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SOME PROPERTIES OF FANO MANIFOLDS THAT ARE ZEROS OF SECTIONS IN HOMOGENEOUS VECTOR BUNDLES OVER GRASSMANNIANS

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Let X be a Fano manifold which is the zero scheme of a general global section s in an irreducible homogeneous vector bundle over a Grassmannian. We prove that the restriction of the Plücker embedding embeds X projectively normal, and that every small deformation of X comes from a deformation of the section s. These results are strengthened in the case of Fano 4-folds.

Introduction.

Let $Gr(k,n) = \mathbf{SL}(n,\mathbb{C})/P_k$ be the Grassmannian of k-dimensional quotients of n-dimensional complex space $\mathbb{C}^n$ considered as quotient of $\mathbf{SL}(n,\mathbb{C})$ by a maximal parabolic subgroup P_k. Then (irreducible) representations of P_k give rise to (irreducible) homogeneous vector bundles over $Gr(k,n)$. The purpose of this note is to prove the following theorems:

Theorem 1. *Let X be a Fano manifold which is the zero scheme of a general global section in a globally generated irreducible homogeneous vector bundle $\mathcal{F}$ over $Gr(k,n)$. Then X is projectively normal.*

Here by a Fano manifold we mean a manifold X with ample anticanonical divisor $-K_X$, and $X \subset Gr(k,n)$ is considered to be embedded by the restriction of the Plücker embedding.

Theorem 2. *Let X be as above. Then every small deformation of X is again the zero scheme of a section in the same homogeneous bundle.*

Moreover it becomes obvious from the proof that the bundle $\mathcal{F}$ in Theorem 1 can be replaced by the sum of one irreducible vector bundle and line bundles.

Concerning Fano 4-folds we have a slightly more general result:

Theorem 3. *Suppose* $\dim(X) = 4$ *and that the Picard group* $\mathrm{Pic}(X)$ *of X is generated by $\mathcal{O}_X(-K_X)$. Then the statements of Theorems* 1 *and* 2 *remain*

true is $\mathcal{F}$ is fully reducible, i.e. direct sum of irreducible homogeneous vector bundles.

The idea of the proofs goes back to Borcea [**Bor**] and Wehler [**We**], who obtained results similar to Theorem 2 in the case of varieties parametrizing linear subspaces on complete intersections of hypersurfaces. Nevertheless it is difficult to make statements for arbitrary irreducible homogeneous vector bundles. Therefore it is worthwhile to point out that the condition of X being Fano is crucial here and sharp in a certain sense (cf. Example 4.11 and Remark 5.5). On the other hand homogeneous vector bundles seem to play an important role in the classification of Fano manifolds (cf. [**Muk**], [**Kü**]), e.g., all Fano 3-folds V of the "main series", i.e. with very ample $-K_V$ and $b_2(V) = 1$ arise as sections of the sum of an irreducible homogeneous vector bundle and line bundles over ordinary or isotropic Grassmannians.

The proofs work as follows: The Theorems will follow from the vanishing of certain cohomology groups of bundles and sheaves on X and $Gr(k,n)$. Via spectral sequence arguments this is reduced to the vanishing of cohomology groups of vector bundles on $Gr(k,n)$ involving only wedge products and tensor products of $\mathcal{F}$, its dual and the tangent bundle $\Theta_{Gr(k,n)}$. Since all these bundles are homogeneous, we can apply Bott's Theorem to obtain the vanishings once we determine the weights of the corresponding representations. This last step is the "ugly" part and consists in combining various estimates, and here is where the Fano-condition comes in to keep control of the shape of the occuring weights.

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1.

Let $Y = Gr(k,n) \subset \mathrm{IP}^N$ be embedded by the Plücker embedding. It is well known that Y is projectively normal of dimension $k(n-k)$ with canonical line bundle $\mathcal{O}_Y(K_Y) = \mathcal{O}_Y(-n)$. Let $X \subset Y$ be a subvariety. Then from the commutativity of

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{J}_X & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{O}_{\mathrm{IP}^N} & = & \mathcal{O}_{\mathrm{IP}^N} & & \\
 & & & & & & \uparrow & & \\
 & & & & & & \mathcal{I}_X & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}
\tag{1.1}
$$

where $\mathcal{J}_X$ is the ideal sheaf of X in Y and $\mathcal{I}_X$ the ideal sheaf of X in IP^N, it is clear that $\mathcal{O}_X(1) := \mathcal{O}_Y(1) \otimes \mathcal{O}_X$ embeds X projectively normal if and only if

$$H^1(Gr(k,n), \mathcal{J}_X(r)) = 0 \quad \forall \quad r \geq 1. \tag{1.2}$$

Now let $s \in H^0(Gr(k,n), \mathcal{F})$ be a general global section in a globally generated vector bundle $\mathcal{F}$ over $Gr(k,n)$, such that X, the variety of zeros of s, is non-empty. Then it is known (cf. [**We**]) that every small deformation of X can be obtained by varying the section s if

$$H^1(Gr(k,n), \mathcal{F} \otimes \mathcal{J}_X) = 0, \quad \text{and} \tag{1.3.a}$$

$$H^1(X, \Theta_{Gr(k,n)}|_X) = 0. \tag{1.3.b}$$

Finally the Koszul complex associated to the section s gives, for any vector bundle $\mathcal{E}$ on $Gr(k,n)$, spectral sequences

$$\begin{aligned} &H^p(Gr(k,n), \mathcal{E} \otimes \textstyle\bigwedge^q \mathcal{F}^*) \implies H^{p-q}(X, \mathcal{E}|X) \\ &H^p(Gr(k,n), \mathcal{E} \otimes \textstyle\bigwedge^{q+1} \mathcal{F}^*) \implies H^{p-q}(Gr(k,n), \mathcal{E} \otimes \mathcal{J}_X),\ q \geq 0. \end{aligned} \tag{1.4}$$

2.

Now we recall the set-up and fix notations for the weight calculations. Let

$$P_k = \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \in \mathbf{SL}(n, \mathbb{C}), A \in \mathbf{GL}(k, \mathbb{C}) \right\},$$

such that $Gr(k,n) = \mathbf{SL}(n, \mathbb{C})/P_k$. Then an irreducible homogeneous vector bundle $\mathcal{F}$ comes from a representation of the reductive part of P_k consisting of matrices with $B = 0$. Such a representation is uniquely determined by its highest weight which can be written as an integral vector $\beta = (\beta_1, ..., \beta_n)$ (cf. [**Kü**]). In this notation the universal quotient bundle $\mathcal{Q}$ on the Grassmannian, which comes from the representation of A, has highest weight $(1, 0, ..., 0)$, and the tangent bundle $\Theta_{Gr(k,n)}$ reads $(1, 0, ..., 0, -1)$. The vector bundle $\mathcal{F}$ is globally generated if and only if $\beta_i \geq \beta_{i+1}$ for all $1 \leq i \leq n-1$, and the highest weight of the representation corresponding to the dual bundle $\mathcal{F}^*$ is $(-\beta_k, ..., -\beta_1, -\beta_n, ..., -\beta_{k+1})$.

The Weyl group acts on the weights by permutations among the first k and the last $n-k$ entries, yielding

$$rk(\mathcal{F}) = \dim(\beta) = \prod_{1 \leq i < j \leq k} \frac{j - i + \beta_i - \beta_j}{j - i} \prod_{k+1 \leq s < t \leq n} \frac{t - s + \beta_s - \beta_t}{t - s} \tag{2.1}$$

for the rank of the corresponding vector bundle. In this notation, the vanishing part of Bott's Theorem (cf. [**Bot**], Theorem IV') can be expressed as follows:

Theorem. *Let $\mathcal{F}$ be an irreducible homogeneous vector bundle over $Gr(k,n)$ with highest weight $(\gamma_1,...,\gamma_n)$. Then $H^p(Gr(k,n),\mathcal{F})$ does not vanish if and only if all entries of the vector $(\alpha_1,...,\alpha_n) = (n+\gamma_1, n-1+\gamma_2,...,1+\gamma_n)$ are distinct and p is the number of pairs $(i<j)$ such that $\alpha_i < \alpha_j$.*

3.

Let X and $\mathcal{F}$ be as in the introduction. Since X has positive dimension, i.e. $rk(\mathcal{F}) < k(n-k)$ and (2.1) we may assume that $\mathcal{F}$ comes from a representation of the A-part of P_k, which means $\beta_{k+1} = ... = \beta_n = 0$ for the corresponding highest weight. Hence we will write $\beta = (\beta_1,...,\beta_k)$ for this weight, and $|\beta| = \beta_1 + ... + \beta_k$. Then by symmetry the weight of $det(\mathcal{F})$ is $rk(\beta)\cdot|\beta|/k$ times the weight $(1,...,1)$ which corresponds to $\bigwedge^k \mathcal{Q}$ defining the Plücker embedding. Therefore, using the adjunction formula, the condition of X being Fano turns out to be

$$n > \frac{rk(\beta)\cdot|\beta|}{k} \tag{3.1}$$

which is usually stronger than $\dim(X) > 0$ which in turn reads

$$n > \frac{rk(\beta)}{k} + k. \tag{3.2}$$

Note that (3.1) shows in particular that the entries of the weights of arbitrary wedge products of $\mathcal{F}$ are smaller than n.

(3.3) Remark. Our results are well known for hypersurfaces in projective space, so we may assume $k, n-k \geq 2$. Moreover we assume that β is not one of the following (cf. [**Kü**]):

(i) $(1,0,...,0)$, since $X = Gr(k,n-1)$ in this case.

(ii) $(2,0,...,0)$, $2k \leq n$, since X parametrizes k-dimensional subspaces on (affine) $(n-1)$-dimensional quadrics in this case. Then X is itself a rational homogeneous manifold (resp. two disjoint copies of those if $2k = n$), hence known to be rigid and projectively normal.

(iii) $(2,0,...,0)$ and $(1,1,0,...,0)$ for $2k > n$, since $X = \emptyset$ is these cases.

4.

Now we prove Theorem 1. By (1.2) and (1.4) it suffices to show

$$H^p(Gr(k,n),(\textstyle\bigwedge^p \mathcal{F}^*)(r)) = 0 \quad \forall \quad p>0, r>0.$$

We will prove a slightly stronger result, namely the vanishing for $p > 0$ and $r \geq 0$, which also implies the connectivity of X. Suppose one of these groups does not vanish and let β be the highest weight of $\mathcal{F}$. Note that $(\bigwedge^p \mathcal{F}^*)(r)$ is fully reducible and apply Bott's theorem. The proof relies on the elementary observation that, since there is no space "between" the last $n-k$ entries, p jumps in steps of $n-k$ and there has to be a $(n-k)$-jump between the entries of a weight of $\bigwedge^p \mathcal{F}$. The latter forces β to have a jump which makes the rank big. But the rank in turn is bounded in terms of n which yields a contradiction.

More formally, there has to be a weight $(b_1, ..., b_k)$ of $\bigwedge^p \mathcal{F}$ and a positive integer $s < k$ such that

$$\text{(4.1)} \qquad \begin{cases} p = s(n-k) \\ n-1 \geq b_1 \geq ... \geq b_s \geq n-k+r+s \\ 0 \leq b_k \leq ... \leq b_{s+1} \leq r+s. \end{cases}$$

Moreover, starting with $s(n-k)|\beta| = b_1 + ... + b_k$, (4.1) together with (3.1) implies a condition on the rank of β:

$$\text{(4.2)} \qquad rk(\beta)|\beta| < \frac{k^2(k-1)}{s(|\beta|-1)} + k^2.$$

Beginning with some special cases we show that such bundles do not exist.
(4.3) Using (3.3) we may assume that $\beta \neq (1,0,...,0)$, $(2,0,...,0)$, $(1,...,1,0)$ or $(t,...,t)$.
(4.4) Suppose $k = 2$. Then $s = 1$, $r = 0$, and $b_2 \leq 1$, so $b_1 + b_2 \leq n$, but $p = n-2$, hence $\beta = (1,0)$, $(1,1)$, or $(2,0)$.

Suppose $k = 3$. If $s = 1$ then $r \leq 1$, $b_2 + b_3 \leq 4$. By $|\beta|(n-3) \leq n+3$, we know $|\beta| \leq 4$, and $|\beta| \geq 3$ by (4.3). If $|\beta| = 3$, then $n \leq 6$ and $\beta = (3,0,0)$ or $(2,1,0)$ contradicting $n > rk(\beta)$. If $|\beta| = 4$, then $n = 5 > 4/3 \cdot rk(\beta)$ shows $\beta = (2,1,1)$.

If $s = 2$, then $r = 0$, so $b_1 + b_2 + b_3 = (2n-6)|\beta| \leq 2n$, but $|\beta| \geq 3$, hence $n \leq 4$.

So we may assume $k \geq 4$.
(4.5) Using (4.4), we conclude $|\beta| \leq k$, in particular $\beta_k = 0$.
(4.6) We may assume $\beta \neq (c,0,...,0)$, since, for $c \geq 3$,

$$rk(c,0,...,0) = \binom{k+c-1}{c} \geq \frac{k^2(k-1)}{c(c-1)} + \frac{k^2}{c}.$$

Suppose $\beta = (1,1,0,...,0)$. By (3.3iii), $n-k \geq k$, but $b_i \leq k-1$ for the entries of wedge products of β.

Suppose $\beta = (2,1,0,...,0)$. Then $3 \cdot rk(\beta) = k(k^2-1) \geq k^2(k-1)/2 + k^2$. For $\beta = (2,2,0,...,0)$, we have $4 \cdot rk(\beta) = k^2(k^2-1)/3 \geq k^2(k-1)/3 + k^2$ since $k \geq 4$. Moreover $rk(t,t,0,..,0) \geq rk(2,2,0,...,0)$ for $t \geq 2$ and $rk(v,w,0,...,0) \geq rk(2,1,0,...,0)$ for $v > w > 0$.

Hence, by (4.2) we may assume $\beta_3 \neq 0$.

(4.7) In the same way it is shown that we may assume $\beta_{k-2} = 0$ and $|\beta| \geq 5$.

(4.8) We are left with $k \geq 6$, $|\beta| \geq 5$ and $rk(\beta) \geq \binom{k}{3}$, which gives a contradiction to (4.2).

This completes the proof of Theorem 1.

(4.9) Corollary. *Under the assumptions of Theorem* 1 X *is connected or* $2k = n$ *and* $\mathcal{F} \simeq S^2\mathcal{Q}$.

Considering a diagram similar to (1.1) and using Kodaira vanishing, we obtain

(4.10) Corollary. *For* X *to be projectively normal it is sufficient to assume that* $\mathcal{F}$ *is the sum of one irreducible homogeneous vector bundle and line bundles.*

The following example shows that one has to impose a non-positivity condition on the canonical bundle.

(4.11) Example. Consider the surface S parametrizing lines on the cubic 3-fold, in this context originally studied by Wehler [**We**]. S is the variety of zeros of a section in $S^3\mathcal{Q}$ over $Gr(2,5)$. Then one can show $H^3(Gr(2,5), \bigwedge^3(S^3\mathcal{F})^*(2)) = 1$ and $H^1(Gr(2,5), \mathcal{J}_S(2)) = 1$, i.e., S is not quadratically normal with respect to $\mathcal{O}_S(1) = \mathcal{O}_S(K_S)$.

5.

The proof of Theorem 2 is similar. By (1.3) and (1.4) it suffices to show

$$H^p(Gr(k,n), \mathcal{F} \otimes \bigwedge^p \mathcal{F}^*) = 0 \quad \forall \quad p \geq 1 \quad \text{and} \tag{5.1.a}$$

$$H^{p+1}(Gr(k,n), \Theta_{Gr(k,n)} \otimes \bigwedge^p \mathcal{F}^*) = 0 \quad \forall \quad p \geq 0. \tag{5.1.b}$$

If (5.1a) does not hold, then there exists an integer $0 < s < k$ and a weight $(a_1,...,a_k)$ of $\mathcal{F}^* \otimes \bigwedge^{s(n-k)} \mathcal{F}$ such that

$$\text{(a)} \qquad \begin{cases} n-1 \geq a_1 \geq ... \geq a_s \geq n-k+s \\ \\ \qquad a_k \leq ... \leq a_{s+1} \leq s. \end{cases}$$

Therefore we get

$$n \leq \frac{1}{s(|\beta|-1)}\big(s(k-s) + |\beta|(sk+1) - s + 1\big), \tag{5.2}$$

where β as always is the highest weight of $\mathcal{F}$. Concerning violations of (5.1.b) the output of Bott's Theorem is more subtle since the weight of $\Theta_{Gr(k,n)}$ is involved. Namely, if (5.1.b) does not hold, then there exists an integer $0 < s < k$ and a weight $(b_1, ..., b_k)$ of $\bigwedge^p \mathcal{F}$ such that either

$$\text{(b)} \qquad \begin{cases} p = s(n-k) - 1 \\ n-1 \geq b_1 \geq ... \geq b_s \geq n-k+s+1 \\ b_k \leq ... \leq b_{s+2} \leq s, \ b_{s+1} \leq s+1, \end{cases}$$

or

$$\text{(b')} \qquad \begin{cases} p = s(n-k) - 2 \\ n-1 \geq b_1 \geq ... \geq b_{s-1} \geq n-k+s \\ n-k+s-1 \leq b_s \leq n-k+s \\ b_k \leq ... \leq b_{s+2} \leq s, \ b_{s+1} \leq s+1. \end{cases}$$

Note that β is not of type $(1, 1, 0, ..., 0)$, since $b_1 \leq n-k-1$ in this case (cf. (3.3iii)). So by (3.3) we may assume $|\beta| \geq 3$ in the following. Now (b) again implies (5.2), whereas from (b') we infer

$$\begin{aligned} n &\leq \tfrac{1}{s(|\beta|-1)}\big(s(k-s) + 2 - k + |\beta|sk + 2|\beta|\big) \\ &\leq \begin{cases} k + 2 + \frac{k+3}{|\beta|-1}, & s = 1 \\ 2k \quad \text{for} \quad s \geq 2. \end{cases} \end{aligned} \tag{5.3}$$

By considering increasing values of $|\beta|$ and using (3.1) we see that the situation $s = 1$ and $2k + 1 \leq n \leq k + 2 + \frac{k+3}{|\beta|-1}$ can be excluded.

Hence, by (5.2) and (5.3) it remains to consider the case $n \leq 2k$. We do this by listing all possible highest weights β and checking case by case that (a), (b) and (b') can not be satisfied.

(5.4) Lemma. *Suppose* $|\beta| \geq 3$ *and*

$$\frac{rk(\beta) \cdot |\beta|}{k} < n \leq 2k.$$

Then $\beta = (\beta_1, ..., \beta_k)$ *is one of the following:*

(i) $(t, ..., t), t \leq n-1$ (ii) $(2, 1, ..., 1)$
(iii) $(2, ..., 2, 1), n = 2k$ (iv) $(1, ..., 1, 0)$
(v) $(1, 1, 1, 0, 0), 7 \leq n \leq 10$ (vi) $(1, 1, 1, 0, 0, 0), n = 11, 12$
(vii) $(1, 1, 1, 1, 0, 0), n = 11, 12.$

The Lemma's proof is obvious.

The exclusion of (i) is immediate and (vii) violates (5.2) and (5.3) since $s \geq 2$ in this case. The remaining types are most economically dealt with by determining the highest weights of the irreducible summands of the relevant representations and comparing these to (a), (b) and (b').

This completes the proof of Theorem 2.

(5.5) Remark. As an illustration of the well known fact that K3 surfaces have nonalgebraic small deformations we have nonvanishing in (5.1.b) for a quartic in IP^3, i.e. $\mathcal{F} = S^4\mathcal{Q}$ over $Gr(1, 4)$.

6.

To prove Theorem 3 note that Fano 4-folds with $\mathrm{Pic}(X) \simeq \mathbb{Z} \cdot K_X$ arising as zeros of sections in fully reducible homogeneous vector bundles over Grassmannians have been classified in [**Kü**]. There are only few cases that are not covered by Theorems 1 and 2, and verifying the assertions in the above way poses no further problems.

7.

Finally we ask some questions arising in this context.

(7.1) Question. Are there examples of Fano-manifolds V $(\dim(V) \geq 4)$ with very ample $-K_V$ spanning $\mathrm{Pic}(V)$ which are not projectively normal?

(7.2) Question. Do Theorems 1 and 2 hold for fully reducible $\mathcal{F}$?

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