DEGREE–ONE MAPS ONTO LENS SPACES

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The paper deals with the question about the existence or non-existence of a degree-one map of a closed orientable 3-manifold M to some lens space. The answer to this question is determined by the cyclic decomposition of $H_1(M)$, except when $H_1(M)$ contains an even number of direct factors isomorphic to $\mathbf{Z}_{2^k}$. In this case one has to calculate the linking matrix of M to get the answer. For every n even, we give a Seifert manifold M_n with $H_1(M_n) \cong \mathbf{Z}_n \oplus \mathbf{Z}_n$ that does not admit a degree-one map to $L(n, m)$ for any m.

1. Introduction.

Motivated by articles by Y. Rong [10] also Y. Rong and S. Wang [11], we consider the question whether a given 3-manifold M admits a degree-one map to a lens space $L(n, m)$ or not. For every n a geometric criterium is given (Theorem 2.2) by the linking pairing $\operatorname{Tor} H_1(M) \otimes \operatorname{Tor} H_1(M) \to \mathbf{Q}/\mathbf{Z}, (\alpha, \beta) \mapsto \alpha \odot \beta$. Moreover it has the advantage of being quite simple and useful for the geometric construction of examples with a negative answer. Because it is a pairing on abelian group it is easy to prove (Theorem 2.10) that the answer to the question about the existence of a degree-one map to a $L(n, m)$ is determined by the cyclic decomposition of $H_1(M)$ except for the case where $n = 2^k \bar{n}$, $k > 0$, $\bar{n}$ odd and $H_1(M)$ contains an even number of direct factors isomorphic to $\mathbf{Z}_{2^k}$. In the end one has to calculate the linking pairing only in this case.

For the case $L(n, m) = L(2, 1) = \mathbf{P}^3$ the often used method depends on the existence of a generator $\xi \in H^1(L(2, 1), \mathbf{Z}_2)$ with the property $\xi \cup \xi \cup \xi \neq 0$. If n is odd or a multiple of 4 (see for example Remark 2.8 (c)) there is no element $\xi \in H^1(L(n, m), \mathbf{Z}_n)$ with $\xi^3 \neq 0$, but there is a generator $\mu \in H_1(L(n, m))$ with linking number $\mu \odot \mu \neq 0$ [16, 14.7.3 (c)]. This is the ingredient for Theorem 2.2. If there is a degree-one map $M \to L(n, m)$ then $H_1(M)$ has a direct factor isomorphic to $\mathbf{Z}_n$, see [2], [16, 14.2.6]. Exploring the geometric description of the linking number we construct for every even n a Seifert manifold M_n with $H_1(M_n) = \mathbf{Z}_n \oplus \mathbf{Z}_n$ such that there is no degree-one map from M_n to a lens space $L(n, m)$. If $n = 2$ the Seifert manifold M_2

does not contain an embedded closed non-orientable surface of odd Euler characteristic (Theorem 4.1).

Degree-one maps to $\mathbf{P}^3$ appear in another context, namely in the theory of kinks which are related to Riemannian 4-manifolds with a Lorentz metric. The homotopy classification of Lorentz metrics on a space-time model reduces to the study of the homotopy classes $[M^3, \mathbf{P}^3]$ where M^3 corresponds to the space part. In [13] it is shown that $[M^3, \mathbf{P}^3]$ is an abelian group isomorphic to $\mathbf{Z} \oplus \mathbf{Z}_2^\ell$. The number of kinks (i.e. the number of tiping over of the light cone) appears in the factor $\mathbf{Z}$ and depends on the existence or non-existence of a degree-one map $M^3 \to \mathbf{P}^3$. In [13] one asks to determine the existence or non-existence of degree-one maps to $\mathbf{P}^3$ for all closed connected orientable 3-manifolds. The theory of kinks has recently again been considered by G. W. Gibbons and S. W. Hawking [4] in the more general theory when the topology of the space changes during the time.

Discussions with Ralph Stöcker have been of great influence to this paper. Jeff Williams and Peter Zvengrowski introduced us to the theory of kinks and Daryl Cooper suggested Theorem 4.1. We thank the referee for his help, in particular for the better formulation of Theorem 2.10.

2. A criterion for the existence of degree-one maps.

Notations 2.1.   For lens spaces we use the usual notation $L(n,m)$ and understand at all times that $n > 1$ and $\gcd(n,m) = 1$.

In the following we use the group $\mathbf{Q}/\mathbf{Z}$. An element of this group is denoted by $\left[\frac{r}{s}\right]$ or, mostly, by $[r/s]$ where $r, s \in \mathbf{Z}$, $s \neq 0$, $\gcd(r,s) = 1$; this stands for the coset $\frac{r}{s} + \mathbf{Z}$. Given $[r/s]$ we denote by $[r^{-1}/s]$ the class $[r'/s]$ with $rr' \equiv 1 \mod s$.

For a closed connected oriented 3-manifold N we use the following homomorphisms:

— $j_n \colon H^1(N, \mathbf{Z}_n) \to H^1(N, \mathbf{Q}/\mathbf{Z})$ induced by the monomorphism $j \colon \mathbf{Z}_n \to \mathbf{Q}/\mathbf{Z}, x + n\mathbf{Z} \mapsto [x/n]$;

— the *Bockstein operator* $B \colon H^1(N, \mathbf{Q}/\mathbf{Z}) \to H^2(N)$, see [16, 14.7.4];

— the Bockstein operator $B_n = \mu_n \circ B \circ j_n \colon H^1(N, \mathbf{Z}_n) \to H^2(N, \mathbf{Z}_n)$ where μ_n is induced by the projection $\mu \colon \mathbf{Z} \to \mathbf{Z}_n$;

— the *Poincaré duality isomorphism* $D \colon H^2(N) \to H_1(N)$ defined by $D(\gamma) = \gamma \cap \{N\}$ where $\{N\}$ is the fundamental class of N.

For any $\xi, \eta \in \operatorname{Tor} H_1(N)$, we denote by $\xi \odot \eta$ the *linking number* of ξ and η. This is an element of $\mathbf{Q}/\mathbf{Z}$. (For the definition and notation see [16, 14.7.2].)

Theorem 2.2. *Let M be a closed connected orientable 3–manifold.*

(a) *Assume that there is an element $\alpha \in H_1(M)$ of order $n > 1$ such that the linking number $\alpha \odot \alpha$ is equal to $[r/n]$ where r is prime to n. Then there exists a degree-one map $f \colon M \to L(n,s)$ where s is the inverse of r modulo n.*

(b) *If there is a degree-one map $f \colon M \to L(n,m)$ then there exists an element $\alpha \in \operatorname{Tor} H_1(M)$ such that $\alpha \odot \alpha = [m^{-1}/n]$, the order of α is n and α generates a direct factor of $H_1(M)$.*

Lemma 2.3. *Let $f \colon M \to L(n,m)$ be a continuous mapping of degree d. For $\ell \in \mathbf{Z}$ there is a mapping $M \to L(n,m)$ of degree $d + \ell n$.*

Proof. Let $g \colon S^3 \to S^3$ be of degree ℓ and $\pi \colon S^3 \to L(n,m)$ be the universal covering. Then $\tilde{g} = \pi \circ g \colon S^3 \to L(n,m)$ is of degree ℓn. Now the composition of maps

$$M \overset{\approx}{\longrightarrow} M \# S^3 \longrightarrow M \vee S^3 \overset{f \vee \tilde{g}}{\longrightarrow} L(n,m)$$

gives the desired mapping of degree $d + \ell n$. $\qquad\square$

Proposition 2.4. *If α and $\alpha \odot \alpha$ have the same finite order n then α is a generator of a direct factor of $H_1(M)$.*

Proof. Otherwise there is an element $\beta \in H_1(M)$ such that $\alpha = q\beta$ and the orders of β and q have a common divisor:

$$\operatorname{order}(\beta) = m^k b, \ \gcd(m,b) = 1, \ q = m^\ell q_0, \ \gcd(m,q_0) = 1,$$
$$\operatorname{order}(\alpha) = m^{k-\ell} a, \ a | b, \ 0 < \ell < k.$$

Since the order of $\beta \odot \beta$ divides the order of β we have $m^k b = \operatorname{order}(\beta \odot \beta) \cdot s$ and $\beta \odot \beta = \left[\frac{st}{m^k b}\right]$ for suitable s and t. Now, for some r relatively prime to $m^{k-\ell} a$

$$\left[\frac{r}{m^{k-\ell}a}\right] = \alpha \odot \alpha = q^2 \beta \odot \beta = \left[q_0^2 \frac{stm^\ell}{m^{k-\ell}b}\right].$$

Thus

$$br \equiv aq_0^2 stm^\ell \quad \bmod \ m^{k-\ell}ab,$$

contradicting $\gcd(m,br) = 1$. $\qquad\square$

Taking another generator of the group generated by α changes the r by multiplying it by a square and Corollary 2.5 follows:

Corollary 2.5. *If $\operatorname{Tor} H_1(M) = \mathbf{Z}_n \oplus A$ where $\gcd(n, |A|) = 1$ then the homotopy type of $L(n,m)$ in Theorem 2.2 is uniquely determined.*

Proof of 2.2. Consider the lens space $L(n,s)$ where $sr \equiv 1 \mod n$; in the following we will write $s = r^{-1}$ and $L = L(n,s)$.

First we recall that the mapping

$$(1) \qquad [M, L] \to \mathrm{Hom}(H_1(M), H_1(L)), \quad f \mapsto f_*$$

is surjective. This is obtained as follows. By the universal coefficient theorem, there is an isomorphism $H^1(M, H_1(L)) = \mathrm{Hom}(H_1(M), H_1(L))$. Thus each element $f_* \in \mathrm{Hom}(H_1(M), H_1(L))$ determines a homotopy class, represented by $\hat{f} \colon M \to K(\pi_1(L), 1)$. Remember that $K(\pi_1(L), 1)$ can be constructed with L as 3-skeleton; let $\rho \colon L \hookrightarrow K(\pi_1(L), 1)$ be the embedding. By the cellular approximation theorem we may assume that $\hat{f}(M) \subset L$. This gives a map $f \colon M \to L$ such that $\rho \circ f = \hat{f}$. The result can also be obtained using standard obstruction arguments.

By [**16**, 15.6.3], there is an $\hat{\alpha} \in H^1(M, \mathbf{Z}_n)$ such that

$$j \langle \hat{\alpha} \cup B_n \hat{\alpha}, \{M\} \rangle = \alpha \odot \alpha = \left[\frac{r}{n} \right].$$

From [**16**, 14.7.3 (c), 15.6.3] it follows for the lens space $L = L(n, s)$ that there are generators $\mu \in H_1(L, \mathbf{Z}_n)$ and $\hat{\mu} \in H^1(L, \mathbf{Z}_n)$ with

$$\mu \odot \mu = j \langle \hat{\mu} \cup B_n \hat{\mu}, \{L\} \rangle = \left[\frac{r}{n} \right].$$

Next we prove that the $f \colon M \to L$ above can be chosen such that $f^*(\hat{\mu}) = \hat{\alpha}$. To see this, define

$$\varphi \colon H_1(M) \to \mathbf{Z}_n, \quad \varphi(a) = \langle \hat{\alpha}, a \rangle,$$
$$\psi \colon H_1(L) \to \mathbf{Z}_n, \quad \psi(b) = \langle \hat{\mu}, b \rangle.$$

Since $\hat{\mu}$ is a generator of $H^1(L, \mathbf{Z}_n)$ the homomorphism ψ is an isomorphism and the homomorphism

$$\psi^{-1} \circ \varphi \colon H_1(M) \to H_1(L)$$

is well defined. By (1) there is an $f \colon M \to L$ with $f_* = \psi^{-1} \circ \varphi$; hence, $\psi \circ f_* = \varphi$. Now, for any $a \in H_1(M)$,

$$\langle \hat{\alpha}, a \rangle = \varphi(a) = \psi f_*(a) = \langle \hat{\mu}, f_* a \rangle = \langle f^* \hat{\mu}, a \rangle,$$

and this implies $\hat{\alpha} = f^*(\hat{\mu})$ by the universal coefficient theorem.

Let d be the degree of f. Then

$$r + n\mathbf{Z} = \langle \hat{\alpha} \cup B_n \hat{\alpha}, \{M\} \rangle = \langle f^* \hat{\mu} \cup B_n f^* \hat{\mu}, \{M\} \rangle$$
$$= \langle \hat{\mu} \cup B_n \hat{\mu}, f_* \{M\} \rangle = \langle \hat{\mu} \cup B_n \hat{\mu}, d\{L\} \rangle = d \cdot (r + n\mathbf{Z}).$$

This shows that $d \equiv 1 \mod n$. Now Lemma 2.3 gives the existence of a mapping $M \to L(n, s)$ of degree 1.

For the proof of the second assertion of Theorem 2.2, let $f \colon M \to L(n, m)$ be a degree-one map. As mentioned at the beginning of the proof, there exists a generator μ of $H_1(L(n, m)) \cong \mathbf{Z}_n$ such that $\mu \odot \mu = [m^{-1}/n]$. Define $\alpha \in \operatorname{Tor} H_1(M)$ by $\alpha = DBj_n f^*(\hat{\mu})$. Then the order of α divides n and $\alpha \odot \alpha = \mu \odot \mu = [m^{-1}/n] \neq 0$ see [16, 15.6.3]; hence order$(\alpha) = n$. By Proposition 2.4, α generates a direct factor of $H_1(M)$. $\qquad\square$

Remark 2.6. Every $\alpha \in H^1(M, \mathbf{Z}_n)$ gives a homotopy class of maps $\bar{\alpha} \colon M \to K(\mathbf{Z}_n, 1)$. It is possible to consider $K(\mathbf{Z}_n, 1)$ as the infinite lens space L_n^∞ with some $L(n, m)$ (usual 3-dimensional lens space) as 3-skeleton. Because the dimension of M is 3 and $\pi_2(L(n, m))$ is 0, there is no obstruction to lifting $\bar{\alpha}$ to obtain a map $f \colon M \to L(n, m)$. Now the problem is: does there exist an m such that f is a degree-one map? When n is prime proceeding geometrically like Greenberg and Harper [4, page 295], or using $H^3(L(n, m), \mathbf{Z}_n) = H^3(K(\mathbf{Z}_n, 1), \mathbf{Z}_n)$ and the well known cohomology of the Eilenberg-MacLane spaces, it is possible to find an element $\hat{\mu} \in H^3(L(n, m), \mathbf{Z}_n)$ such that $\hat{\mu} \cup B_n(\hat{\mu}) \neq 0$. The criterium may be obtained taking the corresponding element in the cohomology of M. If this element is not 0 it determines m. In this case the criterion is an easy generalization of Edmonds' result [3].

Y. Rong and S. Wang [11, Cor. 6] gave another proof of the following corollary.

Corollary 2.7. *There exists a degree-one map $f \colon L(p, q) \to L(n, m)$ if and only if $p = kn$ and $m \equiv kqc^2 \mod n$ where k and c are integers. In particular, there is a degree-one map $f \colon L(p, q) \to \mathbf{P}^3 \approx L(2, 1)$ if and only if $p \equiv 2 \mod 4$.*

Proof. Since $L(p, q) \approx L(p, q^{-1})$ we may replace $f \colon L(p, q) \to L(n, m)$ by $f \colon L(p, q^{-1}) \to L(n, m^{-1})$. Choose $\alpha \in H_1(L(p, q^{-1}))$ with the property that $\alpha \odot \alpha = [q/p]$. Then each generator of $H_1(L(p, q^{-1}))$ is of the form $c\alpha$ with $\gcd(p, c) = 1$.

Suppose that $f \colon L(p, q^{-1}) \to L(n, m^{-1})$ is a degree-one map. By 2.2 (b), there is $\beta \in H_1(L(p, q^{-1}))$ of order n such that $\beta \odot \beta = [m/n]$; hence, $p = kn$, $k \in \mathbf{Z}$, and $\beta = kc\alpha$ for some $c \in \mathbf{Z}$. Then

$$\left[\frac{m}{n}\right] = \beta \odot \beta = kc\alpha \odot kc\alpha = \left[\frac{k^2 c^2 q}{p}\right] = \left[\frac{kc^2 q}{n}\right] \quad \Longrightarrow \quad m \equiv kqc^2 \mod n.$$

The other direction follows from 2.2 (a) using the same calculation. $\qquad\square$

Remarks 2.8.

(a) Since for $\mathbf{Z}_2$-coefficients the Bockstein operator B_2 satifies $B_2(\hat{\alpha}) = \hat{\alpha} \cup \hat{\alpha}$ it follows that

$$\frac{1}{2}\langle \hat{\alpha} \cup \hat{\alpha} \cup \hat{\alpha}, \{M\}\rangle = \alpha \odot \alpha \quad \text{where } \alpha = DB\hat{\alpha}.$$

Hence, to determine the existence of a degree-one map $M \to \mathbf{P}^3$, the criteria $\xi^3 \neq 0$, $\xi \in H^1(M, \mathbf{Z}_2)$ from [**13**], [**17**] and $\alpha \odot \alpha \neq 0$, $\alpha \in \mathrm{Tor}\, H_1(M)$ from 2.2 are equivalent.

(b) In the criterion of [**13**] the condition $\hat{\alpha} \cup \hat{\alpha} \cup \hat{\alpha} \neq 0$ cannot be changed to $\hat{\alpha} \cup \hat{\alpha} \neq 0$ as the following example shows. Let $N(K)$ be the twisted orientable I-bundle over the Klein bottle. Glue two copies together to give $M = N(K) \cup_T N(K)$, $T = \partial N(K)$ such that $H_1(M) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. (Attention: M is not the double of $N(K)$.) Then $H^1(M, \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ where the Poincaré duals of the generators of the two factors are the two Klein bottles central in the two copies of $N(K)$. If $\hat{\alpha} \in H^1(M, \mathbf{Z}_2)$ is one of these generators then $\hat{\alpha} \cup \hat{\alpha} \neq 0$ but $\hat{\alpha} \cup \hat{\alpha} \cup \hat{\alpha} = 0$, see Fig. 1 for the intersection of the duals.
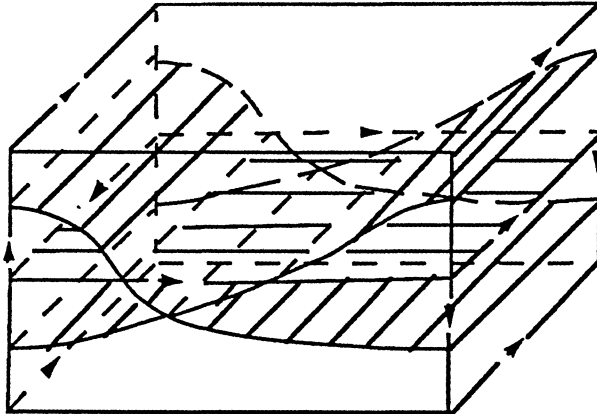

Figure 1.

(c) For a lens space $L = L(n, m)$, n even, there is a convenient geometric way to calculate $\hat{\alpha} \cup \hat{\alpha} \cup \hat{\alpha}$ for $\hat{\alpha} \in H^1(L, \mathbf{Z}_n) = \mathbf{Z}_n$. This is a special case of the calculation in [**8**, Lemma 3.4] where the arguments are used for arbitrary manifolds. Consider the Heegaard splitting $L = V_1 \cup V_2$ of genus 1 and let c_i be the core of V_i. Each c_i defines a generator of $H_1(L) \cong \mathbf{Z}_n$ and $c_2 \sim mc_1$. Let $\hat{\alpha}$ be the Hom-dual of c_1. Taking the meridian disk $D_1 \subset V_1$

and extending it by an annulus parallel to ∂D_1 in V_2 and by the core c_2 we obtain a singular surface F_1 such that $\partial F_1 = nc_2$, $|F_1 \cap c_1| = 1$. Then the homology class defined by F_1 is the Poincaré dual of $\hat{\alpha}$. Let F_1' be a parallel copy of F_1 where the singular curve is a curve in V_2 parallel but disjoint to c_2. Furthermore we take a singular surface F_2 with the roles of the two solid tori interchanged. Then $D_2 \cap (F_1 \cup F_1') \subset V_2$ consists of two points with stars of n segments which intersect as shown in the figure. The number of intersection points of these stars equals :

$$2\left(1 + 2 + \ldots + \left(\frac{n}{2} - 1\right)\right) + \frac{n}{2} = \left(\frac{n}{2}\right)^2 .$$

See Fig. 2. Hence $\hat{\alpha} \cup \hat{\alpha} \cup \hat{\alpha} = m^{-1}\left(\frac{n}{2}\right)^2 \{\hat{L}\} \in H^3(L, \mathbf{Z}_n) = \mathbf{Z}_n$ and this is different from 0 if and only if $n \equiv 2 \mod 4$.
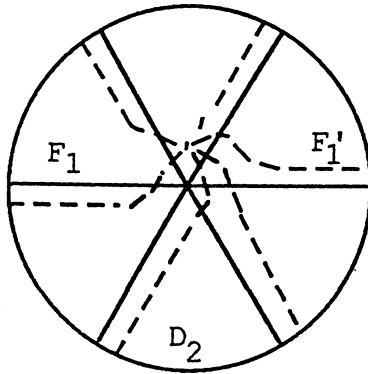

Figure 2.

For the next theorem we need the following lemma.

Lemma 2.9. *Let M be a closed orientable 3–manifold and let n_1, n_2 be two coprime integers. If there are degree-one maps $M \to L(n_1, m_1)$ and $M \to L(n_2, m_2)$ then there is a degree-one map $M \to L(n_1 n_2, m)$.*

Proof. By Theorem 2.2, there exist

$$\alpha_i \in H_1(M) \quad \text{with} \quad n_i \alpha_i = 0, \ \alpha_i \odot \alpha_i = [r_i/n_i], \ r_i m_i \equiv 1 \mod n_i, \ i = 1, 2.$$

Let $\alpha = n_2\alpha_1 + n_1\alpha_2$. Then

$$n_1n_2\alpha = 0,$$
$$\alpha \odot \alpha \equiv n_2^2\alpha_1 \odot \alpha_1 + 2n_1n_2\alpha_1 \odot \alpha_2 + n_1^2\alpha_2 \odot \alpha_2 \quad \mod \mathbf{Z}$$
$$\equiv n_2^2[r_1/n_1] + n_1^2[r_2/n_2] \quad \mod \mathbf{Z}$$
$$\equiv [r/n_1n_2] \quad \mod \mathbf{Z},$$

where $r = n_2^3r_1 + n_1^3r_2$ is prime to n_1n_2. By 2.2(a), there is a degree-one map $M \to L(n,m)$ where $mr \equiv 1 \mod n$. $\qquad\square$

Theorem 2.10. *Let M be an orientable closed $3-$manifold.*

(a) *If there is a degree one map $f\colon M \to L(n,m)$, then $H_1(M) = \mathbf{Z}_n \oplus A$.*

(b) *If $H_1(M) = \mathbf{Z}_n \oplus A$ where $n > 1$ is odd, then there is a degree one map $f\colon M \to L(n,m)$ for some m with $\gcd(n,m) = 1$.*

(c) *If $H_1(M) = \mathbf{Z}_n \oplus A$ where $n = 2^k\bar{n}$, for some $k > 0$ and $\bar{n}$ odd, then there exists a degree-one map $f\colon M \to L(n,m)$ for some m, $\gcd(n,m) = 1$ provided A contains an even number of factors $\mathbf{Z}_{2^k}$.*

The additional assumption on A in (c) is necessary, as examples in the next section will show.

Proof. (a) The result follows from Theorem 2.2 (b).

(b) By Lemma 2.9, it is enough to consider the case where $n = p^k$, where p is an odd prime. By [18, Theorem (4)], see also [6, page 29], there is a $\alpha \in \text{Tor}\, H_1(M)$ such that $\alpha \odot \alpha = [r(p)/p^k]$ with $\gcd(r(p),p) = 1$. The conclusion follows from Theorem 2.2 (a).

(c) Since $\mathbf{Z}_n = \mathbf{Z}_{2^k} \oplus \mathbf{Z}_{\bar{n}}$, by Lemma 2.9, it is enough to show that there is a degree-one map $f\colon M \to L(2^k,m)$ for some odd m. By [18, Theorem (4)], there is a basis of $H_1(M)$ that contains an odd number of generators of order 2^k such that parts of the matrix of the linking pairing for these generators consist of diagonal blocks of the form

$$[m/2^k], \quad \begin{pmatrix} 0 & [1/2^k] \\ [1/2^k] & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} [1/2^{k-1}] & [1/2^k] \\ [1/2^k] & [1/2^{k-1}] \end{pmatrix}.$$

Hence, for at least one of the generators the self-linking number is $[m/2^k] \neq 0$, and now the assertion follows from Theorem 2.2. $\qquad\square$

Remarks 2.11.

(a) Parts of the above results have been known. For $H_1(M) = \mathbf{Z}_n$ Luft-Sjerve [7] gave an answer by proving the existence of a special surface. Using

the argument of [**7**], E. Luft and Y. Rong independently observed that there exists a degree-one map $f\colon M \to L(n,m)$ if $H_1(M) = \mathbf{Z}_n \oplus A$ and the order of Tor A is relatively prime to n. This is a consequence of 2.10.

(b) For many Seifert fiber spaces M one can easily prove the existence of a degree-one map onto some lens space $L(2^k, m)$, proceeding from the canonical presentation of the fundamental group of M to the homology of M with $\mathbf{Z}_2$ coefficients (see below). But this approach does not give negative answers to the existence of degree-one maps onto a fixed lens space or any lens space.

3. On the non-existence of degree-one maps.

Construction 3.1. In the following we denote curves, their homotopy and homology classes by the same symbol. Consider the Seifert manifold $M = M(\varepsilon_1/n_1, \varepsilon_2/n_2, \varepsilon_3/n_3), \varepsilon_i = \pm 1$ obtained from $\left(S^2 - \overset{\circ}{D}_1 \cup \overset{\circ}{D}_2 \cup \overset{\circ}{D}_3 \right) \times S^1$ by attaching the union of three solid tori $T_1 \cup T_2 \cup T_3$ with a glueing map described below; here $\overset{\circ}{D}_1, \dots$ are open discs and $T_1, \dots$ fibered solid tori. Denote ∂D_i by x_i and the fiber S^1 by h. The meridian of the fibered solid torus T_i is denoted by a_i and its longitude by b_i. The glueing map is defined, up to isotopy, by the relations $n_i x_i + \varepsilon_i h = a_i$ and $x_i = b_i$ (such that the determinant is ± 1). Then

$$\pi_1(M(\varepsilon_1/n_1, \varepsilon_2/n_2, \varepsilon_3/n_3))$$
$$= \langle x_1, x_2, x_3, h \mid x_i^{n_i} h^{\varepsilon_i} = 1, 1 \le i \le 3, \ x_1 x_2 x_3 = 1 \rangle.$$

Notice that these relations imply that h is in the center of $\pi_1(M)$.

Theorem 3.2. *For n even the Seifert manifold $M = M(-1/n, -1/n, 1/n)$ admits no degree-one map to a lens space $L(n,m)$ (m arbitrary) and for $n = 2^k$ to any lens space, but $H_1(M) = \mathbf{Z}_n \oplus \mathbf{Z}_n$.*

Proof. Abelianizing $\pi_1(M)$ gives

$$H_1(M) = \langle x_1, x_2 \mid nx_1 = nx_2 = 0 \rangle^{ab} = \mathbf{Z}_n \oplus \mathbf{Z}_n$$
$$= \langle \xi, \eta \mid n\xi = n\eta = 0 \rangle^{ab} \quad \text{where} \ \xi = x_1, \ \eta = x_1 - x_2.$$

(For $n = 2$ the order of $\pi_1(M)$ equals 8, for $n > 2$ the order is infinite.) To calculate $\eta \odot \eta$, we have to find a singular surface $F_{1,2}$ such that $\partial F_{1,2} = n\eta$. It is possible to describe this surface in three parts. One part lies in T_2 and is obtained from an annulus by identifying one boundary component with the curve $h = -a_2 + nb_2$ on ∂T_2 while the other boundary component runs n times along the core $-b_2$ of T_2. (Here b_2 denotes both the longitude and

the core of the solid torus which are homotopic in T_2.) Another part is a singular annulus in T_1 with one boundary component running n times along b_1 and the other being the curve $h = -a_1 + nb_1$ on ∂T_1. These two parts are connected by an annulus $I \times h$. Then

$$\partial F_{1,2} = nb_1 - nb_2 = n(x_1 - x_2) = n\eta.$$

Now we have to calculate the intersection number of $F_{1,2}$ with x_2 and x_1. We recall that $x_2 \subset \left(S^2 - \overset{\circ}{D}_1 \cup \overset{\circ}{D}_2 \cup \overset{\circ}{D}_3 \right) \times S^1$ is the boundary of the open disc $\overset{\circ}{D}_2$ and that it cuts once the curve $-a_2 + nb_2$ on ∂T_2 which is homologous to h. The same argument applies to the intersection of x_1 with $F_{1,2}$. Therefore

$$\eta \odot x_2 = [\pm 1/n], \quad \eta \odot x_1 = [\pm 1/n] \quad \text{so} \quad \eta \odot \eta = [\pm 1/n - \pm 1/n].$$

By the same construction as above we find a singular surface $F_{2,3}$ with

$$\partial F_{2,3} = nb_2 + nb_3 = n(x_2 + x_3) = -nx_1 = -n\xi.$$

Since $F_{2,3}$ intersects ∂T_3 on $-h$ and h intersects x_3 once, we obtain $\xi \odot x_3 = [\pm 1/n]$; similarly the intersection of $F_{2,3}$ and ∂T_2 is h and $\xi \odot x_2 = [\pm 1/n]$. Therefore

$$\xi \odot \xi = \xi \odot x_1 = \xi \odot (-x_3 - x_2) = -\xi \odot x_3 - \xi \odot x_2 = [\mp 1/n + \mp 1/n].$$

For each element $\alpha \in H_1(M) = \mathbf{Z}_n \oplus \mathbf{Z}_n$ we have $\alpha = a\xi + b\eta$ and $\alpha \odot \alpha$ is of the form $[2k/n]$. Therefore an equation $\alpha \odot \alpha = [r/n]$ with $\gcd(r, n) = 1$ is impossible if n is even. By Theorem 2.2, M admits no degree-one map to $L(n, m)$ and hence, if $n = 2^k$, to any lens space (since otherwise the homology group of this lens space is a free factor of $H_1(M)$). $\qquad\square$

Example 3.3.(due to A. Shastri and P. Zvengrowski.) Shastri-Zvengrowski [14] completely determined the existence of a degree-one map of any quotient of S^3 by a free action of a finite group to $\mathbf{P}^3$. The most difficult case was the manifold $M = S^3/Q_{4n}$ where Q_{4n} is the generalized quaternion group $n \geq 1$. (The double cover $S^3 \to SO(3)$ lifts the dihedral group with $2n$ elements to a group Q_{4n} of order $4n$.) It has the presentation

$$Q_{4n} = \langle x, y \mid x^n = y^2, xyx = y \rangle.$$

For $n = 2m$ the homology groups do not suffice to solve the problem and Shastri-Zvengrowski used a free $\mathbf{Z}Q_{8m}$-resolution of $\mathbf{Z}$ to calculate explicitly

the ring structure of $H^*(K(Q_{8m}, 1); \mathbf{Z}_2)$. We will show next that their result can also be obtained by our geometric method.

Consider the following presentation of Q_{8m}.

$$\pi_1(M) = \langle x, y, h \mid y^2 = h, x^{2m} = h, xyx = y \rangle$$
$$= \langle x, y, z, h \mid y^2 = h, x^{2m} = h, z^2 = h^{-1}, xyz = 1 \rangle.$$
$$= \pi_1(M(-1/2m, -1/2, 1/2)).$$

Since the fundamental groups of the Seifert fiber spaces M and $M(-1/2m, -1/2, 1/2)$ are isomorphic to Q_{8m} the spaces are homeomorphic, see [**9**, p. 110]. Clearly, h is in the center of the group and a presentation of the abelian group $H_1(M)$ is as follows:

$$H_1(M) = \langle x, y \mid 2(mx - y) = 0, 2x = 0 \rangle^{ab}$$
$$= \langle \xi, \eta \mid 2\xi = 0, 2\eta = 0 \rangle^{ab} \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$$

with $\xi = x$ and $\eta = mx - y$.

To calculate $\eta \odot \eta$ in the same manner as above we find a singular surface F_{xy} such that $\partial F_{xy} = 2\eta$. Now we have to calculate the intersection number of F_{xy} with y and x (there is no need to worry about the sign of the intersection numbers). We recall that $y \subset \left(S^2 - \overset{\circ}{D}_x \cup \overset{\circ}{D}_y \cup \overset{\circ}{D}_z \right) \times S^1$ is the boundary of the open disc $\overset{\circ}{D}_y$ and that it cuts once the curve $-a_y + 2b_y$ on ∂T_y which is homologous to h. The same argument applies to the intersection of x with F_{xy}. Therefore

$$\eta \odot y = [1/2], \ \eta \odot x = [1/2] \quad \text{so} \quad \eta \odot \eta = [(m-1)/2].$$

We may apply the first part of Theorem 2.2 to get a degree-one map $S^3/Q_{8m} \to L(2,1) \approx \mathbf{P}^3$ if m is even.

Next we calculate $\xi \odot \xi$. Take a singular surface F_{yz} (constructed in the same manner as above) such that $\partial F_{yz} = 2y - 2z$. Then we have $(y-z) \odot y = [1/2]$, $(y-z) \odot z = [1/2]$. From $z = -x - y$ we obtain $(2y + x) \odot y = [1/2]$ and $(2y + x) \odot (x + y) = [1/2]$. Substituting $x = \xi$, $y = -\eta + m\xi$ it follows from $2\eta = 0 = 2\xi$ that

$$-\xi \odot \eta + m\xi \odot \xi = [1/2], \ -\xi \odot \eta + (m+1)\xi \odot \xi = [1/2].$$

By adding these two equations we obtain $\xi \odot \xi = 0$ and $\xi \odot \eta = [1/2]$. If m is odd we have

$$(\xi + \eta) \odot (\xi + \eta) = \xi \odot \xi + \xi \odot \eta + \eta \odot \xi + \eta \odot \eta = 0 + [1/2] + [1/2] + 0 = 0.$$

Since ξ, η, $\xi + \eta$ are the only non-zero elements of $H_1(M) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ it follows from Theorem 2.2 (b) that the manifold S^3/Q_{8m} does not admit a degree-one map to a lens space if m is odd.

4. Embedding closed surface with odd Euler caracteristic.

By G. Bredon and J. Wood [1, page 88], the existence of an element of $H^1(M, \mathbf{Z}_2)$ whose cube-cup is not zero is equivalent to the existence of a closed non-orientable surface of odd Euler characteristic embedded in M. The proof in [1] is based on the theory of L-equivalence. Here we will give a geometric proof (similar to Rubinstein's proof [12, Theorem 18]) that the existence of a degree-one map $M \to \mathbf{P}^3$ is equivalent to the existence of an embedding of closed non-orientable surface of odd Euler characteristic to M.

Theorem 4.1. *Let M be a closed orientable 3-manifold. The following three assertions are equivalent:*

(a)   *There exists a degree-one map $f: M \to \mathbf{P}^3$;*

(b)   *There exists an embedded closed surface F in M with odd Euler characteristic;*

(c)   *There exists $\zeta \in H^1(M; \mathbf{Z}_2)$ with $\zeta \cup \zeta \cup \zeta \neq 0$.*

Proof. For $(a) \Longleftrightarrow (c)$ see [13, Theorem 4.4] or Remark 2.8.

$(b) \Longrightarrow (a)$: If the Euler characteristic of F is odd then $F = F' \# \mathbf{P}^2 = F_0 \cup_{\partial F_0 = \partial \mathcal{M}} \mathcal{M}$, with F' closed, orientable, $\overset{\circ}{D}$ an open disc, $F_0 = F' - \overset{\circ}{D}$, and $\mathcal{M} = \mathbf{P}^2 - \overset{\circ}{D}$. The surface $\mathcal{M}$ is the Möbius band. Let $N(F)$ be a regular neighbourhood of F in M. Then $N(F)$ is an I-bundle over F. Let $p: N(F) \to F$ be the projection of this bundle. The I-bundle $p^{-1}(F_0)$ is a product $F_0 \times I$, otherwise there is an orientation reversing loop in M (which is assumed orientable). The I-bundle $p^{-1}(\mathcal{M})$ is twisted (otherwise the center curve of $\mathcal{M}$ is an orientation reversing loop in M).

Let $N(\mathbf{P}^2)$ be a regular neighbourhood of $\mathbf{P}^2 = D^2 \cup_{\partial D^2 = \partial \mathcal{M}} \mathcal{M}$ in $\mathbf{P}^3$ where $\mathcal{M}$ again is the Möbius band and $N(\mathcal{M})$ is the twisted I-bundle over $\mathcal{M}$, considered as part of $N(\mathbf{P}^2)$:

$$N(\mathbf{P}^2) = (D^2 \times I) \cup N(\mathcal{M}).$$

Define $f: N(F) \to N(\mathbf{P}^2)$ as follows: $f|p^{-1}(\mathcal{M})$ sends $p^{-1}(\mathcal{M})$ to $N(\mathcal{M})$ homeomorphically and $f|p^{-1}(F_0)$ sends $p^{-1}(F_0) = F_0 \times I$ to $D^2 \times I$, inducing a proper degree-one map on the boundary. Now extend f to a map $f: \left(M - N(\overset{\circ}{F})\right) \to \left(\left(\mathbf{P}^3 - N(\overset{\circ}{\mathbf{P}}^2)\right) = D^3\right)$ by mapping a collar of $\partial N(F)$ onto D^3 and the remaining part to the center point of D^3. The map f is a degree-one map $M \to \mathbf{P}^3$.

$(a) \Longrightarrow (b)$: Suppose $f: M \to \mathbf{P}^3$ is a degree-one map. Deforme f to be transverse to $\mathbf{P}^2 \subset \mathbf{P}^3$ such that $F = f^{-1}(\mathbf{P}^2)$ is connected. Let $N(\mathbf{P}^2)$ be a regular neighbourhood of $\mathbf{P}^2$ in $\mathbf{P}^3$. Then $\partial N(\mathbf{P}^2)$ is a 2-sphere. Now both

maps $f|N(F)\colon N(F) \to N(\mathbf{P}^2)$ and $f|\partial N(F)\colon \partial N(F) \to \partial N(\mathbf{P}^2)$ are of degree 1. If $N(F) = F \times I$ then the degrees of the maps $f|F \times \{0\}\colon F \times \{0\} \to \partial N(\mathbf{P}^2)$ and $f|F \times \{1\}\colon F \times \{1\} \to \partial N(\mathbf{P}^2)$ have the same absolute value; hence $f|\partial N(F)\colon \partial N(F) \to \partial N(\mathbf{P}^2)$ is of even degree. Therefore $N(F)$ is a twisted I-bundle and, hence, F is non-orientable.

Suppose the Euler characteristic of F is even. Then $F = F'\#K$ with F' closed orientable and K a Klein bottle. Let $N(F)$ be a regular neighbourhood of F in M. It is an I-bundle. This I-bundle over F' is a product because M is orientable. The I-bundle over K is twisted over every orientation reversing curve. Let α and β represent the two generators of the fundamental group of K, where the loop α is two-sided and β is one-sided. Then α is the only non-trivial element in $\mathrm{Tor}\, H_1(F) \cong \mathrm{Tor}\, H_1(N(F))$ and, hence, the only candidate for $\alpha \odot \alpha \neq 0$. However one can push F off itself in $N(F)$ so that the self-intersection of F is α. Since the I-bundle over α is trivial, α can be pushed off F. Then the intersection of α and F is empty; thus $\alpha \odot \alpha = 0$. (Compare Fig. 1.) By Remark 2.8 (a), this contradicts the existence of $\zeta \in H^1(M; \mathbf{Z}_2)$ with $\zeta \cup \zeta \cup \zeta \neq 0$. $\qquad\square$

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