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HADAMARD-FRANKEL TYPE THEOREMS FOR MANIFOLDS WITH PARTIALLY POSITIVE CURVATURE

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In this paper we prove some theorems that two minimal submanifolds satisfying a condition for the dimensions of the submanifolds in a Riemannian manifolds with partially positive curvature or a Kaehler manifold with partially positive holomorphic sectional curvature must intersect. Our results show that the famous Frankel theorem about intersections of minimal submanifolds in a manifold with positive curvature is generalized to the very wide class of manifolds with partially positive curvature.

1. Introduction.

In 1966, Frankel [**F2**] showed that if N is an n-dimensional complete connected Riemannian manifold with strictly positive sectional curvature and if V is an r-dimensional compact totally geodesic immersed submanifold of N with $2r \geq n$, then the homomorphism of fundamental groups : $\pi_1(V) \to \pi_1(N)$ is surjective. This important theorem follows from an earlier result proved by himself in [**F1**] : Two compact totally geodesic submanifolds P and Q in a Riemannian manifold N of positive sectional curvature must necessarily intersect if their dimension sum is at least that of N. Unfortunately, the set of manifolds with positive sectional curvature is not so big. We don't even know whether the product of two 2-spheres $S^2 \times S^2$ admits a metric with positive sectional curvature or not.

Frankel also showed in [**F1**] that if N is a complete connected Kaehler manifold with positive sectional curvature, then any two compact analytic submanifolds must intersect if their dimension sum is at least that of N. Goldberg and Kobayashi proved in [**GK**] that the same conclusions also hold if N is only assumed to have positive bisectional curvature. We remark that since a complete connected Kaehler manifold of positive bisectional curvature which contains a compact Kaehler submanifold is compact (see Theorem 3.1 below), the above N is actually compact, thus it is biholomorphic to a complex projective space by the settlement of Frankel's conjecture (see [**M**] and [**SY**]). The topology of N is therefore very simple.

The purpose of the present paper is to consider the situation where N can only have either partially positive sectional curvature or partially positive bisectional curvature. Such manifolds cover many known examples. We generalize Frankel's theorems quoted above to this class of manifolds and therefore give some topological obstructions for the existence of higher dimensional totally geodesic submanifolds in a Riemannian manifold with partially positive sectional curvature. Also, we prove an intersection theorem for totally geodesic submanifolds and Kaehler submanifolds in a Kaehler manifold with partially positive holomorphic bisectional curvature.

2. Riemannian manifolds with partially positive sectional curvature.

We recall the definition of Riemannian manifolds with partially positive curvature (cf. [**W2**]). Let N^n be an n-dimensional Riemannian manifold with metric $\langle , \rangle$ and $p \in N^n$ be a point of N^n. If, for any $(k+1)$ mutually orthogonal unit tangent vectors $e, e_1, \cdots, e_k \in T_pN^n$, we have $\Sigma_{i=1}^k K(e \wedge e_i) > 0$(resp., ≥ 0), we say N^n has k-positive (resp., k-nonnegative) Ricci curvature at p. If N^n has k-positive (resp., k-nonnegative)-Ricci curvature at every point of it, we call N^n has k-positive (resp., $k-$nonnegative)-Ricci curvature and denote this fact by $\mathrm{Ric}_{(k)}(N^n) > 0$ (resp., ≥ 0). Here $K(e \wedge e_i)$ denotes the sectional curvature of the plane spanned by e and $e_i (1 \leq i \leq k)$. Thus, 1-positive (1-nonnegative)-Ricci curvature is equivalent to positive (nonnegative) sectional curvature and $(n-1)$-positive ($(n-1)$-nonnegative)-Ricci curvature is equivalent to positive (nonnegative) Ricci curvature. Compact locally symmetric spaces of rank ≥ 2 are examples of manifolds with positive $k(> 1)$-Ricci curvature. Slight purturbations of these metrics give non-symmetric examples. Further examples of manifolds of positive k-Ricci curvature can be found among the compact homogeneous spaces with a bi-invariant metric (cf. [**B**]).

Our first result in this paper generalizes the Frankel's intersection theorem for totally geodesic submanifolds in a manifold of positive curvature to manifolds with positive k-Ricci curvature.

Theorem 2.1. *Let N be an n-dimensional complete connected Riemannian manifold with $k-$nonnegative Ricci curvature and let V and W be two complete immersed totally geodesic submanifolds of dimensions r and s, respectively, each immersed as a closed subset, and let one of V and W be compact. Assume N has $k-$positive Ricci curvature either at all points of V or at all points of W. If $r + s \geq n + k - 1$, then V and W must intersect.*

Proof. We can assume that V and W are embedded, otherwise the proof given will then hold using any "sheet" of the immersion. If $r < k$, we have

$s = n$ by the assumption on the dimensions. We know that $W = N$ and we have $V \cap W \neq \emptyset$. Therefore we may assume that $r \geq k$. Suppose then that V and W do not intersect. Let $\gamma : [0, \ell] \to N$ be a normal geodesic from $p \in V$ to $q \in W$ that realizes the minimum distance between these submanifolds (since both V and W are closed subsets and one of them is compact, such γ exists at least one). An argument using the first variation formula of arc-length shows that γ strikes both V and W orthogonally. Now, take a unit orthonormal basis $e_1, \cdots, e_r$ of T_pV. Parallel translating them along γ gives rise to r mutually orthogonal unit vector fields $E_1, \cdots, E_r$ along γ. From $\dim V + \dim W \geq \dim N + k - 1$, we know that at least k of the vector fields $E_1, \cdots, E_r$ are tangent to W at q. Without loss of generality, we may assume that $E_1, \cdots, E_k$ are tangent to W at q. Each vector field $E_i (i = 1, \cdots, k)$ gives rise to a variation with lengh $L_{E_i}(u)$ of the variational curves of the geodesic γ keeping endpoints on V and W. The first variation of arc-length $L'_{E_i}(0)$ is 0. By the second variation formula of arc-length [**K**, p. 99], we find, for $i = 1, \cdots, k$, that

$$\begin{aligned} (2.1) \qquad L''_{E_i}(0) &= \langle \sigma_W(E_i(q), E_i(q)), \gamma'(\ell) \rangle - \langle \sigma_V(e_i, e_i), \gamma'(0) \rangle \\ &\quad - \int_0^\ell K(\gamma'(t) \wedge E_i(t)) dt, \end{aligned}$$

where σ_V and σ_W denote the second fundamental forms of V and W, respectively. Since both V and W are totally geodesic, we have that $\sigma_V = \sigma_W = 0$. Also, it follows from $\mathrm{Ric}_{(k)}(N) \geq 0$ that

$$\sum_{i=1}^{k} K(\gamma'(t) \wedge E_i(t)) \geq 0.$$

Moreover, since N has positive k-th Ricci curvature either at all points of V or at all points of W, we know that either

$$\sum_{i=1}^{k} K(\gamma'(0) \wedge E_i(0)) > 0,$$

or

$$\sum_{i=1}^{k} K(\gamma'(\ell) \wedge E_i(\ell)) > 0.$$

Substituting these formulas into (2.1), we get

$$\sum_{i=1}^{k} L''_{E_i}(0) = - \int_0^\ell \sum_{i=1}^{k} K(\gamma'(t) \wedge E_i(t)) dt < 0.$$

Thus we know that $L''_{E_i}(0) < 0$, for some i, which contradicts to the assumption that γ is of minimal length from V to W. Hence V and W must intersect. This completes the proof of Theorem 2.1. $\square$

Theorem 2.2. *Let N be an n-dimensional complete, connected Riemannian manifold with $k-$positive Ricci curvature and V be an r-dimensional totally geodesic submanifold with $2r \geq n + k - 1$, then the homomorphism of the fundamental groups: $\pi_1(V) \to \pi_1(N)$ is surjective.*

Proof. We note that $k \leq n - 1$ and the condition $2r \geq n + k - 1$ implies that $r \geq k$. It then follows from Theorem 1 in [**G**] that N is compact. From $\mathrm{Ric}_{(k)}(N) > 0$, we conclude that the Ricci curvature of N is positive, because we have that

$$\sum_{i=1}^{n-1} K(e \wedge e_i) = \frac{n-1}{kC_{n-1}^k} \sum_{1 \leq i_1 < \ldots < i_k \leq n-1} \sum_{j=1}^{k} K(e \wedge e_{i_j}).$$

Hence $\pi_1(N)$ is finite by Bonnet-Myers Theorem [**CE**]. Also, it follows from Theorem 2.1 that V is connected. Let $\widetilde{N}$ be the universal covering manifold of N. $\widetilde{N}$ has again $k-$positive Ricci curvature for the lifted metric. Denote by $\widetilde{V} = \pi^{-1}(V)$ the inverse image of V under the projection map $\pi : \widetilde{N} \to N$. Obviously, $\widetilde{V}$ is a compact totally geodesic submanifold (may not connected). By Theorem 1 all components of $\widetilde{V}$ must intersect and hence $\widetilde{V}$ is indeed connected. Thus $\widetilde{V}$ is a covering space of V. It then follows from the same arguments as in the proof of the Main Theorem in [**F2**] that the homomorphism of the fundamental groups : $\pi_1(V) \to \pi_1(N)$ is surjective. $\square$

The following Corollary is an immediate consequnce of Theorem 2.2 and it strengthens the first result by Frankel stated in Introduction in the case that $\dim N$ is odd.

Corollary 1. *Let N be a complete connected odd dimensional Riemannian manifold with $\mathrm{Ric}_{(2)}(N) > 0$ and let V be a compact totally geodesic submanifold with $2 \dim V \geq \dim N$. Then the homomorphism of the fundamental groups : $\pi_1(V) \to \pi_1(N)$ is surjective.*

3. Kaehler manifolds with partially positive bisectional curvature.

Let N be a Kaehler manifold of complex dimension n. Following [**W1**], we denote by J, G and R the complex structure, the (complex valued) Kaehler metric tensor and the curvature tensor of the Riemannian metric which is the real part of G. This Riemannian metric will always be denoted by $\langle\ ,\rangle$

and its associated norm by $|\ .\ |$. If X and Y are vectors in T_xN, $x \in N$, denote by $H(X,Y)$ the bisectional curvature determined by X and Y, that is, we have that

$$H(X,Y) = \begin{cases} 0 & \text{if } X \text{ or } Y = 0, \\ \frac{R(X,JX,Y,JY)}{|X|^2|Y|^2} & \text{if } X \neq 0 \text{ and } Y \neq 0. \end{cases}$$

It is easy to see that if $\mathrm{span}_R\{X, JX\} = \mathrm{span}_R\{X', JX'\}$ and $\mathrm{span}_R\{Y, JY\} = \mathrm{span}_R\{Y', JY'\}$, then we have

$$H(X,Y) = H(X',Y').$$

Also, by Bianchi's identity, we know that

$$H(X,Y) = \frac{1}{|X|^2|Y|^2}\{R(X,Y,X,Y) + R(X,JY,X,JY)\}.$$

The Kaehler manifold N is said to have q-positive (resp., nonnegative) bisectional curvature ($1 \leq q \leq n$) at a point $x \in N$ if, for every $2q$ orthonormal vectors $\{e_1, Je_1, \cdots, e_q, Je_q\}$ of T_xN, $\Sigma_{i=1}^q H(X, e_i) > 0$(resp., ≥ 0) for all unit vectors $X \in T_pN$. N is said to have q-positive (resp., nonnegative) bisectional curvature if it has q-positive (resp., nonnegative) bisectional curvature at every point of it. Note that 1-positive (resp., nonnegative) bisectional curvature is the same as positive (resp., nonnegative) bisectional curvature in the usual sense, and n-positive (resp., nonnegative) bisectional curvature is the same as positive (resp., nonnegative) Ricci curvature.

The following result is a counterpart in Kaehler geometry of Theorem 1 in [**G**].

Theorem 3.1. *Let N be a complete connected Kaehler manifold of complex dimension $n \geq 2$. Let M be a compact immersed Kaehler submanifold of complex dimension $r \geq 1$. If, for any $x \in M$ and any orthonormal basis of the form $\{e_1, Je_1, \cdots, e_r, Je_r\}$ of T_xM, we have*

$$\liminf_{t\to\infty} \int_0^t \left\{ \sum_{i=1}^r H(\gamma'(s), E_i(s)) \right\} ds > 0, \tag{3.1}$$

then N is necessarily compact, where $\gamma : [0, \infty) \to N$ is the geodesic which starts from x and is orthogonal to M at x and E_i denotes the vector field obtained by the parallel translation of e_i along γ for $i = 1, ..., r$.

Proof. Suppose N is not compact. By the same argument as in the proof of Theorem 1 in [**G**], we can find a point $x \in M$ and a geodesic $\gamma : [0, \infty) \to N$

issuing orthogonally from M at x such that the length of the segment $\gamma|_{[0,t]}$ realizes the distance from $\gamma(t)$ to M, and so γ has no focal point.

Now, take an orthonormal basis $\{e_1, Je_1, \cdots, e_r, Je_r\}$ of T_xM and denote by $E_1(t)$, $JE_1(t)$, $\cdots$, $E_r(t)$, $JE_r(t)$ the vector fields obtained by the parallel translations of them along γ. Define $K(t) = \Sigma_{i=1}^r H(\gamma'(t), E_i(t))/2r$. The condition (3.1) is equivalent to say that we have

$$\liminf_{t\to\infty} \int_0^t K(s)\mathrm{d}s > 0. \tag{3.2}$$

It then follows from [**T**] that the scalar Jacobi equation defined by

$$f'' + K(t)f = 0 \tag{3.3}$$

has a solution on $[0,\infty)$ which satisfies the following initial conditions

$$f(0) = 1, \quad f'(0) = 0 \tag{3.4}$$

and which has at least one zero in $[0,\infty)$. Let $\phi : [0,\infty) \to R$ be such a solution and we can assume $\phi(t_0) = 0$ for some $t_0 > 0$.

Let C be the collection of nonzero smooth vector fields X along $\gamma|_{[0,t_0]}$ which are perpendicular to γ, tangent to M at $\gamma(0)$, and which satisfy $X(t_0) = 0, X'(0) = 0$. We introduce the index form I of the geodesic γ along $\gamma|_{[0,t_0]}$ as follows: for $X, Y \in C$ we have (see [**BC**, p. 221]),

$$I(X,Y) = \langle \sigma_M(X(0), Y(0)), \gamma'(0)\rangle - \int_0^{t_0} \langle X'' + R(X,\gamma')\gamma', Y\rangle \mathrm{d}t, \tag{3.5}$$

where σ_M is the second fundamental form of M in N.

For each $i = 1, \cdots, r$, define vector fields $X_i(t)$ and $Y_i(t)$ along $\gamma|_{[0,t_0]}$ by

$$X_i(t) = \phi(t)E_i(t) \quad \text{and} \quad Y_i(t) = \phi(t)JE_i(t). \tag{3.6}$$

Note that each X_i and Y_i are elements of C. Substituting (3.6) into (3.5) gives, for $i = 1, \cdots, r$,

$$\begin{aligned} I(X_i, X_i) &= \langle \sigma_M(e_i, e_i), \gamma'(0)\rangle \\ &\quad - \int_0^{t_0} (\phi''(t) + K(\gamma'(t) \wedge E_i(t))\phi(t))\phi(t)\mathrm{d}t \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} I(Y_i, Y_i) &= \langle \sigma_M(Je_i, Je_i), \gamma'(0)\rangle \\ &\quad - \int_0^{t_0} (\phi''(t) + K(\gamma'(t) \wedge JE_i(t))\phi(t))\phi(t)\mathrm{d}t. \end{aligned} \tag{3.8}$$

Since M is a Kaehler submanifold of N, we have that $\sigma_M(Je_i, Je_i) = -\sigma_M(e_i, e_i)$. From the definition of bisectional curvature, we know that

$$K(\gamma' \wedge E_i) + K(\gamma' \wedge JE_i) = H(\gamma', E_i).$$

Thus, we have, from (3.3), (3.7) and (3.8), that

$$\sum_{i=1}^{r}\{I(X_i, X_i) + I(Y_i, Y_i)\} = -2r\int_0^{t_0}(\phi''(t) + K(t)\phi(t))\phi(t)\mathrm{dt} = 0.$$

Therefore, we have either $I(X_i, X_i) \leq 0$, or $I(Y_i, Y_i) \leq 0$ for some i. However, by the standard results of the index form (see [**BC**, p. 228]) we must have, for any $X \in C$, $I(X, X) > 0$ unless there is some point on $\gamma|_{[0,t_0]}$ which is a focal point to M along $\gamma|_{[0,t_0]}$. This is a contradiction. The proof of Theorem 3.1 is thus completed. □

Theorem 3.2. *Let N be a complete connected Kaehler manifold with k-nonnegative bisectional curvature of complex dimension n. Let V and W be two compact complex analytic submanifolds in N of complex dimension r and s, respectively. Assume N has k-positive bisectional curvature either at all points of V or at all points of W. If $r + s \geq n + k - 1$, then V and W must intersect.*

Proof. Suppose that V and W do not intersect. Let $\gamma : [0, \ell] \to N$ be a normal geodesic from $p_0 \in V$ to $q_0 \in W$ that realizes the minimum distance between them. Since V is a Kaehler submanifold of N, we can choose an orthonormal basis $e_1, \cdots, e_r, e_{r+1}, \cdots, e_{2r}$ of $T_{p_0}V$ such that $e_{r+i} = Je_i (i = 1, \cdots, r)$. Parallel translating $e_1, Je_1, \cdots, e_r, Je_r$ along γ gives rise to $2r$ orthonormal vector fields $E_1, JE_1, \cdots, E_r$ and JE_r along the geodesic. From $\dim_R V + \dim_R W \geq \dim_R N + 2(k-1)$, at least $2k-1$ vector fields of $E_1, JE_1, \cdots, E_r$ and JE_r are tangent to W at q_0. We arrange them in the following form: $E_{i_1}, \cdots, E_{i_p}$, $JE_{j_1}, \cdots$, JE_{j_q}, where we have $i_1 \neq \cdots \neq i_p$, $j_1 \neq \cdots \neq j_q \in \{1, \cdots, r\}$ and $p + q = 2k - 1$. Thus one of p and q is greater than or equal to k. If $p \geq k$, then $E_{i_1}, \cdots, E_{i_k}$ are tangent to W at q_0. Since W is a Kaehler submanifold, $JE_{i_1}, \cdots, JE_{\imath_k}$ are also tangent to W at q_0. Similarily, if $q \geq k$, then $JE_{j_1}, \cdots, JE_{j_k}$ are tangent to W at q_0, and so $J(JE_{j_1}) = -E_{J_1}, \cdots, J(JE_{J_k}) = -E_{J_k}$ are also tangent to V at q_0. Thus $E_{j_1}, \cdots, E_{j_k}$ are also tangent to V at q_0. In either case, we can find some indices $t_1 \neq \cdots \neq t_k \in \{1, \cdots, r\}$ such that E_{t_1}, $JE_{t_1}, \cdots E_{t_k}$ and JE_{t_k} are tangent to V at q_0. Without loss of generality, we may assume $E_1, JE_1, \cdots, E_k$ and JE_k are tangent to V at q_0. The vector fields $E_1, JE_1, \cdots, E_k$ and JE_k give rise to $2k$ variations of the geodesic γ keeping endpoints on V and W. We see that the first variation of arc-length $L'_{E_i}(0) = L'_{JE_\imath}(0) = 0$ for $i = 1, \cdots, k$.

Using the second variation formula of arc-length, we get, for $i = 1, \cdots, k$, that

$$L''_{E_i}(0) = \langle \sigma_W(E_i(q_0), E_i(q_0)), \gamma'(\ell)\rangle - \langle \sigma_V(e_i, e_i), \gamma'(0)\rangle \\ - \int_0^\ell K(E_i \wedge \gamma')\mathrm{dt},$$

and

$$L''_{JE_i}(0) = \langle \sigma_W(JE_i(q_0), JE_i(q_0)), \gamma'(\ell)\rangle - \langle \sigma_V(Je_i, Je_i), \gamma'(0)\rangle \\ - \int_0^\ell K(JE_i \wedge \gamma')\mathrm{dt},$$

where σ_V and σ_W denote the second fundamental forms of V and W, respectively. Since V and W are Kaehler submanifolds of N, we know, for $i = 1, \cdots, k$, that

$$\sigma_V(Je_i, Je_i) = -\sigma_V(e_i, e_i), \quad \sigma_W(JE_i(q_0), JE_i(q_0)) = -\sigma_W(E_i(q_0), E_i(q_0)).$$

Hence, we have that

$$\sum_{i=1}^k (L''_{E_i}(0) + L''_{JE_i}(0)) = -\int_0^\ell \sum_{i=1}^k \{K(E_i(t) \wedge \gamma'(t)) + K(JE_i(t) \wedge \gamma'(t))\}\mathrm{dt}$$

$$= -\int_0^\ell \sum_{i=1}^k H(\gamma'(t), E_i(t))\mathrm{dt}. \tag{3.9}$$

From the assumption that N has k-nonnegative bisectional curvature and k-positive bisectional curvature either at all points of V or at all points of W, we see that

$$\sum_{i=1}^k H(\gamma'(t), E_i(t)) \geq 0$$

for any $t \in [0, \ell]$, and

$$\max\left\{\sum_{i=1}^k H(\gamma'(0), E_i(0)), \sum_{i=1}^k H(\gamma'(\ell), E_i(\ell))\right\} > 0.$$

Substituting these formulas into (3.9), we get that

$$\sum_{i=1}^k (L''_{E_i}(0) + L''_{JE_i}(0)) < 0.$$

Hence the second variation corresponding to at least one of the vector fields $E_1, JE_1, \cdots, E_k$ and JE_k is strictly negative, contradicting to the assumption that γ is of minimal length from V to W. Thus V and W must intersect. This completes the proof of Theorem 3.2. $\square$

Theorem 3.3. *Let $N^{2n}(n \geq 2)$ be a complete connected Kaehler manifold with k-nonnegative bisectional curvature of real dimension $2n$. Let W^{2r} and V^t be a complete immersed complex analytic submanifold of real dimension $2r$ and a $t(\geq 2n-r+k-1)$-dimensional complete immersed totally geodesic submanifold, respectively, each immersed as a closed subset, and let one of W^{2r} and V^t be compact. Assume N^{2n} has k-positive bisectional curvature either at all points of W^{2r} or at all points of V^t. Then W^{2r} and V^t must intersect.*

Proof. The theorem and proof are variations of a situation dealt with in a previous result. Suppose then that W^{2r} and V^t do not intersect. Let $\gamma : [0, \ell] \to N^{2n}$ be a normal geodesic from $p_0 \in W^{2r}$ to $q_0 \in V^t$ that realizes the minimum distance between them. We choose an orthonormal basis $\{e_1, \cdots, e_r, e_{r+1}, \cdots, e_{2r}\}$ of $T_{p_0}W^{2r}$ such that $e_{r+i} = Je_i (i = 1, \cdots, r)$. Parallel translating them along γ gives rise to $2r$ orthonomal vector fields $E_1, JE_1, \cdots, E_r$ and JE_r along the geodesic. From $\dim_R W^{2r} + \dim V^t \geq 2n+k+r-1$, we know that at least $r+k$ of the vector fields $E_1, JE_1, \cdots, E_r$ and JE_r are tangent to V^t at q_0. Thus there exist some indices $i_1 \neq \cdots \neq i_k \in \{1, \cdots, r\}$ such that $E_{i_1}, JE_{i_1}, \cdots, E_{i_k}$ and JE_{i_k} are tangent to V^t at q_0. The vector fields $E_{i_1}, JE_{i_1}, \cdots, E_{i_k}$ and JE_{i_k} give rise to $2k$ variations of the geodesic γ keeping endpoints on W^{2r} and V^t. By the first variation of arc-length, we have that $L'_{E_{i_j}}(0) = L'_{JE_{i_j}}(0) = 0$, $j = 1, \cdots, k$, and using the second variation formula of arc-length, we get, for $j = 1, \cdots, k$, that

$$L''_{E_{i_j}}(0) = \langle \sigma_{V^t}(E_{i_j}(q_0), E_{i_j}(q_0)), \gamma'(\ell) \rangle - \langle \sigma_{W^{2r}}(e_{i_j}, e_{i_j}), \gamma'(0) \rangle \\ - \int_0^\ell K(E_{i_j} \wedge \gamma')\mathrm{dt},$$

and

$$L''_{JE_{i_j}}(0) = \langle \sigma_{V^t}(JE_{i_j}(q_0), JE_{i_j}(q_0)), \gamma'(\ell) \rangle - \langle \sigma_{W^{2r}}(Je_{i_j}, Je_{i_j}), \gamma'(0) \rangle \\ - \int_0^\ell K(JE_{i_j} \wedge \gamma')\mathrm{dt},$$

where $\sigma_{W^{2r}}$ and σ_{V^t} denote the second fundamental forms of W^{2r} and V^t in N^{2n}, respectively. Since V^t is totally geodesic and W^{2r} is a Kaehler submanifold, we have that

$$\sigma_{V^t} = 0 \text{ and } \sigma_{W^{2r}}(Je_{i_j}, Je_{i_j}) = -\sigma_{W^{2r}}(e_{i_j}, e_{i_j}), \text{ for } j = 1, .., k.$$

Hence, we have that

$$\begin{aligned}\sum_{j=1}^{k}\left(L''_{E_{i_j}}(0) + L''_{JE_{i_j}}(0)\right) &= -\int_0^{\ell}\sum_{j=1}^{k}\{K(E_{i_j}\wedge\gamma') + K(JE_{i_j}\wedge\gamma')\}\,\mathrm{dt} \\ &= \int_0^{\ell}\left\{\sum_{j=1}^{k} H(E_{i_j},\gamma')\right\}\mathrm{dt} \\ &< 0,\end{aligned}$$

where the last inequality follows from the assumption on the bisectional curvature of N^{2n}. Therefore the second variation corresponding to at least one of the vector fields $E_{i_1}, JE_{i_1}, \cdots, E_{i_k}$ and JE_{i_k} is negative, contradicting to the assumption that γ is of minimal length from W^{2r} to V^t. Thus W^{2r} and V^t must intersect. $\square$

Remark. In a previous paper [**KX**], we proved a special case of Theorem 3.3, that is, $k = 1$.

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