A UNIQUENESS THEOREM FOR THE MINIMAL SURFACE EQUATION

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In 1991, Collin and Krust proved that if $u$ satisfies the minimal surface equation in a strip with linear Dirichlet data on two sides, then $u$ must be a helicoid. In this paper, we give a simpler proof of this result and generalize it.

1. Introduction.

Let $\Omega_{\alpha} \subset \mathbb{R}^2$ be a sector domain with angle $0 < \alpha < \pi$. Consider the minimal surface equation

$$\text{div} Tu = 0$$

where $Tu = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}$ and $\nabla u$ is the gradient of $u$. In 1965, Nitsche [7] announced the following results:

1. Given a continuous function $f$ on $\partial \Omega_{\alpha}$, there always exists a solution $u$ which satisfies the minimal surface equation in $\Omega_{\alpha}$ with Dirichlet data $f$ on $\partial \Omega_{\alpha}$;

2. If $u$ satisfies the minimal surface equation with vanishing boundary value in $\Omega_{\alpha}$, then $u \equiv 0$.

Nitsche thus raised the following question: Let $\Omega \subset \Omega_{\alpha}$ and let $f$ be an arbitrary continuous function on $\partial \Omega$. If the Dirichlet problem

$$\begin{cases} 
\text{div} Tu = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega
\end{cases}$$

has a solution, is it unique?

We notice that similar questions for higher dimensions are raised in [6]. Results in this direction were obtained by Miklyukov [5] and Hwang [4] independently, in which the following result was established:

**Theorem 1.** Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain and let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$. For every $R > 0$, set $B_R = \{x \in \mathbb{R}^2 \mid |x| < R\}$ and $\Gamma_R = \partial(\Omega \cap B_R) \cap \overline{B_R}$. If $u$ and $v$ are both solutions of

$$\begin{cases} 
\text{div} Tu = \text{div} Tv & \text{in } \Omega, \\
u = \nu & \text{on } \partial \Omega
\end{cases}$$

then $u \equiv v$. 

357
\[ \partial B_R. \text{ Denote } |\Gamma_R| \text{ as the length of } \Gamma_R. \text{ And suppose that} \]

\[
\begin{align*}
(i) & \quad \text{div } Tu = \text{div } Tv \quad \text{ in } \Omega, \\
(ii) & \quad u = v \quad \text{ on } \partial \Omega, \\
(iii) & \quad \max_{\Omega \cap B_R} |u - v| = O \left( \sqrt{\int_{R_0}^{R} \frac{1}{|\Gamma_r|} \, dr} \right) \quad \text{as } R \to \infty, \text{ for some positive constant } R_0.
\end{align*}
\]

Then \( u \equiv v \text{ in } \Omega. \)

A stronger version of Theorem 1 was discovered by Collin and Krust [2] independently, which is the following:

**Theorem 1*.** Let \( \Omega, u, v, B_R, \Gamma_r \text{ and } |\Gamma_r| \text{ as in Theorem 1.} \text{ And suppose that} \]

\[
\begin{align*}
(i) & \quad \text{div } Tu = \text{div } Tv \quad \text{ in } \Omega, \\
(ii) & \quad u = v \quad \text{ on } \partial \Omega, \\
(iii) & \quad \max_{\Omega \cap B_R} |u - v| = o \left( \int_{R_0}^{R} \frac{1}{|\Gamma_r|} \, dr \right) \quad \text{as } R \to \infty, \text{ for some positive constant } R_0.
\end{align*}
\]

Then \( u \equiv v \text{ in } \Omega. \)

In fact, for any unbounded domain \( \Omega \), we have \( |\Gamma_R| = O(R) \), and condition (iii) in Theorem 1* becomes

\[
\max_{\Omega \cap B_R} |u - v| = o(\log R) \quad \text{as } R \to \infty.
\]

In the special case when \( \Omega \) is a strip, then \( |\Gamma_R| \leq \text{constant} \), and condition (iii) becomes

\[
\max_{\Omega \cap B_R} |u - v| = o(R).
\]

On the other hand, in a strip domain \( \Omega \), Collin [1] showed that there exist two different solutions for the minimal surface equation such that \( u = v \text{ on } \partial \Omega \) and \( \max_{\Omega \cap B_R} |u - v| = O(R) \text{ as } R \to \infty. \text{ So condition (iii) is necessary.} \)

This counterexample also answers Nitsche’s question in the negative.

In contrast, the following result is also given in [2].

**Theorem 2.** Let \( \Omega = (0,1) \times \mathbb{R} \text{ be a strip. Suppose that} \)

\[
\begin{align*}
\text{div } Tu &= 0 \quad \text{ in } \Omega, \\
u(0,y) &= ay + b, \\
u(1,y) &= cy + d
\end{align*}
\]

where \( a, b, c, d \text{ are constant.} \text{ Then } u \text{ must be a helicoid.} \)

The following inequality was discovered by Miklyukov [5, p. 265], Hwang [4, p. 342] and Collin and Krust [2, p. 452]:

\[
(Tu - Tv) \cdot (\nabla u - \nabla v) \geq \frac{\sqrt{1 + |\nabla u|^2} + \sqrt{1 + |\nabla v|^2}}{2} |Tu - Tv|^2
\]

It seems that the method of proof of Theorem 1* can not be used to prove Theorem 2, and so Collin and Krust [2] resorted to the theory of Gauss maps instead.

In this paper, we will point out that the method of proof of Theorem 1 and Theorem 1* could be use to give a simpler proof of Theorem 2. Moreover, we shall generalize Theorem 1* and Theorem 2 to get the more general results as stated in Theorem 3 and Theorem 4. And we will make a remark after Theorem 3 to point out why Collin and Krust [2] could get a better result then Miklyukov [5] and Hwang [4].


Without loss of generality, we may rephrase Theorem 2 in the following form:

**Theorem 2***. Let \( \Omega = (b, a) \times \mathbb{R} \) be a strip domain in \( \mathbb{R}^2 \) where \( a, b \) are two constants with \(-\pi/2 < b < a < \pi/2\), and let \( u \in C^0(\overline{\Omega}) \cap C^2(\Omega) \). Suppose that

\[
\begin{align*}
\text{div}Tu &= 0 \quad \text{in} \quad \Omega, \\
u u &= y \tan x \quad \text{on} \quad \partial\Omega.
\end{align*}
\]

Then \( u \equiv y \tan x \) in \( \Omega \); in other words, \( u \) must be a helicoid.

**Proof.** For any \( y > 0 \), let

\[
\begin{align*}
\Omega_y &= (b, a) \times (-y, y), \\
\Gamma_y &= \{(b, a) \times \{y\}\} \cup \{(b, a) \times \{-y\}\}
\end{align*}
\]

and, set

\[
g(y) = \int_{\Gamma_y} (u - v)(Tu - Tv) \cdot \nu ds = \int_{\partial\Omega_y} (u - v)(Tu - Tv) \cdot \nu ds = \int_{\Omega_y} (\nabla u - \nabla v) \cdot (Tu - Tv)
\]

where \( v \equiv y \tan x \) and \( \nu \) is the unit outward normal of \( \Gamma_y \) and \( \partial\Omega_y \). Since \((\nabla u - \nabla v) \cdot (Tu - Tv) \geq 0\), Fubini's Theorem yields that the derivative \( g'(y) \)
exists for almost all \( y > 0 \) and
\[
g'(y) = \int_{\Gamma_y} (\nabla u - \nabla v) \cdot (Tu -Tv)
\]
whenever \( g'(y) \) exists. Thus, in view of (2), for these \( y \),
\[
g'(y) \geq \int_{\Gamma_y} \frac{\sqrt{1 + |\nabla u|^2} + \sqrt{1 + |\nabla v|^2}}{2} |Tu - Tv|^2
\]
\[
\geq \left( \min_{\Gamma_y} \frac{\sqrt{1 + |\nabla v|^2}}{2} \right) \int_{\Gamma_y} |Tu - Tv|^2,
\]
in which, as \( v_x = y \sec^2 x \), we have
\[
\frac{\sqrt{1 + |\nabla v|^2}}{2} \geq \frac{y \sec^2 x}{2} \geq \frac{y}{2}.
\]
Furthermore, by means of Schwarz's inequality,
\[
|\Gamma_y| \int_{\Gamma_y} |Tu - Tv|^2 \geq \left( \int_{\Gamma_y} |Tu - Tv| \right)^2,
\]
and \(|\Gamma_y| = 2(a - b)\) (in virtue of the special geometry of \( \Omega \)), thus
\[
\int_{\Gamma_y} |Tu - Tv|^2 \geq \frac{1}{2(a - b)} \left( \int_{\Gamma_y} |Tu - Tv| \right)^2.
\]
Hence, for any \( y \) where \( g'(y) \) exists,
\[
g'(y) \geq \frac{y}{4(a-b)} \left( \int_{\Gamma_y} |Tu - Tv| \right)^2
\]
\[
\geq \frac{y}{4(a-b)} \left( \frac{1}{\pi} \int_{\Gamma_y} \tan^{-1}(u - v)(Tu - Tv) \cdot \nu \right)^2.
\]
Now, for all \( y > 0 \), set
\[
h(y) = \int_{\Gamma_y} \tan^{-1}(u - v)(Tu - Tv) \cdot \nu
\]
\[
= \int \int_{\Omega_y} \frac{(\nabla u - \nabla v) \cdot (Tu - Tv)}{1 + (u - v)^2}.
\]
We note that \( h \geq 0 \) and \( h(y) \) increases as \( y \) increases. Thus, if \( h \equiv 0 \), it is easy to see that Theorem 2* holds. Hence we may assume that \( h \neq 0 \) and
that there exist two positive constants $y_1$ and $c_1$ such that $h(y) \geq c_1$ for all $y \geq y_1$.

Substituting this into (3), we obtain $g'(y) \geq \frac{c^2}{4(a-b)^2} y$ for almost all $y \geq y_1$, which yields $g(y) - g(y_1) \geq \frac{c^2}{4(a-b)^2}(y - y_1)^2$. Since $|u| = O(|y|)$ on $\partial \Omega$ as $|y| \to \infty$, by [7, p. 256], we have $|u| = O(|y|)$ in $\Omega$ as $|y| \to \infty$. Since for all $y > 0$, $g(y) = \int_y^\infty (u-v)(Tu-Tv) \cdot \nu$ and $|Tu-Tv| \leq 2$, we have $g(y) = O(y)$ as $y \to \infty$, which gives a contradiction and completes our proof. \qed

By modifying the proof of Theorem 2*, we can derive the following

**Theorem 3.** Let $\Omega \subseteq \mathbb{R}^2$ be an unbounded domain and let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Let $B_R, \Gamma_R$ and $|\Gamma_R|$ be as in Theorem 1. Suppose that

\begin{align*}
(i) \quad \text{div} \, Tu &= \text{div} \, Tv \quad \text{in} \quad \Omega, \\
(ii) \quad u &= v \quad \text{on} \quad \partial \Omega, \\
(iii) \quad \max_{\Omega \cap B_R} |u-v| &= o\left(\frac{1}{|\Gamma_R|} \min_{\Gamma_R} \sqrt{1 + |\nabla v|^2} \, dR\right) \quad \text{as} \quad R \to \infty,
\end{align*}

where $R_0$ is a positive constant. Then we have $u \equiv v$ in $\Omega$.

**Remark.**

(a) Notice that condition (iii) depends on $|\nabla v|$ only, without assuming any condition on $|\nabla u|$.

(b) In Theorem 2*, since $\text{div} \, Tu = 0$ in $\Omega$ and $u = y \tan x$ on $\partial \Omega$, by [7, p. 256], we have $u = O(|y|)$ in $\Omega$ as $|y| \to \infty$. And so, condition (iii) of Theorem 3 holds.

**Proof of Theorem 3.** The proof is similar to that of Theorem 2*. For every $R > 0$, let

\begin{align*}
M(R) &= \max_{\Omega \cap B_R} |u-v| = \max_{\Gamma_R} |u-v|, \\
Q(R) &= \min_{\Gamma_R} \frac{\sqrt{1 + |\nabla v|^2}}{2}, \\
g(R) &= \int_{\Gamma_R} (u-v)(Tu-Tv) \cdot \nu = \int_{\Omega_R} (\nabla u - \nabla v) \cdot (Tu-Tv) \\
\text{and} \\
h(R) &= \int_{\Gamma_R} \tan^{-1}(u-v)(Tu-Tv) \cdot \nu.
\end{align*}

As in the proof of Theorem 2*, we may assume that $h \neq 0$ and that there exist two positive constants $R_1$ and $C_1$ such that $R_1 > R_0$ and

\begin{equation}
4
h(R) \geq C_1 \quad \text{for all} \quad R \geq R_1.
\end{equation}
For almost all $R > 0$, we have

$$g'(R) = \int_{\gamma_R} (\nabla u - \nabla v) \cdot (Tu - Tv)$$

$$\geq \int_{\gamma_R} Q(R)|Tu - Tv|^2$$

$$\geq Q(R)|\Gamma_R|^{-1} \left( \int_{\Gamma_R} |Tu - Tv| \right)^2.$$ 

Thus $g'(R) \geq (\frac{\pi}{2})^2 C_1^2 |\Gamma_R|^{-1} Q(R)$, for almost all $R > R_1$. Hence, for every $R$ and $R_2$ such that $R > R_2 > R_1$, we have

$$g(R) - g(R_2) \geq \left( \frac{2C_1}{\pi} \right)^2 \int_{R_2}^R \frac{Q(r)}{|\Gamma_r|} dr.$$ 

By (4), we have $M(R) > 0$ for all $R \geq R_1$, hence (5) yields, for almost all $R \geq R_1$,

$$g'(R) \geq Q(R)|\Gamma_R|^{-1} \int |Tu - Tv|^2$$

$$\geq g^2(R)Q(R)$$

$$M^2(R)|\Gamma_R|^{-1};$$

and so, for every $R$ and $R_2$ such that $R > R_2 > R_1$,

$$- \frac{1}{g} \bigg|_{R_2}^R \geq \int_{R_2}^R \frac{g'}{g^2} \geq \int_{R_2}^R \frac{Q(r)}{M^2(r)|\Gamma_r|} dr \geq \frac{1}{M^2(R)} \int_{R_2}^R \frac{Q(r)}{|\Gamma_r|} dr,$$

and then

$$\frac{1}{g(R_2)} \geq \frac{1}{M^2(R)} \int_{R_2}^R \frac{Q(r)}{|\Gamma_r|} dr.$$ 

Now, since $M(R) > 0$ for all $R \geq R_1$, $M(R)$ is an increasing function of $R$ and, in view of condition (iii),

$$(M(R))^{-1} \int_{R_1}^R \frac{Q(r)}{|\Gamma_r|} dr \to \infty \quad \text{as } R \to \infty,$$

and also

$$\int_{R_1}^R \frac{Q(r)}{|\Gamma_r|} dr \to \infty \quad \text{as } R \to \infty,$$

hence we can choose a constant $R_3 > R_1$ such that

$$(M(R))^{-1} \int_{R_1}^R \frac{Q(r)}{|\Gamma_r|} \geq \sqrt{2\pi} C_1^{-1}, \quad \text{for every } R \geq R_3,$$
UNIQUENESS FOR MINIMAL SURFACE EQUATION

and a constant $R_4$, $R_4 > R_3$, which depends on $R_3$, such that
\[ \int_{R_1}^{R_4} \frac{Q(r)}{|\Gamma_r|} \, dr = 2 \int_{R_1}^{R_3} \frac{Q(r)}{|\Gamma_r|} \, dr. \]

With this choice of $R_3$ and $R_4$, we have
\[
1 \geq \frac{g(R_3) - g(R_1)}{g(R_3)} \\
\geq \left[ \left( \frac{2C_1}{\pi} \right)^2 \int_{R_1}^{R_3} \frac{Q(r)}{|\Gamma_r|} \right] \left[ (M^2(R_4))^{-1} \int_{R_3}^{R_4} \frac{Q(r)}{|\Gamma_r|} \right] \quad \text{(by (6), (7))} \\
= \left[ \left( \frac{2C_1}{\pi} \right)^2 (M^2(R_4))^{-1} \right] \frac{1}{4} \left( \int_{R_1}^{R_3} \frac{Q(r)}{|\Gamma_r|} \right)^2 \quad \text{(by the choice of } R_3, R_4) \\
\geq \frac{C_1^2}{\pi^2} (2\pi^2) C_1^{-2} \quad \text{(again by the choice of } R_3 \text{ and } R_4) \\
\geq 2,
\]
which is desired contradiction. \( \square \)

**Remark.** The above proof is to show (6), which is the lower bound of $g(R)$, and (7), which is the upper bound of $g(R)$. And from (6) and (7), we get contradiction and so prove the theorem. Miklyukov [5] and Hwang [4] only observed the upper bound of $g(R)$, and so could not derive the better result as in Collin and Krust [2].

Let $\Omega$ be a domain in $\mathbb{R}^2$. Consider the following equation in divergence form
\[
\text{div} A(x, u, \nabla u) = f(x, u, \nabla u),
\]
where
\[
A = (A_1, A_2), \quad A_i : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}, \quad i = 1, 2, \\
f : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R},
\]
and
\[
A_i \in C^0 \left( \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \right) \cap C^1(\Omega \times \mathbb{R} \times \mathbb{R}^2), \quad i = 1, 2, \quad f \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^2).
\]

We rewrite $A(x, u, \nabla u)$ briefly as $Au$.

Suppose that $Au$ satisfies the following structural condition:
\[
\begin{cases} 
(Au - Av) \cdot (\nabla u - \nabla v) \geq |Au - Av|^2 Q(R), \\
\text{ where } R = \sqrt{x^2 + y^2} \text{ and } Q(R) \text{ is a positive function,} \\
(\nabla u - \nabla v) \cdot (Au - Av) = 0, \quad \text{iff } \nabla u = \nabla v.
\end{cases}
\]

(8)
Now we have the following result:

**Theorem 4.** Let $\partial \Omega = \Sigma^\alpha + \Sigma^\beta$ be a decomposition of $\partial \Omega$ such that $\Sigma^\beta \in C^1$. Let $u, v \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma^\beta) \cap C^0(\overline{\Omega})$ and let $M(R) = \max_{\Omega \cap B_R}(u - v, 0)$. Suppose that

\[
\begin{align*}
(i) & \quad A \\
(ii) & \quad \text{div } Au \geq \text{div } Av \quad \text{ in } \Omega \\
(iii) & \quad u \leq v \quad \text{ on } \Sigma^\alpha \\
(iv) & \quad Au \cdot \nu \leq Av \cdot \nu \quad \text{ on } \Sigma^\beta \\
v) & \quad M(R) = o \left( \int_{R_0}^R \frac{|Q_r|}{|r|} \, dr \right) \quad \text{as } R \to \infty, \text{ where } R_0 \text{ is a positive constant.}
\end{align*}
\]

Then, if $\partial \Omega = \Sigma^\beta$, we have either $u(x) = v(x) + a$ positive constant or else $u(x) \leq v(x)$. Otherwise, $u(x) = v(x)$.

The proof of Theorem 4 is exactly the same as that of Theorem 3. The interested readers may consult [4].

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**References**


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