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INITIAL BOUNDARY VALUE PROBLEM FOR 3D BOUSSINESQ SYSTEM WITH THE THERMAL DAMPING

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Abstract

In this paper we consider the initial boundary value problem for the 3D Boussinesq system with the velocity dissipation and weak damping effect to instead of the dissipation effect for the thermal conductivity and establish the global existence of weak solutions. Furthermore, we prove that the global weak solution is strong and unique under some small initial data condition.

1. Introduction

The purpose of this paper is to investigate the initial-boundary value problem of the three-dimensional Boussinesq system with the velocity dissipation and thermal damping effect which can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla \Pi = \theta e_3, & x \in \Omega, t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa \theta = 0, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t \geq 0, \end{cases} \tag{1.1}$$

with the initial condition

$$(u, \theta)|_{t=0} = (u_0, \theta_0), \ \ x \in \Omega \tag{1.2}$$

and the natural boundary condition

$$u|_{\partial\Omega} = 0, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary. $u = (u_1(t, x), u_2(t, x), u_3(t, x))$ denotes the velocity of the fluid, θ and Π stand for the scalar temperature and pressure, respectively. θe_3 is buoyancy force with $e_3 = (0, 0, 1)$, and the damping coefficient κ is a positive number.

The Boussinesq equations arise from a zero order approximation to the coupling between Navier-Stokes equations and the thermodynamic equations and describe many geophysical phenomena in atmospheric and oceanographic sciences [28, 30]. There has been a huge amount of literature on the study of the Boussinesq system by many physicists and mathematicians due to its physical importance and mathematical challenges, for example, see [1, 3, 4, 6, 8, 13, 14, 19, 20, 21, 29, 36, 37, 41] and the references therein.

Here we recall some of the recent progresses in terms of the following generalized Boussi-

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nesq system on the domain $\Omega \subset \mathbb{R}^d$

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\begin{cases}
\partial_t u + u \cdot \nabla u - \mathrm{div}(\mu(\theta)\nabla u) + \nabla \Pi = \theta e_d, \\
\partial_t \theta + u \cdot \nabla \theta - \mathrm{div}(\nu(\theta)\nabla \theta) = 0, \\
\nabla \cdot u = 0, \\
(u(0,x), \theta(0,x)) = (u_0(x), \theta_0(x)).
\end{cases} \tag{1.4}
$$

Recently, big progresses have been made on the global well-posedness of the system (1.4) especially for the case $\Omega = \mathbb{R}^2 (d = 2)$. Chae [7], Hou and Li [15] independently proved the global existence of smooth solutions to (1.4) in $\mathbb{R}^2$ for $\nu(\theta) = 0$ and $\mu(\theta) = \mu > 0$. When $\nu(\theta) = \nu\theta$ with $\nu > 0$ and $\mu(\theta) = 0$, Hmidi and Keraani [12] proved the global existence and uniqueness of solutions to (1.4) in $\mathbb{R}^2$ with the suitable initial data. Wang and Zhang [35] obtained the global existence of smooth solutions to (1.4) under the assumptions that both $\mu(\cdot)$ and $\nu(\cdot)$ belong to $L^\infty(\mathbb{R}^+)$ and have positive lower bounds. Very recently, Abidi and Zhang [2] addressed the global well-posedness of the system (1.4) in $\mathbb{R}^2$ for $-\mathrm{div}(\nu(\theta)\nabla\theta) = (-\Delta)^{\frac{1}{2}}\theta$ under the framework of Besov space and $\|\mu(\cdot) - 1\|_{L^\infty} \leq \varepsilon$ for some sufficiently small ε. Here the fractional Laplacian has been found wide applications in mathematical physics [38], in control and optimization [16, 17] and so on.

Subsequently, we consider the two-dimensional Boussinesq system with variable kinematic viscosity in the velocity equation and with weak damping, and establish the global well-posedness for the two-dimensional Boussinesq system with general initial data in [42].

The results we mentioned above are obtained for the whole space $\mathbb{R}^2$. In many real-world applications, the flows are often restricted to bounded domains with suitable constraints imposed on the boundaries and these applications naturally lead to the studies of the initial-boundary value problems. In addition, solutions of the initial-boundary value problems may exhibit much richer phenomena than those of the whole space counterparts. When μ and ν depend on the temperature, Lorca and Boldrini [26, 27] proved the global existence of weak solution with small initial data and the local existence of strong solution for general data to the problem (1.4). Recently, Sun and Zhang [33] extend the global existence of smooth solution to the Boussinesq system (1.4) with full dissipations (both μ and ν have positive lower bound) in [35] to the case of bounded domain. For the partial dissipation cases, when $\mu(\theta) = \nu > 0$ and $\nu(\theta) = 0$ in (1.4), Lai, Pan and Zhao [19] obtained the unique classical solution for H^3 initial value and the non-slip boundary condition for a bounded smooth domain $\Omega \subset \mathbb{R}^2$. Subsequently, Li, Pan and Zhang [24] obtained the unique classical solution to the system (1.4) with $\mu = 0$ and $\kappa(\theta) > 0$ for H^3 initial value and the slip boundary condition for a bounded smooth domain $\Omega \subset \mathbb{R}^2$.

However, to the best of our knowledge, there are few results for partial dissipation cases $\mu(\theta) = \mu > 0$ and $\nu(\theta) = 0$ which corresponds to our system (1.1) with $\kappa = 0$ in $\Omega = \mathbb{R}^3$. The local smooth solution to the system (1.1) with $\kappa = 0$ has been established. The global regularity or finite time singularity for strong solutions of the system (1.1) with $\kappa = 0$ in $\mathbb{R}^3$ with large initial data is still a challenging open problem just like the three-dimensional Navier-Stokes equations. Some regularity criterions for the system (1.1) have been obtained (see e.g. [10, 31]). It is mentioning that different regularized 3D Boussinesq system have been proposed(we refer interested readers to [5, 39, 40] and the references therein).

On one hand, the system (1.1) without damping mechanism can be viewed as that the Navier-Stokes system which is forced by the temperature. On the other hand, we have no way to capture the regularity effect or decay property of θ which will bring us the main obstacle consists in the propagation of the regularity of u since the temperature equation without a damp term is a pure transport equation. Except for the one [42] mentioned above, very little result is known which studies thermal damping for the Boussinesq system as we know. That is why in the present paper we consider the initial boundary value problem to the system (1.1) with damping mechanism in a bounded smooth domain $\Omega \subset \mathbb{R}^3$.

To state our main result clearly, we define

$$\mathcal{A}_0 := \frac{\|\theta_0\|_{L^2}^2}{\kappa^2} + \|u_0\|_{L^2}^2 \quad \text{and} \quad \mathcal{B}_0 := \frac{\|\theta_0\|_{L^2}^2}{\kappa} + \|\nabla u_0\|_{L^2}^2.$$

Now, we state our main results.

Theorem 1.1. *Let $0 < \alpha < \frac{1}{2}$, $3 < p < \frac{6}{1+2\alpha}$, the initial data $\theta_0 \in L^\infty \cap H^1$ and $u_0 \in D^1_{0,\sigma}$. There exists a sufficiently small positive constant ε_0 depending only on the initial data such that if*

$$\mathcal{A}_0^{2\alpha} \mathcal{B}_0^{1-2\alpha} \le \varepsilon_0, \tag{1.5}$$

then the system (1.1)-(1.3) *has a unique global strong solution $(u(t,x), \theta(t,x), \Pi(t,x))$ satisfying that for any given $0 < \tau < T < \infty$,*

$$\begin{cases} \theta \in C([0,T]; H^1), \\ \nabla u \in L^\infty((0,T); L^2) \cap L^\infty((\tau,T); W^{1,p}) \cap C([\tau,T]; H^1 \cap W^{1,p}), \\ \Pi \in L^\infty((\tau,T); W^{1,p}) \cap C([\tau,T]; H^1 \cap W^{1,p}), \\ u_t \in L^2((0,T); L^2) \cap L^\infty((\tau,T); L^2),\ \nabla \Pi_t \in L^2((\tau,T); L^2), \\ \nabla u_t \in L^2((\tau,T); H^1) \cap L^\infty((\tau,T); L^2),\ u_{tt} \in L^2((\tau,T); L^2). \end{cases}$$

Furthermore, there holds for all $t \ge 0$

$$\|\nabla \theta(\cdot,t)\|_{L^2}^2 \le C e^{-\kappa t} \tag{1.6}$$

and for all $t \ge 1$

$$\|\nabla u_t(\cdot,t)\|_{L^2}^2 + \|\nabla u(\cdot,t)\|_{H^1 \cap W^{1,p}}^2 + \|\nabla \Pi(\cdot,t)\|_{L^2 \cap L^p}^2 \le C e^{-\delta t}, \tag{1.7}$$

where $\delta := \min\{\lambda, \kappa^-\}$ and $\kappa^- < \kappa$ is any positive constant closed enough to κ, the constant C depends only on the initial norms $\|\theta_0, u_0, \nabla u_0\|_{L^2}^2$, the damping constant κ and the Poincaré constant λ.

Remark 1.1. In Theorem 1.1, we do not need to impose any additional initial compatibility conditions unlike in [19, 24] for the two-dimensional case. On the other hand, the exponential decay-in-time properties in (1.6)-(1.7) involve the higher-order derivatives of the strong solutions (θ, u, Π) to the system (1.1)-(1.3).

Remark 1.2. It is easy to see from (1.5) that problem (1.1)-(1.3) has a unique strong solution (θ, u, Π) on $\Omega \times [0, \infty)$, provided $\mathcal{A}_0$ is sufficiently small or $\mathcal{B}_0$ is small enough. It is worth mentioning that the damping coefficient κ just has to be sufficiently large such that it does not need to require any smallness condition on the initial norm $\|\theta_0\|_{L^2}$.

Remark 1.3. The obvious advantage of the damping term lies in that it provides exponential decay of $\|\theta\|_{L^p}$, which has been sufficiently explored in the paper. However, the global well-posedness of system (1.1) without damping effect for the temperature equation is an interesting open problem.

The rest of this paper is organized as follows. In Section 2 we present some notions and basic tools. In Section 3 we first establish the global existence of weak solutions, then obtain the higher regularity for the weak solutions, where the crucial ingredient is to utilize the polynomial decay-in-time estimate for small time and the exponential decay-in-time estimate for large time in terms of the velocity field which is inspired by the recent work given by He, Li and L*ü* [11]. In Section 4 we complete the proof of Theorem 1.1.

2. Preliminaries

In this section, we introduce some notations and conventions, and recall some basic tools which will be used throughout this paper. In particular, we provide the Gagliardo-Nirenberg type inequalities and the regularity estimates for elliptic equations in bounded domains.

First of all, we use the convention that C (or $C(s)$) to denote strictly positive constants depending on the index s, respectively, whose values are insignificant and may change from line to line. We denote $\kappa^- < \kappa$ is any positive constant closed enough to κ. For X a Banach space and I an interval of $\mathbb{R}$, we denote by $C(I;X)$ the set of continuous functions on I with values in X, and by $L^p(I;X)$ with $p \in [1,\infty]$ stands for the set of measurable functions on I with values in X, such that $t \mapsto \|f(t)\|_X \in L^p(I)$. Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. For $r \in [1,\infty]$ and $k \in \mathbb{N}$, the homogeneous and nonhomogeneous Sobolev spaces are defined in a standard way,

$$\begin{aligned}
&L^r := L^r(\Omega), \quad W^{k,r} := \{f \in L^r | D^\alpha f \in L^r, \forall|\alpha| \le k\}, \quad H^k := W^{k,2},\\
&D^{k,r} := \{f \in L^r_{loc} | D^k f \in L^r\}, \quad D^1 := \{f \in L^6 | Df \in L^2\},\\
&C^\infty_{0,\sigma} := \{f \in C^\infty_0 | \nabla\cdot f = 0\},\ D^1_{0,\sigma} := \overline{C^\infty_{0,\sigma}} \text{ closure in the norm of } D^1,\\
&\|\cdot\|_{X_1\cap X_2} := \|\cdot\|_{X_1} + \|\cdot\|_{X_2}, \quad \text{for two Banach spaces } X_1 \text{ and } X_2,\\
&\|f_1,\cdots,f_k\|_X := \|f_1\|_X + \cdots + \|f_k\|_X, \quad \text{for Banach space } X.
\end{aligned}$$

We start with the well-known Gagliardo-Nirenberg inequality for bounded domains which will be used frequently later.

Lemma 2.1 ([18])**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We have*

1. $H^1 \hookrightarrow L^p$ *for* $p \in (1,6]$, *particularly*, $\|f\|_{L^6} \le C\|\nabla f\|_{L^2}$ *for* $f \in H^1_0(\Omega)$*;*
2. $W^{1,p} \hookrightarrow L^\infty$ *for* $p \in (3,\infty)$;
3. $\|f\|_{L^3} \le C\|f\|_{L^2}^{\frac12}\|\nabla f\|_{L^2}^{\frac12}$ *for* $f \in H^1(\Omega)$*;*
4. $\|f\|_{L^4} \le C\|f\|_{L^2}^{\frac14}\|\nabla f\|_{L^2}^{\frac34}$ *for* $f \in H^1(\Omega)$*;*
5. $\|f\|_{L^\infty} \le C\|f\|_{L^2}^{\frac12}\|\nabla f\|_{H^1}^{\frac12}$ *for* $f \in H^1_0(\Omega)\cap H^2(\Omega)$.

We will need the Poincar*é* type inequality.

Lemma 2.2 ([9]). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and $f \in H_0^1(\Omega)$. Then, there exists a constant λ depending only on Ω such that*

$$\|f\|_{L^2} \le \lambda \|\nabla f\|_{L^2},$$

where λ is Poincaré constant which depends only on Ω.

Finally, we recall the following regularity estimate for elliptic equations in a bounded domain.

Lemma 2.3 ([32]). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Consider the Stokes problem*

$$\begin{cases} -\mu\Delta u + \nabla\Pi = F, & x \in \Omega, \\ \nabla \cdot u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

If $F \in W^{m,p}$ with $p \in (1,\infty)$ and $m \ge -1$, then it holds $u \in W^{m+2,p}$ and $\Pi \in W^{m+1,p}$.

Furthermore, we have

$$\|u\|_{W^{m+2,p}} + \|\Pi\|_{W^{m+1,p}} \le C\|F\|_{W^{m,p}},$$

where the constant C depends only on Ω and the indices p, μ, m.

3. Global existence of weak solutions

The goal of this section is to prove the global existence of weak solutions to the system (1.1)-(1.3), which is an important step in the proof of our main theorem. We first recall the definition of weak solutions.

Definition 3.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Assume $(u_0, \theta_0) \in L^2(\Omega)$. A pair of measurable functions (u, θ) is called a weak solution of (1.1)-(1.3) if for any $T > 0$, $u \in C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega)), \theta \in C([0,T]; L^p(\Omega))$ with $1 \le p < \infty$ and

$$\int_\Omega u_0 \cdot \phi_0 dx + \int_0^T \int_\Omega (u \cdot \phi_t + u \cdot \nabla\phi \cdot u - \nabla u \cdot \nabla\phi + \theta\phi_3) dxdt = 0,$$
$$\int_\Omega \theta_0\psi_0 dx + \int_0^T \int_\Omega (\theta\psi_t + (u \cdot \nabla\psi)\theta - \kappa\theta\psi) dxdt = 0,$$

hold for any $\phi = (\phi_1, \phi_2, \phi_3) \in C_0^\infty(\Omega \times [0,T])^3$ satisfying $\phi(x,T) = 0$ and $\nabla \cdot \phi = 0$, and for any $\psi \in C_0^\infty(\Omega \times [0,T])$ satisfying $\psi(x,T) = 0$.

Next, we state the main result of this section as a proposition.

Proposition 3.1. *Under the assumptions of* Theorem 1.1*, there exists a global weak solution (u, θ) of* (1.1)-(1.3) *such that for any $T > 0$, $u \in C([0,T); L^2(\Omega)) \cap L^2([0,T); H_0^1(\Omega))$, and $\theta \in C([0,T); L^p(\Omega))$ with $1 \le p < \infty$.*

Before proving the above proposition, we need to establish two lemmas which involve the basic estimates of θ and u.

By the standard energy method, it is easy to derive the following exponential decay estimate for $\|\theta\|_{L^p}$, which plays an essential role in the proof of our main theorem.

Lemma 3.1. *Under the assumptions of* Theorem 1.1*, we have for all* $1 \le p \le \infty$

$$\|\theta\|_{L^p} = e^{-\kappa t}\|\theta_0\|_{L^p} \le \|\theta_0\|_{L^p}. \tag{3.1}$$

Proof. Taking the L^2 inner product of θ equation in (1.1) with $|\theta|^{p-2}\theta$ and using the divergence-free condition $\nabla\cdot u = 0$, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\theta\|_{L^p}^p + \kappa\|\theta\|_{L^p}^p = 0,$$

which reduces to $\frac{d}{dt}\|\theta\|_{L^p} + \kappa\|\theta\|_{L^p} = 0$. Solving this differential equation leads to (3.1). The proof of Lemma 3.1 is ended. □

With the aid of the above exponential decay estimate for $\|\theta\|_{L^p}$, we can obtain the basic energy estimate involving the velocity u.

Lemma 3.2. *Under the assumptions of* Theorem 1.1*, we have*

$$\sup_{t\in[0,T]}\|u\|_{L^2}^2 + \int_0^T \|\nabla u\|_{L^2}^2 dt \le 3\mathcal{A}_0. \tag{3.2}$$

Proof. Taking the L^2 inner product of the velocity equation in (1.1) with u and using the fact $\nabla\cdot u = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = \int_\Omega \theta u_3 dx \le \|\theta\|_{L^2}\|u\|_{L^2}, \tag{3.3}$$

due to (3.1), it gives that

$$\|u\|_{L^2} \le \frac{1}{\kappa}\|\theta_0\|_{L^2} + \|u_0\|_{L^2}. \tag{3.4}$$

Inserting (3.4) into (3.3) and using (3.1), we get

$$\frac{d}{dt}\|u\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2 \le 2e^{-\kappa t}\|\theta_0\|_{L^2}\Big(\frac{\|\theta_0\|_{L^2}}{\kappa} + \|u_0\|_{L^2}\Big). \tag{3.5}$$

Integrating (3.5) with respect to time t over $[0, T]$ yields the desired result (3.2). This completes the proof of Lemma 3.2. □

We now prove the Proposition 3.1.

Proof of Proposition 3.1. The proof is a consequence of Schauder's fixed point argument which follows [19, 25]. For the sake of completeness, we prove the Proposition 3.1 in detail. For any fixed $T \in (0,\infty)$, we consider the problem (1.1)-(1.3) in $\Omega\times[0, T]$. For notational convenience, we write

$$X := C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H_0^1(\Omega))$$

with $\|V\|_X^2 := \|V\|_{C([0,T];L^2(\Omega))}^2 + \|V\|_{L^2([0,T];H_0^1(\Omega))}^2$ and define

$$\mathcal{B} = \{V \in X : \|V\|_X \le R_0\},$$

where R_0 will be specified later. Clearly, $\mathcal{B} \subset X$ is closed and convex.

We fix $\epsilon \in (0,1)$ and define a continuous map on $\mathcal{B}$. For any $V \in \mathcal{B}$, we regularize it and the initial data (u_0, θ_0) via the standard mollifying process, we first mollify V by the standard procedure (see [25]) to get

$$V_\epsilon = \bar{V}_\epsilon * \rho_{\epsilon/2},$$

where $\bar{V}_\epsilon$ is the truncation of V in $\Omega_\epsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \epsilon\}$ (extended by 0 to Ω), and $\rho_{\epsilon/2}$ is the standard mollifier. Then v_ϵ satisfies

$$V_\epsilon \in C([0,T]; C^\infty_{0,\sigma}(\bar{\Omega})) \text{ with } C^\infty_{0,\sigma}(\bar{\Omega}) := \{f \in C^\infty_0(\bar{\Omega}) : \nabla \cdot f = 0\}; \tag{3.6}$$

$$\|V_\epsilon\|_{C([0,T];L^2(\Omega))} \le C\|V\|_{C([0,T];L^2(\Omega))}; \tag{3.7}$$

$$\|V_\epsilon\|_{L^2([0,T];H^1_0(\Omega))} \le C\|V\|_{L^2([0,T];H^1_0(\Omega))}, \tag{3.8}$$

for some constant $C > 0$ which is independent of ϵ. Similarly, we regularize the initial data to obtain the smooth approximation θ^ϵ_0 for θ_0 and u^ϵ_0 for u_0 respectively, such that

$$\theta^\epsilon_0 \in C^\infty_0(\bar{\Omega}) \text{ and } \|\theta^\epsilon_0 - \theta_0\|_{H^1(\Omega)} < \epsilon,$$
$$u^\epsilon_0 \in C^\infty_{0,\sigma}(\bar{\Omega}) \text{ and } \|u^\epsilon_0 - u_0\|_{H^1(\Omega)} < \epsilon.$$

Then the transport equation with smooth initial data

$$\begin{cases} \partial_t\theta + V_\epsilon \cdot \nabla\theta + \kappa\theta = 0, \\ \theta(0,x) = \theta^\epsilon_0(x), \end{cases} \tag{3.9}$$

has a unique solution $\theta^\epsilon(t,x)$.

We next solve the nonhomogeneous (linearized) Navier-Stokes equation with smooth initial data $u^\epsilon_0(x)$

$$\begin{cases} \partial_t u + V_\epsilon \cdot \nabla u - \Delta u + \nabla\Pi = \theta^\epsilon e_3, \\ \nabla \cdot u = 0, u|_{\partial\Omega} = 0, \\ u(0,x) = u^\epsilon_0(x), \end{cases} \tag{3.10}$$

and denote its solution by u^ϵ. This process allows us to define a map

$$T_\epsilon(V) = u^\epsilon.$$

The solvability of (3.9) and (3.10) follow easily from [19, 25]. We then apply Schauder's fixed point theorem to construct a sequence of approximate solutions to (1.1)-(1.3). It suffices to show that, for any fixed $\epsilon \in (0,1)$, $T_\epsilon : \mathcal{B} \to \mathcal{B}$ is continuous and compact. More precisely, we need to show the following dissertations:
(1) $\|u^\epsilon\|_{\mathcal{B}} \le R_0$, namely, T_ϵ maps $\mathcal{B}$ into $\mathcal{B}$ for any fixed $\epsilon \in (0,1)$;
(2) $\|u^\epsilon\|_{C([0,T];H^1_0(\Omega))} + \|u^\epsilon\|_{L^2([0,T];H^2(\Omega))} \le C$;
(3) $\|T_\epsilon(V_1) - T_\epsilon(V_2)\|_{\mathcal{B}} \le C\|V_1 - V_2\|_{\mathcal{B}}$ for C independent of ϵ and any $V_1, V_2 \in \mathcal{B}$.

It is easy to verify (1). In fact, due to Lemma 3.1, we have for any $t \in [0,T]$,

$$\|\theta^\epsilon\|^2_{L^p} + \int_0^T \|\theta^\epsilon\|^2_{L^p} dt \le \|\theta_0\|^2_{L^p} + \epsilon C(\Omega, p) \quad \text{for} \quad p \in [2,\infty) \tag{3.11}$$

and

$$\sup_{t\in[0,T]} \|u^\epsilon\|_{L^2}^2 + \int_0^T \|\nabla u^\epsilon\|_{L^2}^2 dt \le 3\Big(\frac{\|\theta_0\|_{L^2}^2 + \epsilon C(\Omega, p)}{\kappa^2} + \|u_0\|_{L^2}^2\Big). \tag{3.12}$$

Next, we prove (2) which implies the compactness of T_ϵ by the Sobolev embedding theorem. Similar to (3.3), we obtain

$$\begin{aligned} \|u_t^\epsilon\|_{L^2}^2 + \frac{d}{dt}\|\nabla u^\epsilon\|_{L^2}^2 &= \int_\Omega (-V_\epsilon \cdot \nabla u^\epsilon + \theta^\epsilon e_3) \cdot u_t^\epsilon dx \\ &\le C\Big(\|V_\epsilon \cdot \nabla u^\epsilon\|_{L^2}^2 + \|\theta^\epsilon\|_{L^2}^2\Big) + \varepsilon \|u_t\|_{L^2}^2 \\ &\le C\Big(\|\theta^\epsilon\|_{L^2}^2 + \|V_\epsilon\|_{L^\infty}^2 \|\nabla u^\epsilon\|_{L^2}^2\Big) + \varepsilon \|u_t\|_{L^2}^2, \end{aligned} \tag{3.13}$$

which follows from (3.11) and the Gronwall inequality that

$$\sup_{t\in[0,T]} \|\nabla u^\epsilon\|_{L^2}^2 + \int_0^T \|u_t^\epsilon\|_{L^2}^2 dt \le C\Big(\|\theta_0\|_{L^2}^2 + \epsilon + \|\nabla u_0\|_{L^2}^2\Big) e^{C\int_0^T \|V_\epsilon\|_{L^\infty}^2 ds}. \tag{3.14}$$

Recalling the elliptic regularity estimate (see Lemma 2.3), we have

$$\begin{aligned} \|\nabla^2 u^\epsilon\|_{L^2}^2 + \|\nabla \Pi^\epsilon\|_{L^2}^2 &\le C\|u_t^\epsilon + V_\epsilon \cdot \nabla u^\epsilon - \theta^\epsilon e_3\|_{L^2}^2 \\ &\le C(\|u_t^\epsilon\|_{L^2}^2 + \|V_\epsilon\|_{L^\infty}^2 \|\nabla u^\epsilon\|_{L^2}^2 + \|\theta^\epsilon\|_{L^2}^2). \end{aligned} \tag{3.15}$$

Combining (3.15) with (3.11) and (3.14) gives that

$$\|\nabla^2 u^\epsilon\|_{L^2([0,T];L^2(\Omega))} + \|u_t^\epsilon\|_{L^2([0,T];L^2(\Omega))} \le C\Big(\|\theta_0\|_{L^2}^2, \|\nabla u_0\|_{L^2}^2, \int_0^T \|V_\epsilon\|_{L^\infty}^2 ds\Big). \tag{3.16}$$

Now, we prove the continuity of T_ϵ. Let $T_\epsilon(v_i) = u_i (i = 1, 2)$. By definition of T_ϵ, we have

$$\begin{cases} \partial_t u_i^\epsilon + V_{i\epsilon} \cdot \nabla u_i^\epsilon - \Delta u_i^\epsilon + \nabla \Pi_i^\epsilon = \theta_i^\epsilon e_3, & x \in \Omega, t > 0, \\ \partial_t \theta_i^\epsilon + V_{i\epsilon} \cdot \nabla \theta_i^\epsilon + \kappa \theta_i^\epsilon = 0, & x \in \Omega, t > 0, \\ \nabla \cdot u_i^\epsilon = 0, u_i^\epsilon|_{\partial\Omega} = 0, & x \in \Omega, t \ge 0, \\ (u_i^\epsilon(0, x), \theta_i^\epsilon(0, x)) = (u_0^\epsilon(x), \theta_0^\epsilon(x)). \end{cases} \tag{3.17}$$

Subtracting the equation for $i = 2$ from the one for $i = 1$, we have

$$\begin{cases} \partial_t u^\epsilon + V_{1\epsilon} \cdot \nabla u^\epsilon - \Delta u^\epsilon + V_\epsilon \cdot \nabla u_2^\epsilon + \nabla \Pi^\epsilon = \theta^\epsilon e_3, & x \in \Omega, t > 0, \\ \partial_t \theta^\epsilon + V_{1\epsilon} \cdot \nabla \theta^\epsilon + V_\epsilon \cdot \nabla \theta_2^\epsilon + \kappa \theta^\epsilon = 0, & x \in \Omega, t > 0, \\ \nabla \cdot u^\epsilon = 0, u^\epsilon|_{\partial\Omega} = 0, & x \in \Omega, t \ge 0, \\ (u^\epsilon(0, x), \theta^\epsilon(0, x)) = (0, 0), \end{cases} \tag{3.18}$$

where $u^\epsilon = u_2^\epsilon - u_1^\epsilon$, $V_\epsilon = V_{2\epsilon} - V_{1\epsilon}$, $\theta^\epsilon = \theta_2^\epsilon - \theta_1^\epsilon$ and $\Pi^\epsilon = \Pi_2^\epsilon - \Pi_1^\epsilon$.

Taking the L^2 inner product of $(3.18)_2$ with θ^ϵ and using the divergence-free condition $\nabla \cdot v_{1\epsilon} = 0$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\theta^\epsilon\|_{L^2}^2 + \kappa \|\theta^\epsilon\|_{L^2}^2 = -\int_\Omega V_\epsilon \cdot \nabla \theta_2^\epsilon \theta^\epsilon dx \\ &\le \|\theta^\epsilon\|_{L^2} \|V_\epsilon\|_{L^2} \|\nabla \theta_2^\epsilon\|_{L^\infty} \le \frac{\kappa}{2}\|\theta^\epsilon\|_{L^2}^2 + C_\kappa \|\nabla \theta_2^\epsilon\|_{L^\infty}^2 \|V_\epsilon\|_{L^2}^2. \end{aligned} \tag{3.19}$$

Since $\theta_2^\epsilon \in C([0, T]; C_0^\infty(\bar{\Omega}))$, we get from (3.19) that

$$\|\theta^\epsilon\|_{L^2}^2 + \kappa \int_0^T \|\theta^\epsilon\|_{L^2}^2 ds \le C_\kappa T \|V_\epsilon\|_{C([0,T];L^2(\Omega))}^2. \tag{3.20}$$

Taking the L^2 inner product of $(3.18)_1$ with u^ϵ and using the divergence-free condition $\nabla \cdot V_{1\epsilon} = 0$, we obtain

$$
(3.21)\qquad \begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u^\epsilon\|_{L^2}^2 + \|\nabla u^\epsilon\|_{L^2}^2 &= \int_\Omega \theta^\epsilon u_3^\epsilon dx - \int_\Omega V_\epsilon \cdot \nabla u_2^\epsilon \cdot u^\epsilon dx \\
&\le \|\theta^\epsilon\|_{L^2}\|u^\epsilon\|_{L^2} + \|V_\epsilon\|_{L^2}\|\nabla u_2^\epsilon\|_{L^6}\|u^\epsilon\|_{L^3} \\
&\le \lambda\|\theta^\epsilon\|_{L^2}\|\nabla u^\epsilon\|_{L^2} + C\lambda^{\frac{1}{2}}\|V_\epsilon\|_{L^2}\|u_2^\epsilon\|_{H^2}\|\nabla u^\epsilon\|_{L^2} \\
&\le C(\lambda)\|\theta^\epsilon\|_{L^2}^2 + C(\kappa,\lambda)\|V_\epsilon\|_{L^2}^2\|u_2^\epsilon\|_{H^2}^2 + \frac{1}{2}\|\nabla u^\epsilon\|_{L^2}^2,
\end{aligned}
$$

which from $u_2^\epsilon \in L^2([0,T]; H^2(\Omega))$ and (3.20) yields that

$$
(3.22)\qquad \|u^\epsilon\|_{C([0,T];L^2(\Omega))}^2 + \|u^\epsilon\|_{L^2([0,T];H_0^1(\Omega))}^2 \le C(\kappa,\lambda,T)\|V_1 - V_2\|_{C([0,T];L^2(\Omega))}^2,
$$

that is,

$$
(3.23)\qquad \|T_\epsilon(V_1) - T_\epsilon(V_2)\|_{\mathcal{B}}^2 \le C\|V_1 - V_2\|_{\mathcal{B}}^2,
$$

which implies that $T_\epsilon : \mathcal{B} \to \mathcal{B}$ is continuous.

Therefore, the Schauder's fixed point theorem implies that for any fixed $\epsilon \in (0,1)$, there exists $u^\epsilon \in \mathcal{B}$ such that $T_\epsilon(u^\epsilon) = u^\epsilon$, namely,

$$
(3.24)\qquad \begin{cases}
\partial_t u^\epsilon + u_\epsilon \cdot \nabla u^\epsilon - \Delta u^\epsilon + \nabla \Pi^\epsilon = \theta^\epsilon e_3, & x \in \Omega, t > 0, \\
\partial_t \theta^\epsilon + u_\epsilon \cdot \nabla \theta^\epsilon + \kappa\theta^\epsilon = 0, & x \in \Omega, t > 0, \\
\nabla \cdot u^\epsilon = 0, u^\epsilon|_{\partial\Omega} = 0, & x \in \Omega, t \ge 0, \\
(u^\epsilon(0,x), \theta^\epsilon(0,x)) = (u_0^\epsilon(x), \theta_0^\epsilon(x)), &
\end{cases}
$$

where u_ϵ is the regularization of u^ϵ. By a bootstrap argument (c.f. [19, 25]) we know that $(\theta^\epsilon, u^\epsilon) \in C^\infty(\bar{\Omega} \times [0,T])$. Then it is obvious that $(\theta^\epsilon, u^\epsilon)$ satisfies the integral identities,

$$
(3.25)\qquad \int_\Omega u_0^\epsilon \cdot \phi_0 dx + \int_0^T \int_\Omega (u^\epsilon \cdot \phi_t + u_\epsilon \cdot \nabla\phi \cdot u^\epsilon - \nabla u^\epsilon \cdot \nabla\phi + \theta^\epsilon \phi_3) dxdt = 0,
$$

$$
(3.26)\qquad \int_\Omega \theta_0^\epsilon \psi_0 dx + \int_0^T \int_\Omega (\theta^\epsilon \psi_t + (u_\epsilon \cdot \nabla\psi)\theta^\epsilon - \kappa\theta^\epsilon\psi) dxdt = 0,
$$

for any $\epsilon > 0$, $\phi = (\phi_1, \phi_2, \phi_3) \in C_0^\infty(\Omega \times [0,T])^3$ satisfying $\phi(x,T) = 0$ and $\nabla \cdot \phi = 0$, and for any $\psi \in C_0^\infty(\Omega \times [0,T])$ satisfying $\psi(x,T) = 0$.

In view of (3.11), (3.12) and from the definition of u^ϵ we know that there exist functions $u \in \mathcal{B}$ and $\theta \in C([0,T]; L^p(\Omega))$ for $2 \le p < \infty$ such that as $\epsilon \to 0^+$,

$$
\begin{aligned}
u^\epsilon &\rightharpoonup u \quad \text{weaky in} \quad C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)), \\
\theta^\epsilon &\rightharpoonup \theta \quad \text{weaky in} \quad C([0,T]; L^p(\Omega)),
\end{aligned}
$$

and

$$
(3.27)\qquad \|u\|_{C([0,T];L^2(\Omega))}^2 + \|u\|_{L^2([0,T];H_0^1(\Omega))}^2 \le C(\theta_0, u_0, \Omega),
$$

$$
(3.28)\qquad \|\theta\|_{C([0,T];L^p(\Omega))} + \|\theta\|_{L^1([0,T];L^p(\Omega))} \le \|\theta_0\|_{C([0,T];L^p(\Omega))}.
$$

Since $u \cdot \nabla\psi \in C([0,T]; L^2(\Omega))$, we have

$$
\left|\int_0^T \int_\Omega (u_\epsilon \cdot \nabla\psi \cdot \theta^\epsilon - u \cdot \nabla\psi \cdot \theta) dxdt\right|
$$

$$
\begin{aligned}
&\leq |\int_0^T \int_\Omega (u_\epsilon - u) \cdot \nabla\psi \cdot \theta^\epsilon dxdt| + |\int_0^T \int_\Omega u \cdot \nabla\psi \cdot (\theta^\epsilon - \theta) dxdt| \\
&\leq \|\theta^\epsilon\|_{L^2([0,T];L^2(\Omega))} \|u^\epsilon - u\|_{L^2([0,T];L^2(\Omega))} + |\int_0^T \int_\Omega u \cdot \nabla\psi \cdot (\theta^\epsilon - \theta) dxdt| \\
&\leq C\|u^\epsilon - u\|_{L^2([0,T];L^2(\Omega))} + |\int_0^T \int_\Omega u \cdot \nabla\psi \cdot (\theta^\epsilon - \theta) dxdt| \to 0 \quad \text{as} \quad \epsilon \to 0^+.
\end{aligned}
$$

Moreover, since

$$
\begin{aligned}
&|\int_0^T \int_\Omega (u_\epsilon \cdot \nabla\phi \cdot u^\epsilon - u \cdot \nabla\phi \cdot u) dxdt| \\
&\leq |\int_0^T \int_\Omega u_\epsilon \cdot \nabla\phi \cdot (u^\epsilon - u) dxdt| + |\int_0^T \int_\Omega (u_\epsilon - u) \cdot \nabla\phi \cdot u dxdt| \\
&\leq \|u_\epsilon\|_{L^2([0,T];L^2(\Omega))} \|u^\epsilon - u\|_{L^2([0,T];L^2(\Omega))} + \|u\|_{L^2([0,T];L^2(\Omega))} \|u_\epsilon - u\|_{L^2([0,T];L^2(\Omega))} \\
&\leq C\|u^\epsilon - u\|_{L^2([0,T];L^2(\Omega))} \to 0 \quad \text{as} \quad \epsilon \to 0^+,
\end{aligned}
$$

letting $\epsilon \to 0^+$ in (3.25) and (3.26), respectively, we verify that (u, θ) is a weak solution to (1.1)-(1.3) in $\Omega \times [0, T]$. We conclude the argument by noticing that T is arbitrary. This combining with (3.27) and (3.28) completes the proof of Proposition 3.1. □

4. Global regularity with small initial data

In this section, we prove that the weak solutions described in Proposition 3.1 are actually strong solutions by establishing the global regularity for small initial data.

Proposition 4.1. *Let* (θ_0, u_0) *satisfy the conditions in* Theorem 1.1 *and* (θ, u, Π) *be the global weak solution obtained in* Proposition 3.1. *Then it holds that*

$$
\int_0^T \|\nabla u\|_{L^2}^4 dt \leq 2\Big(\mathcal{B}_0 + \mathcal{B}_0^\alpha\Big). \tag{4.1}
$$

Before proving Proposition 4.1, we first establish an auxiliary result, the time-weighted estimate on the gradient of velocity.

Lemma 4.1. *Assume that* (θ_0, u_0) *satisfies the conditions in* Theorem 1.1. *Let* (θ, u, Π) *be the global weak solution obtained in* Proposition 3.1. *It holds that*

$$
\sup_{t\in[0,T]} \Big(t^\alpha \|\nabla u\|_{L^2}^2\Big) + \int_0^T t^\alpha \|u_t\|_{L^2}^2 dt \leq C\mathcal{A}_0^\alpha \mathcal{B}_0^{1-\alpha} \exp\Big\{C \int_0^T \|\nabla u\|_{L^2}^4 dt\Big\}. \tag{4.2}
$$

Proof. Firstly, taking the L^2 inner product of u equation in (1.1) with u_t and using the divergence-free condition $\nabla \cdot u = 0$, we obtain

$$
\|u_t\|_{L^2}^2 + \frac{d}{dt}\|\nabla u\|_{L^2}^2 = \int_\Omega (-u \cdot \nabla u + \theta e_3) \cdot u_t dx. \tag{4.3}
$$

Recalling that (u, Π) satisfies the following Stokes system

$$
\begin{cases}
-\Delta u + \nabla\Pi = -u_t - u \cdot \nabla u + \theta e_3, & x \in \Omega, \\
\nabla \cdot u = 0, & x \in \Omega, \\
u = 0, & x \in \partial\Omega.
\end{cases} \tag{4.4}
$$

It follows from Lemma 2.3 and Gagliardo-Nirenberg's inequality that

$$\begin{aligned}\|\nabla^2 u\|_{L^2} + \|\nabla\Pi\|_{L^2} &\le C\|u_t + u\cdot\nabla u - \theta e_3\|_{L^2}\\ &\le C(\|u_t\|_{L^2} + \|u\|_{L^6}\|\nabla u\|_{L^3} + \|\theta\|_{L^2})\\ &\le C(\|u_t\|_{L^2} + \|\nabla u\|_{L^2}^3 + \|\theta\|_{L^2}) + \frac{1}{2}\|\nabla^2 u\|_{L^2},\end{aligned}$$

which gives that

$$\|\nabla^2 u\|_{L^2} + \|\nabla\Pi\|_{L^2} \le C(\|u_t\|_{L^2} + \|\nabla u\|_{L^2}^3 + \|\theta\|_{L^2}). \tag{4.5}$$

Combining with (4.3) and using the Hölder and Young inequalities, we obtain

$$\begin{aligned}\|u_t\|_{L^2}^2 + \frac{d}{dt}\|\nabla u\|_{L^2}^2 &\le C\Big(\|u\cdot\nabla u\|_{L^2}^2 + \|\theta\|_2^2\Big) + \varepsilon\|u_t\|_{L^2}^2\\ &\le C\Big(\|u\|_{L^6}^2\|\nabla u\|_{L^3}^2 + \|\theta\|_{L^2}^2\Big) + \varepsilon\|u_t\|_{L^2}^2\\ &\le C\Big(\|\nabla u\|_2^3\|\nabla^2 u\|_{L^2} + \|\theta\|_{L^2}^2\Big) + 2\varepsilon\|u_t\|_{L^2}^2\\ &\le C\Big(\|\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^6\Big) + 3\varepsilon\|u_t\|_{L^2}^2,\end{aligned}$$

which gives that

$$\|u_t\|_{L^2}^2 + \frac{d}{dt}\|\nabla u\|_{L^2}^2 \le C\Big(\|\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^6\Big). \tag{4.6}$$

Utilizing the Gronwall inequality to (4.6) gives that

$$\sup_{t\in[0,T]}\|\nabla u\|_{L^2}^2 + \int_0^T \|u_t\|_{L^2}^2 dt \le C\mathcal{B}_0 \exp\Big\{C\int_0^T \|\nabla u\|_{L^2}^4 dt\Big\}. \tag{4.7}$$

Multiplying (4.6) by t, one has

$$t\|u_t\|_{L^2}^2 + \frac{d}{dt}\Big(t\|\nabla u\|_{L^2}^2\Big) \le \|\nabla u\|_{L^2}^2 + C\Big(t\|\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 t\|\nabla u\|_{L^2}^2\Big), \tag{4.8}$$

which together with (3.3) and the Gronwall inequality yields that

$$\sup_{t\in[0,T]} t\|\nabla u\|_{L^2}^2 + \int_0^T t\|u_t\|_{L^2}^2 dt \le C\mathcal{A}_0 \exp\Big\{C\int_0^T \|\nabla u\|_{L^2}^4 dt\Big\}. \tag{4.9}$$

Consequently, it follows from (4.7)-(4.9) that

$$\begin{aligned}&\sup_{t\in[0,T]}\Big(t^\alpha\|\nabla u\|_{L^2}^2\Big) + \int_0^T t^\alpha\|u_t\|_{L^2}^2 dt \qquad (4.10)\\ &= \sup_{t\in[0,T]}\Big[\|\nabla u\|_{L^2}^{2(1-\alpha)}\Big(t\|\nabla u\|_{L^2}^2\Big)^\alpha\Big] + \int_0^T \|u_t\|_{L^2}^{2(1-\alpha)}\Big(t\|u_t\|_{L^2}^2\Big)^\alpha dt\\ &\le \Big(\sup_{t\in[0,T]}\|\nabla u\|_{L^2}^2\Big)^{1-\alpha}\Big[\sup_{t\in[0,T]}\Big(t\|\nabla u\|_{L^2}^2\Big)\Big]^\alpha + \Big(\int_0^T \|u_t\|_{L^2}^2 dt\Big)^{1-\alpha}\Big(\int_0^T t\|u_t\|_{L^2}^2 dt\Big)^\alpha\\ &\le C\mathcal{A}_0^\alpha\mathcal{B}_0^{1-\alpha}\exp\Big\{C\int_0^T \|\nabla u\|_{L^2}^4 dt\Big\}.\end{aligned}$$

This ends the proof of Lemma 4.1. □

Next, we recall the following abstract bootstrap argument or continuity argument which will be needed to prove Proposition 4.1.

Lemma 4.2 ([34])**.** *Let $T > 0$. For each $t \in [0, T]$, we have two statements, a "Hypothesis" $H(t)$ and a "Conclusion" $C(t)$. Suppose we can verify the following four assertions:*

(1) *(Hypothesis implies Conclusion) If $H(t)$ holds for some time $t \in [0, T]$, then $C(t)$ also holds for the same t;*

(2) *(Conclusion is stronger than Hypothesis) If $C(t)$ holds for some time $t_0 \in [0, T]$, then $H(t)$ holds for t in a neighborhood of t_0;*

(3) *(Conclusion is closed) If $C(t_n)$ holds for the time sequences $\{t_n\} \subset [0, T]$ and $t_n \to t$, then $C(t)$ holds;*

(4) *(Base case) If $H(t)$ holds for at least one $t_1 \in [0, T]$.*

Then $C(t)$ holds for all $t \in [0, T]$.

Proof of Proposition 4.1. If $T \leq 1$, due to (4.2), we have

$$
\begin{aligned}
\int_0^T \|\nabla u\|_{L^2}^4 dt &\leq \sup_{t\in[0,T]} \left(t^\alpha \|\nabla u\|_{L^2}^2\right)^2 \int_0^T t^{-2\alpha} dt \\
&\leq C\mathcal{A}_0^{2\alpha}\mathcal{B}_0^{2(1-\alpha)} \exp\left\{C\int_0^T \|\nabla u\|_{L^2}^4 dt\right\} \\
&= C\mathcal{A}_0^{2\alpha}\mathcal{B}_0^{1-2\alpha}\mathcal{B}_0 \exp\left\{C\int_0^T \|\nabla u\|_{L^2}^4 dt\right\}.
\end{aligned} \tag{4.11}
$$

On the other hand, if $T > 1$, due to (3.2) and (4.2), we have

$$
\begin{aligned}
&\int_0^T \|\nabla u\|_{L^2}^4 dt \\
&\leq \sup_{t\in[0,1]} \left(t^\alpha \|\nabla u\|_{L^2}^2\right)^2 \int_0^1 t^{-2\alpha} dt + \sup_{t\in[1,T]} \left(t^\alpha \|\nabla u\|_{L^2}^2\right) \int_1^T \|\nabla u\|_{L^2}^2 dt \\
&\leq C\mathcal{A}_0^{2\alpha}\mathcal{B}_0^{2(1-\alpha)} \exp\left\{C\int_0^T \|\nabla u\|_{L^2}^4 dt\right\} + C\mathcal{A}_0^{2\alpha}\mathcal{B}_0^{1-\alpha} \exp\left\{C\int_0^T \|\nabla u\|_{L^2}^4 dt\right\} \\
&\leq C\mathcal{A}_0^{2\alpha}\mathcal{B}_0^{1-2\alpha}\left(\mathcal{B}_0 + \mathcal{B}_0^\alpha\right) \exp\left\{C\int_0^T \|\nabla u\|_{L^2}^4 dt\right\}.
\end{aligned} \tag{4.12}
$$

For the sake of simplicity, we define function $E(t)$ on [0, T] as follows

$$
E(t) := \int_0^t \|\nabla u\|_{L^2}^4 ds.
$$

In summary, for both the cases $t \leq 1$ and $t > 1$, respectively, there is a positive constant C such that

$$
E(t) \leq C\mathcal{A}_0^{2\alpha}\mathcal{B}_0^{1-2\alpha}\left(\mathcal{B}_0 + \mathcal{B}_0^\alpha\right) \exp\left\{CE(t)\right\}. \tag{4.13}
$$

Next, we use the bootstrap argument. Setting

$$
\text{Hypothesis } H(t):\ E(t) \leq 4\left(\mathcal{B}_0 + \mathcal{B}_0^\alpha\right) \quad \text{for} \quad t \in [0, T] \tag{4.14}
$$

and

$$\text{Conclusion } C(t): \ E(t) \le 2\Big(\mathcal{B}_0 + \mathcal{B}_0^{\alpha}\Big) \quad \text{for} \quad t \in [0, T]. \tag{4.15}$$

The conditions (2)-(4) in Lemma 4.2 are clearly true and it remains to verify (1) under the small initial data condition (1.5). Once this is verified, then the bootstrap argument would imply that (4.15) actually holds for any $t \in [0, T]$.

It follows from (4.13) and (4.14) that,

$$E(t) \le C\mathcal{A}_0^{2\alpha}\mathcal{B}_0^{1-2\alpha}\Big(\mathcal{B}_0 + \mathcal{B}_0^{\alpha}\Big)\exp\Big\{4C\Big(\mathcal{B}_0 + \mathcal{B}_0^{\alpha}\Big)\Big\} \le 2\Big(\mathcal{B}_0 + \mathcal{B}_0^{\alpha}\Big),$$

provided that (1.5) holds.

Thus, this ends the proof of Proposition 4.1. □

Proposition 4.2. *Let* (θ_0, u_0) *satisfy the conditions in* Theorem 1.1 *and* (θ, u, Π) *be the global weak solution obtained in* Proposition 3.1. *Then it holds that*

$$\sup_{t\in[0,\sigma(T)]} t^{1+\alpha}\|u_t\|_{L^2}^2 + \int_0^{\sigma(T)} t^{1+\alpha}\|\nabla u_t\|_{L^2}^2 dt \le C\Big(\|\theta_0, u_0, \nabla u_0\|_{L^2}, \|\theta_0\|_{L^\infty}, \kappa\Big), \tag{4.16}$$

where $\sigma(t) := \min\{1, t\}$.

Proof. Taking t-derivative of the u equations in (1.1), one has

$$u_{tt} + u\cdot\nabla u_t + u_t\cdot\nabla u - \Delta u_t + \nabla\Pi_t = \theta_t e_3. \tag{4.17}$$

Taking the L^2 inner product of (4.17) with u_t and using the fact $\nabla\cdot u = 0$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \\
&= -\int_\Omega u_t\cdot\nabla u\cdot u_t dx + \int_\Omega \theta_t e_3\cdot u_t dx \\
&= -\int_\Omega u_t\cdot\nabla u\cdot u_t dx - \int_\Omega (u\cdot\nabla\theta e_3)\cdot u_t dx - \kappa\int_\Omega \theta e_3\cdot u_t dx \\
&=: \sum_{i=1}^{3} I_i,
\end{aligned} \tag{4.18}$$

where we have used the θ equation, namely, $\theta_t = -u\cdot\nabla\theta - \kappa\theta$.

We estimate the above three terms one by one as follows

$$\begin{aligned}
I_1 &= -\int_\Omega u_t\cdot\nabla u\cdot u_t dx \le C\|u_t\|_{L^4}^2\|\nabla u\|_{L^2} \\
&\le C\|u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2} \le \varepsilon\|\nabla u_t\|_{L^2}^2 + C\|u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4,
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
I_2 &= -\int_\Omega (u\cdot\nabla\theta e_3)\cdot u_t dx = \int_\Omega u\cdot\nabla u_t\cdot\theta e_3 dx \\
&\le C\|u\|_{L^2}\|\theta\|_{L^\infty}\|\nabla u_t\|_{L^2} \le \varepsilon\|\nabla u_t\|_{L^2}^2 + C\|\theta\|_{L^\infty}^2\|u\|_{L^2}^2,
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
I_3 &= -\kappa\int_\Omega \theta e_3\cdot u_t dx \le C\kappa\|\theta\|_{L^2}\|u_t\|_{L^2} \\
&\le \kappa\|\theta\|_{L^2} + C\kappa\|\theta\|_{L^2}\|u_t\|_{L^2}^2.
\end{aligned} \tag{4.21}$$

Substituting (4.19)-(4.21) into (4.18) and absorbing the ε- terms, one has

$$\frac{d}{dt}\|u_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2\le C(\|\nabla u\|_{L^2}^4+\kappa\|\theta\|_{L^2})\|u_t\|_{L^2}^2+C\|\theta\|_{L^\infty}^2\|u\|_{L^2}^2+\kappa\|\theta\|_{L^2}. \tag{4.22}$$

Multiplying (4.22) by $t^{1+\alpha}$ yields that

$$\begin{aligned}\frac{d}{dt}\Big(t^{1+\alpha}\|u_t\|_{L^2}^2\Big)+t^{1+\alpha}\|\nabla u_t\|_{L^2}^2\le{}&(1+\alpha)t^{\alpha}\|u_t\|_{L^2}^2+C(\|\nabla u\|_{L^2}^4+\kappa\|\theta\|_{L^2})t^{1+\alpha}\|u_t\|_{L^2}^2\\&+Ct^{1+\alpha}\|\theta\|_{L^\infty}^2\|u\|_{L^2}^2+C\kappa t^{1+\alpha}\|\theta\|_{L^2}.\end{aligned} \tag{4.23}$$

Due to (3.1) and (3.2), we have

$$\begin{aligned}\int_0^{\sigma(T)}\|u\|_{L^2}^2t^{1+\alpha}\|\theta\|_{L^\infty}^2dt&\le\Big(\sup_{t\in[0,\sigma(T)]}\|u\|_{L^2}^2\Big)\int_0^{\sigma(T)}t^{1+\alpha}e^{-2\kappa t}\|\theta_0\|_{L^\infty}^2dt\\&\le\frac{C\mathcal{A}_0\|\theta_0\|_{L^\infty}^2}{\kappa^2},\end{aligned} \tag{4.24}$$

and

$$\int_0^{\sigma(T)}t^{1+\alpha}\kappa\|\theta\|_{L^2}dt=\int_0^{\sigma(T)}t^{1+\alpha}\kappa e^{-\kappa t}\|\theta_0\|_{L^2}dt\le\frac{\|\theta_0\|_{L^2}}{\kappa}, \tag{4.25}$$

where the following basic calculation has been performed for $m\in\mathbb{Z}^+$

$$\int_0^{\sigma(T)}t^{1+\alpha}e^{-m\kappa t}dt=-\frac{1}{m\kappa^2}\int_0^{\sigma(T)}t^{1+\alpha}\kappa de^{-m\kappa t}\le\frac{1+\alpha}{m\kappa^2}\int_0^{\sigma(T)}t^{\alpha}\kappa e^{-m\kappa t}dt\le\frac{1+\alpha}{m^2\kappa^2}.$$

Invoking the Gronwall inequality to (4.23) and using (4.24)-(4.25) yields that (4.16). □

Next, we will prove the following exponential decay estimates in time with respect to the velocity for large time, which plays a crucial role in our analysis.

Proposition 4.3. *Let (θ_0,u_0) satisfy the conditions in* Theorem 1.1 *and (θ,u,Π) be the global weak solution obtained in* Proposition 3.1. *Then it holds that*

$$\sup_{t\in[0,T]}e^{\delta t}\|u\|_{L^2}^2+\int_0^Te^{\delta t}\|\nabla u\|_{L^2}^2dt\le C, \tag{4.26}$$

$$\sup_{t\in[0,T]}e^{\delta t}\|\nabla u\|_{L^2}^2+\int_0^Te^{\delta t}\|u_t\|_{L^2}^2dt\le C, \tag{4.27}$$

$$\sup_{t\in[\sigma(T),T]}e^{\delta t}\|u_t\|_{L^2}^2+\int_{\sigma(T)}^Te^{\delta t}\|\nabla u_t\|_{L^2}^2dt\le C, \tag{4.28}$$

$$\sup_{t\in[\sigma(T),T]}e^{\delta t}\Big(\|\nabla^2u\|_{L^2}^2+\|\nabla\Pi\|_{L^2}^2\Big)\le C, \tag{4.29}$$

where $\delta:=\min\{\lambda,\kappa^-\}$ and $\kappa^-<\kappa$ is any positive constant closed enough to κ as mentioned above, the constant C depends only on the initial norm $\|\theta_0,u_0,\nabla u_0\|_{L^2}$, $\|\theta_0\|_{L^\infty}$ and κ,λ.

Proof. First, coming back to (3.5), and using the Poincaré inequality (see Lemma 2.2), we have

$$\frac{d}{dt}\|u\|_{L^2}^2+\delta\|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\le e^{-\kappa t}\|\theta_0\|_{L^2}\Big(\frac{1}{\kappa}\|\theta_0\|_{L^2}+\|u_0\|_{L^2}\Big). \tag{4.30}$$

Integrating (4.30) with respect to time variable over $[0,T]$ yields the desired result (4.26).

Next, multiplying (4.6) by $e^{\delta t}$, one has

$$(4.31)\quad e^{\delta t}\|u_t\|_{L^2}^2+\frac{d}{dt}\Big(e^{\delta t}\|\nabla u\|_{L^2}^2\Big)\le \delta e^{\delta t}\|\nabla u\|_{L^2}^2+C\Big(e^{\delta t}\|\theta\|_{L^2}^2+\|\nabla u\|_{L^2}^4 e^{\delta t}\|\nabla u\|_{L^2}^2\Big),$$

Integrating (4.31) with respect to time variable over $[0,T]$, using (3.1), (4.1), (4.26) and the Gronwall inequality yield that (4.27).

Similarly, multiplying (4.22) by $e^{\delta t}$, one has

$$(4.32)\qquad \frac{d}{dt}\Big(e^{\delta t}\|u_t\|_{L^2}^2\Big)+e^{\delta t}\|\nabla u_t\|_{L^2}^2\le \delta e^{\delta t}\|u_t\|_{L^2}^2+C(\|\nabla u\|_{L^2}^4+\kappa\|\theta\|_{L^2})e^{\delta t}\|u_t\|_{L^2}^2 + C\|\theta\|_{L^\infty}^2 e^{\delta t}\|u\|_{L^2}^2+\kappa e^{\delta t}\|\theta\|_{L^2}.$$

Integrating (4.32) with respect to time variable over $[\sigma(T),T]$, using (3.1), (4.1), (4.27) and Gronwall inequality yield that (4.28).

Due to (4.5), one has

$$\sup_{t\in[\sigma(T),T]} e^{\delta t}\Big(\|\nabla^2 u\|_{L^2}^2+\|\nabla\Pi\|_{L^2}^2\Big)\le C\sup_{t\in[\sigma(T),T]} e^{\delta t}\Big(\|u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^6+\|\theta\|_{L^2}^2\Big),$$

which together with (3.1), (4.27) and (4.28) gives that (4.29). □

Proposition 4.4. *Let* (θ_0,u_0) *satisfy the conditions in* Theorem 1.1 *and* (θ,u,Π) *be the global weak solution obtained in* Proposition 3.1. *It holds that*

$$(4.33)\qquad \int_0^T \|\nabla u\|_{L^\infty}dt \quad\le\quad C,$$

where the constant C depends only on the initial norm $\|\theta_0,u_0,\nabla u_0\|_{L^2}^2$, $\|\theta_0\|_{L^\infty}$ *and* κ,λ.

Proof. It follows from Lemma 2.3 and Gagliardo-Nirenberg's inequality that

$$\begin{aligned}
&\|\nabla^2 u\|_{L^p}+\|\nabla\Pi\|_{L^p}\\
&\le C\|u_t+u\cdot\nabla u-\theta e_3\|_{L^p}\\
&\le C\Big(\|u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}}+\|u\|_{L^6}\|\nabla u\|_{L^{\frac{6p}{6-p}}}+\|\theta\|_{L^p}\Big)\\
&\le C\Big(\|u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}}+\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{p}{5p-6}}\|\nabla^2 u\|_{L^p}^{\frac{4p-6}{5p-6}}+\|\theta\|_{L^p}\Big)\\
&\le C\Big(\|u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}}+\|\nabla u\|_{L^2}^{\frac{6p-6}{p}}+\|\theta\|_{L^p}\Big)+\frac12\|\nabla^2 u\|_{L^p},
\end{aligned}$$

which gives that

$$(4.34)\qquad \|\nabla^2 u\|_{L^p}+\|\nabla\Pi\|_{L^p}\le C\Big(\|u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}}+\|\nabla u\|_{L^2}^{\frac{6p-6}{p}}+\|\theta\|_{L^p}\Big).$$

From (4.34) and Sobolev inequality, we have

$$(4.35)\qquad \begin{aligned}\|\nabla u\|_{L^\infty}&\le C\|\nabla u\|_{L^2}+C\|\nabla^2 u\|_{L^p}\\ &\le C\Big(\|\nabla u\|_{L^2}+\|u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}}+\|\nabla u\|_{L^2}^{\frac{6p-6}{p}}+\|\theta\|_{L^p}\Big).\end{aligned}$$

Due to (3.1), (3.2), (4.16) and (4.27), we have

$$(4.36)\int_0^{\sigma(T)}\|\nabla u\|_{L^\infty}dt \quad\le\quad C\int_0^{\sigma(T)}\Big(\|\nabla u\|_{L^2}+\|u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}}+\|\nabla u\|_{L^2}^{\frac{6p-6}{p}}+\|\theta\|_{L^p}\Big)dt$$

$$
\begin{aligned}
&\le C\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 dt \\
&\quad + \sup_{t\in[0,\sigma(T)]} \left(t^{1+\alpha}\|u_t\|_{L^2}^2\right)^{\frac{6-p}{4p}} \int_0^{\sigma(T)} t^{-\frac{1+\alpha}{2}} \left(t^{1+\alpha}\|\nabla u_t\|_{L^2}^2\right)^{\frac{3p-6}{4p}} dt \\
&\quad + \sup_{t\in[0,\sigma(T)]} \|\nabla u\|_{L^2}^{\frac{2p-6}{p}} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^4 dt + \frac{\|\theta_0\|_{L^2\cap L^\infty}}{\kappa} \\
&\le C\Big(\|\theta_0, u_0, \nabla u_0\|_{L^2}^2, \|\theta_0\|_{L^\infty}, \kappa, \lambda\Big),
\end{aligned}
$$

where we have used the fact $\int_0^{\sigma(T)} t^{-2p(1+\alpha)/(p+6)}dt < \infty$ which requires the condition $p < \frac{6}{1+2\alpha}$.

On the other hand, we have from (3.1), (4.27) and (4.28)

$$
\begin{aligned}
\int_{\sigma(T)}^T \|\nabla u\|_{L^\infty} dt &\le C\int_{\sigma(T)}^T \Big(\|\nabla u\|_{L^2} + \|u_t\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2}^{\frac{6p-6}{p}} + \|\theta\|_{L^p}\Big)dt \\
&\le C \sup_{t\in[\sigma(T),T]} e^{\frac{\delta t}{2}}\|\nabla u, u_t\|_{L^2} \int_{\sigma(T)}^T e^{-\frac{\delta t}{2}} dt \\
&\quad + C\Big(\int_{\sigma(T)}^T e^{-\delta t}dt\Big)^{\frac12}\Big(\int_{\sigma(T)}^T e^{\delta t}\|\nabla u_t\|_{L^2}^2 dt\Big)^{\frac12} \\
&\quad + \sup_{t\in[\sigma(T),T]} \|\nabla u\|_{L^2}^{\frac{2p-6}{p}} \int_{\sigma(T)}^T \|\nabla u\|_{L^2}^4 dt + \frac{\|\theta_0\|_{L^2\cap L^\infty}}{\kappa} \\
&\le C\Big(\|\theta_0, u_0, \nabla u_0\|_{L^2}^2, \|\theta_0\|_{L^\infty}, \kappa, \lambda\Big).
\end{aligned} \tag{4.37}
$$

Combining this with (4.36) gives (4.33) and finishes the proof of Proposition 4.4. □

Proposition 4.5. *let* (θ_0, u_0) *satisfy the conditions in* Theorem 1.1 *and* (θ, u, Π) *be the global weak solution obtained in* Proposition 3.1. *Then it holds that*

$$
\|\nabla\theta\|_{L^2} \le Ce^{-\kappa t}\|\nabla\theta_0\|_{L^2}, \tag{4.38}
$$

where the constant C depends only on the initial norm $\|\theta_0, u_0, \nabla u_0\|_{L^2}^2$, $\|\theta_0\|_{L^\infty}$ *and* κ, λ.

Proof. Taking the x_i-derivative of the θ equations in (1.1), one has

$$
\partial_i\theta_t + u\cdot\nabla\partial_i\theta + \kappa\partial_i\theta = -\partial_i u\cdot\nabla\theta. \tag{4.39}
$$

Taking the L^2 inner product of θ equation in (4.39) with $\partial_i\theta$, using the divergence-free condition $\nabla\cdot u = 0$ and summing over i, we obtain

$$
\frac12\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 + \kappa\|\nabla\theta\|_{L^2}^2 \le \|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2}^2,
$$

which together with (4.33) and the Gronwall inequality lead to (4.38). □

The following lemma is necessary for further estimating on the higher-order derivatives of the strong solution (θ, u, Π).

Proposition 4.6. *Assume that* (θ_0, u_0) *satisfies the conditions stated in* Theorem 1.1. *Let* (θ, u, Π) *be the global weak solution obtained in* Proposition 3.1. *It holds that*

$$\sup_{t\in[0,T]} \sigma(t)e^{\delta t}(\|u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla\Pi\|_{L^2}^2) + \int_0^T e^{\delta t}(\sigma(t)\|\nabla u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla\Pi\|_{L^2}^2)dt \le C, \tag{4.40}$$

where the constant C depends only on the initial norm $\|\theta_0, u_0, \nabla u_0\|_{L^2}^2$, $\|\theta_0\|_{L^\infty}$ and κ, λ.

Proof. Multiplying (4.22) by $\sigma(t)e^{\delta t}$, one has

$$\begin{aligned}
&\frac{d}{dt}\Big(\sigma(t)e^{\delta t}\|u_t\|_{L^2}^2\Big) + \sigma(t)e^{\delta t}\|\nabla u_t\|_{L^2}^2 \\
&\le \frac{d}{dt}\Big(\sigma(t)e^{\delta t}\Big)\|u_t\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^4 + \kappa\|\theta\|_{L^2})\Big(\sigma(t)e^{\delta t}\|u_t\|_{L^2}^2\Big) \\
&\quad + C\|\theta\|_{L^\infty}^2 e^{\delta t}\|u\|_{L^2}^2 + \kappa e^{\delta t}\|\theta\|_{L^2}.
\end{aligned} \tag{4.41}$$

which together with (3.1), (4.1), (4.26), (4.27) and the Gronwall inequality yield that

$$\sup_{t\in[0,T]} \sigma(t)e^{\delta t}\|u_t\|_{L^2}^2 + \int_0^T \sigma(t)e^{\delta t}\|\nabla u_t\|_{L^2}^2 dt \le C. \tag{4.42}$$

Due to (4.5), (4.26), (4.27) and (4.42), one has

$$\begin{aligned}
&\sup_{t\in[0,T]} \sigma(t)e^{\delta t}\Big(\|\nabla^2 u\|_{L^2}^2 + \|\nabla\Pi\|_{L^2}^2\Big) + \int_0^T e^{\delta t}\Big(\|\nabla^2 u\|_{L^2}^2 + \|\nabla\Pi\|_{L^2}^2\Big)dt \\
&\le C \sup_{t\in[0,T]} \sigma(t)e^{\delta t}\Big(\|u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^6 + \|\theta\|_{L^2}^2\Big) \\
&\quad + C\int_0^T e^{\delta t}\Big(\|u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^6 + \|\theta\|_{L^2}^2\Big)dt \le C.
\end{aligned} \tag{4.43}$$

Thus, the proof of Proposition 4.6 is completed. □

The following proposition is concerned with the estimates on the higher-order derivatives on the strong solutions u, Π which in particular imply the continuity in time of both $\nabla^2 u$ and $\nabla\Pi$ in the $L^2 \cap L^p$ norm.

Proposition 4.7. *Let (θ_0, u_0) satisfy the conditions in* Theorem 1.1 *and (θ, u, Π) be the global weak solution obtained in* Proposition 3.1. *Then it holds that*

$$\sup_{t\in[0,T]} \sigma(t)e^{\delta t}\|\nabla u_t\|_{L^2}^2 + \int_0^T \sigma(t)e^{\delta t}\|u_{tt}\|_{L^2}^2 dt \le C, \tag{4.44}$$

where the constant C depends only on the initial norm $\|\theta_0, u_0, \nabla u_0\|_{L^2}^2$, $\|\theta_0\|_{L^\infty}$ and κ, λ.

Proof. Taking the L^2 inner product of (4.17) with u_{tt} and using the divergence-free condition $\nabla \cdot u = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u_t\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2 = -\int_\Omega u\cdot\nabla u_t\cdot u_{tt}dx - \int_\Omega u_t\cdot\nabla u\cdot u_{tt}dx - \int_\Omega \theta_t e_3\cdot u_{tt}dx \\
&\le \varepsilon\|u_{tt}\|_{L^2}^2 + \|u\|_{L^\infty}^2\|\nabla u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2\|\nabla u\|_{L^\infty}^2 + \|\theta_t\|_{L^2}^2 \\
&\le \varepsilon\|u_{tt}\|_{L^2}^2 + C\|\nabla u\|_{H^1}^2\|\nabla u_t\|_{L^2}^2 + C(\|\nabla u\|_{H^1}^2 + \|\nabla u_t\|_{L^2}^2 + \|\theta\|_{L^p}^2)\|u_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2 \\
&\le \varepsilon\|u_{tt}\|_{L^2}^2 + C(\|\nabla u\|_{H^1}^2 + \|u_t\|_{L^2}^2)\|\nabla u_t\|_{L^2}^2 + C(\|\nabla u\|_{H^1}^2 + \|\theta\|_{L^p}^2)\|u_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2
\end{aligned} \tag{4.45}$$

$$\leq \varepsilon\|u_{tt}\|_{L^2}^2 + \|\theta_t\|_{L^2}^2 + C(\|\nabla u\|_{H^1}^2 + \|u_t\|_{L^2}^2 + \|\theta\|_{L^p}^2)\|\nabla u_t\|_{L^2}^2,$$

where we have used the facts $\|u\|_{L^\infty} \leq C\|\nabla u\|_{H^1}$ and

$$\|\nabla u\|_{L^\infty} \leq C(\|\nabla u\|_{L^2} + \|\nabla u_t\|_2 + \|\theta\|_{L^p}), \tag{4.46}$$

(4.34) and the Poincaré inequality in the last step due to $u_t|_\Omega = 0$.

In view of the θ equation (1.1), one has

$$\|\theta_t\|_{L^2}^2 \leq \|u\cdot\nabla\theta\|_{L^2}^2 + \kappa\|\theta e_3\|_{L^2}^2 \leq C\|\nabla u\|_{H^1}^2 + \kappa\|\theta\|_{L^2}^2. \tag{4.47}$$

Inserting (4.47) into (4.45), absorbing the ε- term, we obtain

$$\begin{aligned}\frac{d}{dt}\|\nabla u_t\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2 \quad &\leq \quad C(\|\nabla u\|_{H^1}^2 + \|u_t\|_{L^2}^2 + \|\theta\|_{L^p}^2)\|\nabla u_t\|_{L^2}^2 \\ &\quad +C\|\nabla u\|_{H^1}^2 + \kappa\|\theta\|_{L^2}^2.\end{aligned} \tag{4.48}$$

Next, multiplying (4.48) by $\sigma(t)e^{\delta t}$, one has

$$\begin{aligned}&\frac{d}{dt}\Big(\sigma(t)e^{\delta t}\|\nabla u_t\|_{L^2}^2\Big) + \sigma(t)e^{\delta t}\|u_{tt}\|_{L^2}^2 \\ &\leq \frac{d}{dt}\Big(\sigma(t)e^{\delta t}\Big)\|\nabla u_t\|_{L^2}^2 + Ce^{\delta t}\|\nabla u\|_{H^1}^2 + \kappa e^{\delta t}\|\theta\|_{L^2}^2 \\ &\quad + C(\|\nabla u\|_{H^1}^2 + \|u_t\|_{L^2}^2 + \|\theta\|_{L^p}^2)\Big(\sigma(t)e^{\delta t}\|\nabla u_t\|_{L^2}^2\Big),\end{aligned} \tag{4.49}$$

together (3.1), (4.26), (4.27), (4.40) and the Gronwall inequality yield that (4.44). □

Proposition 4.8. *Let* (θ_0, u_0) *satisfy the conditions in* Theorem 1.1 *and* (θ, u, Π) *be the global weak solution obtained in* Proposition 3.1. *It holds that*

$$\sup_{t\in[0,T]} \sigma(t)e^{\delta t}\Big(\|\nabla^2 u\|_{L^p}^2 + \|\nabla\Pi\|_{L^p}^2\Big) + \int_0^T \sigma(t)e^{\delta t}\Big(\|\nabla^2 u_t\|_{L^2}^2 + \|\nabla\Pi_t\|_{L^2}^2\Big)dt \leq C, \tag{4.50}$$

where the constant C depends only on the initial norm $\|\theta_0, u_0, \nabla u_0\|_{L^2}^2$, $\|\theta_0\|_{L^\infty}$ *and* κ, λ.

Proof. Recalling that (u_t, Π_t) satisfies the following Stokes system

$$\begin{cases} -\Delta u_t + \nabla\Pi_t = -u_{tt} - u\cdot\nabla u_t - u_t\cdot\nabla u + \theta_t e_3, & x\in\Omega, \\ \nabla\cdot u_t = 0, & x\in\Omega, \\ u_t = 0, & x\in\partial\Omega. \end{cases} \tag{4.51}$$

It follows from Lemma 2.3 and (4.45)-(4.47) that

$$\begin{aligned}\|\nabla^2 u_t\|_{L^2}^2 + \|\nabla\Pi_t\|_{L^2}^2 \quad &\leq \quad C\|u_{tt} + u\cdot\nabla u_t + u_t\cdot\nabla u - \theta_t e_3\|_{L^2}^2 \\ &\leq \quad C(\|u_{tt}\|_{L^2}^2 + \|u\|_{L^\infty}^2\|\nabla u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2\|\nabla u\|_{L^\infty}^2 + \|\theta_t\|_{L^2}^2) \\ &\leq \quad C\|u_{tt}\|_{L^2}^2 + C(\|\nabla u\|_{H^1}^2 + \|u_t\|_{L^2}^2 + \|\theta\|_{L^p}^2)\|\nabla u_t\|_{L^2}^2 \\ &\quad +C\|\nabla u\|_{H^1}^2 + C\kappa\|\theta\|_{L^2}^2,\end{aligned} \tag{4.52}$$

from which, (4.40) and (4.44), it follows that

$$\int_0^T \sigma(t)e^{\delta t}(\|\nabla^2 u_t\|_{L^2}^2 + \|\nabla\Pi_t\|_{L^2}^2)dt \leq C. \tag{4.53}$$

Due to (4.34), we have

$$\|\nabla^2 u\|_{L^p}^2 + \|\nabla\Pi\|_{L^p}^2 \le C(\|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^{\frac{12(p-1)}{p}} + \|\theta\|_{L^p}^2), \tag{4.54}$$

which combining (4.40) and (4.44) implies that

$$\begin{aligned} &\sup_{t\in[0,T]} \sigma(t)e^{\delta t}(\|\nabla^2 u\|_{L^p}^2 + \|\nabla\Pi\|_{L^p}^2) \\ &\le C \sup_{t\in[0,T]} \sigma(t)e^{\delta t}(\|\nabla u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\theta\|_{L^p}^2) \le C. \end{aligned} \tag{4.55}$$

Thus, the proof of Proposition 4.8 is completed. □

Finally, we give the proof of our main Theorem 1.1.

Proof of Theorem 1.1. From (4.26), (4.27), (4.29), (4.40) and (4.50), we deduce that

$$\nabla u, \Pi \in C([\tau, T]; L^2) \cap C([\tau, T] \times \overline{\Omega}), \tag{4.56}$$

where we have used the standard embedding

$$L^\infty([\tau, T]; H^1 \cap W^{1,p}) \cap H^1([\tau, T]; L^2) \hookrightarrow C([\tau, T]; L^2) \cap C([\tau, T] \times \overline{\Omega}).$$

Furthermore, it follows from (3.1) and (4.38) that

$$\theta \in C([0, T]; H^1). \tag{4.57}$$

Due to (4.27) and (4.40), one obtains that

$$u_t \in H^1((\tau, T); L^2) \hookrightarrow C([\tau, T]; L^2), \tag{4.58}$$

which together with (4.56) yields that

$$u_t + u\cdot\nabla u \in C([\tau, T]; L^2). \tag{4.59}$$

Since (u, θ) fulfills (4.4), we deduce from (4.50), (4.56)-(4.59) that for any $p \in (3, 6/(1+2\alpha))$

$$\nabla u, \Pi \in C([\tau, T]; D^1 \cap D^{1,p}). \tag{4.60}$$

With the global regularity established at our hand, we can prove the uniqueness of the solution. This idea is borrowed from [22] which is introduced in [20, 23].

Let $(u_i, \theta_i)(i = 1, 2)$ be two global smooth solutions to system (1.1)-(1.3) with the same initial data (u_0, θ_0). Denote

$$(u, \theta, \Pi) = (u_1 - u_2, \theta_1 - \theta_2, \Pi_1 - \Pi_2).$$

Obviously, the difference (u, θ, Π) satisfies

$$\begin{cases} \partial_t u + \nabla\Pi - \Delta u = -\mathrm{div}(u_1 \otimes u + u \otimes u_2) + \theta e_3, \\ \partial_t\theta + \kappa\theta = -\mathrm{div}(u_1\theta + u\theta_2), \\ \mathrm{div} u = 0, \\ (u(0, x), \theta(0, x)) = (0, 0), \\ u(t, x)|_\Omega = 0. \end{cases} \tag{4.61}$$

Introduce some new notations $(\overline{u}, \overline{\theta}, \overline{\Pi}) = (I - \Delta)^{-1}(u, \theta, \Pi)$, from (4.61), we have

$$
\begin{cases}
\partial_t\overline{u} + \nabla\overline{\Pi} - \Delta\overline{u} = -(I-\Delta)^{-1}\mathrm{div}(u_1\otimes u + u\otimes u_2) + \overline{\theta}e_3,\\
\partial_t\overline{\theta} + \kappa\overline{\theta} = -(I-\Delta)^{-1}\mathrm{div}(u_1\theta + u\theta_2),\\
\mathrm{div}\overline{u} = 0,\\
(\overline{u}(0,x), \overline{\theta}(0,x)) = (0,0),\\
\overline{u}(t,x)|_{\Omega} = 0.
\end{cases}
\tag{4.62}
$$

Taking the L^2 inner product of $(4.62)_1$ and $(4.62)_2$ with $(I-\Delta)\overline{u}$ and $(I-\Delta)\overline{\theta}$, respectively, we obtain

(4.63)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\overline{u}, \nabla\overline{u}, \overline{\theta}, \nabla\overline{\theta}\|_{L^2}^2 + \|\nabla\overline{u}, \Delta\overline{u}, \kappa\overline{\theta}, \kappa\nabla\overline{\theta}\|_{L^2}^2\\
&= -\int_\Omega (u_1\otimes u + u\otimes u_2)\cdot\nabla\overline{u}dx + \int_\Omega \overline{\theta}(I-\Delta)\overline{u}_3dx - \int_\Omega u_1\theta\cdot\nabla\overline{\theta}dx - \int_\Omega u\theta_2\cdot\nabla\overline{\theta}dx\\
&= \sum_{i=1}^{4} J_i.
\end{aligned}
$$

Next, we estimate the above four terms one by one.

$$
\begin{aligned}
J_1 &= -\int_\Omega (u_1\otimes u + u\otimes u_2)\cdot\nabla\overline{u}dx\\
&\le \int_\Omega \Big(|u_1| + |u_2|\Big)\Big(|\overline{u}| + |\Delta\overline{u}|\Big)|\nabla\overline{u}|dx\\
&\le \|u_1, u_2\|_{L^6}\Big(\|\overline{u}\|_{L^3}\|\nabla\overline{u}\|_{L^2} + \|\Delta\overline{u}\|_{L^2}\|\nabla\overline{u}\|_{L^3}\Big)\\
&\le \varepsilon\|\nabla\overline{u}, \Delta\overline{u}\|_{L^2}^2 + C\|\nabla u_1, \nabla u_2\|_{L^2}^4\|\overline{u}, \nabla\overline{u}\|_{L^2}^2,
\end{aligned}
\tag{4.64}
$$

$$
\begin{aligned}
J_2 &= \int_\Omega \overline{\theta}(I-\Delta)\overline{u}_3dx\\
&\le \|\overline{\theta}\|_{L^2}\Big(\|\overline{u}\|_{L^2} + \|\Delta\overline{u}\|_{L^2}\Big) \le \varepsilon\|\Delta\overline{u}\|_{L^2}^2 + C\|\overline{u}, \overline{\theta}\|_{L^2}^2,
\end{aligned}
\tag{4.65}
$$

$$
\begin{aligned}
J_3 &= -\int_\Omega u_1\theta\cdot\nabla\overline{\theta}dx = -\int_\Omega u_1(I-\Delta)\overline{\theta}\cdot\nabla\overline{\theta}dx\\
&= -\sum_{k=1}^{3}\int_\Omega \partial_k u_1\partial_k\overline{\theta}\cdot\nabla\overline{\theta}dx \le \|\nabla u_1\|_{L^\infty}\|\nabla\overline{\theta}\|_{L^2}^2,
\end{aligned}
\tag{4.66}
$$

$$
\begin{aligned}
J_4 &= -\int_\Omega u\theta_2\cdot\nabla\overline{\theta}dx \le \int_\Omega |\theta_2|\Big(|\overline{u}| + |\Delta\overline{u}|\Big)|\nabla\overline{\theta}|dx\\
&\le \|\theta_2\|_{L^\infty}\Big(\|\overline{u}\|_{L^2} + \|\Delta\overline{u}\|_{L^2}\Big)\|\nabla\overline{\theta}\|_{L^2}\\
&\le \varepsilon\|\nabla\overline{\theta}, \Delta\overline{u}\|_{L^2}^2 + C\|\theta_2\|_{L^\infty}^2\|\overline{u}, \nabla\overline{\theta}\|_{L^2}^2.
\end{aligned}
\tag{4.67}
$$

Plugging (4.63)-(4.67) into (4.63) and absorbing the ε-terms, we have

$$
\frac{1}{2}\frac{d}{dt}\|\overline{u}, \nabla\overline{u}, \overline{\theta}, \nabla\overline{\theta}\|_{L^2}^2 \le C\Big(1 + \|\nabla u_1, \nabla u_2\|_{L^2}^4 + \|\theta_2\|_{L^\infty}^2\Big)\|\overline{u}, \nabla\overline{u}, \overline{\theta}, \nabla\overline{\theta}\|_{L^2}^2,
\tag{4.68}
$$

which together with the zero initial data and the Gronwall inequality yields that $\overline{u} = \overline{\theta} = 0$.

By definition of $(\overline{u}, \overline{\theta})$, one obtain that $u = \theta = 0$ which implies the uniqueness.

Thus, we have finished the proof of Theorem 1.1. $\square$

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