Sato, H. and Suyama, Y.
Osaka J. Math.
57 (2020), 51–59

EXAMPLES OF SINGULAR TORIC VARIETIES WITH CERTAIN NUMERICAL CONDITIONS

Hiroshi SATO and Yusuke SUYAMA

(Received August 6, 2018)

Abstract

We give various examples of $\mathbb{Q}$-factorial projective toric varieties such that the sum of the squared torus invariant prime divisors is positive. We also determine the generators for the cone of effective 2-cycles on a toric variety of Picard number two. This result is convenient to explain our examples.

Contents

1. Introduction

In [10], the following concepts were introduced:

Definition 1.1 [10, Definition 3.1]. Let X be a $\mathbb{Q}$-factorial projective toric d-fold. Put

$$\gamma_2 = \gamma_2(X) := D_1^2 + \cdots + D_n^2 \in \mathrm{N}^2(X),$$

where $D_1, \ldots, D_n$ be the torus invariant prime divisors.

If $\gamma_2 \cdot S > 0$ (resp. ≥ 0) for any subsurface $S \subset X$, then we say that X is γ_2-*positive* (resp. γ_2-*nef*).

When X is smooth, it is expected that γ_2-positive or γ_2-nef toric varieties have good geometric properties (see [5], [8] and [9]. Also see Questions 1.2 and 1.3 below). We should remark that $\frac{1}{2}\gamma_2(X)$ is the second Chern character $\mathrm{ch}_2(X)$ of X in this case. It was confirmed that these properties hold for the case where X is a $\mathbb{Q}$-factorial terminal toric Fano 3-fold in [10]. Therefore, [10] posed the following questions:

Question 1.2 [10, Question 5.4]. Does there exist a $\mathbb{Q}$-factorial terminal projective γ_2-positive toric variety X of $\rho(X) \geq 2$?

2010 Mathematics Subject Classification. Primary 14M25; Secondary 14J45.

Question 1.3 [10, Question 5.6]. For any $\mathbb{Q}$-factorial terminal projective γ_2-nef toric d-fold of $\rho(X) \geq 2$, does one of the following hold?

(1) There exists a Fano contraction $\varphi : X \to \overline{X}$ such that $\overline{X}$ is a γ_2-nef toric $(d-1)$-fold.
(2) There exists a toric finite morphism $\pi : X' \to X$ such that X' is a direct product of lower-dimensional γ_2-nef toric varieties.

In this paper, we give answers for these questions by giving certain explicit examples (see Examples 3.2, 3.3 and 3.5, and Theorem 3.4). According to these examples, we see that higher-dimensional γ_2-positive or γ_2-nef singular toric varieties do *not* have good geometric properties like smooth cases.

2. Preliminaries

In this section, we introduce some basic results and notation of toric varieties. For the details, please see [1], [3] and [6]. For the toric Mori theory, see also [2], [4, Chapter 14] and [7].

Let $X = X_\Sigma$ be the toric d-fold associated to a fan Σ in $N = \mathbb{Z}^d$ over an algebraically closed field k of arbitrary characteristic. We will use the notation $\Sigma = \Sigma_X$ to denote the fan associated to a toric variety X. We denote the Picard number of X by $\rho(X)$. Put $N_\mathbb{R} := N \otimes \mathbb{R}$. There exists a one-to-one correspondence between the r-dimensional cones in Σ and the torus invariant subvarieties of dimension $d-r$ in X. Let $\mathrm{G}(\Sigma)$ be the set of primitive generators for 1-dimensional cones in Σ. Thus, for $v \in \mathrm{G}(\Sigma)$, we have the torus invariant prime divisor corresponding to $\mathbb{R}_{\geq 0}v \in \Sigma$.

Let X be a projective toric d-fold. For $1 \leq r \leq d$, we put

$$\mathrm{Z}_r(X) := \{\text{the } r\text{-cycles on } X\}, \text{ while } \mathrm{Z}^r(X) := \{\text{the } r\text{-cocycles on } X\}.$$

We introduce the numerical equivalence $\equiv$ on $\mathrm{Z}_r(X)$ and $\mathrm{Z}^r(X)$ as follows: For $C \in \mathrm{Z}_r(X)$, we define $C \equiv 0$ if $D \cdot C = 0$ for any $D \in \mathrm{Z}^r(X)$, while for $D \in \mathrm{Z}^r(X)$, we define $D \equiv 0$ if $D \cdot C = 0$ for any $C \in \mathrm{Z}_r(X)$. We put

$$\mathrm{N}_r(X) := (\mathrm{Z}_r(X) \otimes \mathbb{R}) / \equiv, \text{ while } \mathrm{N}^r(X) := (\mathrm{Z}^r(X) \otimes \mathbb{R}) / \equiv .$$

We denote the cone of effective r-cycles of X by $\mathrm{NE}_r(X) \subset \mathrm{N}_r(X)$. $\mathrm{NE}_r(X)$ is a strongly convex rational polyhedral cone in $\mathrm{N}_r(X)$.

For $\mathrm{NE}_1(X) = \mathrm{NE}(X)$, that is, the ordinary *Kleiman-Mori cone*, there is a good description of 1-cycles. So, let X be a $\mathbb{Q}$-factorial projective toric d-fold. Let $C = C_\tau$ be the torus invariant curve corresponding to a $(d-1)$-dimensional cone τ generated by $x_1, \ldots, x_{d-1}$, where $x_1, \ldots, x_{d-1} \in \mathrm{G}(\Sigma)$. Then, there exist exactly two maximal cone $y_1 + \tau$ and $y_2 + \tau$ which contain τ as a face, where $y_1, y_2 \in \mathrm{G}(\Sigma)$. So, we have the linear relation

$$a_1 y_1 + a_2 y_2 + b_1 x_1 + \cdots + b_{d-1} x_{d-1} = 0,$$

where $a_1, a_2, b_1, \ldots, b_{d-1} \in \mathbb{Q}$ and $a_1, a_2 > 0$. We call this equality the *wall relation* for τ. The wall relation is determined up to multiple of positive rational numbers. If C spans an extremal ray of $\mathrm{NE}(X)$, we say that the wall relation for τ is *extremal*.

We end this section by determining the structure of $\mathrm{NE}_2(X)$, which is useful to describe the examples in Section 3.

Theorem 2.1. *If $X = X_\Sigma$ is a $\mathbb{Q}$-factorial projective toric d-fold of $\rho(X) = 2$, then* $\mathrm{NE}_2(X)$ *is generated by at most* 3 *torus invariant surfaces.*

Proof. First, we remark that [5, Proposition 3.2] says that $\mathrm{NE}_2(X)$ is generated by torus invariant surfaces.

Reid's wall description of extremal rays of toric varieties tells us that there exist exactly two extremal wall relations

$$a_1x_1 + \cdots + a_mx_m = c_1y_1 + \cdots + c_{n-1}y_{n-1},$$

$$b_1y_1 + \cdots + b_ny_n = d_1x_1 + \cdots + d_{m-1}x_{m-1},$$

where $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$, $m, n \geq 2$, $m + n = d + 2$, $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{Q}_{>0}$, $c_1, \ldots, c_{n-1}, d_1, \ldots, d_{m-1} \in \mathbb{Q}_{\geq 0}$. Without loss of generality, we may assume that

$$0 \leq \frac{d_1}{a_1} \leq \frac{d_2}{a_2} \leq \cdots \leq \frac{d_{m-1}}{a_{m-1}} \text{ and } 0 \leq \frac{c_1}{b_1} \leq \frac{c_2}{b_2} \leq \cdots \leq \frac{c_{n-1}}{b_{n-1}}.$$

By a $\mathbb{R}$-basis $\{x_1, \ldots, x_{m-1}, y_1, \ldots, y_{n-1}\}$ for $N_\mathbb{R}$, we obtain linear relations

$$D_i - \frac{a_i}{a_m}D_m + \frac{d_i}{b_n}E_n = 0\ (1 \leq i \leq m-1),\ E_j - \frac{b_j}{b_n}E_n + \frac{c_j}{a_m}D_m = 0\ (1 \leq j \leq n-1)$$

in $\mathrm{N}^1(X)$, where D_i and E_j are the torus invariant prime divisors corresponding to x_i and y_j, respectively. First, we show the following:

Claim. *For any $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$, D_m and E_n are contained in the cone $\mathbb{R}_{\geq 0}D_i + \mathbb{R}_{\geq 0}E_j \subset \mathrm{N}^1(X)$.*

Proof of Claim. If $d_i = 0$, then we have $\frac{a_i}{a_m}D_m = D_i$. So, we may assume $d_i \neq 0$. By the above equalities, we have

$$\frac{b_j}{d_i}\left(D_i - \frac{a_i}{a_m}D_m + \frac{d_i}{b_n}E_n\right) + E_j - \frac{b_j}{b_n}E_n + \frac{c_j}{a_m}D_m = 0$$

$$\Longleftrightarrow \quad \frac{b_j}{d_i}D_i + E_j = \left(\frac{a_ib_j}{a_md_i} - \frac{c_j}{a_m}\right)D_m,$$

where $\frac{a_ib_j}{a_md_i} - \frac{c_j}{a_m}$ has to be positive since X is complete. The proof for E_n is completely similar. □

For $1 \leq i_1 < i_2 \leq m-1$ and $1 \leq j_1 < j_2 \leq n-1$, we have

$$\frac{a_m}{a_{i_1}}D_{i_1} = \frac{a_m}{a_{i_2}}D_{i_2} + \frac{a_m}{b_n}\left(\frac{d_{i_2}}{a_{i_2}} - \frac{d_{i_1}}{a_{i_1}}\right)E_n \text{ and } \frac{b_n}{b_{j_1}}E_{j_1} = \frac{b_n}{b_{j_2}}E_{j_2} + \frac{b_n}{a_m}\left(\frac{c_{j_2}}{b_{j_2}} - \frac{c_{j_1}}{b_{j_1}}\right)D_m.$$

These equalities mean that $D_{i_1} \in \mathbb{R}_{\geq 0}D_{i_2} + \mathbb{R}_{\geq 0}E_n \subset \mathrm{N}^1(X)$, while $E_{j_1} \in \mathbb{R}_{\geq 0}E_{j_2} + \mathbb{R}_{\geq 0}D_m \subset \mathrm{N}^1(X)$. Therefore, any 2-cycle $D_{i_1} \cdots D_{i_k} \cdot E_{j_1} \cdots E_{j_l}$ $(k < m,\ l < n,\ k+l = d-2)$ is contained in the cone generated by

$$D_p \cdots D_{m-1} \cdot E_q \cdots E_{n-1}\ (p \geq 1,\ q \geq 1,\ p+q = 4)$$

in $\mathrm{NE}_2(X)$. One can easily see that the possibilities for (p, q) are $(1, 3)$, $(2, 2)$ and $(3, 1)$. Thus, $\mathrm{NE}_2(X)$ is generated by the three 2-cycles

$$S_1 := D_1 \cdots D_{m-1} \cdot E_3 \cdots E_{n-1},\ S_2 := D_2 \cdots D_{m-1} \cdot E_2 \cdots E_{n-1},$$

$$\text{and } S_3 := D_3 \cdots D_{m-1} \cdot E_1 \cdots E_{n-1},$$

where $S_1 = 0$ (resp. $S_3 = 0$) if $n = 2$ (resp. $m = 2$). These 2-cycles are obtained by multiplying some torus invariant surfaces by positive rational numbers. □

By Theorem 2.1, in order to prove the positivity (resp. non-negativity) of $\gamma_2(X)$, it is sufficient to check the positivity (resp. non-negativity) for the above three 2-cycles. Furthermore, [10, Proposition 3.4] says that $\gamma_2(X) \cdot S_1 > 0$ and $\gamma_2(X) \cdot S_3 > 0$. So, only we have to do is to check the positivity (resp. non-negativity) for S_2. We remark that $\rho(S_2) = 2$. So, we can apply [10, Proposition 3.5]. We describe them here for the reader's convenience: Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial projective toric d-fold, and $S \subset X$ a torus invariant subsurface of $\rho(S) = 2$. Let $\tau \in \Sigma$ be a $(d-2)$-dimensional cone associated to S and $\tau \cap \mathrm{G}(\Sigma) = \{x_1, \ldots, x_{d-2}\}$. There exist exactly 4 maximal cones

$$\mathbb{R}_{\geq 0}y_1 + \mathbb{R}_{\geq 0}y_3 + \tau,\ \mathbb{R}_{\geq 0}y_2 + \mathbb{R}_{\geq 0}y_3 + \tau,\ \mathbb{R}_{\geq 0}y_1 + \mathbb{R}_{\geq 0}y_4 + \tau,\ \mathbb{R}_{\geq 0}y_2 + \mathbb{R}_{\geq 0}y_4 + \tau$$

in Σ, where $\{y_1, y_2, y_3, y_4\} \subset \mathrm{G}(\Sigma)$. Let

$$b_1y_1 + b_2y_2 + c_3y_3 + a_1x_1 + \cdots + a_{d-2}x_{d-2} = 0 \text{ and}$$

$$b_3y_3 + b_4y_4 + c_1y_1 + e_1x_1 + \cdots + e_{d-2}x_{d-2} = 0$$

be the wall relations corresponding to $(d-1)$-dimensional cones $\mathbb{R}_{\geq 0}y_3 + \tau$ and $\mathbb{R}_{\geq 0}y_1 + \tau$, respectively, where $a_1, \ldots, a_{d-2}, b_1, b_2, b_3, b_4, c_1, c_3, e_1, \ldots, e_{d-2} \in \mathbb{Q}$ and $b_1, b_2, b_3, b_4 > 0$. Then, the following holds:

Proposition 2.2 ([10, Proposition 3.4])**.** *There exists a positive rational number α such that*

$$\begin{aligned}\alpha\gamma_2(X) \cdot S = & -b_3c_1\left(b_1^2 + b_2^2 + c_3^2 + a_1^2 + \cdots + a_{d-2}^2\right) \\ & + 2b_1b_3\left(b_1c_1 + b_3c_3 + a_1e_1 + \cdots + a_{d-2}e_{d-2}\right) \\ & - b_1c_3\left(b_3^2 + b_4^2 + c_1^2 + e_1^2 + \cdots + e_{d-2}^2\right).\end{aligned}$$

3. Examples of γ_2-positive toric varieties

We need the following lemma to explain the singularities in the examples below.

Lemma 3.1. *Let $d \geq 3$ and $e_1, \ldots, e_d$ the standard basis for N. Put*

$$x_1 := e_1, \ldots, x_{d-1} := e_{d-1}, x_d := ce_d - \sum_{i=p}^{d-1} e_i,$$

where $1 \leq p \leq d-1$, $c \in \mathbb{Z}$ and $0 < c < d-p+1$. Then, the cone $\mathbb{R}_{\geq 0}x_1 + \cdots + \mathbb{R}_{\geq 0}x_d \subset N_{\mathbb{R}}$ is terminal.

Proof. The hyperplane passing through $x_1, \ldots, x_d$ is

$$\left\{ (t_1, \ldots, t_d) \in N_{\mathbb{R}}^d \;\middle|\; t_1 + \cdots + t_{d-1} + \frac{d-p+1}{c} t_d = 1 \right\}.$$

For $(a_1, \ldots, a_d) \in \mathbb{Q}_{\geq 0}^d$, suppose that

$$x := a_1 x_1 + \cdots + a_d x_d = a_1 e_1 + \cdots + a_{p-1} e_{p-1} + (a_p - a_d) e_p + \cdots + (a_{d-1} - a_d) e_{d-1} + c a_d e_d \in \mathbb{Z}^d$$

and that

$$a_1 + \cdots + a_{p-1} + (a_p - a_d) + \cdots + (a_{d-1} - a_d) + \frac{d-p+1}{c} \times c a_d = a_1 + \cdots + a_d \leq 1.$$

If $a_i = 1$ for $1 \leq i \leq d$, then $x = x_i$. So we may assume $a_1, \ldots, a_d < 1$. Then, since $a_1, \ldots, a_{p-1} \in \mathbb{Z}$, $a_1 = \cdots = a_{p-1} = 0$. So, we have $0 \leq a_p + \cdots + a_d \leq 1$. For any $p \leq i \leq d-1$, we have $-1 < a_i - a_d < 1$. However, $a_i - a_d \in \mathbb{Z}$ means that $a_i - a_d = 0$. If $a_d \neq 0$, then $c a_d \geq 1$ holds because $c a_d \in \mathbb{Z}$. This is impossible, since

$$a_p + \cdots + a_d = (d-p+1) \times a_d \geq \frac{d-p+1}{c} > 1.$$

Therefore, $a_p = \cdots = a_d = 0$. Thus, $x \in \{x_1, \ldots, x_d, 0\}$. □

The following is an answer to Question 1.2. Moreover, this is a counterexample to Question 1.3, too.

EXAMPLE 3.2. Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial terminal toric Fano 4-fold such that the primitive generators of 1-dimensional cones in Σ are

$$x_1 = (1,0,0,0),\; x_2 = (0,1,0,0),\; x_3 = (0,0,1,0),$$

$$x_4 = (0,0,0,1), x_5 = (-1,-2,-1,0), x_6 = (0,-1,-2,-1).$$

The singular locus of X is $S_{1,5} \cup S_{4,6}$, where $S_{1,5}$ and $S_{4,6}$ are the torus invariant surfaces corresponding to $\mathbb{R}_{\geq 0} x_1 + \mathbb{R}_{\geq 0} x_5$ and $\mathbb{R}_{\geq 0} x_4 + \mathbb{R}_{\geq 0} x_6$, respectively. One can easily see that X is terminal by Lemma 3.1. The extremal wall relations of Σ are

$$2x_1 + 3x_2 + 2x_5 = x_4 + x_6 \text{ and } 3x_3 + 2x_4 + 2x_6 = x_1 + x_5.$$

Let $D_1, \ldots, D_6$ be the torus invariant prime divisors corresponding to $x_1, \ldots, x_6$, respectively. Theorem 2.1 tells us that it is sufficient to show the positivity for $D_5 D_6$. The wall relations associated to $\mathbb{R}_{\geq 0} x_1 + \mathbb{R}_{\geq 0} x_5 + \mathbb{R}_{\geq 0} x_6$ and $\mathbb{R}_{\geq 0} x_3 + \mathbb{R}_{\geq 0} x_5 + \mathbb{R}_{\geq 0} x_6$ are

$$3x_3 + 2x_4 - x_1 - x_5 + 2x_6 = 0 \text{ and } x_1 + 2x_2 + x_3 + x_5 = 0,$$

respectively. By Proposition 2.2, there exists a positive rational number α such that

$$\alpha \gamma_2(X) \cdot D_5 D_6 = -1 \times 1 \times (3^2 + 2^2 + (-1)^2 + (-1)^2 + 2^2) + 2 \times 3 \times 1 \times (3 \times 1 + 1 \times (-1) + (-1) \times 1)$$

$$-3 \times (-1) \times (1^2 + 2^2 + 1^2 + 1^2) = 8 > 0.$$

Therefore, X is γ_2-positive, but $\rho(X) = 2$. We should remark that $\mathrm{G}(\Sigma)$ has no centrally symmetric pair.

For any dimension $d \geq 4$, there exists a toric d-fold satisfying the condition of Question 1.2:

Example 3.3. Let $d \geq 4$ and $\{e_1, \ldots, e_d\}$ the standard basis for $N = \mathbb{Z}^d$. Put

$$x_1 := e_1, \ \ldots, \ x_{d-2} := e_{d-2}, \ x_{d-1} := -\left(e_1 + \cdots + e_{d-2} + (d-2)e_{d-1}\right), \ x_d := e_{d-1},$$

$$y_1 := -(e_{d-1} + e_d), \ y_2 = e_d.$$

Let $X = X_\Sigma$ be the $\mathbb{Q}$-factorial terminal toric Fano d-fold of $\rho(X) = 2$ such that $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_d, y_1, y_2\}$. The singular locus of X is the torus invariant curve corresponding to the cone $\mathbb{R}_{\geq 0}x_1 + \cdots + \mathbb{R}_{\geq 0}x_{d-1}$. One can easily confirm that this singularity is terminal by Lemma 3.1. The extremal wall relations of Σ are

$$x_1 + \cdots + x_{d-1} + (d-2)x_d = 0 \text{ and } (d-2)y_1 + (d-2)y_2 = x_1 + \cdots + x_{d-1}.$$

By Theorem 2.1, all we have to do is to show $\gamma_2(X) \cdot D_2 \cdots D_{d-1} > 0$, where $D_1, \ldots, D_d, E_1$, E_2 are the torus invariant prime divisors corresponding to $x_1, \ldots, x_d, y_1, y_2$, respectively. The wall relations associated to

$$\mathbb{R}_{\geq 0}x_1 + \mathbb{R}_{\geq 0}x_2 + \cdots + \mathbb{R}_{\geq 0}x_{d-1} \text{ and } \mathbb{R}_{\geq 0}y_1 + \mathbb{R}_{\geq 0}x_2 + \cdots + \mathbb{R}_{\geq 0}x_{d-1}$$

are

$$(d-2)y_1 + (d-2)y_2 - x_1 - x_2 - \cdots - x_{d-1} = 0 \text{ and } x_1 + (d-2)x_d + x_2 + \cdots + x_{d-1} = 0,$$

respectively. Proposition 2.2 says that for $\alpha \in \mathbb{Q}_{>0}$, we have

$$\begin{aligned}
&\alpha\gamma_2(X) \cdot D_2 \cdots D_{d-1} \\
&= 2 \times (d-2) \times 1 \times (-1) \times (d-1) - (d-2) \times (-1) \times \left(1^2 + (d-2)^2 + 1^2 \times (d-2)\right) \\
&= (d-2)^3 - (d-2)(d-1) = (d-2)((d-3)^2 + (d-4)) > 0.
\end{aligned}$$

Thus, X is γ_2-positive. Moreover, $\mathrm{G}(\Sigma)$ has no centrally symmetric pair in this case, too.

Next, we consider Question 1.2 for *Gorenstein* $\mathbb{Q}$-factorial projective toric d-folds. We remark that there exists a counterexample to Question 1.3 in this situation (see [10, Remark 5.7]).

The following is the answer to Question 1.2 for $d = 2$.

Theorem 3.4. *Let S be a Gorenstein projective toric surface. Then, S is γ_2-positive if and only if $\rho(S) = 1$.*

Proof. If S is nonsingular, then the statement is obviously true (for example, see [9, Proposition 4.3]).

Suppose $\rho(S) \geq 2$. Only we have to do is to show that S is not γ_2-positive.

First, we remark that for a blow-up $\psi : S_1 \to S_2$ between smooth projective toric surfaces S_1 and S_2, we have $\gamma_2(S_2) - \gamma_2(S_1) = 3$.

Next, we investigate primitive crepant contractions. So, let $\psi : S_1 \to S_2$ be a toric morphism between Gorenstein projective toric surfaces S_1 and S_2 such that $\mathrm{G}(\Sigma_{S_1}) = \mathrm{G}(\Sigma_{S_2}) \cup \{y\}$ and $ax_1 + bx_2 = qy$ for some 2-dimensional cone $\mathbb{R}_{\geq 0}x_1 + \mathbb{R}_{\geq 0}x_2 \in \Sigma_{S_2}$, where

a, b, q are coprime positive integers and $x_1, x_2 \in \mathrm{G}(\Sigma_{S_2})$. Then, [10, Proposition 4.2] says that

$$\gamma_2(S_1) = \gamma_2(S_2) - \frac{a^2 + b^2 + q^2}{abq}.$$

Since ψ is crepant if and only if $a + b = q$, this equality is equivalent to

$$\gamma_2(S_2) - \gamma_2(S_1) = \frac{a^2 + b^2 + (a+b)^2}{ab(a+b)} = 2\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{a+b}\right).$$

Put

$$f(a, b) := \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{a+b}\right).$$

Then,

$$\begin{aligned} f(a+1, b) - f(a, b) &= \left(\frac{1}{a+1} - \frac{1}{a}\right) - \left(\frac{1}{a+b+1} - \frac{1}{a+b}\right) \\ &= -\frac{1}{a(a+1)} + \frac{1}{(a+b)(a+b+1)} < 0. \end{aligned}$$

This means that $f(a, b)$ takes the maximum value at $(a, b) = (1, 1)$. Thus, we have

$$\gamma_2(S_2) - \gamma_2(S_1) \le \frac{1^2 + 1^2 + 2^2}{1 \times 1 \times 2} = 3.$$

There exists the crepant resolution $\pi : \overline{S} \to S$ which is a finite succession of primitive crepant contractions as above. On the other hand, there exists a toric morphism $\varphi : \overline{S} \to S'$ which is a finite succession of blow-ups such that S' is a smooth projective toric surface of $\rho(S') = \rho(S)$. Thus, we have

$$\gamma_2(S') - \gamma_2(\overline{S}) = 3\left(\rho(\overline{S}) - \rho(S')\right),$$

while

$$\gamma_2(S) - \gamma_2(\overline{S}) \le 3\left(\rho(\overline{S}) - \rho(S)\right).$$

Therefore, $\gamma_2(S) \le \gamma_2(S') \le 0$, that is, S is not γ_2-positive. □

However, there exists a Gorenstein $\mathbb{Q}$-factorial projective γ_2-positive toric 3-fold X of $\rho(X) = 2$:

Example 3.5. Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial Gorenstein toric Fano 3-fold such that the primitive generators of 1-dimensional cones in Σ are

$$x_1 = (1, 0, 0),\ x_2 = (0, 1, 0),\ x_3 = (0, 0, 1),\ x_4 = (0, -2, -1),\ x_5 = (-1, -1, 0).$$

The singular locus of X is the torus invariant curve corresponding to the cone $\mathbb{R}_{\ge 0}x_3 + \mathbb{R}_{\ge 0}x_4$. The hyperplane passing through x_1, x_3, x_4 and x_3, x_4, x_5 are

$$\left\{(t_1, t_2, t_3) \in N^3_{\mathbb{R}} \mid t_1 - t_2 + t_3 = 1\right\} \text{ and } \left\{(t_1, t_2, t_3) \in N^3_{\mathbb{R}} \mid -t_2 + t_3 = 1\right\},$$

respectively. Thus, X is Gorenstein. There exist exactly two extremal wall relations

$$2x_1 + 2x_5 = x_3 + x_4 \text{ and } 2x_2 + x_3 + x_4 = 0.$$

Let $D_1, \ldots, D_5$ be the torus invariant prime divisors corresponding to $x_1, \ldots, x_5$, respectively. By Theorem 2.1, it is sufficient to check the positivity for D_4. The wall relations associated to $\mathbb{R}_{\geq 0}x_1 + \mathbb{R}_{\geq 0}x_4$ and $\mathbb{R}_{\geq 0}x_2 + \mathbb{R}_{\geq 0}x_4$ are

$$2x_2 + x_3 + x_4 = 0 \text{ and } x_1 + x_5 + x_2 = 0,$$

respectively. By Proposition 2.2, there exists a positive rational number α such that

$$\alpha\gamma_2(X) \cdot D_4 = -1 \times 1 \times (2^2 + 1^2 + 1^2) + 2 \times 2 \times 1 \times (2 \times 1) - 2 \times 0 \times (1^2 + 1^2 + 1^2) = 2 > 0.$$

Therefore, X is γ_2-positive, but $\rho(X) = 2$.

Acknowledgements. The first author was partly supported by JSPS KAKENHI Grant Number JP18K03262. The second author was partly supported by JSPS KAKENHI Grant Number JP18J00022.

References

[1] D.A. Cox, J.B. Little and H.K. Schenck: Toric varieties, Graduate Studies in Mathematics **124**, American Mathematical Society, Providence, RI, 2011.
[2] O. Fujino and H. Sato: *Introduction to the toric Mori theory*, Michigan Math. J. **52** (2004), 649–665.
[3] W. Fulton: Introduction to toric varieties, Annals of Mathematics Studies **131**, The William H. Roever Lectures in Geometry, Princeton University Press, Princeton, NJ, 1993.
[4] K. Matsuki: Introduction to the Mori program, Universitext, Springer-Verlag, New York, 2002.
[5] E. Nobili: *Classification of Toric 2-Fano 4-folds*, Bull. Braz. Math. Soc., New Series **42** (2011), 399–414.
[6] T. Oda: Convex bodies and algebraic geometry, An introduction to the theory of toric varieties, Translated from the Japanese, Results in Mathematics and Related Areas (3) **15**, Springer-Verlag, Berlin, 1988.
[7] M. Reid: *Decomposition of toric morphisms*; in Arithmetic and geometry, Vol. II, 395–418, Progr. Math. **36**, Birkhäuser Boston, Boston, MA, 1983.
[8] H. Sato: *The numerical class of a surface on a toric manifold*, Int. J. Math. Math. Sci. 2012, 9 pp.
[9] H. Sato: *Toric 2-Fano manifolds and extremal contractions*, Proc. Japan Acad. Ser. A Math. Sci. **92** (2016), 121–124.
[10] H. Sato and R. Sumiyoshi: *Terminal toric Fano three-folds with certain numerical conditions*, to appear in Kyoto J. Math., arXiv:1806.03784.

Hiroshi Sato
Department of Applied Mathematics
Faculty of Sciences
Fukuoka University
8–19–1, Nanakuma, Jonan-ku
Fukuoka 814–0180
Japan
e-mail: hirosato@fukuoka-u.ac.jp

Yusuke Suyama
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560–0043
Japan
e-mail: y-suyama@cr.math.sci.osaka-u.ac.jp