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GORENSTEIN T-SPREAD VERONESE ALGEBRAS

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Abstract

Let $S = K[x_1, x_2, \dots, x_n]$ be the polynomial ring in n variables over a field K. We fix integers d and t. A monomial $x_{i_1}x_{i_2}\cdots x_{i_d}$ with $i_1 \le i_2 \le \cdots \le i_d$ is t-spread if $i_j - i_{j-1} \ge t$, for any $2 \le j \le n$. Let $I_{n,d,t}$ be the ideal generated by all t-spread monomials of degree d and let $K[I_{n,d,t}]$ be the toric algebra generated by the monomials v with $v \in G(I_{n,d,t})$. This generalizes the classical (squarefree)Veronese algebras. The aim of this paper is to characterize the algebras $K[I_{n,d,t}]$ which are Gorenstein.

Introduction

Let K be a field and let $S = K[x_1, x_2, \dots, x_n]$ be the polynomial ring in n indeterminates over K. In the paper [7], Ene, Herzog and Qureshi introduced the concept of t-spread monomials. We fix integers d and t. A monomial $x_{i_1}x_{i_2}\cdots x_{i_d}$ with $i_1 \le i_2 \le \cdots \le i_d$ is t-spread if $i_j - i_{j-1} \ge t$, for any $2 \le j \le n$. Thus any monomial is 0-spread and a squarefree monomial is 1-spread. A monomial ideal in S is called a t-spread monomial ideal if it is generated by t-spread monomials. For example, $I = (x_1x_3x_7, x_1x_4x_7, x_1x_5x_8) \subset K[x_1, x_2, \dots, x_8]$ is a 2-spread monomial ideal.

Let $d \ge 1$ be an integer. A monomial ideal in S is called a t-spread Veronese ideal of degree d if it is generated by all t-spread monomials of degree d. We denote it by $I_{n,d,t}$. Note that $I_{n,d,t} \neq 0$ if and only if $n > t(d-1)$. For example, if $n = 5, d = 2$ and $t = 2$, then

$$I_{5,2,2} = (x_1x_3, x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_3x_5) \subset K[x_1, x_2, \dots, x_5].$$

We consider the toric algebra generated by the monomials v with $v \in G(I_{n,d,t})$, here, for a monomial ideal I, $G(I)$ denotes the minimal system of monomial generators of I. This is called a t-spread Veronese algebra and we denote it by $K[I_{n,d,t}]$. It generalizes the classical (squarefree)Veronese algebras. By [7, Corollary 3.4], the t-spread Veronese algebra is a Cohen-Macaulay domain.

We fix an integer d and a sequence $\mathbf{a} = (a_1, \dots, a_n)$ of integers with $1 \le a_1 \le a_2 \le \cdots \le a_n \le d$ and $d = \sum_{i=1}^n a_i$. The K-subalgebra of $S = K[x_1, x_2, \dots, x_n]$ generated by all monomials of the form $t_1^{c_1}t_2^{c_2}\cdots t_n^{c_n}$ with $\sum_{i=1}^n c_i = d$ and $c_i \le a_i$ for each $1 \le i \le n$ is called an algebra of Veronese type and it is denoted by $A(\mathbf{a}, d)$. If each $a_i = 1$, then $A(\mathbf{1}, d)$ is generated by all the squarefree monomials of degree d in S.

De Negri and Hibi proved in [8, Theorem 2.4] that, in the squarefree case, the algebra of Veronese type $A(\mathbf{1}, d)$ is Gorenstein if and only if

(i) $d = n$, or

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(ii) $d = n - 1$, or

(iii) $d < n - 1$ and $n = 2d$.

The aim of this paper is to characterize the toric algebras $K[I_{n,d,t}]$ which have the Gorenstein property. Our approach is rather geometric. We identify the t-spread Veronese algebra, $K[I_{n,d,t}]$, with the Ehrhart ring $\mathcal{A}(\mathcal{P})$ associated to a suitable polytope $\mathcal{P}$, and next we employ Hibi's results in [12] which caracterize the Gorenstein property of $\mathcal{A}(\mathcal{P})$.

The main result of this paper, Theorem 3.4, classifies the t-spread Veronese algebras which are Gorenstein. Namely, we show that, for $d, t \geq 2$, $K[I_{n,d,t}]$ is Gorenstein if and only if $n \in \{(d-1)t+1, (d-1)t+2, dt, dt+1, dt+d\}$. We illustrate all our results with suitable examples. We also see that, in this cases, the h^*-vector of the t-spread Veronese algebra $K[I_{n,d,t}]$ is unimodal.

1. The Ehrhart ring of a rational convex polytope

Let $\mathcal{P} \subset \mathbb{R}^N$ be a convex polytope of dimension d and let $\partial\mathcal{P}$ be the boundary of $\mathcal{P}$. Then $\mathcal{P}$ is called *of standard type* if $d = N$ and the origin of $\mathbb{R}^N$ is contained in the interior of $\mathcal{P}$. We call a polytope $\mathcal{P}$ *rational* if every vertex of $\mathcal{P}$ has rational coordinates and *integral* if every vertex of $\mathcal{P}$ has integral coordinates. The *Ehrhart ring* of $\mathcal{P}$ is $\mathcal{A}(\mathcal{P}) = \bigoplus_{n\geq 0} \mathcal{A}(\mathcal{P})_n$, where $\mathcal{A}(\mathcal{P})_n$ is the K-vector space generated by the monomials $\{x^a y^n : a \in n\mathcal{P} \cap \mathbb{Z}^n\}$. Here $n\mathcal{P}$ denotes the dilated polytope $\{(na_1, na_2, \ldots, na_d) : (a_1, a_2, \ldots, a_d) \in \mathcal{P}\}$. It is known that $\mathcal{A}(\mathcal{P})$ is a finitely generated K-algebra and a normal domain ([16, Theorem 9.3.6]). The reader can find more about Ehrhart rings of rational convex polytopes in [2] and [16].

Let $\mathcal{P} \subset \mathbb{R}^d$ be a d-dimensional convex polytope of standard type. Then the *dual polytope of* $\mathcal{P}$ is

$$\mathcal{P}^* = \{(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d : \textstyle\sum_{i=1}^d \alpha_i\beta_i \leq 1, \text{ for all } (\beta_1, \ldots, \beta_d) \in \mathcal{P}\}.$$

One can check that $\mathcal{P}^*$ is a convex polytope of standard type and $(\mathcal{P}^*)^* = \mathcal{P}$;(see [4, Exercise 1.14] or [17, Chapter 2]). It is known the fact that if $(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ and if $H \subset \mathbb{R}^d$ is the hyperplane defined by the equation $\sum_{i=1}^d \alpha_i x_i = 1$, then $(\alpha_1, \ldots, \alpha_d)$ is a vertex of $\mathcal{P}^*$ if and only if $H \cap \mathcal{P}$ is a facet of $\mathcal{P}$, see [17, Chapter 2]. Therefore, the dual polytope of a rational convex polytope is rational. In order to classify the t-spread Veronese algebras which are Gorenstein, we will show that any t-spread Veronese algebra coincides with the Ehrhart ring of an integral convex polytope, so we need a criterion for the Ehrhart ring $\mathcal{A}(\mathcal{P})$ to be Gorenstein.

Let $\mathcal{P}$ be an integral polytope in $\mathbb{R}^d_+$ of $\dim \mathcal{P} = d$. We consider the toric ring $K[\mathcal{P}]$ which is generated by all the monomials $x_1^{a_1} \ldots x_n^{a_n} s^q$ with $a = (a_1, a_2, \ldots, a_n) \in \mathcal{P} \cap \mathbb{Z}^n$ and $q = a_1 + a_2 + \cdots + a_n$. It is known that if $K[\mathcal{P}]$ is a normal ring, then $K[\mathcal{P}]$ is Cohen-Macaulay ([3, Theorem 6.3.5]).

Theorem 1.1 (Stanley, Danilov [14], [6])**.** *Let $\mathcal{P} \subset \mathbb{R}^d_+$ be an integral convex polytope and suppose that its toric ring $K[\mathcal{P}]$ is normal, thus $K[\mathcal{P}] = \mathcal{A}(\mathcal{P})$. Then the canonical module $\Omega(K[\mathcal{P}])$ of $K[\mathcal{P}]$ coincides with the ideal of $K[\mathcal{P}]$ which is generated by those monomials $x^a s^q$ with $a \in q(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^d$.*

By [9, Proposition A.6.6], the Cohen-Macaulay type of a Cohen-Macaulay graded S-module M of dimension d coincides with $\beta^S_{n-d}(M)$. In particular, a Cohen-Macaulay ring $R = S/I$ is Gorenstein if and only if $\beta^S_{n-d}(R) = 1$. Let $\mathcal{P}$ be a polytope as in Theorem 1.1 and

$\delta \geq 1$ be the smallest integer such that $\delta(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^d \neq \emptyset$. Then, by [13], the **a**-invariant is

$$\mathbf{a}(K[\mathcal{P}]) = -\min(\omega_{K[\mathcal{P}]}) \neq 0 = -\delta.$$

REMARK 1. By [9, Corollary A.6.7], $K[\mathcal{P}]$ is Gorenstein if and only if $\Omega(K[\mathcal{P}])$ is a principal ideal. In particular, if $K[\mathcal{P}]$ is Gorenstein, then $\delta(\mathcal{P}-\partial\mathcal{P})$ must posses a unique interior vector.

Theorem 1.2 (Hibi,[12])**.** *Let $\mathcal{P}$ be a integral convex polytope of dimension d and let $\delta \geq 1$ be the smallest integer for which $\delta(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^d \neq \emptyset$. Fix $\alpha \in \delta(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^d \neq \emptyset$ and denote by $\mathcal{Q}$ the rational convex polytope of standard type $\mathcal{Q} = \delta\mathcal{P} - \alpha \subset \mathbb{R}^d$. Then the Ehrhart ring of $\mathcal{P}$ is Gorenstein if and only if the dual polytope $\mathcal{Q}^*$ of $\mathcal{Q}$ is integral.*

A sketch of a proof of the Theorem 1.2 can be found in [9, Section 12.5] and an algebraic proof of the same theorem can be found in [13].

2. t-spread Veronese algebras

Let $\mathcal{M}_{n,d,t}$ be the set of t-spread monomials of degree d in n variables.

To begin with, we study when the t-spread Veronese algebra, $K[I_{n,d,t}]$, is a polynomial ring. If $t = 1$, then the t-spread Veronese algebra coincides with the classical squarefree Veronese algebra and those which are Gorestein are studied by De Negri and Hibi in [8]. Assume $t \geq 2$. If $n = (d-1)t + 1$, then $K[I_{n,d,t}]$ has only one generator, thus it is Gorenstein. Therefore, in what follows, we always consider $t \geq 2$ and $n \geq (d-1)t + 2$.

In order to study when the t-spread Veronese algebra $K[I_{n,d,t}]$ is a polynomial ring, we need to study sorted sets of monomials, a concept introduced by Sturmfels ([15]). Let S_d be the K-vector space generated by the monomials of degree d in S and let $u, v \in S_d$ be two monomials. We write $uv = x_{i_1}x_{i_2}\ldots x_{i_{2d}}$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{2d} \leq n$ and we define

$$u' = x_{i_1}x_{i_3}\ldots x_{i_{2d-1}}, v' = x_{i_2}x_{i_4}\ldots x_{i_{2d}}.$$

The pair (u', v') is called the *sorting* of (u, v) and the map

$$\text{sort} : S_d \times S_d \to S_d \times S_d, (u, v) \mapsto (u', v')$$

is called the *sorting operator*. A pair (u, v) is *sorted* if $\text{sort}(u, v) = (u, v)$. For example, $(x_1^2x_2x_3, x_1x_2x_3^2)$ is a sorted pair. Notice that if (u, v) is sorted, then $u >_{lex} v$ and $\text{sort}(u, v) = \text{sort}(v, u)$. If $u_1 = x_{i_1}\ldots x_{i_d}, u_2 = x_{j_1}\ldots x_{j_d}, \ldots, u_r = x_{l_1}\ldots x_{l_d}$, then the r-tuple $(u_1, \ldots, u_r)$ is sorted if and only if

$$i_1 \leq j_1 \leq \cdots \leq l_1 \leq i_2 \leq j_2 \leq \cdots \leq l_2 \leq \cdots \leq i_d \leq j_d \leq \cdots \leq l_d,$$

which is equivalent to (u_i, u_j) being sorted, for all $i > j$.

Proposition 2.1. *Let $u_1, \ldots, u_q$ be the generators of $K[I_{n,d,t}]$. If $n = (d-1)t + 2$, then any $r-$tuple $(u_1, \ldots, u_r)$ with $u_1 \geq_{lex} u_2 \geq_{lex} \cdots \geq_{lex} u_r$ is sorted.*

Proof. It suffices to show that any pair (u_i, u_j) with $u_i >_{lex} u_j$ is sorted. Let $u_i = x_{i_1}x_{i_2}\cdots x_{i_d}$ with $i_k - i_{k-1} \geq t$, for any $k \in \{2, \ldots, d\}$ and $v_j = x_{j_1}x_{j_2}\cdots x_{j_d}$ with $j_k - j_{k-1} \geq t$, for any $k \in \{2, \ldots, d\}$. Since $n = (d-1)t + 2$, the smallest monomial is $x_2x_{t+2}\cdots x_{(d-1)t+2}$ and the largest monomial is $x_1x_{t+1}\cdots x_{(d-1)t+1}$, with respect to lexicographic order. Then $1 + jt \leq i_{j+1} \leq 2 + jt$, for any $j \in \{0, 1, \ldots, d-1\}$. Since $u_i >_{lex} u_j$, we have

$$u_i = x_1\ldots x_{1+(k-1)t}x_{2+kt}\ldots x_{2+(d-1)t} \text{ and}$$

$$u_j = x_1 \dots x_{1+(l-1)t} x_{2+lt} \dots x_{2+(d-1)t}, \text{ for some } k > l.$$

Then one easily sees that (u_i, u_j) is always sorted. □

Corollary 2.2. *Let $n \geq (d-1)t+2$. The t-spread Veronese algebra $K[I_{n,d,t}]$ is a polynomial ring if and only if $n = (d-1)t+2$. In particular, $K[I_{n,d,t}]$ is Gorestein if $n = (d-1)t+2$.*

Proof. Let $u_1, \dots, u_q$ be the generators of $K[I_{n,d,t}]$. We want to show that these elements are algebraically independent over the field K. Let $f = \sum_\alpha a_\alpha y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_q^{\alpha_q}$ be a polynomial such that $f(u_1, \dots, u_q) = 0$. By Proposition 2.1, any r-tuple $(u_1, \dots, u_r)$ of generators with $u_1 \geq_{lex} \cdots \geq_{lex} u_r$ is sorted, which implies that the monomials $u_1^{\alpha_1} \dots u_q^{\alpha_q}$ are all pairwise distinct. Then the coefficients a_α are all zero, which implies that $u_1, \dots, u_q$ are algebraically independent over K.
For the converse part, assume that there exists $n \geq (d-1)t+3$ such that $K[I_{n,d,t}]$ is a polynomial ring. Then it is clear that if $u = x_1 x_{t+1} \cdots x_{(d-2)t+1} x_n$ and $v = x_2 x_{t+2} \cdots x_{(d-2)t+2} x_{n-1}$, then (u, v) is unsorted and the pair (u', v'), where $u' = x_1 x_{t+1} \cdots x_{(d-2)t+1} x_{n-1}$ and $v' = x_2 x_{t+2} \cdots x_{(d-2)t+2} x_n$ is the sorting pair of (u, v) and the equality $uv - u'v'$ gives a non-zero polynomial in the defining ideal of $K[I_{n,d,t}]$, contradicting the fact that $K[I_{n,d,t}]$ is a polynomial ring. □

Moreover, we can make a stronger reduction. Let $n < dt$. Then the smallest t-spread monomial of degree d is $x_{n-(d-1)t} x_{n-(d-2)t} \dots x_n$. As $n - (d-1)t < t$, the generators of $I_{n,d,t}$ can be viewed in a polynomial ring in the variables $\{x_1, \dots, x_n\} \setminus \cup_{l=1}^{d-1} \{x_{n-dt+lt+1}, \dots, x_{lt}\}$. Thus $K[I_{n,d,t}] \subset S'$, where

$$S' = K[\{x_1, \dots, x_n\} \setminus \cup_{l=1}^{d-1} \{x_{n-dt+lt+1}, \dots, x_{lt}\}],$$

which is a polynomial ring in $n' = n - (d-1)(dt-n) = d(n-(d-1)t)$ variables. Note that, in S', $I_{n,d,t}$ is a t'-spread ideal, where $t' = n - (d-1)t$. Thus, $n' = dt'$. This discussion shows that, in what follows, we may consider $n \geq dt$.

Theorem 2.3. (i) *If $n \geq dt+1$, then* $\dim K[I_{n,d,t}] = n$.
(ii) *If $n = dt$, then* $\dim K[I_{n,d,t}] = n - d + 1$.

Proof. (i). We denote by y_i the d-th power of the variable x_i, for any $1 \leq i \leq n$. Let $A = K[I_{n,d,t}]$. We prove that $y_1, y_2, \dots, y_n$ belong to the quotient field of A, denoted by $Q(A)$. We first show by induction on $0 \leq k \leq d-1$ that $y_{kt+j} \in Q(A)$, for any $1 \leq j \leq t$.

We check for $k = 0$: it is clear that

$$y_1 = x_1^d = \frac{\prod_{j=1}^d x_1 x_{t+1} \dots \widehat{x_{jt+1}} \dots x_{td+1}}{(x_{t+1} \dots x_{dt+1})^{d-1}} \in Q(A).$$

Here, by $\widehat{x_{jt+1}}$, we mean that the variable x_{jt+1} is missing.
Since $y_1 y_{t+j} \dots y_{(d-1)t+j} \in Q(A)$, for $1 \leq j \leq t$, we get $y_{t+j} \dots y_{(d-1)t+j} \in Q(A)$, for $1 \leq j \leq t$. But also $y_j y_{t+j} \dots y_{(d-1)t+j} \in Q(A)$, so we obtain $y_j \in Q(A)$, for any $1 \leq j \leq t$. Therefore, it follows that $y_1, \dots, y_t$ belong to $Q(A)$.

Assume that $y_1, y_2, \dots, y_t, \dots, y_{kt+1}, \dots, y_{(k+1)t} \in Q(A)$. We want to prove that $y_{(k+1)t+1}, \dots, y_{(k+2)t}$ are also in $Q(A)$. Firstly, let us check if $y_{(k+1)t+1}$ belongs to $Q(A)$. Notice that, since $y_{t+1} y_{2t+1} \dots y_{kt+1} y_{(k+1)t+1} \dots y_{dt+1}$ and $y_{t+1}, y_{2t+1}, \dots, y_{kt+1} \in Q(A)$ by our assumption, it follows that $y_{(k+1)t+1} \dots y_{dt+1} \in Q(A)$.
Also, since $y_1 y_{2t+1} \dots y_{kt+1} y_{(k+2)t+1} \dots y_{dt+1} \in Q(A)$, using our assumption, we get

$y_{(k+2)t+1} \dots y_{dt+1} \in Q(A)$. But since $y_{(k+1)t+1} \dots y_{dt+1}$ is in $Q(A)$, it follows that $y_{(k+1)t+1} \in Q(A)$.

Now we check that $y_{(k+1)t+s} \in Q(A)$, for any $2 \leq s \leq t$. Using the monomials $y_s \dots y_{kt+s} y_{(k+1)t+s} \dots y_{(d-1)t+s} \in Q(A)$ and $y_s, \dots, y_{kt+s} \in Q(A)$ by our assumption, we get

$$y_{(k+1)t+s} \dots y_{(d-1)t+s} \in Q(A).$$

Moreover, $y_1 \dots y_{kt+1} y_{(k+1)t+1} y_{(k+2)t+s} \dots y_{(d-1)t+s}$ is in $Q(A)$, so by our assumption and by the fact that $y_{(k+1)t+1} \in Q(A)$, we obtain

$$y_{(k+2)t+s} y_{(k+3)t+s} \dots y_{(d-1)t+s} \in Q(A).$$

Therefore, using $y_{(k+1)t+s} \dots y_{(d-1)t+s}$ and $y_{(k+2)t+s} \dots y_{(d-1)t+s}$ in $Q(A)$, we get

$$y_{(k+1)t+s} \in Q(A),$$

for any $2 \leq s \leq t$. So far, we have seen that $y_{kt+j} \in Q(A)$, also for any $0 \leq k \leq d-1$ and $1 \leq j \leq t$. Let now $dt+1 \leq m \leq n$. Then $y_1 y_{t+1} \dots y_{(d-1)t+1} y_m \in Q(A)$. Since $y_1, y_{t+1}, \dots, y_{(d-1)t+1} \in Q(A)$, it follows that $y_m \in Q(A)$ as well. Therefore, $Q(A) \supset \{x_1^d, \dots, x_n^d\}$. It follows that $\dim A = \operatorname{trdeg} Q(A) \geq n$, since $x_1^d, \dots, x_n^d$ are obviously algebraic independent over K. But since A is a subalgebra of $K[x_1, \dots, x_n]$, by [10, Proposition 3.1], $\dim A \leq n$. Therefore, $\dim A = n$.
(ii). It follows from [1, Corollary 3.2]. □

REMARK 2. The result from part (i) of Theorem 2.3 also follows from [1, Corollary 3.2], but we preferred to give a completely different proof here.

Let $\mathcal{P} \subset \mathbb{R}^n$ denote the rational convex polytope

$$\mathcal{P} = \{(a_1, \dots, a_n) \in \mathbb{R}^n : \textstyle\sum_{i=1}^n a_i = d, a_i \geq 0, \text{ for } 1 \leq i \leq n, \text{ and } a_i + \cdots + a_{i+t-1} \leq 1, \text{ for } 1 \leq i \leq n-t+1\}.$$

Clearly, $K[I_{n,d,t}] = K[\mathcal{P}]$, since $K[I_{n,d,t}]$ is generated by the monomials of $G(I_{n,d,t})$, that is, by monomials $x_1^{a_1} \dots x_n^{a_n}$ with $\sum_{i=1}^n a_i = d, a_i \geq 0$, for $1 \leq i \leq n$ and $a_i + \cdots + a_{i+t-1} \leq 1$, for $1 \leq i \leq n-t+1$.

Since $K[I_{n,d,t}]$ is a normal ring, by [10, Lemma 4.22], we get the following

Theorem 2.4. *The t-spread Veronese algebra $K[I_{n,d,t}]$ is the Ehrhart ring $\mathcal{A}(\mathcal{P})$.*

3. Gorenstein t-spread Veronese algebras

In this section we classify the Gorenstein t-spread Veronese algebras. We split the classification in several theorems.

Theorem 3.1. *If $n = dt + k$, $2 \leq k \leq d-1$, then in $(t+d)P$ there exist d interior lattice points. Therefore, $K[I_{n,d,t}]$ is not Gorenstein.*

Proof. By Theorem 2.3, $\dim K[I_{n,d,t}] = n$, thus $\dim(\mathcal{P}) = n-1$. Let H be the hyperplane in $\mathbb{R}^n$ defined by the equation $a_1 + \cdots + a_n = d$ and let $\phi : \mathbb{R}^{n-1} \to H$ denote the affine map defined by

$$\phi(a_1, \dots, a_{n-1}) = (a_1, \dots, a_{n-1}, d - (a_1 + \cdots + a_{n-1})),$$

for $(a_1,\dots,a_{n-1})\in\mathbb{R}^{n-1}$. Then ϕ is an affine isomorphism and $\phi(\mathbb{Z}^{n-1}) = H\cap\mathbb{Z}^n$. Therefore, $\phi^{-1}(\mathcal{P})$ is an integral convex polytope in $\mathbb{R}^{n-1}$ of $\dim\phi^{-1}(\mathcal{P}) = \dim\mathcal{P} = n-1$. The Ehrhart ring $\mathcal{A}(\phi^{-1}(\mathcal{P}))$ is isomorphic with $\mathcal{A}(\mathcal{P})$ as graded algebras over K. Thus, we want to see if $\mathcal{A}(\phi^{-1}(\mathcal{P}))$ is Gorenstein, and, by abuse of notation, we write $\mathcal{P}$ instead of $\phi^{-1}(\mathcal{P})$. Thus,

$$\mathcal{P} = \{(a_1,\dots,a_{n-1})\in\mathbb{R}^{n-1} : a_i\geq 0, \text{ for } 1\leq i\leq n-1, \text{ and } a_i+a_{i+1}+\cdots+a_{i+t-1}\leq 1, \text{ for } 1\leq i\leq n-t, a_1+a_2+\cdots+a_{n-t}\geq d-1\}.$$

In our hypothesis on n, we show that there are no interior lattice points at lower levels than $t+d$. It is enough to see that there are no interior lattice points at level $t+d-1$. Let $(x_1,\dots,x_{n-1})\in(t+d-1)(\mathcal{P}-\partial\mathcal{P})$. We have

$$(t+d-1)(\mathcal{P}-\partial\mathcal{P}) = \{(a_1,\dots,a_{n-1})\in\mathbb{R}^{n-1} : a_i>0, 1\leq i\leq n-1, a_i+a_{i+1}+\cdots+a_{i+t-1} < (t+d-1), 1\leq i\leq n-t, a_1+a_2+\cdots+a_{n-t} > (d-1)(t+d-1)\}.$$

Since $x_1+\cdots+x_{(d-1)t+k}\geq(d-1)(t+d-1)$, we have $x_1+\cdots+x_{(d-1)t+k-1}\geq(d-1)(t+d-1)-x_{(d-1)t+k}$, thus

$$(d-1)(t+d-2)+\sum_{i=1}^{k-1}x_{(d-1)t+i}\geq\sum_{i=1}^{(d-1)t+k-1}x_i\geq(d-1)(t+d-1)-x_{(d-1)t+k}.$$

It follows that $\sum_{i=1}^{k-1}x_{(d-1)t+i}\geq d-1-x_{(d-1)t+k}$ and, since $x_{(d-1)t+k}<1$, we obtain

$$k-1>\sum_{i=1}^{k-1}x_{(d-1)t+i}\geq d-1,$$

which implies that $k\geq d+1$, a contradiction. Thus, there are no interior lattice points in the dilated polytope at lower levels than $t+d$.

Let $\delta\geq 1$ be the smallest integer such that $\delta(\mathcal{P}-\partial\mathcal{P})\neq\emptyset$. We show that $\delta=t+d$. Indeed, in the dilated polytope $(t+d)(\mathcal{P}-\partial\mathcal{P})$ there are d interior lattice points of the form $\alpha_r=(x_1^{(r)},\dots,x_{n-1}^{(r)})$, $1\leq r\leq d$, where

$$x_j^{(r)} = \begin{cases} d, \text{if } j=it+1,\\ 1, \text{if } j=it+l \text{ with } 0\leq i\leq d-1, 1<l\leq t\\ \quad\text{or } j=dt+l \text{ with } 1\leq l\leq dt+k-2,\\ r, \text{if} j=dt+k-1.\end{cases}$$

It is clear that, for any $1\leq j\leq n-1$, $x_j^{(r)}>0$. For any $1\leq i\leq(d-1)t+1$, $x_i^{(r)}+x_{i+1}^{(r)}+\cdots+x_{i+t-1}^{(r)}=d+t-1<(t+d)(d-1)$. If $k<t$, then $x_{(d-1)t+t-k}^{(r)}+\cdots+x_{dt+k-1}^{(r)}=(t-k)+(k-1)+r=t-1+r<t+d$ and, if $k\geq t$, then $x_{dt+k-t}^{(r)}+\cdots+x_{dt+k-1}^{(r)}=t-1+r<t+d$. Also, $x_1^{(r)}+\cdots+x_{n-t}^{r}=(d+t-1)(d-1)+d+k-1=(d+t)(d-1)+k>(d-1)(t+d)$, since $k\geq 2$. Therefore, these are interior lattice points in $(t+d)\mathcal{P}$. Thus, in this case, the t-spread Veronese algebra $K[I_{n,d,t}]$ is not Gorenstein, by Remark 1. □

Example 1. Let $n=8$, $d=3$ and $t=2$. The smallest level where there are interior lattice points in the dilated polytope is $\delta=5$. In $5(\mathcal{P}-\partial\mathcal{P})$ there are 3 interior lattice points: $(3,1,3,1,3,1,1)$, $(3,1,3,1,3,1,2)$ and $(3,1,3,1,3,1,3)$.
Thus, the 2-spread Veronese algebra $K[I_{8,3,2}]$ is not Gorenstein.

Theorem 3.2. *If $n\geq(t+1)d+1$, then $K[I_{n,d,t}]$ is not Gorenstein.*

Proof. Let $n = kd + q$ with $k \geq t + 1$ and $q \geq 1$. By Theorem 2.3, $\dim K[I_{n,d,t}] = n$, thus $\dim(\mathcal{P}) = n - 1$. Using similar arguments as in Theorem 3.1,

$$\mathcal{P} = \{(a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1} : a_i \geq 0, 1 \leq i \leq n-1, a_i + a_{i+1} + \cdots + a_{i+t-1} \leq 1, 1 \leq i \leq n-t, a_1 + a_2 + \cdots + a_{n-t} \geq d - 1\}.$$

We show that the smallest integer $\delta \geq 1$ such that $\delta(\mathcal{P} - \partial\mathcal{P})$ contains lattice points is $t + 1$. Assume that there are interior lattice points at lower levels than $t + 1$. It is enough to show that there are no interior lattice points at level t. In this case, for each lattice point $(a_1, \dots, a_{n-1}) \in t(\mathcal{P} - \partial\mathcal{P})$, we have $a_i + a_{i+1} + \cdots + a_{i+t-1} < t$, for any $1 \leq i \leq n - t$. Since each $a_i \geq 1$, for any $1 \leq i \leq n - 1$, we have

$$a_i + a_{i+1} + \cdots + a_{i+t-1} \geq t,$$

which is a contradiction.

We show that $(t+1)(\mathcal{P} - \partial\mathcal{P})$ contains only one lattice point which has all the coordinates equal to 1. The interior of the $(t + 1)$-dilated polytope is

$$(t+1)(\mathcal{P} - \partial\mathcal{P}) = \{(a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1} : a_i > 0, 1 \leq i \leq n-1, a_i + a_{i+1} + \cdots + a_{i+t-1} < t+1, 1 \leq 1 \leq n-t, a_1 + a_2 + \cdots + a_{n-t} > (d-1)(t+1)\}.$$

We know that, for each lattice point $(a_1, \dots, a_{n-1}) \in (t + 1)(\mathcal{P} - \partial\mathcal{P})$, we have $a_i + \cdots + a_{i+t-1} \leq t$, for any $1 \leq i \leq n - t$ and, since $a_j \geq 1$, for any $i \leq j \leq i + t - 1$, we obtain $a_i + \cdots + a_{i+t-1} \geq t$, thus we have equality which implies that $a_j = 1$, for any $1 \leq j \leq n - 1$. Hence, $(1, 1, \dots, 1) \in \mathbb{Z}^{n-1}$ is the unique interior lattice point in the dilated polytope $(t+1)\mathcal{P}$. Let us consider $Q = (t + 1)\mathcal{P} - (1, 1, \dots, 1)$. We will show that $K[\mathcal{P}]$ is not Gorenstein by using Theorem 1.2. In fact, we show that the dual Q^* of

$$Q = \{(y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1} : y_i \geq 0, 1 \leq i \leq n-1, y_i + y_{i+1} + \cdots + y_{i+t-1} \leq 0, 1 \leq i \leq n-t, y_1 + \cdots + y_{n-t} \geq (d-1)(t+1) - (n-t)\}.$$

is not an integral polytope. The vertices of Q^* are of the form $(a_1, a_2, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$ such that the hyperplane H of equation $\sum_{i=1}^{n-1} a_i y_i = 1$ has the property that $H \cap Q$ is a facet of Q. In other words, H is a supporting hyperplane of Q. As the hyperplane $\sum_{i=1}^{n-t} y_i = (d-1)(t+1) - (n-t)$, that is,

$$\sum_{i=1}^{n-t} \frac{1}{(d-1)(t+1) - (n-t)} y_i = 1$$

does not have integral coefficients, it follows that Q is not an integral polytope. Thus, by Theorem 1.2, we conclude that $K[I_{n,d,t}]$ is not Gorenstein. □

EXAMPLE 2. Let $n = 10$, $d = 3$ and $t = 2$. Then

$$\mathcal{P} = \{(a_1, \dots, a_9) \in \mathbb{R}^9 : a_i \geq 0, 1 \leq i \leq 9, a_i + a_{i+1} \leq 1, 1 \leq i \leq 8, a_1 + \cdots + a_8 \geq 2\}.$$

For $\delta = 3$, in $3\mathcal{P}$, there exists a unique interior lattice point, namely $(1, 1, 1, 1, 1, 1, 1, 1)$. Let us compute $Q = 3\mathcal{P} - (1, 1, 1, 1, 1, 1, 1, 1, 1)$. We have

$$Q = \{(y_1, \dots, y_9) \in \mathbb{R}^9 : y_i \geq -1, 1 \leq i \leq 9, y_i + y_{i+1} \leq -1, 1 \leq i \leq 8, y_1 + \cdots + y_8 \geq -6\},$$

thus, the dual polytope Q^* is not integral. Therefore, the 2-spread Veronese algebra $K[I_{10,3,2}]$ is not Gorenstein.

Theorem 3.3. *If $n = dt$, then $K[I_{n,d,t}]$ is Gorenstein.*

Proof. In our hypothesis, by Theorem 2.3, $\dim K[I_{n,d,t}] = n - d + 1 = d(t-1) + 1$, thus $\dim \mathcal{P} = d(t-1)$. Using similar arguments as in Theorem 3.1,

$$\mathcal{P} = \{(a_1, a_2, \ldots, a_{n-1}) \in \mathbb{R}^{n-1} : a_i \geq 0, a_i + a_{i+1} + \cdots + a_{i+t-1} \leq 1, 1 \leq i \leq n-t, a_1 + a_2 + \cdots + a_{n-t} \geq d-1\}.$$

We show that the smallest integer $\delta \geq 1$ such that $\delta(\mathcal{P} - \partial\mathcal{P})$ contains lattice points is $t+d-1$. Assume that there are interior lattice points at lower levels than $t+d-1$. It is enough to show that there are no interior lattice points at level $t+d-2$. In this case, for each lattice point $(a_1, \ldots, a_{n-1}) \in (t+d-2)(\mathcal{P} - \partial\mathcal{P})$, we have $a_1 + a_2 + \cdots + a_{n-t} > (d-1)(t+d-2)$. Since $a_i + a_{i+1} + \cdots + a_{i+t-1} < t+d-2$, for any $1 \leq i \leq n-t$, we obtain $a_1 + a_2 + \cdots + a_{n-t} < (d-1)(t+d-2)$, which leads to contradiction. Thus $\delta \geq t+d-1$.

We show that $(t+d-1)\mathcal{P}$ contains a unique interior lattice point. For each lattice point $(a_1, a_2, \ldots, a_{n-1}) \in (t+d-1)\mathcal{P}$, we have $a_1 + a_2 + \cdots + a_{n-t} \geq (d-1)(t+d-1)$. Since $a_i + a_{i+1} + \cdots + a_{i+t-1} \leq t+d-1$, for any $1 \leq i \leq n-t$, we obtain $a_1 + a_2 + \cdots + a_{n-t} \leq (d-1)(t+d-1)$, thus $a_1 + a_2 + \cdots + a_{n-t} = d+t-1$. Hence we obtain $a_{kt+1} + a_{kt+2} + \cdots + a_{kt+t} = t+d-1$, for any $0 \leq k \leq d-2$. So, $a_i + a_{i+1} + \cdots + a_{i+t-1} = t+d-1$, for any $1 \leq i \leq t(d-1)$, with $i \equiv 1 (\mathrm{mod}\, t)$. Since $a_{kt+1} + a_{kt+2} + \cdots + a_{kt+t} = t+d-1$, for any $0 \leq k \leq d-2$, we have

$$a_{kt+2} = t+d-1 - \textstyle\sum_{j \neq kt+2} a_j, \text{ for any } 0 \leq k \leq d-2.$$

Since $a_{kt+2} + a_{kt+3} + \cdots + a_{kt+t+1} \leq t+d-1$, for any $0 \leq k \leq d-2$, we obtain

$$(t+d-1) - \sum_{j \neq kt+2} a_j + a_{kt+3} + \cdots + a_{(k+1)t+1} \leq (t+d-1).$$

Hence, $a_{(k+1)t+1} - a_{kt+1} \leq 0$, for any $0 \leq k \leq d-2$, thus $a_{kt+t+1} \leq a_{kt+1}$, for any $0 \leq k \leq d-2$. Therefore, for each lattice point $(x_1, x_2, \ldots, x_{d(t-1)}) \in (t+d-1)(\mathcal{P} - \partial\mathcal{P})$, we obtain

$$0 < x_{(d-1)+1} < \cdots < x_{t+1} < x_1 < t+d-1$$

and, since there are d consecutive terms in this chain, we have $x_1 \geq d$. If $x_1 > d$, and since each $x_i > 1$, for any $1 \leq i \leq n-1$, then $x_1 + x_2 + \cdots + x_t > d+t-1$, which is a contradiction. Thus, $x_1 = d$. Since $x_1 + x_2 + \cdots + x_t = d+t-1$, $x_1 = d$ and $x_i \geq 1$, for any $1 \leq i \leq t$, we obtain $x_2 = \cdots = x_t = 1$.

Now, since $0 < x_{(d-1)t+1} < \cdots < x_{t+1} < x_1 = d$ and $x_i + x_{i+1} + \cdots + x_{i+t-1} < t+d-1$, we obtain $x_{kt+1} = d-k$, for any $0 \leq k \leq d-2$ and $x_{kt+j} = 1$, for any $0 \leq k \leq d-2$ and $0 \leq j \leq t$, $j \neq 1$, $j \neq 2$. Therefore, $\alpha = (x_1, x_2, \ldots, x_{d(t-1)})$, where

$$x_j = \begin{cases} d-k, & \text{if } j = kt+1, \text{with } 0 \leq k \leq d-2, \\ 1, & \text{if } j = kt+l, \text{with } 0 \leq k \leq d-2, 0 \leq l \leq t-1, l \neq 1, l \neq 2 \end{cases}$$

is the unique interior lattice point in $(t+d-1)\mathcal{P}$. But, for any $0 \leq k \leq d-2$ and $kt+1 \leq j \leq kt+t$,

$$\begin{aligned} x_{kt+2} &= t+d-1 - \sum_{j \neq kt+2} x_j \\ &= t+d-1-(d-k+t-2) = k+1. \end{aligned}$$

So, the unique interior lattice point α in $(t+d-1)\mathcal{P}$ is $(x_1, \ldots, x_{n-1})$, where

$$x_j = \begin{cases} d-k, & j = kt+1, 0 \le k \le d-2, \\ k+1, & j = kt+2, 0 \le k \le d-2, \\ 1, & j = kt+l, 0 \le k \le d-2, 0 \le l \le t-1, l \ne 1, l \ne 2. \end{cases}$$

Using Theorem 1.2, we show that $K[I_{n,d,t}]$ is Gorenstein. Let us compute $\mathcal{Q} = (t+d-1)\mathcal{P}-\alpha$. We have

$$\mathcal{Q} = \{(y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1} : y_i = a_i - x_i, 1 \le i \le n-1, \text{ where } (a_1, a_2, \dots, a_{n-1}) \in (t+d-1)\mathcal{P}\}.$$

Thus, we obtain

$\mathcal{Q} = \{(y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1} : y_i \ge -x_i, 1 \le i \le n-1; y_i + y_{i+1} + \cdots + y_{i+t-1} \le t+d-1-(x_i + x_{i+1} + \cdots + x_{i+t-1}), 1 \le i \le n-t; y_1 + \cdots + y_{n-t} = 0; y_{kt+1} + y_{kt+2} + \cdots + y_{kt+t} = 0, 0 \le k \le d-2\}$.

For $1 \le i \le n-t$, we have $y_i + y_{i+1} + \cdots + y_{i+t-1} \le t+d-1-(x_i + x_{i+1} + \cdots + x_{i+t-1})$, suppose $i = kt+r$, where $1 \le r \le t$. Then $y_{kt+r} + y_{kt+r+1} + \cdots + y_{(k+1)t+r-1} \le t+d-1-(x_{kt+r} + \cdots + x_{(k+1)t+r-1})$, for any $0 \le k \le d-2, 1 \le r \le t$. If $r = 1$, we already have $y_{kt+1} + \cdots + y_{kt+t} = 0$. If $r = 2$, then

$$y_{kt+2} + y_{kt+3} + \cdots + y_{(k+1)t+1} \le (t+d-1) - (k+1+(t-2)+d-(k+1)) = 1.$$

But $y_{kt+2} = -y_{kt+1} - \cdots - y_{kt+t}$, thus

$$y_{(k+1)t+1} - y_{kt+1} \le 1.$$

If $r \ge 3$, then

$$y_{kt+r} + y_{kt+r+1} + \cdots + y_{(k+1)t+r-1} \le (t+d-1) - (t-2+d-(k+1)+k+2) = 0.$$

But $y_{kt+r} = -y_{kt+1} - \dots y_{kt+r-1} - y_{kt+r+1} - \cdots - y_{kt+t}$, thus

$$y_{(k+1)t+1} + \cdots + y_{(k+1)t+r-1} - y_{kt+1} - y_{kt+2} - \cdots - y_{kt+r-1} \le 0.$$

Therefore,

$$\mathcal{Q} = \{(y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1} : y_{(k+1)t+1} + \cdots + y_{(k+1)t+r-1} - y_{kt+1} - \cdots - y_{kt+r-1} \le 0, 3 \le r \le t, y_{(k+1)t+1} - y_{kt+1} \le 1, y_{kt+2} = -y_{kt+1} - y_{kt+3} - \cdots - y_{kt+t}, 0 \le k \le d-2\}.$$

Thus, since the supporting hyperplanes of the polytope $\mathcal{Q}$ have integral coefficients, we conclude that $\mathcal{Q}$ is an integral polytope. Hence, by Theorem 1.2, $K[I_{n,d,t}]$ is Gorenstein. □

EXAMPLE 3. Let $n = 10$, $d = 5$ and $t = 2$. In this case, $\delta = 6$ and in the dilated polytope $6\mathcal{P}$ there is a unique interior lattice point, namely $(5, 1, 4, 2, 3, 3, 2, 4, 1)$. The dual polytope of $\mathcal{Q} = 6\mathcal{P} - (5, 1, 4, 2, 3, 3, 2, 4, 1)$ is an integral polytope, thus $K[I_{10,5,2}]$ is Gorenstein.

We state and prove the main theorem of this paper.

Theorem 3.4. *The t-spread Veronese algebra, $K[I_{n,d,t}]$, is Gorenstein if and only if $n \in \{(d-1)t+1, (d-1)t+2, dt, dt+1, dt+d\}$.*

Proof. If $n = dt+k$ with $2 \le k \le d-1$ and $n \ge (t+1)d+1$, then, by Theorem 3.1 and Theorem 3.2, $K[I_{n,d,t}]$ is not Gorenstein. Hence, it remains to study the cases when $n \in \{(d-1)t+1, (d-1)t+2, dt, dt+1, dt+d\}$.

If $n = (d-1)t+1$, then $K[I_{n,d,t}]$ is a polynomial ring, thus it is Gorenstein. If $n = (d-1)t+2$, by Theorem 2.2, $K[I_{n,d,t}]$ is Gorenstein. If $n = dt$, by Theorem 3.3, we obtain the same conclusion.

Let $n = dt + 1$. In our hypothesis, by Theorem 2.3, $\dim K[I_{n,d,t}] = dt+1$, thus $\dim \mathcal{P} = dt$. Using similar arguments as in Theorem 3.1,

$$\mathcal{P} = \{(a_1, a_2, \ldots, a_{n-1}) \in \mathbb{R}^{n-1} : a_i \geq 0, 1 \leq i \leq n-1, a_i + a_{i+1} + \cdots + a_{i+t-1} \leq 1, 1 \leq i \leq n-t, a_1 + a_2 + \cdots + a_{n-t} \geq d-1\}.$$

We show that the smallest integer $\delta \geq 1$ such that $\delta(\mathcal{P} - \partial\mathcal{P})$ contains lattice points is $t + d$. Assume that there are interior lattice points at lower levels than $t+d$. It is enough to see that there are no interior lattice points at level $t + d - 1$. The interior of the dilated polytope is

$$(t+d-1)(\mathcal{P} - \partial\mathcal{P}) = \{(a_1, a_2, \ldots, a_{n-1}) \in \mathbb{R}^{n-1} : a_i > 0, 1 \leq i \leq n-1, a_i + a_{i+1} + \cdots + a_{i+t-1} < t+d-1, 1 \leq i \leq n-t, a_1 + a_2 + \cdots + a_{n-t} > (d-1)(t+d-1)\}.$$

In this case, for each lattice point $(a_1, a_2, \ldots, a_{n-1}) \in (t+d-1)(\mathcal{P} - \partial\mathcal{P})$, we have $a_i + a_{i+1} + \cdots + a_{i+t-1} \leq t+d-2$, for any $1 \leq i \leq n-t$, thus $a_1 + a_2 + \cdots + a_{(d-1)t} \leq (t+d-2)(d-1)$. But $a_1 + a_2 + \cdots + a_{n-t} \geq (d-1)(t+d) + 1$, thus we obtain

$$(d-1)(t+d-1) + 1 \leq \textstyle\sum_{i=1}^{n-t} a_i \leq (t+d-2)(d-1) + a_{(d-1)t+1},$$

hence, $a_{(d-1)t+1} \geq d$. But, since $a_{(d-2)t+2} + a_{(d-2)t+3} + \cdots + a_{(d-1)t+1} \leq t+d-2$, we obtain $a_{(d-2)t+2} + \cdots + a_{(d-1)t} \leq t-2$, which is the sum of $t-1$ terms and each $a_{(d-2)t+j} > 1$, for any $2 \leq j \leq t$. We show that $(t+d)(\mathcal{P} - \partial\mathcal{P})$ contains only one lattice point. The interior of the dilated polytope is

$$(t+d)(\mathcal{P} - \partial\mathcal{P}) = \{(a_1, a_2, \ldots, a_{n-1}) \in \mathbb{R}^{n-1} : a_i > 0, 1 \leq i \leq n-1, a_i + a_{i+1} + \cdots + a_{i+t-1} < t+d, 1 \leq i \leq n-t, a_1 + a_2 + \cdots + a_{n-t} > (d-1)(t+d)\}.$$

Let $(x_1, x_2, \ldots, x_{n-1}) \in (t+d)(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^{n-1}$. Thus $x_1 + x_2 + \cdots + x_{(d-1)t+1} \geq (d-1)(t+d)+1$.

Claim: $x_{kt+1} \geq d$, for any $0 \leq k \leq d-1$.

Since $x_i + x_{i+1} + \cdots + x_{i+t-1} \leq t+d-1$, for any $1 \leq i \leq k-1$ and for any $k \leq i \leq d-2$, we obtain

$$x_1 + x_2 + \cdots + x_{kt} + x_{kt+2} + \cdots + x_{(d-1)t+1} \leq (t+d-1)(d-1).$$

Hence,

$$(d-1)(t+d) + 1 \leq \sum_{i=1}^{n-t} x_i \leq (d-1)(t+d-1) + x_{kt+1},$$

thus $x_{kt+1} \geq d$, for any $0 \leq k \leq d-1$, as we claimed.

But, $x_{kt+1} + x_{kt+2} + \cdots + x_{(k+1)t} \leq d+t-1$ and $x_{kt+1} \geq d$, $x_{kt+j} \geq 1$, for any $2 \leq j \leq t$, thus $x_{kt+1} + x_{kt+2} + \cdots + x_{(k+1)t} = d+t-1$. The equality holds if and only if, for any $0 \leq k \leq d-1$, $x_{kt+1} = d$ and $x_{kt+j} = 1$, for any $2 \leq j \leq t$. Therefore, $\alpha = (x_1, x_2, \ldots, x_{n-1})$, where

$$x_j = \begin{cases} d, & j = kt+1, 0 \leq k \leq d-1 \\ 1 & j = kt+l, 0 \leq k \leq d-1, 0 \leq l \leq t-1, l \neq 1. \end{cases}$$

is the unique interior lattice point in $(t+d)\mathcal{P}$. Using Theorem 1.2, we show that $K[I_{n,d,t}]$ is Gorenstein. Let us compute $\mathcal{Q} = (t+d)\mathcal{P} - \alpha$.

$$\mathcal{Q} = \{(y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1} : y_i \geq -d, i = kt+1, 0 \leq k \leq d-1, y_i \geq -1, i = kt+l, 0 \leq k \leq d-1, 0 \leq l \leq t-1, l \neq 1, y_i + y_{i+1} + \cdots + y_{i+t-1} \leq 1, 1 \leq i \leq n-t, y_1 + \cdots + y_{n-t} \geq -1\}.$$

In fact, we show that the dual $\mathcal{Q}^*$ of $\mathcal{Q}$ is an integral polytope, by showing that the independent hyperplanes which determine the facets of $\mathcal{Q}$ are

$$y_i = -1, i = kt + l, 0 \le k \le d-1, 0 \le l \le t-1, l \ne 1,$$
$$y_i + y_{i+1} + \cdots + y_{i+t-1} = 1, 1 \le i \le n-t,$$
$$y_1 + \cdots + y_{n-t} = -1.$$

Thus, we need to show that all the hyperplanes $y_i = -d, i = kt+1, 0 \le k \le d-1$ are redundant. Let $0 \le k \le d-1$. Since $y_i + y_{i+1} + \cdots + y_{i+t-1} \le 1$, for any $1 \le i \le k-1$ and $y_{it+2} + \cdots + y_{(i+1)t+1} \le 1$, for any $k \le i \le d-2$, we obtain

$$y_1 + y_2 + \cdots + y_{(k-1)t+1} + \cdots + y_{kt} + y_{kt+2} + \cdots + y_{n-t} \le k - 1 + [d-1-(l-1)] = d-1,$$

and, since $y_1 + y_2 + \cdots + y_{(d-1)t+1} \ge -1$, we obtain $y_i \ge -d, i = kt+1$, for any $0 \le k \le d-1$. Thus, since the supporting hyperplanes of the polytope $\mathcal{Q}$ have integral coefficients, we conclude that $\mathcal{Q}$ is an integral polytope. Hence, by Theorem 1.2, $K[I_{n,d,t}]$ is Gorenstein.

Let $\underline{n = dt + d}$. In our hypothesis, by Theorem 2.3, $\dim K[I_{n,d,t}] = dt + d$, thus $\dim \mathcal{P} = dt + d - 1$. We have,

$$\mathcal{P} = \{(a_1, a_2, \ldots, a_{n-1}) \in \mathbb{R}^{n-1} : a_i \ge 0, 1 \le i \le n-1, a_i + a_{i+1} + \cdots + a_{i+t-1} \le 1, 1 \le i \le n-t, a_1 + a_2 + \cdots + a_{n-t} \ge d-1\}.$$

We show that there are no interior lattice points at lower levels than $t+1$. It is enough to see that there are no interior lattice points at level t. Let $(a_1, a_2, \ldots, a_{n-1}) \in t(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^{n-1}$. We have

$$t(\mathcal{P} - \partial\mathcal{P}) = \{(a_1, a_2, \ldots, a_{n-1}) \in \mathbb{R}^{n-1} : a_i > 0, 1 \le i \le n-1, a_i + a_{i+1} + \cdots + a_{i+t-1} < t, 1 \le i \le n-t, a_1 + a_2 + \cdots + a_{n-t} > (d-1)t\}.$$

Since each $a_i > 0$, for any $1 \le i \le n-1$, we obtain $t > a_i + a_{i+1} + \cdots + a_{i+t-1} \ge t$, which is a contradiction. Thus, there are no interior lattice points in the dilated polytope at lower levels than $t+1$. We show that $(t+1)(\mathcal{P} - \partial\mathcal{P})$ contains only one interior lattice point which has all the coordinates equal to 1. The interior of the dilated polytope is

$$(t+1)(\mathcal{P} - \partial\mathcal{P}) = \{(a_1, a_2, \ldots, a_{n-1}) \in \mathbb{R}^{n-1} : a_i > 0, 1 \le i \le n-1, a_i + a_{i+1} + \cdots + a_{i+t-1} < t+1, 1 \le i \le n-t, a_1 + a_2 + \cdots + a_{n-t} > (d-1)(t+1)\}.$$

We know that, for each lattice point $(a_1, \ldots, a_{n-1}) \in (t+1)(\mathcal{P} - \partial\mathcal{P})$, we have $a_i + \cdots + a_{i+t-1} \le t$, for any $1 \le i \le n-t$ and, since $a_j \ge 1$, for any $i \le j \le i+t-1$, we obtain $a_i + \cdots + a_{i+t-1} \ge t$, thus we have equality which implies that $a_j = 1$, for any $1 \le j \le n-1$. Hence, $(1, 1, \ldots, 1) \in \mathbb{Z}^{n-1}$ is the unique interior lattice point in the dilated polytope $(t+1)\mathcal{P}$.

Let us consider $\mathcal{Q} = (t+1)\mathcal{P} - (1, 1, \ldots, 1)$. We will show that $K[\mathcal{P}]$ is Gorenstein by using Theorem 1.2. In fact, we show that the dual $\mathcal{Q}^*$ of

$$\mathcal{Q} = \{(y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1} : y_i \ge 0, 1 \le i \le n-1, y_i + y_{i+1} + \cdots + y_{i+t-1} \le 0, 1 \le i \le n-t, y_1 + \cdots + y_{n-t} \ge -1\}.$$

is an integral polytope. As the hyperplanes $\sum_{j=i}^{i+t-1} y_j = 0$, for any $1 \le i \le n-t$, and $\sum_{i=1}^{n-t} y_i = -1$, have integral coefficients, it follows that $\mathcal{Q}$ is an integral polytope. Thus, by Theorem 1.2, we conclude that $K[I_{n,d,t}]$ is Gorenstein. □

Example 4. Let $n = 11$, $d = 3$ and $t = 4$. In this case, $\delta = 5$ and in the dilated polytope

$$5(\mathcal{P} - \partial\mathcal{P}) = \{(a_1, \dots, a_{10}) \in \mathbb{R}^{10} : a_i > 0, 1 \le i \le 10, i \ne 4, 8, a_9 < a_5 < a_1, a_9 + a_{10} < a_5 + a_6 < a_1 + a_2 < 5, a_4 = a_8 = 0\}$$

there is a unique interior lattice point, namely $(3, 1, 1, 0, 2, 1, 2, 0, 1, 1)$. Let us compute the polytope $\mathcal{Q} = 5\mathcal{P} - (3, 1, 1, 0, 2, 1, 2, 0, 1, 1)$. Then

$$\mathcal{Q} = \{(y_1, y_2, \dots, y_{10}) \in \mathbb{R}^{10} : y_i > -1, 1 \le i \le 10, i \ne 4, 8, y_9 - y_5 < 1, y_5 - y_1 < 1, y_9 + y_10 - y_5 - y_6 < 1, y_5 + y_6 - y_1 - y_2 < 1, y_1 + y_2 < 1\}.$$

Thus, the dual polytope of $\mathcal{Q}$ is integral. Therefore, $K[I_{11,3,4}]$ is Gorenstein.

Example 5. Let $n = 10$, $d = 3$ and $t = 3$. In this case, $\delta = 6$ and in the dilated polytope $6(\mathcal{P} - \partial\mathcal{P}) = \{(a_1, \dots, a_9) \in \mathbb{R}^9 : a_i > 0, 1 \le i \le 9, a_1 + a_2 + a_3 < 6, a_2 + a_3 + a_4 < 6, a_3 + a_4 + a_5 < 6, a_4 + a_5 + a_6 < 6, a_5 + a_6 + a_7 < 6, a_6 + a_7 + a_8 < 6, a_7 + a_8 + a_9 < 6, a_1 + a_2 + \cdots + a_7 > 12\}$ there is a unique interior lattice point, namely $(3, 1, 1, 3, 1, 1, 3, 1, 1)$. Let us compute the polytope $\mathcal{Q} = 6\mathcal{P} - (3, 1, 1, 3, 1, 1, 3, 1, 1)$. Then

$$\mathcal{Q} = \{(y_1, y_2, \dots, y_9) \in \mathbb{R}^9 : y_2 \ge -1, y_3 \ge -1, y_5 \ge -1, y_6 \ge -1, y_8 \ge -1, y_9 \ge -1, y_1 + y_2 + y_3 \le 1, y_2 + y_3 + y_4 \le 1, y_3 + y_4 + y_5 \le 1, y_4 + y_5 + y_6 \le 1, y_5 + y_6 + y_7 \le 1, y_6 + y_7 + y_8 \le 1, y_7 + y_8 + y_9 \le 1, y_1 + y_2 + \cdots + y_7 \ge -1\}.$$

Thus, the dual polytope of $\mathcal{Q}$ is integral. Therefore, $K[I_{10,3,3}]$ is Gorenstein.

Example 6. Let $n = 8$, $d = 2$ and $t = 3$. In this case, $\delta = 4$ and in the dilated polytope

$$4(\mathcal{P} - \partial\mathcal{P}) = \{(a_1, \dots, a_7) \in \mathbb{R}^7 : a_i > 0, 1 \le i \le 7, a_1 + a_2 + a_3 < 4, a_2 + a_3 + a_4 < 4, a_3 + a_4 + a_5 < 4, a_4 + a_5 + a_6 < 4, a_5 + a_6 + a_7 < 4, a_1 + a_2 + a_3 + a_4 + a_5 > 4\}$$

there is a unique interior lattice point, $(1, 1, \dots, 1)$. Let us compute the polytope $\mathcal{Q} = 4\mathcal{P} - (1, 1, \dots, 1)$. Then

$$\mathcal{Q} = \{(y_1, y_2, \dots, y_7) \in \mathbb{R}^7 : y_i > -1, 1 \le i \le 7, y_1 + y_2 + y_3 < 1, y_3 + y_4 + y_5 < 1, y_4 + y_5 + y_6 < 1, y_5 + y_6 + y_7 < 1, y_1 + y_2 + y_3 + y_4 + y_5 > -1\}.$$

Thus, the dual polytope $\mathcal{Q}^*$ is integral. Therefore, $K[I_{8,2,3}]$ is Gorenstein.

Let R be the polynomial ring $K[t_v : v \in G(I_{n,d,t})]$ and $\varphi : R \to K[I_{n,d,t}]$ be the K-algebra morphism which maps t_v to v, for all $v \in G(I_{n,d,t})$.

Proposition 3.5 ([7, Theorem 3.2])**.** *The set of binomials $\mathcal{G} = \{t_u t_v - t_{u'} t_{v'} : (u, v)$ unsorted, $(u', v') = \mathrm{sort}(u, v)\}$ is a Gröbner basis of the toric ideal $Ker\varphi$.*

As a consequence of it, we have the following result:

Corollary 3.6. *The polytope $\mathcal{P}$ possesses a regular unimodular triangulation.*

Proposition 3.7 ([5])**.** *Let $\mathcal{P} \in \mathbb{R}^d$ be a d-dimensional polytope of standard type such that its dual is a lattice polytope. If $\mathcal{P}$ admits a regular unimodular triangulation, then $h^*(\mathcal{P}, x)$ is unimodal.*

Proposition 3.8. *If $n \in \{(d-1)t + 1, (d-1)t + 2, dt, dt + 1, dt + d\}$, then the h^*-vector of the t-spread Veronese algebra $K[I_{n,d,t}]$ is unimodal.*

Proof. By Theorem 3.4, $K[I_{n,d,t}]$ is Gorenstein. Thus by Proposition 3.7 and Corollary 3.6 the desired result follows. □

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