PACIFIC JOURNAL OF MATHEMATICS
Vol. 176, No. 1, 1996

# ON A THEOREM OF KOCH

FARSHID HAJIR

**We give a short proof of a slightly stronger version of a theorem of Koch: A complex quadratic field whose ideal class group contains a subgroup of type $(4, 4, 4)$ possesses an infinite unramified Galois pro-2 extension.**

## 1. Koch's Theorem.

If $K$ is a finite extension of $\mathbb{Q}$ and $p$ is a prime number, let $K^{(0)} = K$ and for $n \geq 1$ define $K^{(n)}$ to be the maximal abelian unramified $p$-extension of $K^{(n-1)}$. The smallest $n$ such that $K^{(n)} = K^{(n+1)}$ is called the length of the $p$-class field tower of $K$; if no such integer $n$ exists, we say that $K$ has infinite $p$-class field tower. By a group of type $(m_1, \dots, m_t)$ we understand a group isomorphic to $\mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_t\mathbb{Z}$. The purpose of this note is to give a short proof of (a slightly strengthened version of) a theorem of Koch [**4**]:

**Theorem 1.** *If $K$ is a complex quadratic field whose ideal class group contains a subgroup of type $(4, 4, 4)$, then the 2-class field tower of $K$ is infinite.*

Koch's proof proceeds by showing that in a minimal presentation of the Galois group of the maximal unramified 2-extension of $K$ by a free pro-2 group $G$, the relations lie deep in the Zassenhaus filtration of $G$. We replace this key ingredient of his proof, which can be thought of as the study of the quadratic unramified extensions of the genus field of $K$ which are central over $K$ [**3**, Satz 1], with a simple result from genus theory. Moreover, Koch's proof requires a generalization of the Vinberg/Gashütz sharpening of the Golod-Shafarevich theorem on the structure of pro-$p$ groups [**4**, Satz 3]; for our proof, the original Vinberg/Gashütz inequality suffices (for an account of these inequalities, see, e.g., Koch's book [**5**]). Indeed, we will need only the following result (see Martinet [**8**]):

**Theorem 2.** *Suppose $F$ is a totally real field of degree $n$, and $E$ is a totally complex quadratic extension of $F$. Let $t$ be the number of prime ideals of $F$ which ramify in $E$. The 2-rank of the ideal class group of $E$ is at least $t - 1$. If*

$$t \geq 3 + 2\sqrt{n+1},$$

*then the* 2*-class field tower of* $E$ *is infinite.*

**Corollary 3.** *Suppose* $F$ *is a totally real degree* 4 *extension of* $\mathbb{Q}$. *If two rational primes that split completely in* $F$ *ramify in a complex quadratic field* $L$, *then* $E = FL$ *has an infinite* 2*-class field tower.*

*Proof.* With notation as in the theorem, we have $t \geq 8 \geq 3 + 2\sqrt{4+1}$. □

*Proof of Theorem* 1. We know that at least four primes divide the discriminant $D$ of $K$. If six or more primes divide $D$, then an application of Theorem 2 to $K/\mathbb{Q}$ already yields the result. Assume first that exaclty four primes divide $D$. By the criterion of Rédei-Reichardt [**9**] on the 4-rank of the class group of $K$, one knows that $D = -p_1 \cdot p_2 \cdot p_3 \cdot p_4$ where $p_2, p_3, p_4$ are odd primes satisfying $\left(\frac{p_i}{p_j}\right) = +1$ for $i, j > 1, i \neq j$, and one of the following is satisfied:

(I) $p_1 = 4; p_j \equiv 1 \pmod 8), j = 2, 3, 4$.
(II) $p_1 = 8; p_j \equiv 1 \pmod 8), j = 2, 3, 4$.
(III) $p_1 = 8; p_2 \equiv 7 \pmod 8); p_j \equiv 1 \pmod 8), j = 3, 4$.
(IV) $p_1 \equiv 3 \pmod 4$ is an odd prime, $p_j \equiv 1 \pmod 4), j = 2, 3, 4$, and $\left(\frac{p_1}{p_j}\right) = +1$ for $j = 2, 3, 4$.

Incidentally, Koch's theorem was originally stated for case (IV) only. Let $F = \mathbb{Q}\left(\sqrt{p_3}, \sqrt{p_4}\right)$ and $E = F\left(\sqrt{-p_1 \cdot p_2}\right)$. In all cases, $(p_2)$ and the unique rational prime divisior of $(p_1)$ split completely in $F$. Hence, by Corollary 3, $E$ has an infinite 2-class field tower. Since $E/K$ is an unramified 2-extension, $K$ has an infinite 2-class field tower as well. Now suppose exactly five primes $p_1, \ldots, p_5$ divide the discriminant of $K$; using the Rédei-Reichardt criterion [**9**], or its equivalent form due to Rédei [**10**], it is straightforward to check that for some $i$, $1 \leq i \leq 5$, we have

$$p_i \equiv 1 \pmod 4, \quad \left(\frac{p_i}{p_j}\right) = 1, j \neq i.$$

Now let $F = \mathbb{Q}\left(\sqrt{p_i}\right), E = K\left(\sqrt{p_i}\right)$; $E/F$ is a CM-extension with 8 ramified primes. By Theorem 2, $E$ has an infinite 2-class field tower, and so does $K$. □

## 2. Further Remarks.

Koch and Venkov [**6**] have proved that a complex quadratic field whose ideal class group has a subgroup of type $(p, p, p)$ for some odd prime $p$ has an infinite $p$-class field tower. Therefore, a complex quadratic field possesses an infinite Hilbert class field tower whenever its ideal class group contains a subgroup of type $(m, m, m)$ with $m \geq 3$. On the other hand, the field $\mathbb{Q}\left(\sqrt{-105}\right)$, whose ideal class group is of type $(2,2,2)$, has a finite class field tower, since its root discriminant is just below the Odlyzko bound (see e.g. [**8**]). I am indebted to the referee for the above remark.

Note that the proof of Koch's theorem we have given relies only on the existence of two primes that split completely in a real biquadratic field. For instance, the primes 31, 89 split completely in $\mathbb{Q}\left(\sqrt{2}, \sqrt{5}\right)$, hence $\mathbb{Q}\left(\sqrt{-2 \cdot 5 \cdot 31 \cdot 89}\right)$ has an infinite 2-class field tower; its 2-ideal class group is of type (4,2,2).

Taussky-Todd [**12**] proved that a number field with 2-ideal class group of type (2,2) has a finite 2-class field tower of length at most 2. It is natural to ask whether there are number fields with infinite 2-class field tower whose 2-class group is of type (4,2) or (2,2,2) (simplest non-cyclic 2-groups after type (2,2)). Using a minor variation on an idea first introduced by Schoof [**11**], we now show that there are complex quadratic fields with these properties. Consider, for example, $K = \mathbb{Q}\left(\sqrt{-5 \cdot 7 \cdot 41 \cdot 61}\right)$, which has 2-ideal class group of type (2,2,2). To show that this field has infinite 2-tower, let $H_0$ be the Hilbert class field of $K_0 = \mathbb{Q}\left(\sqrt{5 \cdot 41 \cdot 61}\right)$, a real quadratic field with class number 16. Since 7 is inert in $K_0$, it splits into 16 prime ideals in $H_0$, all of which ramify in the CM extension $L = H_0\left(\sqrt{-7}\right)$. Theorem 2 shows that $L$, an unramified 2-extension of $K$, has an infinite 2-class field tower, proving the claim. In fact, for any prime $q$ satisfying $q \equiv 7 \pmod{5 \cdot 41 \cdot 61}$ (there are infinitely many such primes by Dirichlet's theorem), the same argument shows that $K_q = \mathbb{Q}\left(\sqrt{-5 \cdot 41 \cdot 61 \cdot q}\right)$ has infinite 2-class field tower; furthermore, by Rédei-Reichardt, $K_q$ has 2-class group of type (2,2,2).

For the second example, let $K = \mathbb{Q}\left(\sqrt{-5 \cdot 11 \cdot 461}\right)$; this field has 2-ideal class group of type (4,2). Observe that the rational prime ideal (11) splits into 16 prime ideals in $H_0$, the Hilbert class field of the real quadratic field $K_0 = \mathbb{Q}\left(\sqrt{5 \cdot 461}\right)$ with class number 16. Therefore, by the same argument as above, $L = H_0\left(\sqrt{-11}\right)$, and thereby $K$, have infinite 2-class field tower. Let $H$ be the 2-Hilbert class field of $K$. Benjamin [**1**] has shown that the 2-class field tower of a complex quadratic field $E$ with 2-class group of type (4,2) has length at most 2 if the 2-Hilbert class field of $E$ has elementary 2-class group $\left(E = \mathbb{Q}\left(\sqrt{-5 \cdot 13}\right) \text{ is an example}\right)$. Since $K$ has infinite 2-tower, we conclude that $H$ does not have elementary abelian 2-class group.

Finally, note that the 2-rank of the ideal class group of $L$ is at least 15. Using Louboutin [**7**], we compute the 2-rank of the ideal class group of the biquadratic field $E = \mathbb{Q}\left(\sqrt{-11}, \sqrt{5 \cdot 461}\right)$ to be 2. The arguments of [**2**] then show that the 2-rank of the ideal class group of $L$ is 15, 16 or 17.

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Received December 10, 1993 and revised December 7, 1994.

California Institute of Technology
Pasadena CA 91125
*E-mail address*: fhajir@cco.caltech.edu