

# SEMISTABLE FIBRATIONS OVER AN ELLIPTIC CURVE WITH ONLY ONE SINGULAR FIBRE

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## Abstract

In this work we describe a construction of semistable fibrations over an elliptic curve with one unique singular fibre and we give effective examples using monodromy of curves.

## 1. Introduction

Let  $X$  be a compact smooth complex algebraic surface. A fibration of  $X$  is a morphism with connected fibres  $\phi : X \rightarrow B$ , where  $B$  is a smooth curve. It is well known that if  $g(B) \leq 1$  and the fibration is not isotrivial (i.e. such that not all smooth fibres are isomorphic) then it has at least one singular fibre (cf. [10, Théorème 4], [9]).

In this short note we describe a construction of semistable fibrations over an elliptic curve with one unique singular fibre and we give effective examples (section 2). We also establish some general properties of such fibrations (section 1). Note that the existence of these semistable fibrations contrasts with Remark 3 in ([9]) which states that any fibration over an elliptic curve has at least two singular fibres. This claim is said to be an immediate consequence of arguments used in the proof of Theorem 4 of ([9]). However it is not clear how those arguments are used. On the other hand there exist fibrations of genus 3 over an elliptic curve with a unique singular reduced fibre (cf. [7]).

Our construction starts from constructing certain ramified (non Galois) covers  $C \rightarrow E$  of elliptic curves using monodromy and showing that under suitable hypothesis the surface  $C \times C$  has a fibration  $\phi : C \times C \rightarrow E$  as required.

**Notations and conventions.** We work over the complex numbers.  $\sim$  denotes numerical equivalence of divisors, whilst  $\equiv$  denotes linear equivalence. The composition of permutations is done right to left.

## 2. Generalities for a fibration over an elliptic curve with one unique singular fibre

Let  $E$  be an elliptic curve with origin  $O_E \in E$ , and let  $K_E$  be the canonical bundle of  $E$ . Let  $X$  be a compact smooth complex surface and  $\phi : X \rightarrow E$  be a relatively minimal semistable fibration with general fibre  $F$  a curve of genus  $g \geq 2$ .

Note that  $X$  is a minimal surface, since any rational curve must be vertical. By the same reason  $X$  is not birational to a ruled surface.

Let  $\omega_{X|E}$  be the relative dualizing sheaf, and let  $K_X$  be the canonical divisor of  $X$ . De-

note  $\Delta(\phi) = \deg(\phi_*(\omega_{X|E}))$ . Since  $K_E$  is trivial, the relative canonical divisor  $K_\phi$  is linearly equivalent to  $K_X$ , that is,

$$K_\phi \equiv K_X - \phi^* K_E \equiv K_X.$$

Since  $E$  is of genus one, one has  $\Delta(\phi) = \chi(\mathcal{O}_X)$  and  $K_{X|E}^2 = K_X^2$ . Set  $\chi := \chi(\mathcal{O}_X)$  and let  $c_2 := c_2(X) = \chi_{\text{top}}(X)$  be the topological Euler characteristic of  $X$ . Remark that for a semistable fibration over an elliptic curve  $c_2$  is exactly the number of nodes of the singular fibres.

Assume also that the general fibres of  $\phi$  are not all isomorphic (since  $X$  is minimal and the fibration is semistable this is the same as saying that the fibration is not trivial). Then, by Arakelov's theorem (see [2]),  $K_X^2 > 0$ . Since  $X$  is not a rational surface, by the classification of surfaces we conclude that  $X$  is of general type and hence  $\chi \geq 1$ .

Define, as usual,  $q = h^1(\mathcal{O}_X)$  and  $p_g = h^0(K_X)$  to be respectively the irregularity and the geometric genus of  $X$ .

By [2],  $1 \leq q < g+1$ , where the strict inequality comes from the fact that we are assuming that the fibration is not trivial. We recall the sharpening of Vojta's inequalities by Tan (see [11], Theorem 2 and Lemma 3.1), that in our case (i.e.  $\phi : X \rightarrow E$  relatively minimal, not trivial and semistable) says that when the number  $s$  of singular fibres is positive then

$$\chi < \frac{g}{2}s \quad \text{and} \quad K_X^2 < (2g-2)s.$$

Finally we recall Beauville's formula for a reducible semistable fibre  $F_0$  (see the proof of Lemme 1 of [1]):

$$n = g - g(N) + c - 1$$

where  $g$  is the genus of the general fibre,  $N$  is the normalisation of  $F_0$ ,  $g(N) = h^1(N, \mathcal{O}_N)$ ,  $c$  is the number of components of  $F_0$  and  $n$  is the number of double points of  $F_0$ .

**Proposition 2.1.** *Let  $X$  be a compact smooth complex surface,  $E$  an elliptic curve and  $\phi : X \rightarrow E$  a semistable fibration with general fibre  $F$  of genus  $g$  and with one unique singular fibre  $F_0$ . Then:*

- i)  $K_X^2 < 2g - 2$ ;
- ii)  $\chi < \frac{g}{2}$ ;
- iii)  $c_2 < 4g + 2$ ;
- iv) if  $\phi$  is stable,  $c_2 \leq 3g - 3$ ;
- v) if  $\phi$  is stable and  $c_2 = 3g - 3$ , then  $q = 1$  and  $\phi$  is the Albanese map;
- vi) if  $\phi$  is stable,  $2 < \frac{12\chi}{5} < g - 1$ .

*In particular if  $\phi$  is stable,  $g \geq 4$ .*

*Proof.* If  $\phi$  is semistable with one unique singular fibre, the inequalities  $K_X^2 < 2g - 2$  and  $\chi < \frac{g}{2}$  follow from the above mentioned sharpening of Vojta's inequalities. The inequality  $c_2 < 4g + 2$  is the proposition of [3].

Let  $F_0$  be the unique singular fibre. Note that  $c_2$  is exactly the number  $n$  of double points of  $F_0$ . If  $\phi$  is stable, every component  $\theta$  of  $F_0$  satisfies  $K_X\theta \geq 1$ . Since  $K_X F_0 = 2g - 2$ , the

number  $c$  of components of  $F_0$  is at most  $2g - 2$ . Using the above mentioned Beauville's formula we obtain

$$c_2 = g - g(N) + c - 1 \leq g - g(N) + 2g - 2 - 1 \leq 3g - 3 - g(N)$$

and so  $c_2 \leq 3g - 3$ .

If equality holds we conclude that  $c = 2g - 2$  and that  $g(N) = 0$ , i.e. every component of  $F_0$  is a rational curve. But then the Albanese map of  $X$  contracts  $F_0$  to a point and therefore also all fibres of  $\phi$ . Hence  $\phi$  is the Albanese map of  $X$  and  $q = 1$ .

Also, we obtain  $12\chi = c_2 + K_X^2 < 5(g - 1)$  and so

$$2 < \frac{12\chi}{5} < g - 1,$$

where the first inequality comes from the fact that  $X$  has  $\chi \geq 1$ . □

**REMARK 2.1.** Note that if  $\phi$  is stable with one unique singular fibre and  $g = 4$ , then  $\chi = 1$ ,  $K_X^2 \leq 5$  and  $c_2 \leq 9$ . Now  $c_2 = 9$  means  $K_X^2 = 3$ . From Debarre's inequality, [6],  $K_X^2 \geq 2p_g$  for irregular surfaces we conclude that  $q = 1$  and  $\phi$  is the Albanese fibration. But (see [5]) for  $K_X^2 = 3$ ,  $p_g = q = 1$  the genus of the general fibre of the Albanese fibration is 2 or 3. Therefore we have  $4 \leq K_X^2 \leq 5$ .

We also remark that  $\chi \geq 2$  implies  $g \geq 6$ .

### 3. Covering maps and semistable fibrations.

Let  $E$  be an elliptic curve and let  $O_E \in E$  be the origin. Let  $C$  be a smooth curve of genus  $g_C$  and let  $f : C \rightarrow E$  be a covering map of degree  $d$ . We have the following

**Construction 3.1.** Set  $S = C \times C$ , and for any map  $f : C \rightarrow E$ , let  $\phi : S \rightarrow E$

$$\phi(a, b) = f(a) - f(b)$$

be the difference map.

Despite the fact that  $S$  is a very simple surface and that the construction is very elementary, the map  $\phi$  can be interesting. We start by giving a condition that assures the connectedness of the fibres of  $\phi$ .

**Proposition 3.1.** Assume that the map  $f : C \rightarrow E$  is primitive of degree  $d$  and  $g_C > 1$ , (that is,  $f$  is not decomposable and not étale). Then the fibres of  $\phi : S \rightarrow E$  are connected and the general fibre  $F$  of the fibration  $\phi$  has genus  $g(F) = 2(g_C - 1)d + 1$ .

*Proof.* We have to show that  $\phi$  has connected fibres. Suppose by contradiction that the general fibre  $F$  of  $\phi$  is not connected. By the Stein factorization theorem, there exists a smooth curve  $Y$ , a finite morphism  $h : Y \rightarrow E$ ,  $\deg h > 1$ , and a morphism  $\tilde{\phi} : S \rightarrow Y$  with connected fibres such that the following diagram is commutative:

$$\begin{array}{ccc} S & \xrightarrow{\phi} & E \\ & \searrow \tilde{\phi} & \uparrow h \\ & & Y. \end{array}$$

Fix a point  $q \in C$  such that  $f(q) = O_E$ , consider the inclusion  $i_q : C \rightarrow C \times C$ ,  $i_q(p) = (p, q)$ , and let  $C_1 := i_q(C)$ . We have  $\phi \circ i_q = f$ . Setting  $\kappa = \tilde{\phi} \circ i_q$ , we get a decomposition  $f = \kappa \circ h$ , but since  $f$  is not decomposable,  $\deg \kappa = 1$  and  $\deg h = d$ . It follows that  $Y$  is isomorphic to  $C$ . Moreover, for the general fibre of  $\phi$  one has

$$F = \sum_i^d F_i$$

and therefore numerically  $F \sim d\tilde{F}$ ,  $\tilde{F} = F_1$ . Since  $F \cdot C_1 = d$  we get  $\tilde{F} \cdot C_1 = 1$ . By symmetry we also have  $\tilde{F} \cdot C_2 = 1$ , where  $C_2 = \{q\} \times C$ .

Since  $K_S \sim (2g_C - 2)C_1 + (2g_C - 2)C_2$ , and  $\tilde{F}^2 = 0$  we obtain by the adjunction formula  $2g(\tilde{F}) - 2 = 4g_C - 4$ , i.e.  $g(\tilde{F}) = 2g_C - 1$ . But on the other hand  $\tilde{F} \cdot C_j = 1$  implies, by considering one of the projections onto  $C$  of  $C \times C$  that  $g(\tilde{F}) = g_C$ , a contradiction.  $\square$

We want to consider now the case when the map  $f : C \rightarrow E$  has a unique branch point, which will assume to be  $O_E \in E$ .

**Proposition 3.2.** *Assume that  $f : C \rightarrow E$  is primitive and has only  $O_E$  as a critical value. Let  $F_0 = \phi^{-1}(O_E)$  be the fibre over the origin. Then:*

- i) *The map  $\phi$  is smooth outside  $F_0$ ;*
- ii)  *$F_0$  decomposes as  $F_0 = \Gamma + \Delta$ , where  $\Delta \subset S$  is the diagonal and  $\Gamma = F_0 - \Delta$ ;*
- iii)  *$\Delta \cdot \Gamma = 2g_C - 2$ .*
- iv) *If the ramification divisor of  $f$  is reduced, then  $F_0$  has only nodes as singularities and  $\phi$  is a stable fibration with only one singular fibre.*

*Proof.* The only critical value of  $f$  is the origin  $O_E$ . Let  $\eta \in H^0(E, K_E)$ ,  $\eta \neq 0$ , be a generator, then  $\phi^*(\eta) = (\eta_1, \eta_2) = (f^*(\eta), -f^*(\eta))$ . The critical locus  $C(\phi)$  of  $\phi$  is then defined by the vanishing of  $\phi^*(\eta)$ , that is,

$$C(\phi) := \{(p, q) \in S : p, q \in \text{zero locus of } f^*(\eta)\}.$$

In particular,  $C(\phi) \subset F_0 = \phi^{-1}(O_E)$  and therefore  $O_E$  is the only critical value of  $\phi$ . Since  $\Delta \subset F_0$ ,  $\Delta \cdot F_0 = 0$  and so

$$\Delta \cdot \Gamma = \Delta \cdot (F_0 - \Delta) = -\Delta^2 = 2g_C - 2.$$

Suppose that the ramification divisor of  $f$  is reduced. Then in any ramification point  $p_j \in C$  of  $f$ ,  $f$  locally is the map  $z \rightarrow z^2$ . So, for  $(p_1, p_2) \in C(\phi)$ , we can find local coordinates  $(z_i, U_{p_i})$ ,  $i = 1, 2$ , on  $C$  such that  $z_i(p_i) = 0$ , and a local coordinate  $(t, W)$  on  $E$ ,  $O_E \in W$  and  $t(O_E) = 0$ , such that locally  $f$  and  $\phi$  are given by

$$f(z_i) = z_i^2, \quad \phi(z_1, z_2) = z_1^2 - z_2^2.$$

Since  $F_0 \cap U_1 \times U_2$  around  $(p_1, p_2)$  is given by  $z_1^2 - z_2^2 = 0$ , this proves that the singularities of  $F_0$  are nodes. Since  $S$  does not contain any rational curve,  $\phi$  is a stable fibration.  $\square$

In view of Proposition 3.2, to obtain with this construction explicit examples of semistable fibrations over an elliptic curve with one unique singular fibre, we need to find primitive

covers  $f : C \rightarrow E$  such that  $O_E$  is the only branch point and the ramification divisor  $R$  of  $f$  is reduced.

We recall (see [8, Ch. III]) that given a smooth complex irreducible projective curve  $E$ , a finite subset  $B$  of  $E$  and any point  $x_0 \in E - B$ , there is a one-to-one correspondence between the set of isomorphism classes of covering maps  $f : C \rightarrow E$  of degree  $d$  whose branch points lie in  $B$  and the set of group homomorphisms  $\rho : \pi_1(E - B, x_0) \rightarrow S_d$  with transitive image (up to conjugacy in  $S_d$ ). The covering map is primitive if  $\text{Im}(\rho)$  is a primitive subgroup of  $S_d$ .

In the case at hand, i.e. where  $E$  is an elliptic curve  $E$  and  $B = O_E$  is a single point, the fundamental group  $\Pi_1 := \pi_1(E - \{O_E\}, x_0)$  is a free group on two generators  $x, y$ . The monodromy homomorphism of groups  $\rho : \Pi_1 \rightarrow S_d$  is given by  $\rho(x) = \alpha, \rho(y) = \beta$  and the ramification type of the covering  $f : C \rightarrow E$  is encoded in the decomposition of the commutator  $[\alpha, \beta]$  in  $S_d$  as a product of disjoint cycles. The ramification divisor  $R$  of  $f$  is reduced if and only if every cycle in such a decomposition has length 2.

So to give explicit examples it is enough to find transitive and primitive subgroups  $G \subset S_d$  generated by two permutations  $\alpha, \beta$ , such that  $[\alpha, \beta]$  is a product of disjoint transpositions. We give here some examples.

EXAMPLES:

1) Consider the subgroup  $G \subset S_4$  generated by the permutations  $\alpha, \beta$ , where:

$$\alpha = (1, 2, 3), \beta = (2, 3, 4).$$

Then  $G$  is obviously a transitive subgroup and

$$[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1} = (1, 4)(2, 3)$$

Since the order of a block divides 4, if  $G$  were not primitive, we could have only two blocks preserved. On the other hand  $\alpha$  has order 3 and therefore  $\alpha$  cannot preserve or interchange blocks.

In this case,  $C$  is a curve of genus 2 and the general fibre of  $\phi$  has genus 9 (cf. Proposition 3.1).

2) This example is due to Pietro Corvaja.

Consider the subgroup  $G \subset S_8$  generated by the permutations  $\alpha, \beta$ , where:

$$\alpha = (1, 2, 3, 4, 5, 6, 7), \quad \beta = (8, 3, 4, 1, 5, 6),$$

then we have

$$[\alpha, \beta] = (1, 5)(2, 6)(3, 4)(7, 8).$$

Since  $G$  is isomorphic to  $PGL_2(F_7)$ ,  $G$  is transitive and primitive. In this case,  $C$  is a curve of genus 3 and the general fibre of  $\phi$  has genus 33 (cf. Proposition 3.1).

3) Consider the subgroup  $G \subset S_d$ ,  $d = 4n + 1$ ,  $n \geq 2$  generated by the permutations  $\alpha, \beta$ , where:

$$\alpha = (1, 2)(3, 4) \cdots (2n - 1, 2n), \beta = (1, 2n + 1)(2, 2n + 2, \cdots 2n, 4n, 4n + 1)$$

Then

$$[\alpha, \beta] = (1, 2)(3, 4) \cdots (4n - 1, 4n).$$

Again  $G$  is clearly a transitive subgroup of  $S_d$ . Let  $\gamma = (1, 2n + 1)$  and  $\delta = (2, 2n + 2, \cdots 2n, 4n, 4n + 1)$ . Then  $\beta = \gamma\delta = \delta\gamma$  and furthermore  $\beta^{4n-1} = \gamma$ . Hence  $G = \langle \alpha, \beta \rangle$  contains  $\gamma$  and  $\delta$ .

Let  $B$  be a block preserved by  $G$  containing 1.  $B$  has cardinality  $k > 1$ , because  $2n + 1 \in B$ . Since  $k$  must divide  $4n + 1$ ,  $k$  is odd and so  $B$  contains another element  $x$ ,  $x \neq 1$ ,  $x \neq 2n + 1$ . Since  $x$  is a fixed element for  $\gamma$ ,  $\gamma(B) = B$ . Similarly, since 1 is a fixed element for  $\delta$ ,  $\delta(B) = B$ . So also  $\beta(B) = B$  and therefore  $B = \{1, 2, \dots, 4n + 1\}$ , i.e.  $G$  is primitive.

Here for each  $n$ ,  $g(C) = n + 1$  and the general fibre of  $\phi$  has genus  $2nd + 1$  (cf. Proposition 3.1).

**REMARK.** We finally remark that recently, motivated by the theory of the mapping class group, Pietro Corvaja and Fabrizio Catanese constructed several new examples.

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