Rudenko, A.
Osaka J. Math.
57 (2020), 425–450

AN ESTIMATE FOR SURFACE MEASURE OF SMALL BALLS IN CARNOT GROUPS

Alexey RUDENKO

(Received September 6, 2018, revised January 24, 2019)

Abstract

We introduce a family of quasidistances in $\mathbb{R}^d$, such that some of them are equivalent to natural distances on Carnot groups. We find the sufficient conditions for the balls w.r.t. a quasidistance from our family to be comparable to ellipsoids. Using comparability to ellipsoids we find asymptotics of surface measure of intersections of small balls with linear submanifolds and the conditions for finiteness of the integral w.r.t. the surface measure of negative power of the distance. We provide several examples of Carnot groups, where comparability to ellipsoids can be shown for natural distances, and therefore we can study the asymptotics and finitness of the integrals explicitly. We also show an example of a Carnot group, where the comparability to ellipsoids does not hold.

1. Introduction

In this paper we study the bounds on the surface measure on linear manifolds of small balls in $\mathbb{R}^d$ with a specific distance, which is a generalization of a natural distance in Carnot groups (for the latter see [1]). The motivation to consider such question comes from a possibility to use these bounds for the investigation of the properties of paths of hypoelliptic diffusions (diffusions with hypoelliptic, but not necessarily elliptic generator, see [8]). In particular we want to study the existence of local times on manifolds for hypoelliptic diffusions as well as intersection and self-intersection local times (one result of this kind was already proven in [9]). The main connection between the topic of this paper and hypoelliptic diffusions can be seen if we recall some known bounds for the density of hypoelliptic diffusion such as Theorem IV.4.2 of [11], or Theorem 4.13 from [3]. These bounds involve the distances that correspond to a set of vector fields, related to the process (detailed treatment of such distances can be found in [7]), which are exactly the kind of distances we want to investigate.

Since our goal is to prove some very specific results about the finiteness of some integrals, we want to give a few more details about how they appear in the study of local times. One approach to the existence of local times is from the point of view of additive functionals, since local time can be viewed as an additive functional which corresponds to a measure (in our case we should take a finite measure, absolutely continuous w.r.t. given surface measure). There are several known sufficient conditions for existence of such functionals, with the simplest being the finiteness of supremum of L_1 norm of the functional over all possi-

2010 Mathematics Subject Classification. Primary 43A80; Secondary 26D15, 58C35.

This work was partially supported by Erasmus+ programme 2015/16-2016/17 hosted by Institute of mathematics of the Friedrich-Schiller University of Jena (Germany).

ble starting points of the process (such supremum was considered in [2], in the theory of W-functionals). Unfortunately this does not always work in our situation (see examples below), and also is not applicable for intersection and self-intersection local times. So instead, we want to use a sufficient condition for L_2 convergence of a sequence of approximating random variables (as it was done, for example, in [9] or [10]). Such a condition usually involves the finitness of an integral, representing L_2 norm of the local time. Both L_2 and L_1 norms of local time can be described as integrals of the density of the process, and aforementioned density bounds produce integrals of the negative power of the distance w.r.t. the surface measure. Unfortunately finding general conditions when such integrals are finite seems to be problematic, even if our submanifold is linear. However it would be possible to find such condition if we have the bounds for the surface measure of the small balls in Carnot group, taking into account the dependence on the center of the ball. This finally leads us to the topic of this paper, which is to give some useful sufficient conditions under which we can find bounds for the surface measure of small balls, and use them to study the corresponding integrals. To our knowledge only some partial results about asymptotics of the surface measure of small balls in Carnot group are known (see [6, 5, 4] and also [7], for investigations most similar to ours).

Given a matrix function $\Lambda(\delta, x)$ for $\delta > 0$ and $x \in \mathbb{R}^d$ we define a function

$$\rho_\Lambda(x, y) = \inf\{\delta | \exists\varphi \in AC([0,1], \mathbb{R}^d), \exists a \in \mathfrak{B}_b([0,1], \mathbb{R}^d) : \varphi(0) = x, \varphi(1) = y,$$
$$\frac{d}{dt}\varphi(t) = \Lambda(\delta, \varphi(t))a(t), |a(t)| < 1, \text{ a.e. } t \in [0,1]\}.$$

We prove that under some conditions on Λ, the functions ρ_Λ is a quasidistance (it is finite, non-zero for different points, and satisfies a weaker form of triangle inequality: $\rho_\Lambda(x,z) \leqslant C(\rho_\Lambda(x,z) + \rho_\Lambda(x,z))$). We assume a relation between this quasidistance and Euclidean structure of $\mathbb{R}^d$, in a form of uniform bound for $\|I - \Lambda^{-1}(x, \delta)\Lambda(y, \delta)\|$ (here and below $\|\cdot\|$ is a standard operator norm for matrices), so that we are able to prove (under some additional conditions) that balls w.r.t. ρ_Λ are comparable to Euclidean ellipsoids of specific form. By Euclidean ellipsoids or just ellipsoids we mean sets of the form $\{x \in \mathbb{R}^d : |Ax| < 1\}$, where $|\cdot|$ is a standard Euclidean norm on $\mathbb{R}^d$, and A is an arbitrary $n \times n$ matrix. We choose a specific form of ellipsoids for comparison by setting A to be a multiple of $\Lambda^{-1}(\delta, x)$ for a ball of radius δ with the center at point x. The reason for considering ellipsoids and not boxes, for example, is that the intersections of ellipsoids with linear manifolds are also ellipsoids (in the natural euclidean coordinate system on the linear manifold), and so we can easily obtain upper and lower bounds for surface measure of intersection of balls with linear manifolds. Under the assumption that balls are comparable to ellipsoids, we obtain results about asymptotics of surface measures on linear submanifolds of small balls w.r.t. ρ_Λ. We also find the conditions for finiteness of integrals of the form

$$\int_K H(\rho_\Lambda(x, y + Bu)\, du$$

and

$$\int_K \int_K f(v) H(\rho_\Lambda(x + Bv, y + Bu))\, du dv,$$

where B is a constant $n \times k$ matrix and K is a compact in $\mathbb{R}^k$.

To be able to study examples we specify several sufficient conditions for comparability to ellipsoids. We show that for images of 2-step Carnot group under any twice continuously differentiable automorphism of $\mathbb{R}^d$, our assumptions hold and comparison to ellipsoids uniformly over any compact is always possible. Therefore we are able to apply our results for specific examples of 2-step Carnot groups, and different choices of submanifolds (not necessarily linear), finding explicitly the asymptotics for surface measure of balls and the condition for the integrals on submanifolds to be finite. We also prove that if Carnot group addition is of the form $x + L(x)y$ (where $L(x)$ is some matrix function, which is set to coincide with L from the definition of Λ, when we want to obtain a natural distance on Carnot group), then balls are always comparable to ellipsoids uniformly over whole space. Using this we provide an example of a specific n-step Carnot group, where ball asymptotics and finitiness of integrals on linear submanifolds can also be studied explicitly. Unfortunately there are a lot of submanifolds on Carnot groups for which our approach does not work, due to the non-trivial structure of balls in general Carnot groups. To illustrate the difficulties we provide an example where we can find a sequence of arbitrarily small balls, such that the intersection of each ball with the given linear manifold can be divided into two disconnected subsets. Moreover we show that in this example it is impossible for all small balls with center at specific point to be comparable to ellipsoids.

In section 2 we provide all definitions and notations and formulate main results of the paper. In section 3 we prove our main result about comparison of balls and ellipsoids. In section 4 we prove results that give more practical sufficient conditions for comparability to ellipsoids. In section 5 we show a number of consequences of comparability to ellipsoids. In section 6 we propose a number of examples, where the results of sections 4 and 5 can be applied, and in section 7 we study a particular example where they can not be applied.

2. Definitions and main results

In this section we gather all necessary definitions and formulate main results of the paper. Most of the proofs are given in other sections.

Let Λ be a real-valued $d \times d$ matrix function on $(0, +\infty) \times \mathbb{R}^d$. We suppose that

(1) $\Lambda(\delta, x) = L(x)T(\delta)$, where $L(x)$ is a real-valued $d \times d$ matrix function on $\mathbb{R}^d$ and $T(\delta)$ is a real-valued $d \times d$ matrix function on $(0, +\infty)$.
(2) $T(\delta)_{ij} = 0$ if $i \neq j$ and $T(\delta)_{ii} = \delta^{p_i}$ for some constants $p_i > 0$, $i = 1, \dots, d$.
(3) $L(x)$ is Lipshitz on any compact set and everywhere invertible.

Denote as $AC([0,1], \mathbb{R}^d)$ a space of absolutely continuous functions from $[0,1]$ to $\mathbb{R}^d$ and as $\mathfrak{B}_b([0,1], \mathbb{R}^d)$ a space of bounded measurable functions from $[0,1]$ to $\mathbb{R}^d$. We say that ρ is a quasidistance if it satisfies all properties of the distance, but the triangle inequality is replaced with: $\rho(x, y) \leqslant C(\rho(x, z) + \rho(z, y))$.

Definition 1. The quasidistance $\rho = \rho_\Lambda$ related to $\Lambda(\delta, x)$ is the following function:

$$\rho_\Lambda(x, y) = \inf\{\delta > 0 | \exists \varphi \in AC([0,1], \mathbb{R}^d), \exists a \in \mathfrak{B}_b([0,1], \mathbb{R}^d) : \varphi(0) = x, \varphi(1) = y,$$
$$\frac{d}{dt}\varphi(t) = \Lambda(\delta, \varphi(t))a(t), |a(t)| < 1, \text{ a.e. } t \in [0,1]\}.$$

The definition of ρ_Λ follows the ideas of similar definitions in [7] (see also Proposition 1 below and the corresponding definition of Γ), with some notational adjustments needed for our investigation. It is interesting, that the distance ρ_Λ is the same for two different Λ, whenever the matrix function $(\Lambda^{-1}(x,\delta))^T\Lambda^{-1}(x,\delta)$ is the same (in such case for any $\varphi(t)$ the corresponding $a(t)$ may be different, but the norm of $a(t) = \Lambda^{-1}(x,\delta)\frac{d}{dt}\varphi(t)$ is the same), however we will not use this fact directly. The correctness of this definition is provided by the following Lemma.

Lemma 1. *For all Λ satisfying our assumptions the function ρ_Λ is a quasidistance on $\mathbb{R}^d$. If $p_i \geqslant 1$ for all $i = 1,\dots,d$ then it is a distance on $\mathbb{R}^d$.*

To explain the connection of ρ_Λ to Carnot groups we need to reproduce a definition of Carnot group and related objects (see [1]).

DEFINITION 2. Lie group $G = (\mathbb{R}^d,\bullet)$ is called a Carnot group (homogeneous Carnot group) if

(1) G as a Euclidean space can be split into a direct product of h Euclidean spaces G_i of fixed dimensions, say $n_1, n_2, \dots, n_h$ (assuming $\sum_{i=1}^h n_i = d$), such that

$$\beta_\lambda(v_1, v_2, \dots, v_h) = (\lambda v_1, \lambda^2 v_2, \dots, \lambda^h v_h), v_i \in G_i$$

is a group automorphism of G for all positive λ.

(2) Let g be a Lie algebra of left-invariant vector fields on G. Fix a euclidean coordinate system $x = (x_1,\dots,x_d) \in G$ such that $x_{\sum_{i=1}^{l-1} n_i+1},\dots,x_{\sum_{i=1}^{l} n_i}$ define a vector in G_l. Denote as $V_1,\dots,V_d$ such left-invariant vector fields on G, i.e. elements of g, that $V_i|_{x=0} = \frac{\partial}{\partial x_i}|_{x=0}$. Then the smallest Lie subalgebra of g containing $V_1,\dots,V_{n_1}$ is g.

There exists a number d_i, called a homogeneous degree of V_i (the same vector field as above), such that $\beta_\lambda(V_i(x)) = \lambda^{d_i} V_i(\beta_\lambda(x))$ (for example $d_i = 1$ for $i = 1,\dots,n_1$). The number $Q = \sum\limits_{i=1}^{d} d_i = \sum\limits_{j=1}^{h} j n_j$ is called a homogeneous dimension of G. The number h is the number of "steps" and so such Carnot group is called h-step Carnot group (assuming $n_h > 0$).

There is a natural distance on Carnot group G (see [7]):

$$\begin{aligned}(1)\quad \Gamma(x,y) = \inf\{\delta > 0 | \exists\varphi \in AC([0,1],\mathbb{R}^d), \exists a \in \mathfrak{B}_b([0,1],\mathbb{R}^d) : \varphi(0) = x, \varphi(1) = y,\\ \frac{d}{dt}\varphi(t) = \sum_{i=1}^{d} V_i(\varphi(t))a_i(t), |a_i(t)| < \delta^{d_i},\ \text{a.e. } t \in [0,1]\},\end{aligned}$$

which is related to ρ_Λ in the following way. Note that there are other natural distances (and quasidistances) on Carnot groups, which are locally equivalent to Γ (again, see [7]). All our results on behaviour of small balls are also true for all such distances, due to this local equivalence.

Proposition 1. *Suppose that we have a Carnot group $G = (\mathbb{R}^d,\bullet)$, and $V_1,\dots,V_d$ are left-invariant vector fields on G used in the definition of Carnot group. If we define $\Lambda(\delta,x) = L(x)T(\delta)$, taking $V_1(x),\dots,V_d(x)$ as columns of $L(x)$ and setting $T(\delta)_{ij} = 0$ if $i \neq j$, $T(\delta)_{ii} = \delta^{d_i}$, where $d_i > 0$ are the corresponding homogeneous degrees of V_i, then such Λ satisfy our* *assumptions and the corresponding distance ρ_Λ is equivalent to the distance Γ on G.*

Proof. Due to the properties of $V_1(x), \dots, V_d(x)$ (shown, for example, in Theorem 2.1.43 of [1]) the matrix $L(x)$ is infinitely differentiable and invertible, which is enough to satisfy our assumptions. After we insert $a_i(t)\delta^{d_i}$ instead of $a_i(t)$ in the definition of Γ we obtain the definition of ρ_Λ, except that Euclidean norm is replaced by the maximum of coordinates. But since these norms are equivalent, the corresponding distances are also equivalent. $\square$

This Proposition allows us to apply our results to the case of Carnot groups. In the following in the context of Carnot groups we always assume that the relation between Λ and specific Carnot group is the same as in this Proposition.

The main idea of this paper is that balls in distance ρ_Λ can sometimes be compared to Euclidean ellipsoids, which are defined as $E_A(y, \varepsilon) = \{y + Az : |z| < \varepsilon\}$ for any $y \in \mathbb{R}^d$, $\varepsilon > 0$ and invertible $d \times d$ matrix A. Such comparison, if it is possible, allows us to study intersections of balls with linear submanifolds. Therefore our main results can be divided in two groups: those that give sufficient conditions for comparison with ellipsoids, and those that give specific properties of intersections of balls with linear submanifolds given that such comparison is possible. We start with the most general sufficient condition and then provide some more practical, applicable conditions for specific Carnot groups.

We need to specify two assumptions for Λ, which are better described as subclasses of matrix functions. We denote the open balls corresponding to ρ_Λ as follows: $B_\Lambda(x, \varepsilon) = \{y : \rho_\Lambda(x, y) < \varepsilon\}$.

Definition 3. If for some $x \in \mathbb{R}^d$, $\delta > 0$, $\gamma \in (0, 1)$, $r > 0$ and some matrix function Λ (satisfying our assumptions) we have for all $y \in E_{\Lambda(\delta,x)}(x, r)$ that $\|I - \Lambda^{-1}(\delta, x)\Lambda(\delta, y))\| \leqslant \gamma$, then we say that Λ belongs to $LU_d(x, \delta, \gamma, r)$.

For each $u \in \mathbb{R}^d$, $x \in \mathbb{R}^d$ consider the following differential equation for continuously differentiable function f from $[0, 1]$ to $\mathbb{R}^d$: $\frac{d}{dt} f(t) = \Lambda(\delta, f(t))u$ with initial condition $f(0) = x$. We denote as $\Phi(\delta, x, u)$ its solution at $t = 1$, i.e. $f(1)$. According to Picard theorem it is well-defined for each x if u or δ is small enough. We call the set $\{\Phi(\delta, x, u) : |u| < r\}$ an exponential ball, since Φ can be seen as an exponential map of the vector field $\Lambda(\delta, \cdot)u$.

Definition 4. If for some $x \in \mathbb{R}^d$, $\delta > 0$ and some matrix function Λ we have that the function $\Phi(\delta, x, u)$ is well-defined and continuous on $|u| \leqslant 1$, and the set $\{\Phi(\delta, x, u) : |u| < 1\}$ is open, then we say that Λ belongs to $LI_d(x, \delta)$.

We also need to state precisely what it means in our context for balls to be comparable to ellipsoids.

Definition 5. We say that balls $B_\Lambda(x, \delta)$ are comparable to ellipsoids at point x, if there exists $\delta_0 > 0$, $r_1 > 0$ and $r_2 > 0$ such that for all $\delta \in (0, \delta_0)$:

$$E_{\Lambda(\delta,x)}(x, r_1) \subset B_\Lambda(x, \delta) \subset E_{\Lambda(\delta,x)}(x, r_2).$$

We say that balls $B_\Lambda(x, \delta)$ are comparable to ellipsoids uniformly over $x \in K$, if there exists $\delta_0 > 0$, $r_1 > 0$ and $r_2 > 0$ such that for all $\delta \in (0, \delta_0)$ and $x \in K$:

$$E_{\Lambda(\delta,x)}(x,r_1) \subset B_\Lambda(x,\delta) \subset E_{\Lambda(\delta,x)}(x,r_2).$$

Note that we use ellipsoids based on $\Lambda(\delta, x)$ rather than arbitrary ellipsoids, in other words the notion of comparability itself also depends on Λ. This is because in our approach the specific form of ellipsoids play an important role and can not be avoided. In rough terms, if we want to approximate balls with ellipsoids, then the shape of ellipsoids has to change as balls are shrinking, to accomodate to the changes in the shape of the balls. Our notion of comparability means that this change of shape is also described in terms of Λ. Unfortunately it seems to be no way to make the comparison precise, meaning that we do not know how to make $r_2 - r_1$ in the comparison small if δ is small. Our estimates are too rough to allow that, but it is also unclear if such comparison is possible at all, even in the examples.

The most general sufficient condition for comparability to ellipsoids is as follows:

Theorem 1. *If for fixed $x \in \mathbb{R}^d$, $\gamma \in (0,1)$, $r > 0$ and $\delta_0 > 0$*

$$\Lambda \in \bigcap_{\delta\in(0,\delta_0)} (LU_d(x,\delta,\gamma,r) \cap LI_d(x,\delta)),$$

then balls $B_\Lambda(x,\delta)$ are comparable to ellipsoids at point x.

If for fixed compact K, $\gamma \in (0,1)$, $r > 0$ and $\delta_0 > 0$

$$\Lambda \in \bigcap_{x\in K} \Big(\bigcap_{\delta\in(0,\delta_0)} (LU_d(x,\delta,\gamma,r) \cap LI_d(x,\delta)) \Big),$$

then balls $B_\Lambda(x,\delta)$ are comparable to ellipsoids uniformly over $x \in K$.

The conditions $LI_d(x,\delta)$ hold in the setting of Carnot groups (if Λ as in Proposition 1) for any x and δ, since the exponential map on Carnot group is globally well-defined and also globally invertible (see, for example, Theorem 2.2.18 in [1]), and therefore exponential balls $\{\Phi(\delta, x, u) : |u| < r\}$ are always open.

Below we present two Propositions with sufficient conditions for Λ to belong to LU_d.

Proposition 2. *Suppose that Λ satisfies our assumptions and*

(1) *for all $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$: $L(x)L(y) = L(x + L(x)y)$,*
(2) *there exists $\delta_0 > 0$, $r > 0$ and $\gamma \in (0,1)$ such that for all $\delta \in (0,\delta_0)$ and $|u| < r$: $\|I - T^{-1}(\delta)L(T(\delta)u)T(\delta)\| \leqslant \gamma$.*

Then $\Lambda \in \bigcap_{x\in\mathbb{R}^d} \bigcap_{\delta\in(0,\delta_0)} LU_d(x,\delta,\gamma,r)$.

The first condition of this Proposition and our assumptions on L (invertibility in particular) can be used to show that operation $(x, y) \to x + L(x)y$ is a group operation on $\mathbb{R}^d$. For the case of Carnot group, that is, if columns of L are the basis of a Lie algebra of Carnot group on $\mathbb{R}^d$ (as in Proposition 1), the condition implies that the group action is linear on the second argument (meaning it is an action of linear operator $L(x)$ plus x). Also it can be checked, that the condition is always true if we assume such linearity. We chose to use such form of the condition in the statement, because in order to check it we do not need to find the group addition explicitly from L beforehand, or even to show that there is a Carnot group.

Also note that for Carnot groups, if p_i are the corresponding homogeneous degrees (again as in Proposition 1), we also have $T^{-1}(\delta)L(T(\delta)u)T(\delta) = L(u)$, so the second condition holds trivially. Since in 2-step Carnot groups the group action is always linear in both arguments (in the same sense as above, for explanation see Theorem 1.3.15 on p.39 of [1]) the assumption of the Proposition above is true for this case. So the balls w.r.t. distance on any 2-step Carnot group are comparable to ellipsoids uniformly on the whole group.

The following Proposition provides a simple and therefore very useful sufficient condition for Λ to belong to LU_d. It always holds for h-step Carnot groups with $h = 2$ and never holds if $h \geqslant 3$.

Proposition 3. *Suppose that Λ satisfies our assumptions and for all $i = 1, \dots, d$ we have $p_i \in [p, p+q]$ for some $0 < q \leqslant p$. Then for any compact $K \subset \mathbb{R}^d$ there exist $\gamma \in (0, 1)$ and $r > 0$ such that $\Lambda \in \bigcap_{x \in K} \bigcap_{\delta \in (0,1)} LU_d(x, \delta, \gamma, r)$.*

It may seem, given the discussion after the Proposition 2, that for Carnot groups the Proposition above does not give us more than we already know. However, it allows us to say a little bit more for 2-step Carnot groups, if we notice that we can also use this Proposition for images of any twice continuously differentiable isomorphic map of 2-step Carnot group.

To this end suppose that we have a differentiable function $f : \mathbb{R}^d \to \mathbb{R}^d$. We seek to define a map f^* on matrix functions Λ, such that ρ_Λ is preserved under f and f^*, i.e. $\rho_\Lambda(x, y) = \rho_{f^*(\Lambda)}(f(x), f(y))$. This can be achieved if the equation $\frac{d}{dt}\varphi(t) = \Lambda(\delta, \varphi(t))a(t)$ is equivalent to $\frac{d}{dt} f(\varphi(t)) = f^*(\Lambda)(\delta, f(\varphi(t)))a(t)$ (it means that all the curves $\varphi(t)$ that may appear in the definition of ρ_Λ are preserved by f, i.e. the action of f gives a one-to-one correspondence between them). We see that it is true if $f^*(\Lambda)(\delta, f(x)) = J_f(x)\Lambda(\delta, x)$, where J_f is a matrix of first-order derivatives of f:

$$(J_f(x))_{ij} = \frac{\partial}{\partial x_j}(f(x))_i; i = 1, \dots, d, j = 1, \dots, d$$

and all $J_f(\varphi(t))$ are invertible. As a result we may define the transformation of Λ under f as follows.

Definition 6. The transformation of Λ under an injection f is a matrix function $f^*(\Lambda)$ defined for each $y = f(x)$ by the equality $f^*(\Lambda)(\delta, y) = J_f(x)\Lambda(\delta, x)$.

If f is one-to-one on any open subset U of $\mathbb{R}^d$ and J_f is non-degenerate on U then, using the arguments outlined above, we obtain that $\rho_\Lambda(x, y) = \rho_{f^*(\Lambda)}(f(x), f(y))$, as long as near-optimal curves from the definition of ρ_Λ are all inside U, which is true for a fixed x for all small enough $\rho_\Lambda(x, y)$ (by obvious upper bounds for Λ). Therefore for any compact $K \subset U$, we can choose δ_0, such that the balls $B_\Lambda(x, \delta)$ for $\delta < \delta_0$ and $x \in K$ are all preserved under such transformation (i.e. $B_\Lambda(x, \delta) = f^{-1}(B_{f^*(\Lambda)}(f(x), \delta))$). Making an additional assumption that f is twice continuously differentiable we can also guarantee that $f^*(\Lambda)$ satisfies our assumptions if Λ does (only local Lipshitz condition is of any concern, which is preserved if J_f is continuously differentiable). This allows us to work with $f^*(\Lambda)$ if we want to obtain bounds for surface measure of $B_\Lambda(x, \delta)$.

It is not hard to see that such transformations preserve classes LI_d, since they are described by the same type of curves, as used in the definition of ρ_Λ. But the condition from the definition of LU_d generally speaking is not preserved by such transformations, in other words it is tied to a given euclidean structure of the space. An example, which proves this, can be found if we consider 3-step Carnot groups, as in the last section of the paper (this fact is not used in the following, so we leave the details to the reader).

Now we can state more general comparison theorem for 2-step Carnot groups.

Theorem 2. *Suppose that we have a 2-step Carnot group $G = (\mathbb{R}^d, \bullet)$, Λ is defined as in Proposition 1 and $f : \mathbb{R}^d \to \mathbb{R}^d$ is twice continuously differentiable. Then for any compact $K \subset \mathbb{R}^d$, such that f is one-to-one on some neighbourhood of K and J_f is non-degenerate on K, the balls $B_{f^*(\Lambda)}(x, \delta)$ are comparable to ellipsoids uniformly over $x \in K$, with Λ replaced by $f^*(\Lambda)$ in the comparison.*

This Theorem can be used to apply our results also to nonlinear manifolds on 2-step Carnot groups by transforming them into linear submanifolds (we give one such example below).

Now we formulate three theorems that show properties of intersections of balls with linear manifolds, given that the comparability to ellipsoids is true. Note that in this paper the surface measures, that we use, are not related to Carnot group structure, and in fact are always defined as corresponding Lebesgue measure on linear submanifold. The first theorem shows the asymptotics of a surface measure of the ball. The second and the third give sufficient conditions for finiteness of specific integrals w.r.t. surface measure that involve distance ρ_Λ (for single and double integrals correspondingly).

Suppose that we have an orthogonal matrix E of size $d \times d$ and define $B_{ij} = E_{ij}$, $i = 1, \dots, d$, $j = 1, \dots, k$, (so that B is $d \times k$ matrix). Then we can describe any k-dimensional linear subpace N of $\mathbb{R}^d$ (with a choice of E or B) as follows:

$$N = \{x \in \mathbb{R}^d : x = Bu, u \in \mathbb{R}^k\}.$$

Denote as $H_{k,d}$ the set of all multiindices $I = (i_1, \dots, i_k) \in \{1, 2, \dots, d\}^k$ with $i_p < i_q$ for $p < q$. For all $I \in H_{k,d}$ and $z \in \mathbb{R}^d$ we can define:

$$G_I(z) = ((\det L(z))^{-1} \det L^{I,E}(z))^2,$$

where $L^{I,E}(z)$ is a matrix obtained from $E^T L(z)$ by removing rows $1, \dots, k$ and columns $i_1, \dots, i_k$.

Also define

$$m_B(x) = \max\{m \,|\, \exists I \in H_{k,d} : \sum_{j=1}^{k} p_{i_j} = m, G_I(x) \neq 0\},$$

$$H^B_{k,d}(x) = \{I \in H_{k,d} \,|\, \sum_{j=1}^{k} p_{i_j} = m_B(x), G_I(x) \neq 0\},$$

$$G(x) = \sum_{I \in H^B_{k,d}(x)} G_I(x).$$

The number $m_B(x)$ is also known as pointwise degree of the manifold, see for example [5].

Theorem 3. *Suppose that balls $B_\Lambda(x, \delta)$ are comparable to ellipsoids at point x, and the constants in the comparison are $r_1 > 0$ and $r_2 > 0$. Then we have*

$$\begin{aligned} S_k r_1^k (G(x))^{-1/2} &\leqslant \varliminf_{\delta \to 0+} \frac{\lambda_k\{u : x + Bu \in B_\Lambda(x, \delta)\}}{\delta^{m_B(x)}} \\ &\leqslant \varlimsup_{\delta \to 0+} \frac{\lambda_k\{u : x + Bu \in B_\Lambda(x, \delta)\}}{\delta^{m_B(x)}} du \leqslant S_k r_2^k (G(x))^{-1/2}, \end{aligned}$$

where S_k is a volume of a unit ball in $\mathbb{R}^k$, and λ_k is a Lebesgue measure on $\mathbb{R}^k$.

Results of this kind for Carnot groups are known (see for example [5]), but our situation is more general. This Theorem shows that the asymptotic behaviour of the surface measure of balls is tied to $m_B(x)$, and so may be different for different x. In fact one of the motivations behind this whole investigation is that in the framework developed in [7] it is unclear how to deal with points, where $m_B(x)$ may be less then maximal possible for given submanifold.

Suppose that we have a measurable function $h(t)$, which is non-negative, bounded on $t \in (s, +\infty)$ for all $s > 0$ and equal to zero for $t > 1$. We denote $H(t) = \int\limits_t^{+\infty} h(s)ds$.

Theorem 4. *Let K be a compact in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. Suppose that the set $\bigcup\limits_{y\in K} B_\Lambda(y, 1)$ is bounded, and the balls $B_\Lambda(y, \delta)$ are comparable to ellipsoids uniformly over $y \in K$. Then the condition*

$$\sup_{y\in K} \int\limits_{\mathbb{R}^k} H(\rho(y, x + Bu))\, du < +\infty \tag{2}$$

holds if and only if for all $y \in K \cap (x + N)$ we have

$$\int\limits_0^1 h(s) s^{m_B(y)} ds < +\infty. \tag{3}$$

The finiteness of the supremum in the Theorem is a sufficient condition for a measure (in our case the restriction of surface measure to a compact set) to produce an additive functional, in the theory of additive functionals of Markov processes (as was shown in [2]), if $H(\rho(y, x + Bu))$ describes the corresponding potential. The latter is true for Brownian motions on Carnot group, if $H(t) = t^{2-Q}$ (for details see for example [1]). Therefore this theorem has direct applications to the construction of additive functionals for Brownian motions on Carnot group. Unfortunately, as it is seen from examples, the condition often fails and therefore it is interesting to look at weaker conditions, with the same goal in mind. This is precisely what motivates us to formulate the following Theorem.

Theorem 5. *Let f be a non-negative locally integrable function, K is a compact on $\mathbb{R}^k$ and $x \in \mathbb{R}^d$. Suppose that the set $\bigcup\limits_{v\in K} B_\Lambda(x + Bv, 1)$ is bounded and the balls $B_\Lambda(y, \delta)$ are comparable to ellipsoids uniformly over $y \in x + BK$. Then*

$$\int\limits_{K^2} f(v) H(\rho(x + Bv, x + Bu))\, dudv < +\infty$$

holds if and only if

$$\int\limits_K \int\limits_0^1 f(v) h(s) \min_{I\in H_{k,d}} \left((G_I(x + Bv))^{-1/2} s^{\sum\limits_{j=1}^{k} p_{i_j}} \right) dsdv < +\infty.$$

Here if $G_I(x + Bv) = 0$ then we assume that $(G_I(x + Bv))^{-1/2} = +\infty$.

In the setting of Carnot groups, the condition is often weak enough to hold with $f = 1$ and $H(t) = t^{2-Q}$, as it is seen from examples. Therefore it can be used to show the existence of local times on the surface for Brownian motions on Carnot group.

3. Quasidistance and comparability of balls with ellipsoids

In this section we prove Lemma 1 about correctness of the definition of ρ_Λ and Theorem 1 that provides a general condition for the balls w.r.t. ρ_Λ to be comparable to ellipsoids.

Proof of Lemma 1. First we show that ρ_Λ is well-defined. Every two points x, y can be connected by a continuously differentiable curve, yielding φ as in the definition of ρ_Λ, but possibly without the condition $|\Lambda^{-1}(\delta, \varphi(t))\frac{d}{dt}\varphi(t)| < 1$ being true for all t. However, by continuity, we know that $\sup\limits_{0<t<1} |\Lambda^{-1}(\delta, \varphi(t))\frac{d}{dt}\varphi(t)| = M < +\infty$. It means that

$$|\Lambda^{-1}(\beta\delta, \varphi(t))\frac{d}{dt}\varphi(t)| \leqslant \|\Lambda^{-1}(\beta\delta, \varphi(t))\Lambda(\delta, \varphi(t))\|M = \|T(\beta)^{-1}\|M$$

and β can be chosen large enough, so that $\|T(\beta)^{-1}\|M < 1$, showing that ρ_Λ is well-defined.

Now we show that $\rho_\Lambda(x, y)$ for two distinct points x and y can not be zero (it is obviously zero for $x = y$). Suppose we have a sequence of numbers $\delta_n \to 0+$, $n \to +\infty$ and corresponding curves $\varphi_n(t)$ joining x, y and satisfying $|\Lambda^{-1}(\delta_n, \varphi_n(t))\frac{d}{dt}\varphi_n(t)| < 1$, $0 < t < 1$. Take some bounded neighbourhood U of x that does not contain y. By the continuity of L we may assume that $\sup_{z\in U} L(z) = M_1 < +\infty$. According to our assumptions $t_n = \inf\{t \in (0, 1] : \varphi_n(t) \notin U\}$ is well-defined and positive, since $\varphi_n(0) = x \in U$ and $\varphi_n(1) = y \notin U$. Then we have

$$|\varphi_n(t_n) - x| \leqslant \sup_{0<s<t_n} |\frac{d}{ds}\varphi_n(s)| \leqslant \sup_{0<s<t_n} \|\Lambda(\delta_n, \varphi_n(s))\| \leqslant M_1\|T(\delta_n)\|.$$

It follows that for large n it is impossible that $\varphi_n(t_n) \notin U$, which is a contradiction.

To prove a weak form of the triangle inequality take three points x, y, z in $\mathbb{R}^d$ and assume that ψ_1 and ψ_2 are some functions satisfying: $\psi_1(0) = x$, $\psi_1(1) = z$, $\frac{d}{dt}\psi_1(t) = \Lambda(\delta_1, \psi_1(t))b_1(t), |b_1(t)| < 1$ $\psi_2(0) = z$, $\psi_2(1) = y$, $\frac{d}{dt}\psi_2(t) = \Lambda(\delta_2, \psi_2(t))b_2(t), |b_2(t)| < 1$ with some $\delta_1 > 0$, $\delta_2 > 0$. We take $\alpha \in (0, 1)$ and define φ as:

$$\varphi(t) = \begin{cases} \psi_1(\frac{t}{\alpha}), & t \in [0, \alpha) \\ \psi_2(\frac{t-\alpha}{1-\alpha}), & t \in [\alpha, 1] \end{cases}.$$

Then for $\delta > 0$ we can write $\frac{d}{dt}\varphi(t) = \Lambda(\delta, \varphi(t))a(t)$ where

$$\begin{aligned} a(t) &= \begin{cases} \frac{1}{\alpha}\Lambda^{-1}(\delta, \psi_1(\frac{t}{\alpha}))\Lambda(\delta_1, \psi_1(\frac{t}{\alpha}))b_1(\frac{t}{\alpha}), & t \in [0, \alpha) \\ \frac{1}{1-\alpha}\Lambda^{-1}(\delta, \psi_2(\frac{t-\alpha}{1-\alpha}))\Lambda(\delta_1, \psi_2(\frac{t-\alpha}{1-\alpha}))b_2(\frac{t-\alpha}{1-\alpha}), & t \in [\alpha, 1] \end{cases} \\ &= \begin{cases} \frac{1}{\alpha}T(\frac{\delta_1}{\delta})b_1(\frac{t}{\alpha}), & t \in [0, \alpha) \\ \frac{1}{1-\alpha}T(\frac{\delta_2}{\delta})b_2(\frac{t-\alpha}{1-\alpha}), & t \in [\alpha, 1] \end{cases}. \end{aligned}$$

In order for $a(t) < 1$ to hold it is enough to have $\|T(\frac{\delta_1}{\delta})\| \leqslant \alpha$ and $\|T(\frac{\delta_2}{\delta})\| \leqslant 1 - \alpha$, which can be achieved for example by choosing $\alpha = \frac{1}{2}$ and $\delta = C(\delta_1 + \delta_2)$, where C is such that $\|T(C^{-1})\| \leqslant \frac{1}{2}$. It follows that for a sequence of δ_1 and δ_2 converging to $\rho(x, z)$ and $\rho(z, y)$ correspondingly we have $\rho(x, y) \leqslant C(\delta_1 + \delta_2)$.

Note, that the condition $\|T(\frac{\delta_1}{\delta_1+\delta_2})\| + \|T(\frac{\delta_2}{\delta_1+\delta_2})\| \leqslant 1$ is sufficient for the triangle inequality to hold (we may take $\alpha = \|T(\frac{\delta_1}{\delta_1+\delta_2})\|$). Then $\|T(\beta)\| \leqslant \beta$, for $\beta < 1$ is, in turn, sufficient for this condition to hold, meaning $p_i \geqslant 1$ is sufficient for ρ_Λ to be a distance on $\mathbb{R}^d$. □

Proof of Theorem 1. We fix some $x \in \mathbb{R}^d$ and $\delta > 0$ and prove the following statements.

(1) Suppose that there exists $\gamma \in (0, 1)$ and $r > 0$ such that $\Lambda \in LU_d(x, \delta, \gamma, r)$. Then $B_\Lambda(x, c_1\delta) \subset E_{\Lambda(\delta,x)}(x, r)$, where $c_1 = c_1(r, \gamma)$ is any positive number that satisfies $\|T(c_1)\| < \frac{r}{1+\gamma}$.

(2) Suppose that there exists $\gamma \in (0, 1)$ and $r > 0$ such that $\Lambda \in LU_d(x, c_1^{-1}\delta, \gamma, r)$. If also $\Lambda \in LI_d(x, \delta)$, then $E_{\Lambda(c_1^{-1}\delta,x)}(x, (1 - \gamma)\|T(c_1^{-1})\|^{-1}) \subset B_\Lambda(x, \delta)$.

(3) For any $\beta > 0$, $r > 0$ we have $E_{\Lambda(\delta,x)}(x, r) \subset E_{\Lambda(\beta\delta,x)}(x, r\|T(\beta^{-1})\|)$.

To prove the first statement let $y \in B_\Lambda(x, c_1\delta)$ then, by the definition of ρ_Λ, there exists $\varphi \in AC([0, 1], \mathbb{R}^d)$ such that $\varphi(0) = x$, $\varphi(1) = y$ and $\frac{d}{dt}\varphi(t) = \Lambda(c_1\delta, \varphi(t))a(t)$ with $|a(t)| < 1$. We obtain that $\frac{d}{dt}\varphi(t) = \Lambda(\delta, \varphi(t))b(t)$ with $b(t) = \Lambda^{-1}(\delta, \varphi(t))\Lambda(c_1\delta, \varphi(t))a(t)$ and, using the properties of Λ, we get $|b(t)| < \|T(c_1)\|$. Suppose that $\varphi([0, 1]) \subset E_{\Lambda(\delta,x)}(x, r)$ does not hold, then we may denote $s = \inf\{t \in [0, 1] : \varphi(t) \notin E_{\Lambda(\delta,x)}(x, r)\}$. We can see that

$$|\Lambda^{-1}(\delta, x)(\varphi(s) - \varphi(0))| = \left| \int_0^s \Lambda^{-1}(\delta, x)\Lambda(\delta, \varphi(t))b(t)\, dt \right| \leqslant (1 + \gamma)\, \|T(c_1)\| < r$$

but it means that $\varphi(s) \in E_{\Lambda(\delta,x)}(x, r)$ and, since φ is continuous and $E_{\Lambda(\delta,x)}(x, r)$ is open, this is a contradiction, meaning that $\varphi([0, 1]) \subset E_{\Lambda(\delta,x)}(x, r)$ is in fact always true.

To prove the second statement we note that exponential balls are always smaller then the corresponding balls w.r.t. distance ρ_Λ, meaning that, for example, $\{\Phi(\delta, x, u) : |u| < 1\} \subset B_\Lambda(x, \delta)$. So we only need to show a relation between ellipsoids and exponential balls: $E_{\Lambda(c_1^{-1}\delta,x)}(x, (1-\gamma)\|T(c_1^{-1})\|^{-1}) \subset \{\Phi(\delta, x, u) : |u| < 1\}$. Take $y \in E_{\Lambda(c_1^{-1}\delta,x)}(x, (1-\gamma)\|T(c_1^{-1})\|^{-1})$. Suppose that $y \in \{\Phi(\delta, x, u) : |u| < 1\}$ does not hold, then we may denote $s = \inf\{t \in [0, 1] : ty + (1 - t)x \notin \{\Phi(\delta, x, u) : |u| < 1\}\}$.

It is clear that $s > 0$. Let $s_n < s$, $s_n \to s$, $n \to +\infty$. For each s_n we can find u_n with $|u_n| < 1$ such that $s_n y + (1 - s_n)x = \Phi(\delta, x, u_n)$. Then there exists u with $|u| \leqslant 1$, such that some subsequence of u_n converges to u. By continuity of $\Phi(\delta, x, \cdot)$ we obtain that $sy + (1 - s)x = \Phi(\delta, x, u)$. If $|u| < 1$ then we have a contradiction with the definition of s and assumption $\Lambda \in LI_d(x, \delta)$ (since $\{\Phi(\delta, x, u) : |u| < 1\}$ is supposed to be open). Therefore we may assume that $|u| = 1$.

By the definition of Φ there is a function f satisfying $f(0) = x$, $f(1) = sy + (1 - s)x$, $\frac{d}{dt}f(t) = \Lambda(\delta, f(t))u$. We can use the bound on $I - \Lambda^{-1}(c_1^{-1}\delta, x)\Lambda(c_1^{-1}\delta, f(t))$ from the assumption $\Lambda \in LU_d(x, c_1^{-1}\delta, \gamma, r)$, since, by the first statement, under the same assumption $\Lambda \in LU_d(x, c_1^{-1}\delta, \gamma, r)$ we have

$$f(t) \in \{\Phi(\delta, x, u) : |u| < 1\} \subset B_\Lambda(x, \delta) \subset E_{\Lambda(c_1^{-1}\delta,x)}(x, r).$$

Therefore we obtain for $v = T(c_1)u$ that $\frac{d}{dt}f(t) = \Lambda(c_1^{-1}\delta, f(t))v$ and

$$|v - \Lambda^{-1}(c_1^{-1}\delta, x)(f(1) - f(0))| = \left| \int_0^1 (I - \Lambda^{-1}(c_1^{-1}\delta, x)\Lambda(c_1^{-1}\delta, f(t))v\, dt \right| \leqslant \gamma|v|.$$

Consequently

$$|v| \leqslant |\Lambda^{-1}(c_1^{-1}\delta, x)(f(1) - f(0))| + \gamma|v|$$

and, using that $y \in E_{\Lambda(c_1^{-1}\delta,x)}(x, (1-\gamma)\|T(c_1^{-1})\|^{-1})$, we obtain that $|u| \leqslant \|T(c_1^{-1})\||v| < s \leqslant 1$, which is a contradiction, meaning that $y \in \{\Phi(\delta, x, u) : |u| < 1\}$ is in fact true.

The last statement is a consequence of the following inequality

$$|\Lambda^{-1}(\beta\delta, x)(y - x)| \leqslant \|\Lambda^{-1}(\beta\delta, x)\Lambda(\delta, x)\||\Lambda^{-1}(\delta, x)(y - x)| = \|T(\beta^{-1})\||\Lambda^{-1}(\delta, x)(y - x)|.$$

To complete the proof of Theorem we use all three statements to conclude that if

$$\Lambda \in \bigcap_{\delta\in(0,\delta_0)} (LU_d(x, \delta, \gamma, r) \cap LI_d(x, \delta))$$

then

$$E_{\Lambda(\delta,x)}(x, (1-\gamma)\|T(c_1^{-1})\|^{-2}) \subset B_\Lambda(x, \delta) \subset E_{\Lambda(\delta,x)}(x, r\|T(c_1^{-1})\|)$$

for all $\delta < \delta_0 = \min(c_1, 1)$, where $c_1 = c_1(r, \gamma)$ is any positive number that satisfies $\|T(c_1)\| < \frac{r}{1+\gamma}$. This proves the Theorem, since no constants depend on x and δ. □

Remark 1. It seem to be possible to prove the results of this section, when the first two conditions on $\Lambda(\delta, x)$ are replaced by another, more general condition, relating values of $\Lambda(\delta, x)$ for different δ in the form of bounds for $\|\Lambda^{-1}(\beta\delta, x)\Lambda(\delta, x)\|$. However such generalization is not entirely straightforward (some extra care should be taken for the lower bound in Theorem 1). Since such generality is not needed for our applications, it is omitted.

4. Sufficient conditions for balls to be comparable to ellipsoids

Here we provide proofs to several sufficient conditions for the balls w.r.t. ρ_Λ to be comparable to ellipsoids. As we already noted, for exponential maps on Carnot groups the existence of the global inverse is known (see, for example, Theorem 2.2.18 in [1], in which the exponential map is used as an isomorphism between abstract stratified Lie groups and Carnot groups). Therefore, in the case described in Proposition 1, we can immediately conclude that Λ belongs to $LI_d(x, \delta)$ for all x and δ. That is why, for discussing examples related to Carnot groups, it is enough to provide some sufficient conditions for the assumptions of class LU_d to hold uniformly over x and δ. Below we provide proofs for two such results.

Proof of Proposition 2. From the first condition we can easily see that $L(0) = I$ (just input $x = y = 0$ and use invertibility of L), and that for any $x \in \mathbb{R}^d$ and $z(x) = -L^{-1}(x)x$ we have $L(x)L(z(x)) = I$, i.e. $L^{-1}(x) = L(z(x))$.

Fix $x \in \mathbb{R}^d$, $\delta \in (0, \delta_0)$ and $y \in E_{\Lambda(x,\delta)}(x, r)$. Then $y = x + \Lambda(x, \delta)u$, where $|u| < r$, and

$$\begin{aligned}\Lambda^{-1}(\delta, x)\Lambda(\delta, y) &= T^{-1}(\delta)L^{-1}(x)L(y)T(\delta) = T^{-1}(\delta)L(z(x))L(y)T(\delta)\\ &= T^{-1}(\delta)L(z(x) + L(z(x))y)T(\delta) = T^{-1}(\delta)L(z(x) + L^{-1}(x)(x + L(x)T(\delta)u))T(\delta)\\ &= T^{-1}(\delta)L(T(\delta)u)T(\delta).\end{aligned}$$

So, under the first condition, $\Lambda^{-1}(\delta, x)\Lambda(\delta, y)$ can be estimated as needed using the second condition. □

Proof of Proposition 3. Since the square of matrix norm is equivalent to the sum of the squares of its elements and each element of $I - \Lambda^{-1}(\delta, x)\Lambda(\delta, y)$ is proportional to the corresponding element of $I - L^{-1}(x)L(y)$ by a factor $\delta^{p_i - p_j}$, where $p_i - p_j \in [-q, q]$, according to our assumption, we can estimate for all x, y and $\delta \in (0, 1)$:

$$\|I - \Lambda^{-1}(\delta, x)\Lambda(\delta, y))\| \leqslant C\delta^{-q}\|I - L^{-1}(x)L(y)\|$$

where C is some constant. Then, using the Lipschitz condition for L, assuming $x \in K$, $y \in K$, we can obtain

$$\|I - L^{-1}(x)L(y)\| \leqslant \|L^{-1}(x)\|\,\|L(x) - L(y)\| \leqslant C_1\|L^{-1}(x)\|\,|x - y|.$$

For $y \in E_{\Lambda(x,\delta)}(x, r)$ we have that $y = x + \Lambda(x, \delta)u$ with $|u| < r$ and therefore:

$$|x - y| = |\Lambda(x, \delta)u| \leqslant r\delta^p\|L(x)\|.$$

Consequently

$$\|I - \Lambda^{-1}(\delta, x)\Lambda(\delta, y))\| \leqslant C_1 r\|L^{-1}(x)\|\,\|L(x)\|.$$

Choosing small enough r completes the proof (we may let $\delta_0 = 1$). □

Proof of Theorem 2. This theorem follows immediately from Theorem 1, Proposition 1 and Proposition 3 (and the fact that the exponential map on Carnot group always exists and globally invertible, as already were noted above). □

5. Balls asymptotics and surface integrals

In this section we find estimates for surface measure of the balls w.r.t. ρ_Λ on linear submanifolds, and use them to prove several theorems regarding its properties, assuming that balls w.r.t. ρ_Λ are comparable to ellipsoids. We start with few lemmas that include some calculations used in the following theorems.

The first lemma finds the surface measure of the intersection of ellipsoid with linear manifold.

Lemma 2. *For all $d \times d$ real-valued invertible matrices A we have*

$$\lambda_k\{u : y + Bu \in E_A(x, 1)\} = S_k(\det C)^{-1/2}(\max(1 - (DA^{-1}(y - x), A^{-1}(y - x)), 0))^{k/2},$$

where $C = B^T(A^{-1})^T A^{-1}B$, $D = I - A^{-1}BC^{-1}B^T(A^{-1})^T$ and S_k is a volume of a unit ball in $\mathbb{R}^k$.

Proof. The condition $y+Bu \in E_A(x, 1)$ is equivalent to the value of the following quadratic function of u being less than 1:

$$\begin{aligned}|A^{-1}(y - x + Bu)|^2 &= |A^{-1}Bu|^2 + 2(A^{-1}(y - x), A^{-1}Bu) + |A^{-1}(y - x)|^2\\ &= |A^{-1}B(u - u_0)|^2 + |A^{-1}(y - x)|^2 - |A^{-1}Bu_0|^2,\end{aligned}$$

where u_0 is a vector in $\mathbb{R}^k$ that satisfies $(A^{-1}(y - x), A^{-1}Bu) = -(A^{-1}Bu_0, A^{-1}Bu)$ for all u. Such vector can be found uniquely since rank of $C = B^T(A^{-1})^T A^{-1}B$ is k (because rank of $A^{-1}B$ is also k) so we can invert C and obtain

$$u_0 = C^{-1}B^T(A^{-1})^T A^{-1}(x-y).$$

Then

$$|A^{-1}Bu_0|^2 = (A^{-1}BC^{-1}B^T(A^{-1})^T A^{-1}(y-x), A^{-1}(y-x))$$

and (note that C is symmetric and positive definite)

$$|A^{-1}B(u-u_0)|^2 = (B^T(A^{-1})^T A^{-1}B(u-u_0), u-u_0) = |C^{1/2}(u-u_0)|^2,$$

so we get

$$|A^{-1}(y-x+Bu)|^2 = |C^{1/2}(u-u_0)|^2 + (DA^{-1}(y-x), A^{-1}(y-x))$$

where $D = I - A^{-1}BC^{-1}B^T(A^{-1})^T$. Now taking $v = C^{1/2}(u-u_0)$ as a new variable of integration we obtain

$$\begin{aligned}\lambda_k\{u : y+Bu \in E_A(x,1)\} &= \int_{\mathbb{R}^k} 1_{|C^{1/2}(u-u_0)|^2+(DA^{-1}(y-x),A^{-1}(y-x))<1}du \\ &= (\det C)^{-1/2}\int_{\mathbb{R}^k} 1_{|v|^2<1-(DA^{-1}(y-x),A^{-1}(y-x))}dv \\ &= S_k(\det C)^{-1/2}(\max(1-(DA^{-1}(y-x),A^{-1}(y-x)),0))^{k/2}.\end{aligned}$$

□

The following lemma is an easy consequence of the previous, but it contains our main estimate for the surface measure of the ball.

Lemma 3. *Suppose that balls $B_\Lambda(x,\delta)$ are comparable to ellipsoids at point x, with corresponding constants $\delta_0 > 0$, $r_1 > 0$ and $r_2 > 0$. Then we have for all $\delta \in (0,\delta_0)$*

$$S_k r_1^k(\det C(\delta,x))^{-1/2} \leqslant \lambda_k\{u : x+Bu \in B_\Lambda(x,\delta)\} \leqslant S_k r_2^k(\det C(\delta,x))^{-1/2},$$

where $C(s,x) = B^T(\Lambda^{-1}(s,x))^T\Lambda^{-1}(s,x)B$.

Proof. The straightforward application of Lemma 2 gives us:

$$\lambda_k\{u : x+Bu \in E_{\Lambda(a,x)}(x,b)\} = S_k(\det C(a,x))^{-1/2}b^k$$

and using comparability to ellipsoids we obtain the statement of the Lemma. □

In order to link our definition of G_I to the value of $\det C(\delta,x)$ from the estimate above we need the following formula.

Lemma 4. *For all $\delta > 0$ and $x \in \mathbb{R}^d$ we have*

$$\det C(\delta,x) = \sum_{I\in H_{k,d}} G_I(x)\delta^{-2\sum_{j=1}^{k} p_{i_j}}.$$

Proof. It is convenient to denote $Q(i,x) = B^T(L^{-1}(x))^T J(i,i)L^{-1}(x)B$, where $J(i,i)$ is a $d\times d$ matrix with all elements equal to zero, except for $J_{ii}(i,i)$ which is equal to 1. Then

$$C(\delta,x) = B^T(\Lambda^{-1}(\delta,x))^T\Lambda^{-1}(\delta,x)B = \sum_{i=1}^{d} Q(i,x)\delta^{-2p_i},$$

since $(T^{-1}(\delta))^T T^{-1}(\delta) = \sum_{i=1}^{d} J(i,i)\delta^{-2p_i}$. Calculating the determinant of the sum of rank 1 matrices we obtain

$$\det C(\delta,x) = \det(\sum_{i=1}^{d} Q(i,x)\delta^{-2p_i}) = \sum_{I\in H_{k,d}} \det(\sum_{j=1}^{k} Q(i_j,x))\delta^{-2\sum_{j=1}^{k} p_{i_j}}.$$

Note that, by the well-known property of inverse matrix:

$$|(\det L(z))^{-1}\det L^{I,E}(z)| = |\det L^{-1}_{I,E}(z)|,$$

where $L^{-1}_{I,E}(z)$ is the matrix obtained from $L^{-1}(z)E$ by keeping rows $i_1,\dots,i_k$ and columns $1,\dots,k$ and removing all others (it is the same if keep rows $i_1,\dots,i_k$ and remove the rest from $L^{-1}(z)B$). But it is not hard to see that $\sum_{j=1}^{k} Q(i_j,z) = (L^{-1}_{I,E}(z))^T L^{-1}_{I,E}(z)$ and the statement of the Lemma follows. □

Recall that we have a measurable function $h(t)$, which is non-negative and bounded on $t\in(s,+\infty)$ for all $s>0$ and equal to zero for $t>1$, and $H(t) = \int_t^{+\infty} h(s)ds$. The following lemma provides an estimate for the integral of $H(\rho(x,y+Bu))$.

Lemma 5. *Suppose that balls $B_\Lambda(x,\delta)$ are comparable to ellipsoids at point x, with corresponding constants $\delta_0>0$, $r_1>0$ and $r_2>0$. Then we have*

$$S_k r_1^k \int_0^{\delta_0} h(s)(\det C(s,x))^{-1/2}\left(1-\frac{g(s)}{r_1^2}\right)^{k/2} 1_{\{g(s)\leqslant r_1^2\}}ds \leqslant \int_{\mathbb{R}^k} H(\rho(x,y+Bu))du$$

$$\leqslant S_k r_2^k \int_0^{\delta_0} h(s)(\det C(s,x))^{-1/2}\left(1-\frac{g(s)}{r_2^2}\right)^{k/2} 1_{\{g(s)\leqslant r_2^2\}}ds + \sup_{s>\delta_0} h(s)\int_{\mathbb{R}^k} 1_{\rho(x,y+Bu)<1}du,$$

where $C(s,x) = B^T(\Lambda^{-1}(s,x))^T\Lambda^{-1}(s,x)B$ and

$$g(s) = ((I-\Lambda^{-1}(s,x)BC^{-1}(s,x)B^T(\Lambda^{-1}(s,x))^T)\Lambda^{-1}(s,x)(y-x),\Lambda^{-1}(s,x)(y-x)).$$

is a non-negative function.

Proof. The integral can be written in terms of measures of balls:

$$\int_{\mathbb{R}^k} H(\rho(x,y+Bu))du = \int_{\mathbb{R}^k}\int_0^1 h(s)1_{\rho(x,y+Bu)<s}duds.$$

Then we can estimate it above and below in terms of ellipsoids:

$$\int_{\mathbb{R}^k}\int_0^{\delta_0} h(s)1_{y+Bu\in E_{\Lambda(s,x)}(x,r_1)}dsdu \leqslant \int_{\mathbb{R}^k} H(\rho(x,y+Bu))du \leqslant$$

$$\leqslant \int_{\mathbb{R}^k}\int_0^{\delta_0} h(s) 1_{y+Bu \in E_{\Lambda(s,x)}(x,r_2)}\, ds du + \sup_{s>\delta_0} h(s) \int_{\mathbb{R}^k} 1_{\rho(x,y+Bu)<1} du.$$

Finally, calculating the measure of ellipsoids using Lemma 2 (note that $g(s)$ is now equal to the $(DA^{-1}(y-x), A^{-1}(y-x))$ from Lemma 2, which is non-negative since D is an orthogonal projection) we obtain in the upper estimate:

$$\int_{\mathbb{R}^k}\int_0^{\delta_0} h(s) 1_{y+Bu \in E_{\Lambda(s,x)}(x,r_2)} ds du \leqslant S_k r_2^k \int_0^1 h(s) (\det C(s,x))^{-1/2} \left(1 - \frac{g(s)}{r_2^2}\right)^{k/2} 1_{\{g(s) \leqslant r_2^2\}} ds.$$

The lower estimate is obtained analogously. □

Below we also need that for all $x \in \mathbb{R}^d$ the ellipsoids $E_{\Lambda(\delta,x)}(x,r)$ are always shrinking to a point as $\delta \to 0+$ uniformly over x on any compact.

Lemma 6. *Suppose that Λ satisfies our assumptions. Then for any $r > 0$, any compact set K and closed set K_1, with $K \cap K_1 = \emptyset$, there is $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ and $x \in K$ we have $K_1 \cap E_{\Lambda(\delta,x)}(x,r) = \emptyset$.*

Proof. Fix $x \in K$ and $y \in K_1$. We have

$$|y - x| \leqslant \|\Lambda(\delta, x)\| \cdot |\Lambda^{-1}(\delta, x)(y - x)| \leqslant \|\Lambda(1, x)\| \cdot \|T(\delta)\| \cdot |\Lambda^{-1}(\delta, x)(y - x)|$$

and so

$$|\Lambda^{-1}(\delta, x)(y - x)| \geqslant C(\|T(\delta)\|)^{-1},$$

where $C > 0$ does not depend on δ, x and y. Since $\|T(\delta)\|$ converges to 0, as $\delta \to 0+$, the statement follows. □

Now we are ready to prove our theorems about properties of intersections of balls with linear manifolds.

Proof of Theorem 3. Using Lemma 3 and Lemma 4 we obtain for small δ:

$$\lambda_k\{u : x + Bu \in B_\Lambda(x,\delta)\} \leqslant S_k r_2^k (\sum_{I \in H_{k,d}} G_I(x) \delta^{-2\sum_{j=1}^k p_{i_j}})^{-1/2} =$$

$$= S_k r_2^k \delta^{m_B(x)} (\sum_{I \in H_{k,d}} G_I(x) \delta^{2(m_B(x) - \sum_{j=1}^k p_{i_j})})^{-1/2},$$

where $\sum_{I \in H_{k,d}} G_I(x) \delta^{2(m_B(x) - \sum_{j=1}^k p_{i_j})}$ converges to $G(x)$ as $\delta \to 0+$ by the definition of $m_B(X)$ and $G(x)$. The upper bound easily follows and the lower bound can be shown in a similar way. □

Proof of Theorem 4. Using Lemma 5 we can bound the supremum in (2) above with (omitting some multiplicative and additive constants)

$$\sup_{y\in K}\int\limits_0^{\delta_0} h(s)(\det C(s,y))^{-1/2}ds$$

or alternatively with

$$\int\limits_{\mathbb{R}^k}\int\limits_0^{\delta_0} h(s)1_{y+Bu\in E_{\Lambda(s,x)}(x,r_2)}\,dsdu$$

and below with

$$\sup_{y\in K\cap(x+N)}\int\limits_0^{\delta_0} h(s)(\det C(s,y))^{-1/2}ds$$

using that $g(s) = 0$ for all s, when points that we consider belong to $x + N$. Applying the formula for the determinant from Lemma 4 and also bounding the sum with maximum of its members (below with constant 1 and above with the constant equal to the number of the members in the sum), we can replace $(\det C(s,y))^{-1/2}$ in both bounds with (multiplied by a constant in the lower bound)

$$\min_{I\in H_{k,d}}((G_I(y))^{-1/2}s^{\sum\limits_{j=1}^{k}p_{i_j}}).$$

Note that if the integral under supremum in (2) is finite for all $y \in K\cap(x+N)$ then the condition (3) is also fulfilled for all $y \in K \cap (x + N)$. Indeed if we take multiindex $I' = (i'_1,\dots,i'_d)$ satisfying $G_{I'}(y) \neq 0$, with the largest $\sum\limits_{j=1}^{k} p_{i'_j}$ (which is equal to $m_B(y)$) then

$$\int\limits_0^1 h(s)s^{\sum\limits_{j=1}^{k}p_{i'_j}}ds < +\infty$$

must hold, because otherwise the integral can not be finite (multiple multiindices with the same $\sum\limits_{j=1}^{k} p_{i_j} = m_B(y)$ and $G_I(y) \neq 0$ do not change anything since $G_I(y) \geqslant 0$).

On the other hand if the condition (3) is fulfilled for each $y \in K \cap (x + N)$, then for all $y \in K \cap (x + N)$ we can choose a multiindex $I' = (i'_1,\dots,i'_d)$ with $G_{I'}(y) \neq 0$, such that $m_B(y) = \sum\limits_{j=1}^{k} p_{i'_j}$, meaning that the integral is finite, and moreover its supremum is finite over some neighbourhood U of y, since

$$\sup_{z\in U}\int\limits_0^{\delta_0} h(s)\min_{I\in H_{k,d}}((G_I(z))^{-1/2}s^{\sum\limits_{j=1}^{k}p_{i_j}})\,ds \leqslant \sup_{z\in U}(G_{I'}(z))^{-1/2}\int\limits_0^{\delta_0} h(s)s^{m_B(y)}ds$$

and $G_I(z)$ is continuous.

To finish the proof we note that it is possible to find a finite number of open sets such that together they cover $K \cap (x + N)$, and on each of them there is a constant multiindex $I' = (i'_1,\dots,i'_d)$, such that

$$\int_0^1 h(s)s^{\sum_{j=1}^{k} p_{i'_j}} ds < +\infty$$

and $G_{I'}(y)$ is separated from zero. Then we fix compact $\tilde{K}$, defined as K without the union of these open sets, and using Lemma 6, we can choose $s_0 > 0$ and $r > 0$, such that for all $y \in \tilde{K}$ and all $s < s_0$ we have $E_{\Lambda(s,y)}(y, r) \cap (x + N) = \emptyset$, which means that the part of the integral

$$\int_{\mathbb{R}^k}\int_0^{\delta_0} h(s)1_{y+Bu\in E_{\Lambda(s,x)}(x,r_2)}\, duds$$

for $s < s_0$ is zero. Then our upper bound is finite on all of K and the statement follows. □

Proof of Theorem 5. We use Lemma 5 on the integral w.r.t. u, and use that $g(s) = 0$ for all s, when points that we consider belong to $x+N$. Since, by the conditions of the Theorem, a set of points within a distance ρ_Λ smaller than 1 to $x + BK$ is bounded, we can see that $\int_{\mathbb{R}^k} 1_{\rho(x+Bv,x+Bu)<1} du$ is uniformly bounded over $v \in K$. Therefore the integral is finite if and only if

$$\int_K\int_0^1 f(v)h(s)(\det C(s, x + Bv))^{-1/2} dsdv < +\infty,$$

where upper limit in the integral by ds can be set to 1, because $\det C(t, y)$ is separated from zero on $(t, y) \in [q, 1] \times (x + BK)$ for all $q > 0$ (since it is continuous and non-zero, and $x + BK$ is compact). Now we use formula from Lemma 4 which gives us the statement of the Theorem. □

6. Applications and examples

In this section we present several examples, where we can find asymptotics of surface measure of small balls (some of these asymptotic results are already known, see [6, 5]), and determine if the integrals of the negative power of the distance w.r.t. the surface measure and their supremum are finite. We use sufficient conditions to verify that balls are comparable to ellipsoids and then apply Theorems 3, 4, 5. We find $m_B(x)$, $G_I(x)$, $G(x)$, and, assuming $h(s) = s^\alpha$, derive the exact integrability conditions. All results obtained are summarized in the corresponding Theorems, in terms of Γ, a natural distance on Carnot group introduced earlier.

EXAMPLE 1. Take a 3-dimensional Heisenberg group, i.e. Carnot group $(\mathbb{R}^3, \bullet)$ with group action $x \bullet y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + y_1x_2 - x_1y_2)$. Using Proposition 1 we get (with $d = 3$, $k = 2$): $L(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & -x_1 & 1 \end{pmatrix}$ and $T(\delta) = \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta^2 \end{pmatrix}$. In this case Theorem 2 tells us that balls are uniformly comparable to ellipsoids, so Theorems 3, 4, 5 are applicable (in Theorems 4, 5 the additional condition of boundedness of the union of balls is trivially satisfied, since all closed balls are bounded). Take $k = 2$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case it is easy to find that: $G_{1,2}(z) = 1$, $G_{1,3}(z) = z_1^2$, $G_{2,3}(z) = z_2^2$. Therefore functions $G(z)$ and $m_B(z)$ defining asymptotics of surface measure of balls in Theorem 3 can be found as follows: $m_B(z) = 2$ if $z_1 = z_2 = 0$ and $m_B(z) = 3$ otherwise, $G(z) = 1$ if $z_1 = z_2 = 0$ and $G(z) = z_1^2 + z_2^2$ otherwise.

The main condition in Theorem 4 is satisfied if and only if $\int_0^1 h(s)s^2 ds < +\infty$ in case the compact $K \cap \{z \,|\, z_3 = x_3\}$ contains points z where $z_1 = z_2 = 0$ and if and only if $\int_0^1 h(s)s^3 ds < +\infty$ in case the compact $K \cap \{z \,|\, z_3 = x_3\}$ does not contain points where $z_1 = z_2 = 0$. In particular for $h(s) = s^\alpha$, $\alpha \in (-4, -3]$ the unboundedness appear at the neighbourhood of points on the plane $\{z|z_3 = x_3\}$ such that $z_1 = z_2 = 0$.

The condition of Theorem 5 is fulfilled if and only if

$$\int_K \int_0^1 f(v_1 - x_1, v_2 - x_2) h(s) \min(1, s|v|^{-1}) s^2 \, ds dv < +\infty.$$

It can be further simplified as

$$\int_K f(v_1 - x_1, v_2 - x_2)(|v|^{-1} \int_0^{|v|} h(s)s^3 ds + \int_{|v|}^1 h(s)s^2 ds) \, dv < +\infty$$

and in case $f = 1$ and $K = \{v : |v| \leqslant 1\}$ it is equivalent to

$$\int_0^1 h(s)s^3 ds < +\infty.$$

If $h(s) = s^\alpha$ then the condition is equivalent to $\alpha > -4$.

Below we write $f(\delta) \asymp g(\delta)$ as $\delta \to 0+$ if there exists positive constants C_1, C_2 such that

$$C_1 \leqslant \varliminf_{\delta \to 0+} \frac{f(\delta)}{g(\delta)} \leqslant \varlimsup_{\delta \to 0+} \frac{f(\delta)}{g(\delta)} \leqslant C_2.$$

Theorem 6. *Suppose that $G = (\mathbb{R}^3, \bullet)$ is a Carnot group with group action $x \bullet y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + y_1 x_2 - x_1 y_2)$ and the corresponding distance is Γ (as in 1). Then the following statements hold*

(1) *Denote $\psi(x, \delta) = \int_{\mathbb{R}^2} 1_{\{\Gamma(x,(x_1+u_1,x_2+u_2,x_3))<\delta\}} du_1 du_2$. If $x_1 = x_2 = 0$ then $\psi(x, \delta) \asymp \delta^2$ as $\delta \to 0+$, otherwise (if $x_1^2 + x_2^2 \neq 0$) $\psi(x, \delta) \asymp \delta^3$ as $\delta \to 0+$.*

(2) *We have for all $x \in \mathbb{R}^d$*

$$\int_{|u|<1,|v|<1} [\Gamma((x_1 + v_1, x_2 + v_2, x_3), (x_1 + u_1, x_2 + u_2, x_3))]^\beta du dv < +\infty$$

if and only if $\beta > -3$.

(3) *We have for all $a \in \mathbb{R}$ and compacts K, such that $(0,0,a) \in K$,*

$$\sup_{y\in K} \int\limits_{|u|<1} [\Gamma((y_1, y_2, y_3), (u_1, u_2, a))]^\beta \, du < +\infty$$

if and only if $\beta > -2$. In the case $(0,0,a) \notin K$ the supremum is finite if and only if $\beta > -3$.

EXAMPLE 2. Let $G = (\mathbb{R}^3, \bullet)$ be a Carnot group with group action $x \bullet y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2y_1x_2)$. Then we have as in Proposition 1: $L(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x_2 & 0 & 1 \end{pmatrix}$ and $T(\delta) = \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta^2 \end{pmatrix}$ (it is interesting to note that function $f(x) = (x_1, x_2, x_3 + x_1x_2)$, which is an isomorphism from Carnot group in the previous example to our Carnot group G, produces the same Λ when applied to Λ from previous example). Again take $k = 2$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case we obtain $G_{1,2}(z) = 1$, $G_{1,3}(z) = 0$, $G_{2,3}(z) = 4z_2^2$ and the following Theorem can be proved, in the same way as in the previous example.

Theorem 7. *Suppose that $G = (\mathbb{R}^3, \bullet)$ is a Carnot group with group action $x \bullet y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2y_1x_2)$ and the corresponding distance is Γ. Then the following statements hold*

(1) *Denote $\psi(x,\delta) = \int\limits_{\mathbb{R}^2} 1_{\{\Gamma(x,(x_1+u_1,x_2+u_2,x_3))<\delta\}} \, du_1 du_2$. If $x_2 = 0$ then $\psi(x,\delta) \asymp \delta^2$ as $\delta \to 0+$, otherwise (if $x_2 \neq 0$) $\psi(x,\delta) \asymp \delta^3$ as $\delta \to 0+$.*

(2) *We have for all $x \in \mathbb{R}^d$*

$$\int\limits_{|u|<1,|v|<1} [\Gamma((x_1 + v_1, x_2 + v_2, x_3), (x_1 + u_1, x_2 + u_2, x_3))]^\beta \, dudv < +\infty$$

if and only if $\beta > -3$.

(3) *We have for all $a \in \mathbb{R}$ and compacts K, such that $(b,0,a) \in K$ for some $b \in \mathbb{R}$,*

$$\sup_{y\in K} \int\limits_{|u|<1} [\Gamma((y_1, y_2, y_3), (u_1, u_2, a))]^\beta du < +\infty$$

if and only if $\beta > -2$. In the case $(b,0,a) \notin K$ for all $b \in \mathbb{R}$ the supremum is finite if and only if $\beta > -3$.

EXAMPLE 3. Once again take $G = (\mathbb{R}^3, \bullet)$ is a Carnot group with group action $x \bullet y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + y_1x_2 - x_1y_2)$, but consider $k = 1$ and a curve $N = \{(t^2, t, t) : t \in \mathbb{R}\}$. Then, applying a map $f(x) = (x_1 - x_3^2, x_2 - x_3, x_3)$ we arrive to a situation, where Theorems 3, 4, 5 are all applicable (according to Theorem 2). In this case, taking $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in new coordinates, we calculate that $G_1(x) = 4x_3^2$, $G_2(x) = 1$ and $G_3(x) = (1 + x_1 - 2x_2x_3)^2$ in original coordinates, but for transformed Λ. Note that $G_3 = (1 - t^2)^2$ on the curve, so its zero only at $t = x_3 = 1$ or $t = x_3 = -1$. Consequently we have $m_B(x) = 1$ at $(1, 1, 1)$ and $(1, -1, -1)$, and $m_B(x) = 2$ for the rest of the points on the curve.

The condition of Theorem 5 is fulfilled if and only if

$$\int_K \int_0^1 f(v - x_3)h(s) \min(1, s|1 - v^2|^{-1})s\, dsdv < +\infty,$$

which is true for $f = 1$ and any compact K if

$$\int_0^1 h(s)s^2 |\ln s| ds < +\infty$$

in particular $\alpha > -3$ for $h(s) = s^\alpha$ is sufficient. On the other hand for $\alpha = -3$ the conditions fails for any K of non-zero Lebesgue measure. Therefore we obtain the following Theorem.

Theorem 8. *Suppose that $G = (\mathbb{R}^3, \bullet)$ is a Carnot group with group action $x \bullet y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + y_1x_2 - x_1y_2)$ and the corresponding distance is Γ. Then the following statements hold*

(1) *Denote $\psi(v, \delta) = \int_{\mathbb{R}} 1_{\{\Gamma((v^2,v,v),(u^2,u,u))<\delta\}} du$. If $v = 1$ or $v = -1$ then $\psi(x, \delta) \asymp \delta$ as $\delta \to 0+$, otherwise $\psi(x, \delta) \asymp \delta^2$ as $\delta \to 0+$.*

(2) *We have for all $a \in \mathbb{R}$*

$$\int_{|u-a|<1, |v-a|<1} [\Gamma((v^2, v, v), (u^2, u, u))]^\beta\, dudv < +\infty$$

if and only if $\beta > -2$.

(3) *We have for any compact K, such that $(1, 1, 1) \in K$ or $(1, -1, -1) \in K$,*

$$\sup_{y \in K} \int_{|u|<2} [\Gamma(y, (u^2, u, u))]^\beta\, du < +\infty$$

if and only if $\beta > -1$. For any compact K, such that $(1, 1, 1) \notin K$ and $(1, -1, -1) \notin K$, the condition is $\beta > -2$.

EXAMPLE 4. Let us take $d \geqslant 3$ and define $L(x)$ as follows: $L_{ii}(x) = 1$ for $i = 1, \ldots, d$, $L_{ij} = \frac{x_1^{i-j}}{(i-j)!}$ for $j = 2, \ldots, d$, $i = j + 1, \ldots, d$ and $L_{ij}(x) = 0$ for all other pairs of indices i, j. We also define $T(\delta)$ by setting $p_1 = p_2 = 1$ and $p_i = i - 1$ for $i = 3, \ldots, d$. The corresponding group, with action $x \bullet y = x + L(x)y$ is a Carnot group and p_i are the homogeneous degrees of the columns of L. Therefore by Proposition 2 we have comparability to ellipsoids and Theorems 3, 5, 4 are all applicable.

We take $k = d - 1$ and linear manifold $N = \{x : x_d = 0\}$ (and choose E to be identity matrix as before). Denote as $I_l = \{1, \ldots, d\} \backslash \{l\}$, a multiindex containing all possible different indices between 1 and d except l. Then we can calculate: $G_{I_1}(x) = 0$, $G_{I_l}(x) = (\frac{x_1^{d-l}}{(d-l)!})^2$ for $l = 2, \ldots, d$. Therefore $m_B(x) = \frac{d(d-1)}{2}$ if $x_1 \neq 0$ and $m_B(x) = \frac{(d-1)(d-2)}{2} + 1$ if $x_1 = 0$.

Let us check the main condition of Theorem 5. It is equivalent to

$$\int_K \int_0^1 f(v)h(s)\min(1, s^{d-2}|x_1+v_1|^{-(d-2)})s^{(d-1)(d-2)/2+1}\,ds\,dv < +\infty,$$

which is true for $f = 1$ if

$$\int_0^1 h(s)s^{\frac{d(d-1)}{2}}\,ds < +\infty$$

and for $h(s) = s^\alpha$ we see that $\alpha > -\frac{d(d-1)}{2} - 1$ is sufficient. On the other hand for $\alpha = -\frac{d(d-1)}{2} - 1$ the conditions fails for any K of non-zero Lebesgue measure.

Theorem 9. *Suppose that $G = (\mathbb{R}^d, \bullet)$ is a Carnot group with group action $x \bullet y = x + L(x)y$ with $L(x)$ defined as follows: $L_{ii}(x) = 1$ for $i = 1, \dots, d$, $L_{ij} = \frac{x_1^{i-j}}{(i-j)!}$ for $j = 2, \dots, d$, $i = j+1, \dots, d$ and $L_{ij}(x) = 0$ for all other pairs of indices i, j. Denote the corresponding distance as Γ. Then the following statements hold*

(1) *Denote $\psi(x,\delta) = \int_{\mathbb{R}^{d-1}} 1_{\{\Gamma(x, x+(u,0))<\delta\}}du$. If $x_1 = 0$ then $\psi(x,\delta) \asymp \delta^{(d-1)(d-2)/2+1}$ as $\delta \to 0+$, otherwise (if $x_1 \neq 0$) $\psi(x,\delta) \asymp \delta^{d(d-1)/2}$ as $\delta \to 0+$.*

(2) *We have for all $x \in \mathbb{R}^d$*

$$\int_{|u|<1, |v|<1} [\Gamma((x_1+v_1, \dots, x_{d-1}+v_{d-1}, x_d), (x_1+u_1, \dots, x_{d-1}+u_{d-1}, x_d))]^\beta\,du\,dv < +\infty$$

if and only if $\beta > -\frac{d(d-1)}{2}$.

(3) *We have for all $a \in \mathbb{R}$ and compacts $K \subset \mathbb{R}^d$, such that $(0, b, a) \in K$ for some $b \in \mathbb{R}^{d-2}$,*

$$\sup_{y \in K} \int_{|u|<1} [\Gamma(y, (u_1, \dots, u_{d-1}, a))]^\beta\,du < +\infty$$

if and only if $\beta > -\frac{(d-1)(d-2)}{2} - 1$. In the case $(0, b, a) \notin K$ for all $b \in \mathbb{R}^{d-2}$ the supremum is finite if and only if $\beta > -\frac{d(d-1)}{2}$.

7. Distortion of balls in Carnot groups

In this section we consider an example of 3-step Carnot group for which we can prove that balls are not comparable to ellipsoids. Additionally we provide another interesting property, explaining this: we show that, in this example, intersection of balls with linear manifolds have at least two connected components. Intuitively, in this case, we may suggest that small balls have "parabolic" shape, meaning that they are distorted along some quadratic function on one of the coordinates (here we do not make the corresponding precise statement, but our arguments below shed some light on the situation).

Let us consider the following situation: $d = 4$,

$$L(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x_1 & 1 & 0 \\ 0 & x_1x_2 & x_2 & 1 \end{pmatrix}, T(\delta) = \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta^2 & 0 \\ 0 & 0 & 0 & \delta^3 \end{pmatrix}.$$

The corresponding distance ρ_Λ is equivalent to the distance Γ on a Carnot group $G = (\mathbb{R}^4, \bullet)$ with its group operation defined by:

$$x \bullet y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1y_2, x_4 + y_4 + y_2x_1x_2 + y_3x_2 + \frac{1}{2}x_1y_2^2)$$

and its dilations acting as $(x_1, x_2, x_3, x_4) \mapsto (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4)$. This equivalence is the consequence of the relation $L_{ij}(x) = \frac{\partial}{\partial y_j}((x \bullet y)_i)|_{y=0}$ which shows that the columns of L are left-invariant basis for Lie algebra of G with homogeneous degrees $1, 1, 2, 3$ – the same as the powers of δ in T_δ, so we can apply Proposition 1. Such group is isomorphic to an abstract Engel group, and the asymptotic of the surface measure of intersections of small balls with submanifolds for Engel group was studied in [4]. This particular instance of Engel group has some interesting properties of intersections of small balls with linear submanifolds, related to the comparison with ellipsoids, which we are about to present.

Theorem 10. *There exists $s_0 > 0$, such that for any real $s \neq 0$ with $|s| < s_0$ there is a constant $\delta_0 > 0$, such that for all $\delta \in (0, \delta_0)$, $t \in \mathbb{R}$ and $x = (1, s\delta, t, 0)$ the intersection of the ball $B_\Lambda(x, \delta)$ with the linear manifold $H = \{z \in G | z_4 = 0\}$ has at least two connected components.*

Proof. Define for all $x \in G$ a transformation $f_x(y) = x^{-1} \bullet y$ on G, where x^{-1} is an inverse of x in G. It is easy to find its formula:

$$f_x(y) = (y_1 - x_1, y_2 - x_2, y_3 - x_3 - x_1(y_2 - x_2), y_4 - x_4 - x_2(y_3 - x_3) - \frac{1}{2}x_1(y_2 - x_2)^2).$$

Note that due to the left-invariance of the columns of L it is preserved under such transformations and so is the distance ρ_Λ, which gives us that $f_x(B_\Lambda(x, \delta)) = B_\Lambda(0, \delta)$. On the other hand if we set $y_1 = x_1 + a_1$, $y_2 = x_2 + a_2$, $y_3 = x_3 + a_3 + x_1a_2$, $y_4 = 0$, then for all $a \in \mathbb{R}^3$ we obtain every $y \in H$ and therefore:

$$f_x(H) = \{(a_1, a_2, a_3, -x_4 - x_2a_3 - x_1x_2a_2 - \frac{1}{2}x_1a_2^2) | (a_1, a_2, a_3) \in \mathbb{R}^3\}.$$

The ellipsoids $E_{\Lambda(\delta,0)}(0, r)$ are bounded above and below with boxes $Q_q(\delta) = \{z \in G : |z_1| < q\delta, |z_2| < q\delta, |z_3| < q\delta^2, |z_4| < q\delta^3\}$. For small enough $q > 0$ and for all $\delta > 0$ and $z \in Q_q(\delta)$ we have

$$\|I - \Lambda^{-1}(0, \delta)\Lambda(z, \delta)\| = \left\| \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \delta^{-1}z_1 & 0 & 0 \\ 0 & \delta^{-2}z_1z_2 & \delta^{-1}z_2 & 0 \end{pmatrix} \right\| \leqslant \frac{1}{2}.$$

Applying Theorem 1 we can find $q_1 > 0$, $q_2 > 0$ such that $Q_{q_1}(\delta) \subset B_\Lambda(0, \delta) \subset Q_{q_2}(\delta)$ for all $\delta > 0$

But it is easy to see, for any t and any fixed small s, that for $x = (1, s\delta, t, 0)$ we have $0 \in Q_q(\delta) \cap f_x(H)$ and $(0, -2s\delta, 0, 0) \in Q_q(\delta) \cap f_x(H)$, and for all z such that $z_2 = -s\delta$

we have $z \notin Q_q(\delta) \cap f_x(H)$ for small δ, since for all such z the conditions to belong to $Q_q(\delta) \cap f_x(H)$ imply $a_2 = -s\delta$, $|a_3| < q\delta^2$ and

$$|x_2 a_3 + x_1 x_2 a_2 + \frac{1}{2} x_1 a_2^2| = |s\delta a_3 - \frac{1}{2} s^2 \delta^2| < q\delta^3,$$

which is a contradiction for all small enough δ. Therefore each $Q_q(\delta) \cap f_x(H)$ has at least two connected components for small δ and moreover, since the separating set $z_2 = -s\delta$ and the points 0 and $(0, -2s\delta, 0, 0)$, that it separates, does not depend on q, $B_\Lambda(0, \delta) \cap f_x(H)$ also has at least two connected components for small δ. □

This interesting fact however is not in direct contradiction with the uniform comparability to ellipsoids. More precisely if the balls are comparable to ellipsoids uniformly over x in some neighbourhood of $(1, 0, t, 0)$, then we can not conclude immediately that the intersections of balls with the manifold H can not be disconnected, since, in theory, it is possible for an extra disconnected part of the ball to be always inside outer ellipsoid but outside inner ellipsoid. It may be possible to exclude this possibility with the further clarification of the actual geometric shape of the balls, but this requires some additional calculations. Fortunately we can arrive to the same conclusion, if we find the asymptotic behaviour of the surface measure of the balls in this case and compare it to the behaviour predicted by Theorem 3. If there is a difference in asymptotics for some fixed point x then the balls with a center at point x can not be comparable to ellipsoids. But such asymptotics are easy to find in our case, as the following theorem shows.

It is interesting to note, that, as the proof of Theorem 10 suggests, for any x in Carnot group we can always find such C^2 automorphism (for example in the form $f_x(y) = x^{-1} \bullet y$), which transforms balls, such that they become comparable to ellipsoids at point x. But then, of course, the submanifold we consider is no longer linear, and it seems to be hard to describe in a reasonably general way a two-way estimate for the surface measure of the intersection of an ellipsoid with arbitrary submanifold in Euclidean space. Nevertheless this idea can be used to find the asymptotics in examples. In fact it is exactly what we do below.

Theorem 11. *Denote* $\psi(x, \delta) = \int_{\mathbb{R}^3} 1_{\{(z_1, z_2, z_3, 0) \in B_\Lambda(x, \delta)\}} dz_1 dz_2 dz_3$. *If* $x = 0$, *then* $\psi(x, \delta) \asymp \delta^4$. *If* $x_2 = 0$, $x_4 = 0$ *and* $x_1 \neq 0$, *then* $\psi(x, \delta) \asymp \sqrt{x_1^{-1} \delta^9}$. *If* $x_1 = 0$, $x_4 = 0$ *and* $x_2 \neq 0$, *then* $\psi(x, \delta) \asymp x_2^{-1} \delta^5$. *If* $x_4 = 0$, $x_1 \neq 0$ *and* $x_2 \neq 0$, *then* $\psi(x, \delta) \asymp (x_1 x_2)^{-1} \delta^6$. *The constants in all asymptotics do not depend on x.*

Proof. Using again $f_x(y) = x^{-1} \bullet y$ we obtain, with the change of variables:

$$\psi(x, \delta) = \int_{\mathbb{R}^3} 1_{\{(a_1, a_2, a_3, -x_4 - x_2 a_3 - x_1 x_2 a_2 - \frac{1}{2} x_1 a_2^2) \in B_\Lambda(0, \delta)\}} da_1 da_2 da_3$$

and recalling that the ball $B_\Lambda(0, \delta)$ can be covered with the box $Q_q(\delta) = \{z \in G : |z_1| < q\delta, |z_2| < q\delta, |z_3| < q\delta^2, |z_4| < q\delta^3\}$ we estimate:

$$\psi(x, \delta) \leqslant C \int_{\mathbb{R}^3} 1_{\{|a_1| < q\delta\}} 1_{\{|a_2| < q\delta\}} 1_{\{|a_3| < q\delta^2\}} 1_{\{|x_4 + x_2 a_3 + x_1 x_2 a_2 + \frac{1}{2} x_1 a_2^2| < q\delta^3\}} da_1 da_2 da_3.$$

The estimate from below is exactly the same with different constants, so it is enough to investigate the last integral. Note that the variable a_1 can be integrated immediately and the asymptotics of the rest of the integral, denoted as

$$J(x,\delta) = \int_{\mathbb{R}^2} 1_{\{|a_2|<q\delta\}} 1_{\{|a_3|<q\delta^2\}} 1_{\{|x_4+x_2a_3+x_1x_2a_2+\frac{1}{2}x_1a_2^2|<q\delta^3\}} da_2 da_3$$

depend significantly on x.

If $x = 0$, then $J(x,\delta) \sim C\delta^3$ (the symbol $\sim$ means that the left hand side divided by the right hand side converges to 1, as $\delta \to 0+$). If $x_2 = 0$, $x_4 = 0$, but $x_1 \neq 0$, then $J(x,\delta) \sim Cx_1^{-1/2}\delta^{7/2}$. If $x_1 = 0$, $x_4 = 0$, but $x_2 \neq 0$, then $J(x,\delta) \sim Cx_2^{-1}\delta^4$. If $x_4 = 0$, but $x_1 \neq 0$ and $x_2 \neq 0$, then we can find the upper bound for small δ as follows (the lower bound can be found similarly):

$$\begin{aligned} J(x,\delta) &= \delta^5 \int_{\mathbb{R}^2} 1_{\{|\delta^2 u|<q\}} 1_{\{|v|<q\}} 1_{\{|x_2v+x_1x_2\delta u+\frac{1}{2}x_1\delta^4u^2|<q\delta\}} du dv \\ &\leqslant \delta^5 \int_{\mathbb{R}^2} 1_{\{|v|<q\}} 1_{\{-q\delta^{-3}-x_2v\delta^{-4}+\frac{1}{2}x_1x_2^2\delta^{-6}<\frac{1}{2}x_1(u+x_2\delta^{-3})^2<q\delta^{-3}-x_2v\delta^{-4}+\frac{1}{2}x_1x_2^2\delta^{-6}\}} du dv \\ &\leqslant C(x_1x_2)^{-1}\delta^5, \end{aligned}$$

where we make changes of variables $a_2 = u\delta^3$, $a_3 = v\delta^2$ and C is some positive constant that depends only on q. □

The most interesting case in this theorem is, of course, the one where the fractional exponent of δ appears in the asymptotics (in fact it is easy to check that all other cases agree with the statement of Theorem 3). As we can see it is a major difference in comparison to the case when balls can be approximated with ellipsoids: the exponent of δ is always integer in Theorem 3 if all p_i are integer. It means that the comparison of balls with ellipsoids with center of the ball at such point x, that $x_2 = 0$, $x_4 = 0$ but $x_1 \neq 0$, is not possible.

References

[1] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni: Stratified Lie groups and potential theory for their sub-Laplacians, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[2] E.B. Dynkin: Markovskie protsessy, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1963.

[3] S. Kusuoka and D. Stroock: *Applications of the Malliavin calculus. III*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), 391–442.

[4] E. Le Donne and V. Magnani: *Measure of submanifolds in the Engel group*, Rev. Mat. Iberoam. **26** (2010), 333–346.

[5] V. Magnani: *Blow-up estimates at horizontal points and applications*, J. Geom. Anal. **20** (2010), 705–722.

[6] V. Magnani and D. Vittone: *An intrinsic measure for submanifolds in stratified groups*, J. Reine Angew. Math. **619** (2008), 203–232.

[7] A. Nagel, El.M. Stein and S. Wainger: *Balls and metrics defined by vector fields. I. Basic properties*, Acta Math. **155** (1985), 103–147.

[8] L.P. Rothschild and E.M. Stein: *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247–320.

[9] A. Rudenko: *Some uniform estimates for the transition density of a Brownian motion on a Carnot group and their application to local times*, Theory of Stoch. Proc. **19** (2014), 62–90.

[10] A.Rudenko: *Some properties of the Ito-Wiener expansion of the solution of a stochastic differential equation and local times*, Stochastic Processes. **122** (2012), 2454–2479.
[11] N.Th. Varopoulos, L. Saloff-Coste and T. Coulhon: Analysis and geometry on groups, Cambridge Tracts in Mathematics **100**, Cambridge University Press, Cambridge, 1992.

Institute of mathematics of National Academy of Sciences of Ukraine
3, Tereschenkivska st.
Kiev-4, 01004
Ukraine
e-mail: arooden@gmail.com