(\Delta^n)$ denote the holomorphic Zygmund spaces over Δ^n (see [**KR2**]). Suppose that $f_1, \dots, f_m \in \mathcal{H}^\infty(\Delta^n)$ are such that

$$0 < \delta^2 \le \sum_{j=1}^{m} |f_j(z)|^2 \le 1, \qquad z \in \Delta^n. \tag{1.1}$$

In case $n = 1$, L. Carleson [**C**] solved the Corona problem and proved that there exist $g_j \in \mathcal{H}^\infty(\Delta)$ such that

$$\sum_{j=1}^{m} f_j(z) g_j(z) \equiv 1, \qquad \|g_j\|_{\mathcal{H}^\infty(\Delta)} \le C(m, \delta).$$

The question of whether the Corona problem can be solved in several complex variables has attracted much attention (for example, see [**Am**], [**An**], [**AC**], [**Ch**], [**FS1, 2**], [**HS**], [**KL**], [**Li**], [**Lin**], [**S**], and [**V1, V2**], etc.). On a strongly pseudoconvex domain, there have been attempts to generalize the method of Hörmander [**H**] and of Wolff [**KO**] to higher dimensions. This entails solving a problem of the form $\bar\partial u = \mu$, with μ a Carleson measure. One seeks a bounded solution u. Such a bounded solution *does not always exist* when the dimension exceeds 1 (see [**V1**]). However it should be noted that the result of [**V1**] does *not* imply that the Corona problem fails in several variables—only that the $\bar\partial$ technique with *that particular definition of Carleson measure* fails.

The point of the present paper is to obtain favorable results for Lipschitz solutions of the Corona problem with the corona data being Lipschitz — using iteration of one variable techniques. We shall construct a Λ_α solution of the Corona problem in one variable that allows us to treat a vector-valued problem, thus allowing induction on the number of variables. We carry out this plan by constructing an explicit formula for an $\Lambda_\alpha(\Delta^n)$ solution of the Corona problem with Corona data $f_j \in \Lambda_\alpha(\Delta^n)$.

We now give a formal statement of our theorem. The construction of our solution will be given in Section 2. The proof of the theorem is completed in Section 3.

Theorem 1.1. *Let $f_1, \ldots, f_m \in \Lambda_\alpha(\Delta^n)$ $(0 < \alpha < \infty)$ satisfy inequality* (1.1) *and*

$$\sum_{j=1}^{m} \|f_j\|_{\Lambda_\alpha(\Delta^n)} \leq 1. \tag{1.2}$$

Then there are functions $g_1, \ldots, g_m \in \Lambda_\alpha(\Delta^n)$ such that

$$\sum_{j=1}^{m} f_j(z) g_j(z) \equiv 1, \quad \textit{and} \quad \sum_{j=1}^{m} \|g_j\|_{\Lambda_\alpha(\Delta^n)} \leq C(n,m)\alpha^{1-2n}\delta^{-3n}. \tag{1.3}$$

The last estimate, in terms of α^{1-2n} and a negative power of δ, gives an indication of how the problem blows up as $\alpha \to 0^+$.

We refer the reader to [**KR2**] for careful definitions and discussion of the Lipschitz spaces Λ_α.

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2. Construction of the Solution.

In this section we shall construct an explicit solution of the Corona problem in Δ^n without yet proving any regularity properties. Let $f_1, \ldots, f_m \in \mathcal{H}^\infty(\Delta^n)$ satisfy (1.1). Without loss of generality, we may use a normal families argument and reduce the proof of Theorem 1.1 to the case of $f_j \in \mathcal{H}^\infty(\overline{\Delta^n})$. We define

$$\phi_j(z) = \overline{f}_j(z) \left[\sum_{k=1}^{m} |f_k(z)|^2 \right]^{-1}, \qquad j = 1, \ldots, m.$$

For $\lambda, \eta \in \Delta$, we let

$$K[g](\lambda) = \int_\Delta K(\lambda, \eta) g(\eta)\, dA(\eta), \tag{2.1.1}$$

where

$$K(\lambda,\eta)=\frac{1}{2\pi i}\frac{1-|\eta|^2}{(1-\bar{\eta}\lambda)(\eta-\lambda)},\quad dA(\eta)=d\eta\wedge d\bar{\eta}. \tag{2.1.2}$$

Notice that K is essentially the Poisson-Szegö kernel for the disc. It is known that $\overline{\partial}_\lambda K[g]=g$ in the sense of distributions. In fact

$$\frac{1-|\eta|^2}{(1-\bar{\eta}\lambda)(\eta-\lambda)}=\frac{1}{\eta-\lambda}-\frac{\bar{\eta}}{1-\lambda\bar{\eta}}.$$

The first kernel on the right is well known (see [**KR1**]) to be a solution operator for $\bar{\partial}$, up to a constant factor; and the second one is holomorphic in $\lambda\in\Delta$. By standard arguments (integration by parts), one can show that this integral operator K maps $C^\infty(\bar{\Delta})$ to $C^\infty(\bar{\Delta})$.

For convenience, when $1\le j\le n$ and $g\in L^2(\Delta^n)$, we let

$$K_j[g](z_j)=K[g(z_1,\cdots,z_{j-1},\cdot,z_{j+1},\cdots,z_n)](z_j). \tag{2.2}$$

The integral acts only on the variable z_j. We set

$$u^1_{jk}(z)=K_1[\phi_k\overline{\partial}_1\phi_j](z_1), \tag{2.3}$$

$j,k=1,\ldots,n$, where

$$\overline{\partial}_j g\equiv\frac{\partial g}{\partial\bar{z}_j},\quad g\in C^1(\Delta^n).$$

Also set

$$g^0_j=\phi_j,\quad g^1_j=\phi_j-\sum_{k=1}^m f_k(u^1_{jk}-u^1_{kj})\in C^\infty(\overline{\Delta^n}). \tag{2.4}$$

It is obvious that $\overline{\partial}_i g^1_j(\cdot,z')$, $g^1_j(\cdot,z')\in\mathcal{H}(\Delta)$ for each fixed $z'\in\Delta^{n-1}$ and $1\le i\le n$.

Suppose that $g^k_j\in C^\infty(\overline{\Delta^n})$ is already defined so that g^k_j and $\overline{\partial}_i g^k_j$ are holomorphic in $z_1,\ldots,z_k$ for all $1\le i\le n$. Inductively, we set

$$g^{k+1}_j(z)=g^k_j(z)-\sum_{\ell=1}^m f_\ell(z)\left(u^{k+1}_{j\ell}(z)-u^{k+1}_{\ell j}(z)\right) \tag{2.5}$$

where

$$u^{k+1}_{j\ell}(z)=K_{k+1}\left[g^k_\ell\left(z^{k+1}\right)\overline{\partial}_{k+1}g^k_j\right](z_{k+1}). \tag{2.6}$$

Then our definitions imply that $g_j^{k+1}(z) \in C^\infty(\overline{\Delta^n})$. Moreover, since the inductive hypothesis implies that $g_j^k(z)$ and $g_\ell^k \cdot \overline{\partial}_{k+1} g_j^k$ are holomorphic in $z_1, \cdots, z_k$ then

$$\overline{\partial}_i(u_{j\ell}^{k+1}) = 0, \quad \text{and} \quad \overline{\partial}_\imath g_j^k = 0, \quad i = 1, \cdots, k.$$

Therefore

$$\overline{\partial}_i g_j^{k+1} = 0, \quad i = 1, \cdots, k+1.$$

This implies that g_j^{k+1} (also $\overline{\partial}_i g_j^{k+1}$) are holomorphic in $z_1, \ldots, z_{k+1}$ for all $1 \le i \le n$. Finally, notice that

$$\sum_{j=1}^m f_j g_j^{k+1} \equiv 1$$

for all $k = 0, \ldots, n-1$. Therefore the functions $g_1^n, \ldots, g_m^n$ form a solution to the Corona problem with data $f_1, \ldots, f_m$. In order to prove Theorem 1.1, it suffices now to prove the following result:

Theorem 2.1. *Let $f_j, \cdots, f_m \in \Lambda_\alpha(\Delta^n)$ satisfy* (1.1) *and* (1.2). *Then*

$$\|g_j^n\|_{\Lambda_\alpha(\Delta^n)} \le C(n,m)\alpha^{1-2n}\delta^{-3n} \tag{2.7}$$

for all $0 < \alpha < \infty$. Notice that the same solution set $\{g_j^n\}$ suffices for all α.

We shall consider the proof of Theorem 2.1 in the next section.

3. The Proof of Theorem 2.1.

In this section, we shall complete the proof of Theorem 2.1. Let us start with the following well-known simple lemma.

Lemma 3.1. *Let $g \in C^k(\Delta^n), \alpha > 0$. Suppose that $[\alpha] + 1 \le k$, and*

$$|D^k g(z)| \le C \operatorname{dist}(z, \partial\Delta^n)^{\alpha - k}.$$

Then $g \in \Lambda_\alpha(\Delta^n)$. Moreover,

$$\|g\|_{\Lambda_\alpha(\Delta^n)} \le Cn\alpha^{-1}.$$

Here D^k denotes any derivatives of g of order k, and $[\alpha] + 1$ denotes the least integer which is greater than α.

The proof of the above lemma and of more general results can be found in [**KR2**].

Corollary 3.2. *Let $0 < \alpha < \infty$, and let $f \in \mathcal{H}(\Delta^n)$. Then $f \in \Lambda_\alpha(\Delta^n)$ if and only if*

$$|\partial_j \partial^\beta f(z)| \leq C(1 - |z_j|^2)^{\alpha-1},$$

for all $|\beta| = \beta_1 + \cdots \beta_n \leq [\alpha]$, $j = 1, \ldots, n$.

For simplicity, we shall prove Theorem 2.1 only for the case $0 < \alpha < 1$ and $n = 2$. For the case $\alpha \geq 1$ and $n > 2$, the proof may be done similarly, but it is much more tedious. For this special purpose, we shall prove the following lemma:

Lemma 3.3. *Let $0 < \alpha < 1$ and let $f,\ g \in C^\infty(\Delta^n)$ satisfy*

$$|D_j g(z)| + |D_j f(z)| \leq C_1(1 - |z_j|^2)^{\alpha-1},\ j = 1, 2, \cdots, n. \tag{3.1}$$

Then

$$|D_j K_j[f\overline{\partial}_j g](z)| \leq CC_1\alpha^{-2}(1 - |z_j|^2)^{\alpha-1}. \tag{3.2}$$

If we also assume that

$$|D_k P_j[g](z)| \leq C_1(1 - |z_k|^2)^{\alpha-1}, \quad k \neq j, \tag{3.3}$$

then

$$|D_k K_j[f\overline{\partial}_j g](z)| \leq CC_1\alpha^{-2}(1 - |z_k|^2)^{\alpha-1} \tag{3.4}$$

where $D_k = \partial_k$ or $\overline{\partial}_k$, $k = 1, \cdots, n$, and P_j denotes the Bergman projection from $L^2(\Delta)$ onto the Bergman space $A^2(\Delta)$ when we restrict attention to functions of z_j.

Proof. By symmetry, it suffices to treat the cases $j = 1$ and $k = 1$, or 2. Let us prove (3.2) first. Since $\overline{\partial}_1 K_1[f\overline{\partial}_1 g] = f\overline{\partial}_1 g$, it suffices to prove (3.2) for $D^1 = \partial_1$. For this case, without loss of generality, we may assume that $f,\ g$ are function only of z_1, in other words, we assume $n = 1$. First of all,

$$\begin{aligned}
\partial_1 K_1[f\overline{\partial}_1 g](z) &= \frac{1}{2\pi i}\partial_1 \int_\Delta \frac{(1 - |\eta|^2) f(\eta)\overline{\partial}_1 g(\eta)}{(\eta - z_1)(1 - z_1\overline{\eta})} d\eta \wedge d\overline{\eta} \\
&= \frac{1}{2\pi i}\partial_1 \int_\Delta \frac{(1 - |\eta|^2)(f(\eta) - f(z_1))\overline{\partial}_1 g(\eta)}{(\eta - z_1)(1 - z_1\overline{\eta})} d\eta \wedge d\overline{\eta} \\
&\quad + \frac{1}{2\pi i}\partial_1 \left[f(z_1) \int_\Delta \frac{(1 - |\eta|^2)\overline{\partial}_1 g(\eta)}{(\eta - z_1)(1 - z_1\overline{\eta})} d\eta \wedge d\overline{\eta} \right] \\
&= I_1(z) + I_2(z).
\end{aligned}$$

Now

$$
\begin{aligned}
-\partial_{\overline{\eta}}\frac{1-|\eta|^2}{(\eta-z_1)(1-z_1\overline{\eta})} &= \frac{\eta}{(\eta-z_1)(1-z_1\overline{\eta})} - \frac{z_1(1-|\eta|^2)}{(\eta-z_1)(1-z_1\overline{\eta})^2} \\
&= \frac{1}{(1-z_1\overline{\eta})^2}.
\end{aligned} \tag{3.5}
$$

Integrating by parts, we then have

$$I_2(z) = \partial_1[f(z)g(z)] + \partial_1\left[f(z)\frac{1}{2\pi i}\int_\Delta \frac{g(\eta)}{(1-z_1\overline{\eta})^2}d\eta\wedge d\overline{\eta}\right].$$

It is easy to see that $|I_2(z)| \le C(1-|z_1|^2)^{\alpha-1}$ since the assumption (3.1) and

$$
\begin{aligned}
\left|\partial_1\left[\frac{1}{2\pi i}\int_\Delta \frac{g(\eta)}{(1-z_1\overline{\eta})^2}d\eta\wedge d\overline{\eta}\right]\right| &= \left|\frac{1}{2\pi i}\int_\Delta \frac{2\overline{\eta}g(\eta)}{(1-z_1\overline{\eta})^3}d\eta\wedge d\overline{\eta}\right| \\
&\le \left|g(z) + \frac{1}{2\pi i}\int_\Delta \frac{\overline{\eta}(g(\eta)-g(z_1))}{(1-z_1\overline{\eta})^3}d\eta\wedge d\overline{\eta}\right| \\
&\le \left|\|g(z)\|_\infty + \int_\Delta C_1\alpha^{-1}\frac{|z_1-\eta|^\alpha}{|1-z_1\overline{\eta}|^3}\,|dA(\eta)\right| \\
&\le CC_1\alpha^{-1}(1-|z_1|^2)^{\alpha-1}.
\end{aligned}
$$

Now we consider $I_1(z)$. It is clear that

$$
\begin{aligned}
2\pi i I_1(z) &= \partial_1\int_\Delta \frac{(1-|\eta|^2)(f(\eta)-f(z_1))\overline{\partial}_1 g(\eta)}{(\eta-z_1)(1-z_1\overline{\eta})}d\eta\wedge d\overline{\eta} \\
&= -\partial_1 f(z)\int_\Delta \frac{(1-|\eta|^2)\overline{\partial}_1 g(\eta)}{(1-z_1\overline{\eta})(\eta-z_1)}d\eta\wedge d\overline{\eta} \\
&\quad + \int_\Delta \frac{(1-|\eta|^2)(f(\eta)-f(z_1))\overline{\partial}_1 g(\eta)}{(1-z_1\overline{\eta})(\eta-z_1)^2}d\eta\wedge d\overline{\eta} \\
&\quad + \int_\Delta \frac{(1-|\eta|^2)\overline{\eta}(f(\eta)-f(z_1))\overline{\partial}_1 g(\eta)}{(1-z_1\overline{\eta})^2(\eta-z_1)}d\eta\wedge d\overline{\eta} \\
&= I_{11}(z) + I_{12}(z) + I_{13}(z).
\end{aligned}
$$

Since

$$(1-|\eta|^2)|\overline{\partial}_1 g| \le C(1-|\eta|^2)^\alpha,$$

we see that by assumption (3.1) and simple calculation

$$|I_{11}(z)| \le C\alpha^{-1}|\partial_1 f(z)| \le C\alpha^{-1}(1-|z_1|^2)^{\alpha-1}.$$

By assumption (3.1), we have

$$|f(\eta)-f(z_1)| \le CC_1\alpha^{-1}|\eta-z_1|^\alpha,$$

we have

$$\begin{aligned}
&\left|\int_\Delta \frac{(1-|\eta|^2)(f(\eta)-f(z_1))\overline{\partial}_1 g(\eta)}{(1-z_1\overline{\eta})(\eta-z_1)^2}d\eta\wedge d\overline{\eta}\right| \\
&\quad\le CC_1\alpha^{-1}\int_\Delta |\overline{\partial}_1 g(\eta)||\eta-z_1|^{-2+\alpha}|dA(\eta)| \\
&\quad\le CC_1^2\alpha^{-1}\int_\Delta (1-|\eta|^2)^{-1+\alpha}|1-z_1\overline{\eta}|^{-2+\alpha}|dA(\eta)| \\
&\quad\le CC_1^2\alpha^{-2}(1-|z_1|^2)^{2\alpha-1}.
\end{aligned}$$

Therefore

$$|I_{12}(z)|\le C\alpha^{-2}(1-|z_1|^2)^{\alpha-1}.$$

Similarly, we have $|I_{13}(z)|\le C\alpha^{-2}(1-|z_1|^2)^{\alpha-1}$. Hence $|I_1(z)|\le C\alpha^{-2}(1-|z_1|^2)^{\alpha-1}$. Therefore, combining the above estimates, the proof of (3.2) is complete.

Next we prove that (3.4) holds for $D^1=\partial_2$. Notice that

$$\partial_2 K_1[f\overline{\partial}_1 g]=K_1[\partial_2(f\overline{\partial}_1 g)]=K_1[\partial_2 f\,\overline{\partial}_1 g]+K_1[f\overline{\partial}_1\partial_2 g]$$

and

$$K_1[f\overline{\partial}_1\partial_2 g]=\frac{1}{2\pi i}\int_\Delta \frac{f\partial_2 g}{(1-z_1\overline{\eta})^2}dA(\eta)+f(z)\,\partial_2 g(z)-K_1[\overline{\partial}_1 f\,\partial_2 g].$$

We shall estimate all these terms. First

$$\begin{aligned}
&|K_1[\overline{\partial}_1 f\,\partial_2 g](z)| \\
&\qquad\le C(1-|z_2|^2)^{\alpha-1}\int_\Delta \frac{C(1-|\eta|^2)^{-1+\alpha}}{|\eta-z_1|}dA(\eta) \\
&\qquad\le C\alpha^{-1}(1-|z_2|^2)^{-1+\alpha}.
\end{aligned}$$

Similarly

$$|K_1[\overline{\partial}_1 g\,\partial_2 f](z)|\le C(1-|z_2|^2)^{\alpha-1}.$$

It is easy to see that

$$|f(z)||\partial_2 g(z)|\le CC_1(1-|z_2|^2)^{\alpha-1}.$$

Moreover, we have

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_\Delta \frac{f(\eta,z')\partial_2 g}{(1-z_1\overline{\eta})^2}dA(\eta)\right| \\
&\quad=\left|\frac{1}{2\pi i}\int_\Delta \frac{(f(\eta,z')-f(z))\partial_2 g(\eta,z')}{(1-z_1\overline{\eta})^2}dA(\eta)+f(z)\partial_2 P_1[g]\right| \\
&\quad\le \int_\Delta C_1|1-z_1\overline{\eta}|^{\alpha-2}|\partial_2 g||dA(\eta)|+C_1(1-|z_2|^2)^{\alpha-1} \\
&\quad\le C\alpha^{-1}C_1(1-|z_2|^2)^{\alpha-1}.
\end{aligned}$$

Therefore (3.4) holds for $D_k = \partial_2$. Similarly, we have (3.4) holds for $D_k = \overline{\partial}_2$. Therefore, the proof of Lemma 3.3 is complete. □

Combining Lemmas 3.1 and 3.3, to complete the proof of Theorem 2.1, it suffices to prove the following lemma:

Lemma 3.4. *If $f_j \in \Lambda_\alpha(\Delta^n)$ satisfies* (1.1) *and* (1.2). *Then*

$$|D_i P_j[g_\ell^k](z)| \le CC_1(1-|z_i|^2)^{\alpha-1},$$

for all $i, j = 1, \cdots, n$, $i \neq j$ $0 \le k \le n-1$, and $1 \le \ell \le m$.

For convenience, we shall prove Lemma 3.4 for $n = 2$. [The case $n > 2$ is similar, but more tedious.] To achieve this special goal, we first prove:

Lemma 3.5. *Suppose that $f_j \in \Lambda_\alpha(\Delta^n)$ satisfies* (1.1) *and* (1.2). *Then Lemma* 3.4 *holds when $k = 0$, i.e. when $g_j^0 = \phi_j$.*

Proof. By symmetry, it suffices to prove Lemma 3.5 with $j = 1$ and $i = 2$. Moreover, we need only consider the case $D_2 = \overline{\partial}_2$ since $D_2 = \partial_2$ is similar. With the notation $|f(z)|^2 = \sum_{k=1}^m |f_k(z)|^2$, we have

$$\begin{aligned}
&|D_2 P_1[\phi_j](z)| \\
&\le \left| \frac{1}{2\pi} \int_\Delta \frac{\overline{\partial}_2 \phi_j(\eta, z')}{(1 - z_1\overline{\eta})^2} dA(\eta) \right| \\
&\le \left| \frac{1}{2\pi} \int_\Delta \frac{\overline{\partial}_2 \overline{f}_j |f(\eta, z')|^2 - \overline{f}_j(\eta, z') \sum_{k=1}^m f_k(\eta, z') \overline{\partial}_2 \overline{f}_k(\eta, z')}{|f(\eta, z')|^4 (1 - z_1\overline{\eta})^2} dA(\eta) \right| \\
&\le \left| \frac{1}{2\pi} \int_\Delta \frac{\overline{\partial}_2 \overline{f}_j}{|f(\eta, z')|^2 (1 - z_1\overline{\eta})^2} dA(\eta) \right| \\
&\quad + \left| \frac{1}{2\pi} \int_\Delta \frac{\overline{f}_j(\eta, z') \sum_{k=1}^m f_k(\eta, z') \overline{\partial}_2 \overline{f}_k(\eta, z')}{|f(\eta, z')|^4 (1 - z_1\overline{\eta})^2} dA(\eta) \right| \\
&= \frac{1}{2\pi} |J_1(z) + J_2(z)|.
\end{aligned}$$

Now we let

$$h(z) = 1/|f(z)|^2, \quad h_{jk}(z) = \overline{f}_j(z) f_k(z)/|f(z)|^4.$$

Then

$$\begin{aligned}
|J_1(z)| &\le \left| \int_\Delta \frac{(h(\eta, z') - h(z)) \overline{\partial}_2 \overline{f}_j}{(1 - z_1\overline{\eta})^2} dA(\eta) \right| + \left| 2\pi i h(z) \overline{\partial}_2 \overline{f}_j(0, z') \right| \\
&\le C(m, n) \alpha^{-1} \delta^{-3} C_1 (1 - |z_2|^2)^{\alpha-1},
\end{aligned}$$

and

$$\begin{aligned}|J_2(z)| &= \left|\sum_{k=1}^{m}\int_\Delta \frac{(h_{jk}(\eta,z')-h_{jk}(z))\overline{\partial}_2\overline{f}_k}{(1-z_1\overline{\eta})^2}dA(\eta)+2\pi h_{jk}(z)\overline{\partial}_2\overline{f}_k(0,z')\right|\\ &\le C(m,n)\delta^{-3}\alpha^{-1}C_1(1-|z_2|^2)^{\alpha-1}.\end{aligned}$$

This completes the proof of Lemma 3.5. □

Corollary 3.6. *Let $f_j\in\Lambda_\alpha(\Delta^n)$ satisfy* (1.1) *and* (1.2). *Then*

$$|D_i g_j^1(z)|\le C(m,n)\alpha^{-2}\delta^{-3}(1-|z_i|^2)^{\alpha-1}$$

for all $0<\alpha<1$, $1\le i\le n$ and $1\le j\le m$.

Proof. This follows from Lemmas 3.1, 3.2, 3.3 and 3.5.

To prove Lemma 3.4 for the case $n=2$, we need only to prove:

$$|D_kP_j[g_\ell^1](z)|\le C\alpha^{-3}\delta^{-5}(1-|z_k|^2)^{\alpha-1} \tag{3.6}$$

for all $k=1$ with $j=2$ and $1\le\ell\le m$.

Notice that

$$\begin{aligned}&|D_kP_2[g_\ell^1](z)|\\ &=\left|\frac{1}{2\pi}D_k\int_\Delta\frac{g_\ell^1(z_1,\eta,z'')}{(1-z_2\overline{\eta})^2}dA(\eta)\right|\\ &=\left|\frac{1}{2\pi}D_k\int_\Delta\frac{\phi_\ell(z_1,\eta,z'')-\sum_{j=1}^m f_j(z_1,\eta,z'')(u_{\ell j}^1-u_{j\ell}^1)(z_1,\eta,z'')}{(1-z_2\overline{\eta})^2}dA(\eta)\right|\\ &\le\left|D_kP_2[\phi_\ell](z)-\frac{1}{2\pi i}D_k\int_\Delta\frac{\sum_{j=1}^m f_j(z_1,\eta,z'')(u_{\ell j}^1-u_{j\ell}^1)(z_1,\eta,z'')}{(1-z_2\overline{\eta})^2}dA(\eta)\right|.\end{aligned}$$

By Lemma 3.5, we have

$$|D_kP_2[\phi_\ell](z)|\le C\alpha^{-3}\delta^{-4}(1-|z_1|^2)^{\alpha-1}.$$

Thus, combining this with $f_j\in\Lambda_\alpha(\Delta^n)$, the estimation of $D_kP_2[g_\ell^1]$ can be reduced to prove:

$$|D_kP_2[u_{j\ell}^1](z)|\le C(n,m)\alpha^{-3}\delta^{-4}(1-|z_1|^2)^{\alpha-1} \tag{3.7}$$

for $k=1$. In order to prove (3.7) for $k=1$, we need the following lemma.

Lemma 3.7. *Let $f_j\in\Lambda_\alpha(\Delta^n)$ satisfy* (1.1) *and* (1.2). *Then*

$$(1-|z_1|^2)|\partial_1\phi_j(z_1,z_2,z'')-\partial_1\phi_j(z_1,w_2,z'')|\le C\delta^{-3}|z_2-w_2|^\alpha. \tag{3.8}$$

Proof. Since

$$\begin{aligned}
&(1-|z_1|^2)|\partial_1 f_k(z_1,z_2,z'')-\partial_1 f_k(z_1,w_2,z'')|\\
&\quad=\left|(1-|z_1|^2)\partial_1\int_T \frac{f_k(\eta,z_2,z'')-f_k(\eta,w_2,z'')}{1-\overline{\eta}z_1}d\sigma(\eta)\right|\\
&\quad\le C|z_2-w_2|^\alpha(1-|z_1|^2)\int_T\frac{1}{|1-z_1\eta|^2}d\sigma(\eta)\\
&\quad\le C|z_2-w_2|^\alpha.
\end{aligned}$$

Now

$$\partial_1\phi_j(z)=\overline{f}_j(z)\left(\sum_{k=1}^m|f_k(z)|^2\right)^{-2}\sum_{k=1}^m\overline{f}_k(z)\partial_1 f_k(z).$$

Thus

$$\begin{aligned}
&(1-|z_1|^2)|\partial_1\phi_j(z_1,z_2,z'')-\partial_1\phi_j(z_1,w_2,z'')|\\
&\quad\le C\delta^{-2}\sum_{k=1}^m(1-|z_1|^2)|\partial_1 f(z_1,z_2,z'')-\partial_1 f_k(z_1,w_2,z'')|\\
&\qquad+(1-|z_1|^2)\sum_{k=1}^m|\partial_1 f_k|\left|\sum_{k=1}^m\left(\overline{f}_j\overline{f}_k\left(\sum_{k=1}^m|f_k|^2\right)^{-2}\right)(z_1,z_2,z'')\right.\\
&\qquad\left.-\left(\overline{f}_j\overline{f}_k\left(\sum_{k=1}^m|f_k|^2\right)^{-2}\right)(z_1,w_2,z'')\right|\\
&\quad\le C\delta^{-3}|z-w|^\alpha.
\end{aligned}$$

This completes the proof of Lemma 3.7. □

Now we are ready to prove (3.7) for $k=1$. Since g_j^1 is holomorphic in z_1, it suffices to prove (3.7) for $D_1=\partial_1$. Observe that

$$\begin{aligned}
&\left|D_1P_2[u_{j\ell}^1](z)\right|\\
&\quad=\left|\partial_1P_2[K_1[\phi_\ell\overline{\partial}_1\phi_j]](z)\right|\\
&\quad\le\left|\partial_1P_2\left[K_1\left[(\phi_\ell(\eta,z')-\phi_\ell(z))\overline{\partial}_1\phi_j(\eta,z')\right]\right]\right|+\left|\partial_1P_2\left[\phi_\ell K_1[\overline{\partial}_1\phi_j]\right](z)\right|\\
&\quad=I_3(z)+I_4(z).
\end{aligned}$$

We consider $I_3(z)$ first. Now

$$\begin{aligned}
I_3(z) &= \left|\partial_1 P_2\left[K_1\left[(\phi_\ell(\eta,z') - \phi_\ell(z))\overline{\partial}_1\phi_j(\eta,z')\right]\right]\right| \\
&= \left|P_2\left[\partial_1 K_1\left[(\phi_\ell(\eta,z') - \phi_\ell(z))\overline{\partial}_1\phi_j(\eta,z')\right]\right]\right| \\
&\leq \left|P_2\left[\int_\Delta \frac{(1-|\eta|^2)(\phi_\ell(\eta,z') - \phi_\ell(z))\overline{\partial}_1\phi_j(\eta,z')}{(\eta-z)^2(1-z_1\overline{\eta})}\right](z)\right| \\
&+ \left|P_2\left[\int_\Delta \frac{(1-|\eta|^2)\overline{\eta}(\phi_\ell(\eta,z') - \phi_\ell(z))\overline{\partial}_1\phi_j(\eta,z')}{(\eta-z)(1-z_1\overline{\eta})^2}\right](z)\right| \\
&+ \left|P_2\left[\int_\Delta \frac{(1-|\eta|^2)\partial_1(\phi_\ell(z))\overline{\partial}_1\phi_j(\eta,z')}{(\eta-z)(1-z_1\overline{\eta})}\right](z)\right| \\
&= I_{31}(z) + I_{32}(z) + I_{33}(z).
\end{aligned}$$

With the notation $B(z_1,\eta) = (1-|\eta|^2)(\eta-z_1)^{-2}(1-z_1\overline{\eta})^{-1}$ and an application of Lemma 3.7, we have

$$\begin{aligned}
&|I_{31}(z)| \\
&\leq \left|\int_\Delta B(z_1,\eta)\int_\Delta \frac{(\phi_\ell(\eta,\lambda,z'') - \phi_\ell(z_1,\lambda,z''))\overline{\partial}_1\phi_j(\eta,\lambda,z'')}{(1-z_2\overline{\lambda})^2}dA(\lambda)dA(\eta)\right| \\
&\leq \Bigg|\int_\Delta B(z_1,\eta) \\
&\quad \int_\Delta \frac{(\phi_\ell(\eta,\lambda,z'') - \phi_\ell(z_1,\lambda,z''))\left[\overline{\partial}_1\phi_j(\eta,\lambda,z'') - \overline{\partial}_1\phi_j(\eta,z')\right]}{(1-z_2\overline{\lambda})^2}dA(\lambda)dA(\eta)\Bigg| \\
&\quad + \left|\int_\Delta B(z_1,\eta)\overline{\partial}_1\phi_j(\eta,,z'')\int_\Delta \frac{(\phi_\ell(\eta,\lambda,z'') - \phi_\ell(z_1,\lambda,z''))}{(1-z_2\overline{\lambda})^2}dA(\lambda)dA(\eta)\right| \\
&\leq C\delta^{-4}\int_\Delta |\eta-z_1|^{-2+\alpha}|1-z_1\overline{\eta}|^{-1}\int_\Delta |1-z_2\lambda|^{-2+\alpha}dA(\lambda)dA(\eta) \\
&\quad + \left|\int_\Delta B(z_1,\eta)\overline{\partial}_1\phi_j(\eta,z')\int_\Delta \frac{(\phi_\ell(\eta,\lambda,z'') - \phi_\ell(z_1,\lambda,z''))}{(1-z_2\overline{\lambda})^2}dA(\lambda)dA(\eta)\right| \\
&\leq C\alpha^{-2}\delta^{-4}(1-|z_1|^2)^{\alpha-1} + I_{311}.
\end{aligned}$$

Notice that:

$$\begin{aligned}
&\int_\Delta \frac{(\phi_\ell(\eta,\lambda,z'') - \phi_\ell(z_1,\lambda,z''))}{(1-z_2\overline{\lambda})^2}dA(\lambda) \\
&\quad = \int_\Delta \frac{(\phi_\ell(\eta,\lambda,z'') - \phi_\ell(\eta,z_2,z'') + \phi_\ell(z) - \phi_\ell(z_1,\lambda,z''))}{(1-z_2\overline{\lambda})^2}dA(\lambda) \\
&\quad + 2\pi i\left[\phi_\ell(\eta,z') - \phi_\ell(z)\right].
\end{aligned}$$

By using integration by parts and Lemmas 3.5 and 3.7, we have

$$\begin{aligned}
&|I_{311}(z)| \\
&\leq \left|\int_\Delta \frac{\overline{\eta}\overline{\partial}_1\phi_j(\eta,z')}{(\eta-z_1)(1-z_1\overline{\eta})}\int_\Delta \frac{(\phi_\ell(\eta,\lambda,z'')-\phi_\ell(z_1,\lambda,z''))}{(1-z_2\overline{\lambda})^2}dA(\lambda)dA(\eta)\right| \\
&\quad+\left|\int_\Delta \frac{(1-|\eta|^2)\partial_1\overline{\partial}_1\phi_j(\eta,z'')}{(\eta-z_1)(1-z_1\overline{\eta})}\int_\Delta \frac{(\phi_\ell(\eta,\lambda,z'')-\phi_\ell(z_1,\lambda,z''))}{(1-z_2\overline{\lambda})^2}dA(\lambda)dA(\eta)\right| \\
&\quad+\left|\int_\Delta \frac{(1-|\eta|^2)\overline{\partial}_1\phi_j(\eta,z'')}{(\eta-z_1)(1-z_1\overline{\eta})}\int_\Delta \frac{\partial_1\phi_\ell(\eta,\lambda,z'')}{(1-z_2\overline{\lambda})^2}dA(\lambda)dA(\eta)\right| \\
&\leq C\delta^{-4}\int_\Delta \frac{(1-|\eta|^2)^{\alpha-1}}{|1-z_1\overline{\eta}|^2}\int_\Delta \frac{|\lambda-z_2|^\alpha}{|1-z_2\overline{\lambda}|^2}dA(\eta)dA(\lambda) \\
&\quad+C\delta^{-5}\int_\Delta \frac{(1-|\eta|^2)^{2\alpha-1}}{|1-z_1\overline{\eta}|^2}\int_\Delta \frac{|\lambda-z_2|^\alpha}{|1-z_2\overline{\lambda}|^2}dA(\lambda)dA(\eta) \\
&\quad+C\delta^{-2}\int_\Delta \frac{(1-|\eta|^2)^\alpha}{|1-z_1\overline{\eta}|^2}\left|\int_\Delta \frac{\partial_1\phi_j(\eta,\lambda,z'')}{(1-z_2\overline{\lambda})^2}dA(\lambda)dA(\eta)\right| \\
&\leq C\alpha^{-2}\delta^{-4}(1-|z_1|^2)^{\alpha-1}+C\delta^{-5}\int_\Delta \frac{(1-|\eta|^2)^\alpha}{|1-z_1\overline{\eta}|^2}|\partial_1P_2\left[\phi_j\right](\eta,z')|dA(\eta) \\
&\leq C\alpha^{-2}\delta^{-4}(1-|z_1|^2)^{\alpha-1}+C\delta^{-5}\int_\Delta \frac{(1-|\eta|^2)^{2\alpha-1}}{|1-z_1\overline{\eta}|^2}dA(\eta) \\
&\qquad\leq C\alpha^{-2}\delta^{-4}(1-|z_1|^2)^{\alpha-1}.
\end{aligned}$$

Since the estimation of $I_{32}(z)$ is similar to and easier than that of $I_{31}(z)$, we therefore have

$$I_{31}(z)+I_{32}(z)\leq C\alpha^{-2}\delta^{-5}(1-|z_1|^2)^{\alpha-1}.$$

Now we consider $I_{33}(z)$. By Lemmas 3.5 and 3.7 again, we have

$$\begin{aligned}
I_{33}(z)&\leq \left|\int_\Delta \frac{(1-|\eta|^2)}{(\eta-z_1)(1-z_1\overline{\eta})}\int_\Delta \frac{\partial_1\phi_\ell\overline{\partial}_1\phi_j(\eta,\lambda,z'')}{(1-z_2\overline{\lambda})^2}dA(\lambda)dA(\eta)\right| \\
&\leq \left|\int_\Delta \frac{(1-|\eta|^2)}{(\eta-z_1)(1-z_1\overline{\eta})}\right. \\
&\qquad\left.\int_\Delta \frac{(\partial_1\phi_\ell(\eta,\lambda,z'')-\partial_1\phi_\ell(\eta,z_2,z''))\overline{\partial}_1\phi_j(\eta,\lambda,z'')}{(1-z_2\overline{\lambda})^2}dA(\lambda)dA(\eta)\right| \\
&\quad+\left|\int_\Delta \frac{(1-|\eta|^2)}{(\eta-z_1)(1-z_1\overline{\eta})}\partial_1\phi_\ell(\eta,z')\int_\Delta \frac{\overline{\partial}_1\phi_j(\eta,\lambda,z'')}{(1-z_2\overline{\lambda})^2}dA(\lambda)dA(\eta)\right|
\end{aligned}$$

$$\leq C\alpha^{-1}\delta^{-4}\int_\Delta \frac{1}{|1-z_1\overline{\eta}|^2}(1-|\eta|^2)^{\alpha-1}dA(\eta)$$
$$+ C\delta^{-4}\int_\Delta \frac{1-|\eta|^2}{|1-z_1\overline{\eta}|^2}(1-|\eta|^2)^{2\alpha-2}dA(\eta)$$
$$\leq C\delta^{-4}\alpha^{-2}(1-|z_1|^2)^{\alpha-1}.$$

Therefore

$$|I_3(z)| \leq C\alpha^{-2}\delta^{-4}(1-|z_1|^2)^{\alpha-1}.$$

Now we estimate $I_4(z)$. Observe that

$$\begin{aligned}
I_4(z) &= \left|\partial_1 P_2\left[\phi_\ell K_1\left[\overline{\partial}_1\phi_j\right]\right]\right| \\
&\leq \left|\partial_1\int_\Delta \frac{\phi_\ell(z_1,\lambda,z'')K_1\left[\overline{\partial}_1\phi_j\right](z_1,\lambda,z'')}{(1-z_2\bar{\lambda})^2}dA(\lambda)\right| \\
&\leq \left|\int_\Delta \frac{\partial_1\phi_\ell(z_1,\lambda,z'')\,K_1\left[\overline{\partial}_1\phi_j\right](z_1,\lambda,z'')}{(1-z_2\bar{\lambda})^2}dA(\lambda)\right| \\
&\quad+\left|\int_\Delta \frac{\phi_\ell(z_1,\lambda,z'')\partial_1 K_1\left[\overline{\partial}_1\phi_j\right](z_1,\lambda,z'')}{(1-z_2\bar{\lambda})^2}dA(\lambda)\right| \\
&= I_{41}(z) + I_{42}(z).
\end{aligned}$$

By Lemmas 3.3 and 3.5, we have

$$\left|\partial_1 K_1\left[\overline{\partial}_1\phi_k\right]\right| \leq C\alpha^{-1}\delta^{-3}(1-|z_1|^2)^{\alpha-1}.$$

Thus

$$\begin{aligned}
I_{42}(z) &\leq \int_\Delta \frac{|\phi_\ell(z_1,\lambda,z'')-\phi_\ell(z)|\left|\partial_1 K_1\left[\overline{\partial}_1\phi_k\right](z_1,\lambda,z'')\right|}{|1-z_2\bar{\lambda}|^2}|dA(\lambda)| \\
&\quad+\left|\phi_\ell(z)\int_\Delta \frac{\partial_1 K_1\left[\overline{\partial}_1\phi_k\right](z_1,\lambda,z'')}{(1-z_2\bar{\lambda})^2}dA(\lambda)\right| \\
&\leq C\delta^{-5}\alpha^{-1}(1-|z_1|^2)^{\alpha-1}\int_\Delta \frac{|\lambda-z_2|^\alpha}{|1-z_2\bar{\lambda}|^2}|dA(\lambda)| + I_{421}(z) \\
&\leq C\delta^{-5}\alpha^{-2}(1-|z_1|^2)^{\alpha-1} + I_{421}(z).
\end{aligned}$$

Since

$$\begin{aligned}
I_{421}(z) &\leq \left|\partial_1 P_2\left[K_1\left[\overline{\partial}_1\phi_k\right]\right]\right| \\
&= \left|\partial_1 K_1\left[P_2\left[\overline{\partial}_1\phi_k\right]\right]\right| \\
&= \left|\partial_1 K_1\left[\overline{\partial}_1 P_2\left[\phi_k\right]\right]\right|.
\end{aligned}$$

By Lemma 3.5, we have

$$|\bar{\partial}_1 P_2 [\phi_k] (z)| \le C\alpha^{-1}(1 - |z_1|^2)^{\alpha-1}.$$

Moreover, we have

$$\begin{aligned}
|D_2 P_2 [\phi_k] (z)| &\le |\phi_k| + \left| \int_\Delta \frac{\phi_k(z_1, \xi, z'') - \phi(z)}{(1 - z_2\bar{\xi})^3} dA(\xi) \right| \\
&\le C\delta^{-1} + C\delta^{-2} \int_\Delta \frac{|\xi - z_2|}{|1 - z_2\bar{\xi}|^3} |dA(\xi)| \\
&\le C\delta^{-2}\alpha^{-1}(1 - |z_2|^2)^{-1+\alpha}.
\end{aligned}$$

Thus (3.1) is satisfied for $f = 1$ and $g = P_2 [\phi_k]$. Now we apply Lemma 3.3, we have

$$I_{421} \le \left| \partial_1 K_1 \left[\bar{\partial}_1 P_2 [\phi_k] \right] \right| \le C\alpha^{-2}\delta^{-2}(1 - |z_1|^2)^{\alpha-1}.$$

Therefore, we have

$$I_{42}(z) \le C\alpha^{-2}\delta^{-5}(1 - |z_1|^2)^{\alpha-1}.$$

Next we estimate $I_{41}(z)$. By Lemma 3.5, we have

$$\begin{aligned}
I_{41}(z) &\le |P_2 [\partial_1 \phi_\ell \phi_j]| + C \left| P_2 \left[\partial_1 \phi_j \int_\Delta \left[\frac{2\phi_k(\eta, z')}{(1 - z_1\bar{\eta})} + \frac{\phi_k(\eta, z')}{(1 - z_1\bar{\eta})^2} \right] dA(\eta) \right] \right| \\
&\le C\alpha^{-2}\delta^{-4}(1 - |z_1|^2)^{\alpha-1} + C \left| P_2 \left[\partial_1 \phi_j \int_\Delta \frac{\phi_k(\eta, z')}{(1 - z_1\bar{\eta})^2} dA(\eta) \right] \right| \\
&\le C\alpha^{-2}\delta^{-4}(1 - |z_1|^2)^{\alpha-1} + C \left| P_2 [\partial_1 \phi_j \, P_1 [\phi_k]] \right| \\
&\le C\alpha^{-2}\delta^{-4}(1 - |z_1|^2)^{\alpha-1} + C|P_1 [\phi_k] (z) P_2 [\partial_1 \phi_j] | \\
&\quad + C \left| P_2 [\partial_1 \phi_j \, (P_1 [\phi_k]) - P_1 [\phi_k] (z_1, z_2, z'')] \right| \\
&\le C\alpha^{-2}\delta^{-4}(1 - |z_1|^2)^{\alpha-1} \\
&\quad + C\alpha^{-1}\delta^{-2}(1 - |z_1|^2)^{\alpha-1} \\
&\quad \times \int_\Delta \frac{|P_1 [\phi_k] (z_1, \lambda, z'') - P_1 [\phi_k] (z_1, z_2, z'')|}{|1 - z_2\bar{\lambda}|^2} |dA(\lambda)| \\
&\le C\alpha^{-2}\delta^{-4}(1 - |z_1|^2)^{\alpha-1}
\end{aligned}$$

Therefore, $I_4(z) \le C\alpha^{-2}\delta^{-4}(1 - |z_1|^2)^{\alpha-1}$. Combining all of these estimates, the proof of (3.7) for the case $k = 1$ is complete. Therefore, the proof of Lemma 3.4 for the case $n = 2$ is complete. □

As a consequence of Lemmas 3.1, 3.2, 3.3, 3.6 and (3.7) , we have that the proof of Theorem 2.1 for the case $n = 2$ and $0 < \alpha < 1$ is complete. The other cases can be done similarly, but the details are tedious.

We note in closing that the solution to the Corona problem presented in this paper is essentially linear in nature. It is well known that solutions to the original $\mathcal{H}^\infty$ Corona problem are perforce non-linear in nature.

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