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REMARKS ON FÖLLMER'S PATHWISE ITÔ CALCULUS

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Abstract

We extend some results about Föllmer's pathwise Itô calculus that have only been derived for continuous paths to càdlàg paths with quadratic variation. We study some fundamental properties of pathwise Itô integrals with respect to càdlàg integrators, especially associativity and the integration by parts formula. Moreover, we study integral equations with respect to pathwise Itô integrals. We prove that some classes of integral equations, which can be explicitly solved in the usual stochastic calculus, can also be solved within the framework of Föllmer's calculus.

1. Introduction

Föllmer's pathwise Itô calculus originated in a seminal paper that introduced the notion of the quadratic variation of a deterministic càdlàg path and proved a non-probabilistic version of Itô's formula [15]. In recent years, some related works have appeared, addressing applications to mathematical finance. Most of these works only deal with continuous paths of quadratic variation while the results of the original paper include discontinuous cases. The main purpose of the present paper is to generalize some of these results so that they can be applied to càdlàg paths with quadratic variation.

Föllmer's pathwise Itô calculus is a deterministic counterpart of the classical Itô calculus. Its methodology is completely analytic and does not need any probabilistic assumptions. Therefore it can be regarded as a useful tool to study financial trading strategies under probability-free settings: see, for example, [17, 38, 9]. Moreover, this theory is understood as a method to construct stochastic integrals in a strictly pathwise manner. It can be applied to stochastic processes having finite quadratic variation. Such a class of processes is strictly wider than that of semimartingales. In this sense, Föllmer's pathwise Itô calculus enables us to extend stochastic integration theory beyond semimartingales.

There are several approaches to pathwise constructions of stochastic integration. We can refer to [3, 27, 46, 47, 34, 35, 28, 29, 43, 37, 14, 7, 6, 8, 1], for example. Among them, we think that Föllmer's approach is intuitively clear (especially from a financial application viewpoint) and needs only elementary arguments, which should be regarded as an advantage. The rough path theory, pioneered by [30] (see, also, [22] and [20]), should be considered as another important approach. Some authors have studied the relation between Föllmer's pathwise Itô calculus and rough path theory: see [35] and [19], for example.

Let us give an outline of Föllmer's pathwise Itô calculus. Let $\Pi = (\pi_n)_{n\in\mathbb{N}}$ be a sequence of partitions of $\mathbb{R}_{\geq 0}$ such that $|\pi_n| \to 0$ as $n \to \infty$. We say that a càdlàg path $X\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$

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has quadratic variation along Π if there exists a càdlàg increasing function $[X,X]$ such that for all $t \in \mathbb{R}_{\geq 0}$

(i) $\sum_{t_i \in \pi_n} (X_{t_{i+1}\wedge t} - X_{t_i \wedge t})^2$ converges to $[X,X]_t$ as $n \to \infty$,

(ii) $\Delta[X,X]_t = (\Delta X_t)^2$.

An $\mathbb{R}^d$-valued càdlàg path $X = (X^1, \dots, X^d)$ has quadratic variation if, for each i and j, the real-valued path $X^i + X^j$ has quadratic variation. In [15], it is proved that if X has quadratic variation, then for any $f \in C^2(\mathbb{R}^d)$ the path $t \mapsto f(X_t)$ satisfies Itô's formula. That is, in 1-dimensional case,

$$\begin{aligned} f(X_t) - f(X_0) = &\int_0^t f'(X_{s-})\,dX_s + \frac{1}{2}\int_0^t f''(X_{s-})\,d[X,X]_s \\ &+ \sum_{0<s\leq t} \left\{ \Delta f(X_s) - f'(X_{s-})\Delta X_s - \frac{1}{2} f''(X_{s-})(\Delta X_s)^2 \right\} \end{aligned}$$

holds for all $t \in \mathbb{R}_{\geq 0}$. Here, the first term of the right-hand side, which we call the Itô-Föllmer integral, is defined as the limit of a sequence of non-anticipative Riemann sums. Föllmer's theorem claims that the Itô-Föllmer integral $\int_0^t f'(X_{s-})\,dX_s$ exists and it satisfies the above formula.

The results of this paper are divided into two parts. Our first aim is to establish several calculation rules for Itô-Föllmer integrals. We slightly extend the theorem of Föllmer to a path of the form $t \mapsto f(A_t, X_t)$ with $f \in C^{1,2}(\mathbb{R}^m \times \mathbb{R}^d)$ and an m-dimensional path A of locally finite variation. Thus we find that for a path of the form $t \mapsto \nabla_x f(A_t, X_t)$ the Itô-Föllmer integral $\int_0^t \langle \nabla_x f(A_{s-}, X_{s-}), dX_s \rangle$ exists. We call a path of the form $\nabla_x f(A_t, X_t)$ an admissible integrand of X. Because a càdlàg path defined as the Itô-Föllmer integral of an admissible integrand has quadratic variation, we can consider the Itô-Föllmer integral by a path of this type. For this case, we prove that Itô-Föllmer integral satisfies the so-called associativity rule (Theorem 2.19). This was already proved by [38] for continuous integrators; we extend it to general càdlàg paths with quadratic variation. An integration by parts formula is then proved as an application of associativity and Föllmer's theorem (Corollary 2.21).

The next aim is to study integral equations with respect to Itô-Föllmer integrals. By using the calculation rules mentioned above, we can solve certain classes of integral equations. First, we solve linear integral equations: the solutions to the homogeneous linear equations are given by the Doleans-Dade exponentials (Proposition 3.1). The solutions to the inhomogeneous linear equations are constructed by using the result for the homogeneous case (Proposition 3.4), which is a re-interpretation of the result by [26] in our pathwise setting. These results are applied to compute a portfolio insurance strategy in finance (Proposition 3.15). Further, we solve a path-dependent equation, which is called the drawdown equation, introduced by [4]: their results are re-interpreted in our pathwise setting (Proposition 3.13).

Let us describe the structure of this paper. In Section 2, we study the basic properties of Itô-Föllmer integrals. In Section 2.1 we define a quadratic variation and show some propositions that will be used in following sections. Itô-Föllmer integrals and the associated Itô formula is introduced in Section 2.2. The associativity of Itô-Föllmer integrals and an integration by parts formula are proved in Section 2.3. Pathwise quadratic variations of semimartingales are discussed in Section 2.4. In Section 3, integral equations with respect to Itô-Föllmer integrals are studied. In Section 3.1, linear integral equations are explicitly solved. In Section 3.2, certain nonlinear integral equations are solved. In Section 3.3, draw-down equations, which are path-dependent equations, are studied. Applications to financial topics satisfying certain kinds of floor constraints are considered in Section 3.4.

Here we give some notation and terminology that are frequently used in this paper. We write $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_{\geq 1} = \{1, 2, 3, \dots\}$. The set of real numbers is denoted by $\mathbb{R}$, and we set $\mathbb{R}_{\geq 0} = [0, \infty[= \{r \in \mathbb{R} \mid r \geq 0\}$. For $x, y \in \mathbb{R}^d$, the standard Euclidean norm and inner product are denoted by $\|x\|$ and $\langle x, y\rangle$, respectively.

A function $X\colon \mathbb{R}_{\geq 0} \to E \subset \mathbb{R}^d$ is called an E-valued càdlàg path if it is right continuous and has left-hand limits in E at all $t \in \mathbb{R}_{\geq 0}$. The set of all E-valued càdlàg paths is denoted by $D(\mathbb{R}_{\geq 0}, E)$. For $X \in D(\mathbb{R}_{\geq 0}, E)$, we write $X_t = X(t)$ and $\Delta X_t = X_t - X_{t-}$. For convenience, we define $X_{0-} = X_0$ and $\Delta X_0 = 0$. Here note that $X \in D(\mathbb{R}_{\geq 0}, \mathbb{R}^d)$ satisfying $X_t \in E$ $(t \geq 0)$ is not necessarily regarded as an E-valued càdlàg path because X_{t-} can be in $\mathbb{R}^d \setminus E$ for some $t \in \mathbb{R}_{\geq 0}$. Moreover, let us recall that if X is a càdlàg path satisfying $\|\Delta X\| \leq c$ on $[0, t]$, then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|s - u| < \delta$ and $s, u \in [0, t]$ implies $\|X_s - X_u\| < \varepsilon + c$.

The symbol FV_{loc} denotes the set of all real-valued càdlàg paths of locally finite variation. The total variation of $A \in FV_{\mathrm{loc}}$ on $[0, t]$ is denoted by $V(A)_t$. If $A \in FV_{\mathrm{loc}}$, the series $\sum_{0<s\leq t} \Delta A_s$ converges absolutely for all $t \in \mathbb{R}_{\geq 0}$. Then $A^{\mathrm{d}}_t := \sum_{0<s\leq t} \Delta A_s$ and $A^{\mathrm{c}} := A - A^{\mathrm{d}}$ are called the purely discontinuous part and the continuous part of A, respectively. For $A \in FV_{\mathrm{loc}}$, we write $\int_0^t f(s)\, dA_s = \int_{]0,t]} f(s)\, dA_s$ when the Lebesgue-Stieltjes integral in the right-hand side is well-defined.

In this paper, we suppose that any partition $\pi = (t_i)_{i\in\mathbb{N}}$ of $\mathbb{R}_{\geq 0}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots \to \infty$. For a partition $\pi = (t_i)_{i\in\mathbb{N}}$, we define $|\pi| = \sup_i |t_{i+1} - t_i|$. We often identify the partition $\pi = (t_i)_{i\in\mathbb{N}}$ with the set $\{t_0, t_1, \dots\}$, and use set notation for partitions. For example, we write $t_i \in \pi$ or $\pi \subset \pi'$ for two partitions π and π'. Given a sequence of partitions $(\pi_n)_{n\in\mathbb{N}} = ((t^n_i)_{i\in\mathbb{N}})_{n\in\mathbb{N}}$ and an $\mathbb{R}^d$-valued càdlàg path X, we write $\delta^n_i X = X_{t^n_{i+1}} - X_{t^n_i}$ for convenience.

Given $E \subset \mathbb{R}^n$, a normed space V, and a function $f : E \to V$, we define

$$\omega(f; \varepsilon) = \sup\{\, \|f(y) - f(x)\|_V \mid x, y \in E,\ \|x - y\|_{\mathbb{R}^n} < \varepsilon \,\}.$$

If f is uniformly continuous, we have $\omega(f; \varepsilon) \to 0$ as $\varepsilon \to 0$.

Recall that a function $f\colon U \to \mathbb{R}^n$ defined on an open subset $U \subset \mathbb{R}^m \times \mathbb{R}^d$ is of $C^{k,l}$-class $(k, l \in \mathbb{N})$ if

(i) for each y, the map $x \mapsto f(x, y)$ is k-times continuously differentiable and all derivatives with respect to x are continuous in $(x, y) \in U$,
(ii) for each x, the map $y \mapsto f(x, y)$ is l-times continuously differentiable and all derivatives with respect to y are continuous in $(x, y) \in U$.

The symbol $C^{k,l}(U)$ stands for the space of real-valued $C^{k,l}$-class function on U. The first and the second order derivatives of f with respect to a variable x are denoted by $\nabla_x f$ and $\nabla^2_x f$, respectively, if they exist. We simply write ∇f for the first order derivative with respect to all the variables.

2. Quadratic variations and Itô-Föllmer integration

2.1. Definition of quadratic variation and its properties. In this subsection, we define the quadratic variation of a càdlàg path along a sequence of partitions, and we study some fundamental properties of it.

The discrete quadratic variation of a real-valued càdlàg path X along a partition π is defined by

$$[X,X]^{\pi}_t = \sum_{t_i \in \pi} (X_{t_{i+1}\wedge t} - X_{t_i \wedge t})^2 .$$

DEFINITION 2.1. Let $\Pi = (\pi_n)_{n\in\mathbb{N}}$ be a sequence of partitions of $\mathbb{R}_{\geq 0}$ such that $|\pi_n| \to 0$, and let $X : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a càdlàg path. We say that X *has quadratic variation along* Π if it satisfies the following conditions.

(i) The sequence of càdlàg paths $([X,X]^{\pi_n})_{n\in\mathbb{N}}$ converges pointwise to some càdlàg increasing function $[X,X]^{\Pi}$.

(ii) $\Delta[X,X]^{\Pi}_t = (\Delta X_t)^2$ holds for all $t \in \mathbb{R}_{\geq 0}$.

The increasing function $[X,X]^{\Pi}$ is called the *quadratic variation* of X along $\Pi = (\pi_n)$. We often omit the symbol Π and simply write $[X,X]$ if there is no ambiguity.

Note that if X has quadratic variation along Π, it satisfies $\sum_{0<s\leq t}(\Delta X_s)^2 < \infty$ for all $t \in \mathbb{R}_{\geq 0}$. The increasing property of $[X,X]^{\Pi}$ indeed follows from conditions (i) and (ii) of Definition 2.1. We can see it from Proposition 2.5 and the fact that the pointwise limit of a sequence of increasing functions is again increasing. A typical example of a quadratic variation is the quadratic variation of a semimartingale. We discuss this case in Section 3.1. Another example is a Dirichlet process: see [15, 16, 41]. Also, paths of certain Gaussian processes have quadratic variation: see, for example, [2]. The existence of quadratic variations under non-probabilistic settings is studied in [44, 45, 29, 21].

REMARK 2.2. In general, the dependence of the quadratic variation $[X,X]^{\Pi}$ on the choice of a sequence of partitions is inevitable. This problem is discussed in [10, Section 7].

DEFINITION 2.3. A d-dimensional càdlàg path $X = (X^1, \dots, X^d)$ has quadratic variation along Π if, for each $i, j \in \{1, \dots, d\}$, $X^i + X^j$ has quadratic variation along Π.

REMARK 2.4. In general, the existence of the quadratic variations of X and Y does not imply that of $X + Y$: see [39].

The symbol $QV(\Pi;\mathbb{R}^d)$ denotes the set of all $\mathbb{R}^d$-valued càdlàg paths that have quadratic variation along Π. We also write $QV(\Pi)$, or QV if $d = 1$. For $(X,Y) \in QV(\Pi;\mathbb{R}^2)$, we define

$$[X,Y]^{\Pi} = \frac{1}{2}\Big([X+Y,X+Y]^{\Pi} - [X,X]^{\Pi} - [Y,Y]^{\Pi}\Big).$$

The path $[X,Y]^{\Pi}$ is of locally finite variation, by definition. $[X,Y]^{\Pi}$ is called the quadratic covariation of X and Y along Π. For $X, Y \in QV(\Pi;\mathbb{R})$, the condition $(X,Y) \in QV(\Pi;\mathbb{R}^2)$ is clearly equivalent to the following two conditions.

(i) The function $t \mapsto \sum_{t_i\in\pi_n}(X_{t_{i+1}\wedge t} - X_{t_i\wedge t})(Y_{t_{i+1}\wedge t} - Y_{t_i\wedge t})$ converges pointwise to some càdlàg path $[X,Y]^{\Pi}$ of locally finite variation.

(ii) $\Delta[X,Y]^{\Pi}_t = \Delta X_t \Delta Y_t$ holds for all $t \in \mathbb{R}_{\geq 0}$.

Let δ_a be the Dirac measure on $\mathbb{R}_{\geq 0}$ concentrated at a, let X and Y be càdlàg paths, and let π be a partition of $\mathbb{R}_{\geq 0}$. We define locally finite measures μ^{π}_X and $\mu^{\pi}_{X,Y}$ by

$$\mu^{\pi}_{X,Y} = \sum_{t_i \in \pi} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})\, \delta_{t_i}$$

and $\mu^{\pi}_X = \mu^{\pi}_{X,X}$. Then,

$$\mu^{\pi}_{X,Y}([0,t]) = \sum_{t_i \in \pi \cap [0,t]} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}).$$

Proposition 2.5. *Let X and Y be real-valued càdlàg paths and let $\Pi = (\pi_n)_{n\in\mathbb{N}}$ be a sequence of partitions such that $|\pi_n| \to 0$. Then, for all $t \in \mathbb{R}_{\geq 0}$, the following two conditions are equivalent.*

(i) *The sequence $([X,Y]^{\pi_n}_t)_{n\in\mathbb{N}}$ converges.*
(ii) *The sequence $(\mu^{\pi_n}_{X,Y}([0,t]))_{n\in\mathbb{N}}$ converges.*

If these conditions are satisfied, both sequences have the same limit.

Proof. It suffices to give a proof when $X = Y$. If $t \in [t_i, t_{i+1}[$ and $t_i \in \pi_n$, we have

$$\begin{aligned} \left| [X,X]^{\pi_n}_t - \mu^{\pi_n}_X([0,t]) \right| &= \left| (X_{t_{i+1}} - X_{t_i})^2 - (X_t - X_{t_i})^2 \right| \\ &= \left| (X_{t_{i+1}} + X_t - 2X_{t_i})(X_{t_{i+1}} - X_t) \right| \\ &\leq 4 \sup_{s\in[0,t+|\pi_n|]} |X_s| \left| X_{t_{i+1}} - X_t \right|. \end{aligned} \tag{2.1}$$

Combining the assumption $|\pi_n| \to 0$, right continuity of X, and (2.1), we obtain the assertion. □

We see that, by Proposition 2.5, there is essentially no difference between the two definitions of quadratic variation, convergence of $\mu^{\pi_n}_X$ and convergence of $[X,X]^{\pi_n}$. Proposition 2.5 yields that, for $X \in QV(\Pi)$, the sequence of distribution functions associated with $(\mu^{\pi_n}_X)_n$ converges pointwise to $[X,X]$. Therefore, the sequence of measures $(\mu^{\pi_n}_X)$ converges vaguely to the Stieltjes measure μ_X generated by $[X,X]$. Here recall that $(\mu^{\pi_n}_X)$ converges to μ_X vaguely if and only if $(\mu^{\pi_n}_X([0,t]))$ converges $\mu_X([0,t])$ for any t satisfying $\mu_X(\{t\}) = 0$. For such a $t \in \mathbb{R}_{\geq 0}$ and for every bounded continuous function h, we have

$$\lim_{n\to\infty} \int_{[0,t]} h\, d\mu^{\pi_n}_X = \int_{[0,t]} h\, d\mu_X.$$

(See, for example, [18, Corollary 1.204].) This implies that, in the continuous path case,

$$\lim_{n\to\infty} \int_{[0,t]} f(X_s)\, \mu^{\pi_n}_X(ds) = \int_{[0,t]} f(X_s)\, d[X,X]_s$$

holds for every bounded continuous function f. The following lemma, which is proved as a part of the proof of the Itô formula in [15], claims that a similar property holds for a general càdlàg path of $QV(\Pi)$. See also [31, Theorem 6.52] for a proof. Here we will give a slightly extended version of this result. This lemma is a key to the proof of Proposition 2.7, Theorems 2.13 and 2.19.

Lemma 2.6. *Assume that* $X = (X^1, \dots, X^d) \in QV(\Pi; \mathbb{R}^d)$ *and* $Y = (Y^1, \dots, Y^m) \in D(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$. *Then for any continuous function* $g : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, $t \in \mathbb{R}_{\geq 0}$, *and* $i, j \in \{1, \dots, d\}$, *we have*

$$\lim_{n\to\infty} \int_{[0,t]} g(X_s, Y_s)\, \mu^{\pi_n}_{X^i, X^j}(ds) = \int_{[0,t]} g(X_{s-}, Y_{s-})\, d[X^i, X^j]_s.$$

Proof. If $d = 1$, the assertion follows directly from Lemma A.1. For the general case, we apply Lemma A.1 to each X^i and $X^i + X^j$, and combine their convergence and the definition of quadratic covariation. □

In semimartingale theory, a process defined as the composition of a semimartingale and a C^1 function again has quadratic variation ([32, Chapitre VI. 5. Theorem]). [15] mentions that a similar result holds within this framework. Here we prove a slightly extended version of that result.

Proposition 2.7. *Let* $X \in QV(\Pi; \mathbb{R}^d)$, *let* $A \in FV^m_{\mathrm{loc}}$, *and let* $f \in C^{0,1}(\mathbb{R}^m \times \mathbb{R}^d)$ *be locally Lipschitz. Then* $f(A, X)$ *has the quadratic variation along* Π *given by*

$$[f(A,X), f(A,X)]_t = \sum_{k,l=1}^{d} \int_0^t \left(\frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_l} \right)(A_{s-}, X_{s-})\, d[X^k, X^l]^{\mathrm{c}}_s + \sum_{0<s\leq t} (\Delta f(A_s, X_s))^2. \tag{2.2}$$

Proof. Fix $t > 0$. Because the image of $[0, t]$ under (A, X) is bounded in $\mathbb{R}^{m+d}$, we can assume, without loss of generality, that f has compact support. First note that there is a positive constant C such that

$$\sum_{0<s\leq t} (\Delta f(A_s, X_s))^2 \leq C \left(\sum_{0<s\leq t} (\Delta X_s)^2 + \sum_{0<s\leq t} (\Delta A_s)^2 \right) < \infty$$

thanks to the Lipschitz continuity of f.

Let

$$D(t) = \{ s \in [0,t] \mid \Delta(A,X)_s \neq 0 \},$$
$$D_p(t) = \left\{ s \in D(t) \,\middle|\, \|\Delta(A,X)_s\|_{\mathbb{R}^{m+d}} \geq \frac{1}{p} \right\}, \quad p \in \mathbb{N}_{\geq 1}.$$

Then each $D_p(t)$ is a finite set and $\bigcup_{p\geq 1} D_p(t) = D(t)$. By convention, we use the following notation.

$$\sum_{(1,n,p)} = \sum_{\substack{t^n_i \in \pi_n \cap [0,t] \\ ]t^n_i, t^n_{i+1}] \cap D_p(t) \neq \emptyset}}, \qquad \sum_{(2,n,p)} = \sum_{t^n_i \in \pi_n \cap [0,t]} - \sum_{(1,n,p)}.$$

Using this notation and Taylor's theorem, we have

$$\begin{aligned} &\sum_{t^n_i \in \pi_n \cap [0,t]} \{\delta^n_i f(A,X)\}^2 \\ &= \sum_{(1,n,p)} \{\delta^n_i f(A,X)\}^2 + \sum_{(2,n,p)} \left\{ f(A_{t^n_{i+1}}, X_{t^n_{i+1}}) - f(A_{t^n_i}, X_{t^n_{i+1}}) \right\}^2 \\ &\quad + 2 \sum_{(2,n,p)} \left\{ f(A_{t^n_{i+1}}, X_{t^n_{i+1}}) - f(A_{t^n_i}, X_{t^n_{i+1}}) \right\} \left\{ f(A_{t^n_i}, X_{t^n_{i+1}}) - f(A_{t^n_i}, X_{t^n_i}) \right\} \end{aligned} \tag{2.3}$$

$$
\begin{aligned}
&+\sum_{t_i^n\in\pi_n\cap[0,t]}\left\langle\nabla_x f(A_{t_i^n},X_{t_i^n}),\delta_i^n X\right\rangle^2-\sum_{(1,n,p)}\left\langle\nabla_x f(A_{t_i^n},X_{t_i^n}),\delta_i^n X\right\rangle^2\\
&+2\sum_{(2,n,p)}\langle r_i^n,\delta_i^n X\rangle\langle\nabla_x f(A_{t_i},X_{t_i}),\delta_i^n X\rangle+\sum_{(2,n,p)}\langle r_i^n,\delta_i^n X\rangle^2\\
=:\ & I_1^{(n)}+I_2^{(n)}+2I_3^{(n)}+I_4^{(n)}-I_5^{(n)}+2I_6^{(n)}+I_7^{(n)},
\end{aligned}
$$

where

$$
r_i^n=\int_{[0,1]}\nabla_x f(A_{t_i^n},X_{t_i^n}+s\,\delta_i^n X)\,ds-\nabla_x f(A_{t_i^n},X_{t_i^n})\in\mathbb{R}^d.
$$

We now consider the behavior of each term of the right-hand side of (2.3).

Since D_1 is finite, it is easy to verify that

$$
\lim_{n\to\infty} I_1^{(n)}=\sum_{s\in D_p(t)}(\Delta f(A_s,X_s))^2,
$$

$$
\lim_{n\to\infty} I_5^{(n)}=\sum_{s\in D_p(t)}\sum_{k,l=1}^d\left(\frac{\partial f}{\partial x_k}\frac{\partial f}{\partial x_l}\right)(A_{s-},X_{s-})\Delta X_s^k\Delta X_s^l.
$$

Lipschitz continuity of f implies that

$$
\varlimsup_{n\to\infty} I_2^{(n)}\le\frac{K}{p}\sum_{k=1}^m V(A^k)_t,\quad \varlimsup_{n\to\infty}\left|I_3^{(n)}\right|\le\frac{K}{p}\sum_{k=1}^m V(A^k)_t
$$

holds for a positive constant K. The convergence

$$
\lim_{n\to\infty} I_4^{(n)}=\sum_{k,l=1}^d\int_0^t\left(\frac{\partial f}{\partial x_k}\frac{\partial f}{\partial x_l}\right)(A_{s-},X_{s-})\,d[X^k,X^l]_s
$$

follows from Lemma 2.6. Moreover, we have

$$
\varlimsup_{n\to\infty}\left|I_6^{(n)}\right|\le\omega\left(\nabla_x f;\frac{2}{p}\right)K'\sum_{k=1}^d[X^k,X^k]_t,\quad \varlimsup_{n\to\infty} I_7^{(n)}\le\omega\left(\nabla_x f;\frac{2}{p}\right)^2\sum_{k=1}^d[X^k,X^k]_t,
$$

where $K'=\sup_{s\in[0,t]}\|\nabla_x f(A_s,X_s)\|$.

From (2.3) and the above, we see that

$$
\begin{aligned}
&\varlimsup_{n\to\infty}\left|\sum_{t_i\in\pi_n\cap[0,t]}\{\delta_i^n f(A,X)\}^2-(\text{RHS of }(2.2))\right|\\
&\qquad\le C\left(\frac{1}{p}+\omega\left(\nabla_x f;\frac{2}{p}\right)+\omega\left(\nabla_x f;\frac{2}{p}\right)^2\right)+\left|\sum_{s\in D(t)\setminus D_p(t)}(\Delta f(A_s,X_s))^2\right|\\
&\qquad\quad+\left|\sum_{s\in D(t)\setminus D_p(t)}\sum_{k,l=1}^d\left(\frac{\partial f}{\partial x_k}\frac{\partial f}{\partial x_l}\right)(A_{s-},X_{s-})\Delta X_s^k\Delta X_s^l\right|
\end{aligned}
\tag{2.4}
$$

holds for some positive constant C. Let $p\to\infty$, and then every term of the right-hand side of (2.4) converges to 0. This completes the proof. □

REMARK 2.8. If the functions f of Proposition 2.7 are defined on only an open subset $U \subset \mathbb{R}^{m+d}$, the assertion is still valid provided that (A, X) can be regarded as a U-valued càdlàg path.

The following results are direct consequences of Proposition 2.7.

Corollary 2.9. (i) *If $A \in FV_{\mathrm{loc}}$, then $A \in QV(\Pi; \mathbb{R})$ holds for any partitions $\Pi = (\pi_n)$ with the condition $|\pi_n| \to 0$, and its quadratic variation is*

$$[A, A] = \sum_{0<s\leq \cdot} (\Delta A_s)^2.$$

(ii) *If $A \in FV_{\mathrm{loc}}$ and $X \in QV(\Pi; \mathbb{R})$, we have $(X, A) \in QV(\Pi; \mathbb{R}^2)$ and*

$$[X + A, X + A] = [X, X] + [A, A] + 2[X, A],$$
$$[X, A] = \sum_{0<s\leq \cdot} \Delta X_s \Delta A_s,$$
$$[X + A, X + A]^{\mathrm{c}} = [X, X]^{\mathrm{c}}.$$

(iii) *If $A \in FV_{\mathrm{loc}}$ and $(X, Y) \in QV(\Pi; \mathbb{R}^2)$, then $(X+A, Y) \in QV(\Pi; \mathbb{R}^2)$ and its quadratic covariation is $[X + A, Y] = [X, Y] + [A, Y]$.*

2.2. Itô-Föllmer integrals and Itô's formula. In this subsection, we introduce the idea of the "integral" $\int_0^t \xi_{s-} dX_s$ with respect to $X \in QV(\Pi)$ and prove the Itô formula. We first define the Itô-Föllmer integral as the limit of a sequence of non-anticipative Riemann sums.

DEFINITION 2.10. Let $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ be a càdlàg path and consider $X \in QV(\Pi; \mathbb{R}^d)$. We call the limit

$$\int_0^t \langle \xi_{s-}, dX_s \rangle := \lim_{n\to\infty} \sum_{t_i \in \pi_n} \left\langle \xi_{t_i}, X_{t_{i+1}\wedge t} - X_{t_i \wedge t} \right\rangle,$$

the Itô-Föllmer integral of ξ with respect to X on $]0, t]$ along the sequence Π, if it converges to a finite number. In the 1-dimensional case, we simply write $\int_0^t \xi_{s-} dX_s$ instead of $\int_0^t \langle \xi_{s-}, dX_s \rangle$.

REMARK 2.11. It is a well-known fact that $\int_0^t \xi_{s-} dX_s$ exists if X or ξ has locally finite variation. When X is in FV_{loc}, the Itô-Föllmer integral coincides with the Stieltjes integral of $s \mapsto \xi_{s-}$ by X.

[15] considers a different summation

$$\sum_{t_i \in \pi_n \cap [0,t]} \left\langle \xi_{t_i}, X_{t_{i+1}} - X_{t_i} \right\rangle$$

to define a pathwise Itô integral. However, they are equivalent under the right continuity of integrators.

Proposition 2.12. *For $\xi, X \in D(\mathbb{R}_{\geq 0}, \mathbb{R}^d)$ and $t \in \mathbb{R}_{\geq 0}$ the following two conditions are equivalent.*

(i) $\sum_{t_i \in \pi_n} \left\langle \xi_{t_i}, X_{t_{i+1}\wedge t} - X_{t_i\wedge t} \right\rangle$ *converges.*

(ii) $\sum_{t_i \in \pi_n \cap [0,t]} \langle \xi_{t_i}, X_{t_{i+1}} - X_{t_i} \rangle$ *converges .*

If the conditions above are satisfied, the limits are equal.

Proof. For $t \in [t_i^n, t_{i+1}^n[$ ($t_i^n \in \pi_n$), we have

$$\left| \sum_{t_i^n \in \pi_n} \left\langle \xi_{t_i^n}, X_{t_{i+1}^n \wedge t} - X_{t_i^n \wedge t} \right\rangle - \sum_{t_i^n \in \pi_n \cap [0,t]} \left\langle \xi_{t_i^n}, X_{t_{i+1}^n} - X_{t_i^n} \right\rangle \right| \leq \sup_{s \in [0,t]} \|\xi_s\|_{\mathbb{R}^d} \left\| X_{t_{i+1}^n} - X_t \right\|_{\mathbb{R}^d} .$$

Letting $n \to \infty$ the last term converges to 0 by the condition $|\pi_n| \to 0$ and the right continuity of X. $\square$

In this paper, we consider integrands of the form $t \mapsto g(X_t)$ for a "nice" function g. [15] proves that the Itô-Föllmer integral of $t \mapsto \nabla f(X_t)$ with respect to X exists for C^2-class f and it satisfies the same relation as that of Itô's formula in stochastic calculus. We can easily extend this result as follows.

Theorem 2.13. *Let $X \in QV(\Pi; \mathbb{R}^d)$ and $A \in FV^m_{\mathrm{loc}}$. Then, for any function $f \in C^{1,2}(\mathbb{R}^m \times \mathbb{R}^d)$ and $t \in \mathbb{R}_{\geq 0}$, the Itô-Föllmer integral of $\nabla_x f(A, X)$ with respect to X exists and satisfies the following formula.*

$$\begin{aligned} f(A_t, X_t) - f(A_0, X_0) = &\sum_{k=1}^m \int_0^t \frac{\partial f}{\partial a_k}(A_{s-}, X_{s-})\, d(A^k)^{\mathrm{c}}_s + \int_0^t \langle \nabla_x f(A_{s-}, X_{s-}), dX_s \rangle \\ &+ \sum_{k,l=1}^d \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x_k \partial x_l}(A_{s-}, X_{s-})\, d[X^k, X^l]^{\mathrm{c}}_s \\ &+ \sum_{0<s\leq t} \left\{ \Delta f(A_s, X_s) - \sum_{k=1}^d \frac{\partial f}{\partial x_k}(A_{s-}, X_{s-}) \Delta X^k_s \right\}. \end{aligned}$$

The proof of this theorem is similar to the proof of Föllmer's pathwise Itô formula for C^2-functions. We consider the first order Taylor expansion with respect to a and the second order Taylor expansion with respect to x, while, in C^2 case, we use the second order Taylor expansion for all the variables.

Remark 2.14. Theorem 2.13 still holds for a $C^{1,2}$ function f defined on an open subset $U \subset \mathbb{R}^m \times \mathbb{R}^d$ and for $t \in [0, T]$ if the restriction of (A, X) to $[0, T]$ is càdlàg as a U-valued path.

2.3. Itô-Föllmer integrals of admissible integrands. In this subsection, we study some more basic properties of the Itô-Föllmer integrals with respect to X for integrands of the form $\nabla_x f(A_t, X_t)$.

Definition 2.15. Let $X \in QV(\Pi; \mathbb{R}^d)$. A d-dimensional càdlàg path ξ is called an *admissible integrand of* X if, for any $T > 0$, there exist an $m \in \mathbb{N}$, a function f of $C^{1,2}$-class defined on an open subset U of $\mathbb{R}^m \times \mathbb{R}^d$, and $A \in FV^m_{\mathrm{loc}}$ such that $\xi_t = \nabla_x f(A_t, X_t)$ holds for all $t \in [0, T]$ and the restriction of (A, X) to $[0, T]$ is U-valued càdlàg path.

For the terminology "admissible integrand", we follow [38]. It follows from Theorem 2.13 that the Itô-Föllmer integral $\int_0^t \langle \xi_{s-}, dX_s \rangle$ exists for ξ and X of Definition 2.15. In this case, the map $t \mapsto \int_0^t \langle \xi_{s-}, dX_s \rangle$ is an $\mathbb{R}$-valued càdlàg path. The representation of ξ in Definition 2.15 is not unique. However, the Itô-Föllmer integral of ξ does not depend of the representation because the integral is just the limit of non-anticipative Riemann sums by definition. (See Definition 2.10.) We can easily see that the space of admissible integrands of X is a vector space and the map $\xi \mapsto \int_0^t \langle \xi_{s-}, dX_s \rangle$ on it is linear.

Remark 2.16. (i) For an admissible integrand $\xi = (\xi^1, \dots, \xi^d)$ of $X \in QV(\Pi; \mathbb{R}^d)$, the equation

$$\int_0^t \langle \xi_{s-}, dX_s \rangle = \sum_{i=1}^d \int_0^t \xi^i_{s-} dX^i_s$$

is not necessarily valid. Note that the existence of the Itô-Föllmer integral of the left-hand side does not imply the existence of those of the right-hand side. If all integrals of the right-hand side exist, then the integral of the left-hand side also exits and the above equality holds.

(ii) Let $X \in QV(\Pi)$, and $\xi_t = g(A_t, X_t)$ $(t \geq 0)$, where $g \in C^1(\mathbb{R}^m \times \mathbb{R})$ and $A \in FV^m_{\text{loc}}$. Then ξ is an admissible integrand of X. Indeed, $\xi_t = \frac{\partial}{\partial x} f(A_t, X_t)$ holds for $f(a, x) = \int_0^x g(a, y)\, dy$.

(iii) Let ξ is an admissible integrand of $X \in QV(\Pi)$ and let $B \in FV_{\text{loc}}$. Then for $Y := X + B \in QV(\Pi)$, the Itô-Föllmer integral $\int_0^t \xi_{s-} dY_s$ exits and it satisfies

$$\int_0^t \xi_{s-} dY_s = \int_0^t \xi_{s-} dX_s + \int_0^t \xi_{s-} dB_s, \quad t \in \mathbb{R}_{\geq 0}.$$

We will now study the quadratic variation of a path given by Itô-Föllmer integration.

Proposition 2.17. *Let $X \in QV(\Pi; \mathbb{R}^d)$, let $\xi^{(1)}, \dots, \xi^{(m)}$ be admissible integrands of X, and let $Y^l_t = \int_0^t \left\langle \xi^{(l)}_{s-}, dX_s \right\rangle$ for all $t \geq 0$ and $l \in \{1, \dots, m\}$. Then $Y = (Y^1, \dots, Y^m)$ belongs to $QV(\Pi; \mathbb{R}^m)$ and its quadratic covariations have the following form.*

$$\left[Y^k, Y^l\right]_t = \sum_{i,j=1}^d \int_0^t \xi^{(k),i}_{s-} \xi^{(l),j}_{s-} d\left[X^i, X^j\right]_s . \tag{2.5}$$

Proof. We first assume $m = 1$. Fix $T > 0$ and an expression $\xi = (\nabla_x f) \circ (A, X)$ on $[0, T]$, as in Definition 2.15. Define

$$B_t = f(A_t, X_t) - f(A_0, X_0) - Y_t, \quad t \in [0, T].$$

Then, by Theorem 2.13, B is a càdlàg path of locally finite variation whose jumps are given by

$$\Delta B_t = \Delta f(A_t, X_t) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(A_{t-}, X_{t-}) \Delta X^i_t.$$

Using Proposition 2.7 and Corollary 2.9, we can directly calculate the quadratic variation of Y as follows.

$$[Y, Y]_t = [f(A, X), f(A, X)]_t + \sum_{0 \leq s \leq t} \left[(\Delta B_s)^2 - 2\Delta B_s \Delta f(A_s, X_s)\right]$$

$$= \sum_{i,j=1}^{d} \int_0^t \left(\frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \right) (A_{s-}, X_{s-})\, d[X^i, X^j]_s.$$

Since $T > 0$ is arbitrarily fixed, this formula is extended to $\mathbb{R}_{\geq 0}$.

For general $m \in \mathbb{N}_{\geq 1}$, we obtain (2.5) by applying the above result to $Y^k + Y^l$ for $k, l \in \{1, \dots m\}$. □

By Proposition 2.17, we see that a path of the form $Y_t = \int_0^t \langle \xi_{s-}, dX_s \rangle$ again has quadratic variation. So we can consider Itô-Föllmer integration by Y. The associativity of Itô-Föllmer integration with respect to continuous integrators are proved in [38, Theorem 13]. In this paper, we will extend this result to càdlàg integrators. For this purpose, we prepare a lemma. This lemma is due to [38, Proof of Theorem 13] for the continuous path case.

Lemma 2.18. *Let $X \in QV(\Pi; \mathbb{R}^d)$, let ξ be an admissible integrand of X, and let $Y_t = \int_0^t \langle \xi_{s-}, dX_s \rangle$. Then, for each $T > 0$, we can choose a $C^{1,2}$ function F such that $\xi_t = \nabla_x F(A_t, X_t)$, $Y_t = F(A_t, X_t)$, and*

$$\text{(2.6)} \qquad \sum_{i=1}^{m} \int_0^t \frac{\partial F}{\partial a_i}(A_{s-}, X_{s-})\, d(A^i)^{\mathrm{c}}_s + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(A_{s-}, X_{s-})\, d[X^i, X^j]^{\mathrm{c}}_s = 0$$

holds for all $t \in [0, T]$.

Proof. Fix $T > 0$ arbitrarily. Take $\widetilde{m} \in \mathbb{N}$, $\widetilde{A} \in FV^{m}_{\mathrm{loc}}$, an open set $U \subset \mathbb{R}^m \times \mathbb{R}^d$, and $f \in C^{1,2}(U, \mathbb{R})$ such that $\xi_t = \nabla_x f(\widetilde{A}_t, X_t)$ for $t \in [0, T]$. Let us define $\widetilde{A}^0_t = f(\widetilde{A}_t, X_t) - f(\widetilde{A}_0, X_0) - Y_t$. Then, the path $\widetilde{A}^0$ is of finite variation on $[0, T]$ by Itô's formula. Moreover, let $m = \widetilde{m} + 1$, $A = (\widetilde{A}^0, \widetilde{A})$, and

$$F(\widetilde{a}_0, \widetilde{a}, x) = f(\widetilde{a}, x) - f(A_0, X_0) - \widetilde{a}_0, \quad (\widetilde{a}_0, \widetilde{a}, x) \in \mathbb{R} \times U.$$

Then, we have $F \in C^{1,2}(\mathbb{R} \times U)$, $Y_t = F(A_t, X_t)$, and

$$\nabla_x F(A_t, X_t) = \nabla_x f(\widetilde{A}_t, X_t) = \xi_t, \quad t \in [0, T].$$

By construction and Itô's formula, F clearly satisfies (2.6). □

The following theorem is the main theorem of this subsection. A corresponding result for continuous integrators is already proved by [38]. Difficulties in discontinuous cases are the facts that Lemma 2.6 is not obvious in discontinuous settings and that the summation of residual terms does not vanish as $n \to \infty$. The latter problem is solved by classifying the jumps as in the proof of Proposition 2.7 and observing carefully big jump parts and small jump parts each.

Theorem 2.19. *Let $X \in QV(\Pi; \mathbb{R}^d)$ and $Y^k_t = \int_0^t \left\langle \xi^{(k)}_{s-}, dX_s \right\rangle$, where $\xi^{(1)}, \dots, \xi^{(\nu)}$ are admissible integrands of X. Suppose that $\eta = (\eta^1, \dots, \eta^\nu)$ is an ν-dimensional càdlàg path. Then, the Itô-Föllmer integral $\int_0^t \langle \eta_{s-}, dY_s \rangle$ exists if and only if the Itô-Föllmer integral $\int_0^t \langle \sum_{k=1}^{\nu} \eta^k_{s-} \xi^{(k)}_{s-}, dX_s \rangle$ exists. If one of them exists, they satisfy*

$$\text{(2.7)} \qquad \int_0^t \langle \eta_{s-}, dY_s \rangle = \int_0^t \left\langle \sum_{k=1}^{\nu} \eta^k_{s-} \xi^{(k)}_{s-}, dX_s \right\rangle, \quad t \in \mathbb{R}_{\geq 0}.$$

Proof. Fix $T > 0$. For each $l \in \{1, \dots, \nu\}$, let $Y^l = F^l(A^l, X)$ be the expression of Lemma 2.18 where $A^l \in FV^{m_l}_{\mathrm{loc}}$. By the second-order Taylor expansion, we have

$$\begin{aligned}\delta^n_i F^l(A^{(l)}, X) = &\left\langle \nabla_a F^l(A^{(l)}_{t^n_i}, X_{t^n_i}), \delta^n_i A^{(l)} \right\rangle + \left\langle r^{n,l}_i, \delta^n_i A^{(l)} \right\rangle + \left\langle \nabla_x F^l(A^{(l)}_{t^n_i}, X_{t^n_i}), \delta^n_i X \right\rangle \\ &+ \frac{1}{2} \left\langle \nabla^2_x F^l(A^{(l)}_{t^n_i}, X_{t^n_i})(\delta^n_i X), \delta^n_i X \right\rangle + \left\langle R^{n,l}_i(\delta^n_i X), \delta^n_i X \right\rangle,\end{aligned}$$

where $r^{n,l}_i$ and $R^{n,l}_i$ are corresponding residual terms. Let $D^l(t)$, $D^l_p(t)$, $\sum_{(1,n,p)_l}$, and $\sum_{(2,n,p)_l}$ be the quantities in the proof of Proposition 2.7 corresponding to the path $(A^{(l)}, X)$. Then,

$$\begin{aligned}&\sum_{\pi_n \cap [0,t]} \langle \eta_{t^n_i}, \delta^n_i Y \rangle - \sum_{\pi_n \cap [0,t]} \left\langle \sum_{l=1}^{\nu} \eta^l_{t^n_i} \xi^{(l)}_{t^n_i}, \delta^n_i X \right\rangle \\
&= \sum_{l=1}^{\nu} \sum_{(1,n,p)_l} \eta^l_{t^n_i} \delta^n_i F^l(A^{(l)}, X) - \sum_{l=1}^{\nu} \sum_{(1,n,p)_l} \eta^l_{t^n_i} \left\langle \nabla_x F^l(A^{(l)}_{t^n_i}, X_{t^n_i}), \delta^n_i X \right\rangle \\
&\quad + \sum_{l=1}^{\nu} \sum_{\pi_n \cap [0,t]} \eta^l_{t^n_i} \left\langle \nabla_a F^l(A^{(l)}_{t^n_i}, X_{t^n_i}), \delta^n_i A^{(l)} \right\rangle - \sum_{l=1}^{\nu} \sum_{(1,n,p)_l} \eta^l_{t^n_i} \left\langle \nabla_a F^l(A^{(l)}_{t^n_i}, X_{t^n_i}), \delta^n_i A^{(l)} \right\rangle \\
&\quad + \frac{1}{2} \sum_{l=1}^{\nu} \sum_{\pi_n \cap [0,t]} \left\langle \eta^l_{t^n_i} \nabla^2_x F^l(A^{(l)}_{t^n_i}, X_{t^n_i})(\delta^n_i X), \delta^n_i X \right\rangle \\
&\quad - \frac{1}{2} \sum_{l=1}^{\nu} \sum_{(1,n,p)_l} \left\langle \eta^l_{t^n_i} \nabla^2_x F^l(A^{(l)}_{t^n_i}, X_{t^n_i})(\delta^n_i X), \delta^n_i X \right\rangle \\
&\quad + \sum_{l=1}^{\nu} \sum_{(2,n,p)_l} \eta^l_{t^n_i} \left\langle r^{n,l}_i, \delta^n_i A^{(l)} \right\rangle + \sum_{l=1}^{\nu} \sum_{(2,n,p)_l} \eta^l_{t^n_i} \langle R^n_i(\delta^n_i X), \delta^n_i X \rangle \\
&=: I^{(n)}_1 - \widetilde{I}^{(n)}_1 + I^{(n)}_2 - \widetilde{I}^{(n)}_2 + \frac{1}{2} I^{(n)}_3 - \frac{1}{2} \widetilde{I}^{(n)}_3 + I^{(n)}_4 + I^{(n)}_5.\end{aligned}$$

Clearly we have

$$\lim_{n \to \infty} \left(I^{(n)}_1 - \widetilde{I}^{(n)}_1 \right) = \sum_{l=1}^{\nu} \sum_{s \in D^l_1(p)} \eta^l_{s-} \left(\Delta Y^l_s - \sum_{k=1}^{d} \xi^{(l),k}_{s-} \Delta X^k_s \right) = 0.$$

By the dominated convergence theorem, we see that

$$\begin{aligned}\lim_{n \to \infty} I^{(n)}_2 &= \sum_{l=1}^{\nu} \sum_{j=1}^{m_l} \int_0^t \eta^l_{s-} \frac{\partial F^l}{\partial a_j}(A^{(l)}_{s-}, X_{s-})\, dA^{(l),j}_s, \\
\lim_{n \to \infty} \widetilde{I}^{(n)}_2 &= \sum_{l=1}^{\nu} \sum_{j=1}^{m_l} \sum_{s \in D^l_p(p)} \eta^l_{s-} \frac{\partial F^l}{\partial a_j}(A^{(l)}_{s-}, X_{s-}) \Delta A^{(l),j}_s.\end{aligned}$$

By Lemma 2.6, we have

$$\begin{aligned}\lim_{n \to \infty} I^{(n)}_3 &= \sum_{l=1}^{\nu} \sum_{j,k=1}^{d} \int_0^t \eta^l_{s-} \frac{\partial^2 F^l}{\partial x_j \partial x_k}(A^{(l)}_{s-}, X_{s-})\, d[X^j, X^k]_s, \\
\lim_{n \to \infty} \widetilde{I}^{(n)}_3 &= \sum_{l=1}^{\nu} \sum_{j,k=1}^{d} \sum_{s \in D^l_p(t)} \eta^l_{s-} \frac{\partial^2 F^l}{\partial x_j \partial x_k}(A^{(l)}_{s-}, X_{s-}) \Delta X^j_s \Delta X^k_s.\end{aligned}$$

By a discussion similar to that of the proof of Theorem 2.13, we can take a constant C, which does not depend on ε and n, such that

$$\varlimsup_{n\to\infty}\left|I_4^{(n)}\right| \le C\sum_{l=1}^{\nu}\omega\left(\nabla_a F^l;\frac{2}{p}\right),\quad \varlimsup_{n\to\infty}\left|I_5^{(n)}\right| \le C\sum_{l=1}^{\nu}\omega\left(\nabla_x^2 F^l;\frac{2}{p}\right).$$

As a consequence of these observations and (2.6) we obtain

$$\begin{aligned}
&\varlimsup_{n\to\infty}\left|\sum_{\pi_n\cap[0,t]}\langle\eta_{t_i^n},\delta_i^n Y\rangle-\sum_{\pi_n\cap[0,t]}\left\langle\sum_{l=1}^{\nu}\eta^l_{t_i^n}\xi^{(l)}_{t_i^n},\delta_i^n X\right\rangle\right|\\
&\le C\sum_{l=1}^{\nu}\left(\omega\left(\nabla_a F^l;\frac{2}{p}\right)+\omega\left(\nabla_x^2F^l;\frac{2}{p}\right)\right)+\left|\sum_{l=1}^{\nu}\sum_{s\in D^l(t)\setminus D^l_p(t)}\sum_{j=1}^{m_l}\eta^l_{s-}\frac{\partial F^l}{\partial a_j}(A^{(l)}_{s-},X_{s-})\Delta A^{(l),j}_s\right|\\
&\quad+\frac{1}{2}\left|\sum_{l=1}^{\nu}\sum_{s\in D^l(t)\setminus D^l_p(t)}\sum_{j,k=1}^{d}\eta^l_{s-}\frac{\partial^2F^l}{\partial x_j\partial x_k}(A^{(l)}_{s-},X_{s-})\Delta X^j_s\Delta X^k_s\right|.
\end{aligned}$$

Letting $p\to\infty$, we see that the right-hand side of this inequality converges to 0. Hence, if one of the two Itô-Föllmer integrals exists, then the other also exists and their values are equal. □

Remark 2.20. If $\eta=(\eta^1,\dots,\eta^\nu)$ is an admissible integrand of Y, associativity in the sense of (2.7) can be shown by direct calculation. Fix $T>0$ and let $\xi^{(l)}_t=\nabla_x F^l(A^{(l)}_t,X_t)$ be those in the proof of Theorem 2.19. Define $m=m_1+\cdots+m_\nu$, $a=(a^{(1)},\dots,a^{(\nu)})\in\mathbb{R}^m$, $\mathbf{F}(a,x)=(F^1(a^{(1)},x),\dots,F^\nu(a^{(\nu)},x))$, and $A_t=(A^{(1)}_t,\dots,A^{(\nu)}_t)$. Then we have $Y_t=\mathbf{F}(A_t,X_t)$. Choose $B\in FV^n_{\mathrm{loc}}$ and g of $C^{1,2}$-class such that $\eta_t=\nabla_y g(B_t,Y_t)$ holds for all $t\in[0,T]$. Let $C=(B,A)$ and $h(c,x)=h(b,a,x)=g(b,\mathbf{F}(a,x))$. Then, we have

$$\nabla_x h(C_t,X_t)=\nabla_y g(B_t,Y_t)\nabla_x\mathbf{F}(A_t,X_t)=\sum_{l=1}^{\nu}\eta^l_t\xi^{(l)}_t.$$

Hence, $\sum_{l=1}^{\nu}\eta^l_t\xi^{(l)}_t$ is an admissible integrand of X. This expression is given by [38] in the proof of Theorem 13. Applying the Itô formula to $h(C_t,X_t)$, we get

$$\begin{aligned}
\text{(2.8)}\qquad \int_0^t\left\langle\sum_{l=1}^{\nu}\eta^l_{s-}\xi^{(l)}_{s-},dX_s\right\rangle&=h(C_t,X_t)-h(C_0,X_0)-\sum_{i=1}^{n+m}\int_0^t\frac{\partial h}{\partial c_i}(C_{s-},X_{s-})\,d(C^i)^{\mathrm{c}}_s\\
&\quad-\frac{1}{2}\sum_{i,j=1}^{d}\int_0^t\frac{\partial^2h}{\partial x_i\partial x_j}(C_{s-},X_{s-})\,d\left[X^i,X^j\right]^{\mathrm{c}}_s\\
&\quad-\sum_{0<s\le t}\left\{\Delta h(C_s,X_s)-\sum_{i=1}^{d}\frac{\partial h}{\partial x_i}(B_{s-},X_{s-})\Delta X^i_s\right\}.
\end{aligned}$$

On the other hand, applying the Itô formula to $g(B,Y)$, we see that

$$\text{(2.9)}\qquad \int_0^t\langle\eta_{s-},dY_s\rangle=g(B_t,Y_t)-g(B_0,Y_0)-\sum_{k=1}^{m}\int_0^t\frac{\partial g}{\partial b_k}(B_{s-},Y_{s-})\,d(B^k)^{\mathrm{c}}_s$$

$$- \frac{1}{2} \sum_{k,l=1}^{\nu} \int_0^t \frac{\partial^2 g}{\partial y_k \partial y_l}(B_{s-}, Y_{s-})\, d\left[Y^k, Y^l\right]_s^{\mathrm{c}}$$
$$- \sum_{0<s\le t} \left\{ \Delta g(B_s, Y_s) - \sum_{l=1}^{\nu} \frac{\partial g}{\partial y_l}(B_{s-}, Y_{s-}) \Delta Y_s^l \right\}.$$

Thanks to Proposition 2.17 and (2.6), we can check that the right-hand sides of (2.8) and (2.9) are equal.

The following integration by parts formula is proved as a corollary of associativity.

Corollary 2.21. *Let $X \in QV(\Pi)$. For admissible integrands ξ, η of X and for $A, B \in FV_{\mathrm{loc}}$, we define*

$$Y_t = \int_0^t \xi_{s-} dX_s + A_t, \quad Z_t = \int_0^t \eta_{s-} dX_s + B_t.$$

Then we have

$$Y_t Z_t = \int_0^t Y_{s-} dZ_s + \int_0^t Z_{s-} dY_s + [Y, Z]_t.$$

Proof. First note that $(Y, Z) \in QV(\Pi; \mathbb{R}^2)$ holds by Corollary 2.9 and Proposition 2.17. Applying Theorem 2.13 to $f(y, z) = yz$, we get

$$Y_t Z_t - Y_0 Z_0 = \int_0^t \langle \nabla f(Y_{s-}, Z_{s-}), d(Y, Z)_s \rangle + [Y, Z]_t, \quad t \ge 0.$$

We can deduce that ξY is an admissible integrand of X from the expression for Y in Lemma 2.18. Hence by Theorem 2.19 and Remark 2.16, the Itô-Föllmer integral $\int_0^t Y_{s-} dZ_s$ exists. Thus we obtain

$$\int_0^t \langle \nabla f(Y_{s-}, Z_{s-}), d(Y, Z)_s \rangle = \int_0^t Y_{s-} dZ_s + \int_0^t Z_{s-} dY_s, \quad t \ge 0.$$

This completes the proof. □

2.4. Quadratic variations of paths and semimartingales. In this subsection, we describe relations between quadratic variations in Föllmer's sense and those of semimartingales. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_{\ge 0}})$ be a filtered measurable space such that $(\mathcal{F}_t)$ is right continuous and universally complete, and let $\mathcal{P}$ be a family of probability measures on $(\Omega, \mathcal{F})$.

Definition 2.22. Let $\pi = (\tau_k)_{k \in \mathbb{N}}$ be a sequence of random variables such that $0 = \tau_0 \le \tau_1 \le \dots$. If for every ω the partition $(\tau_k(\omega))_{k \in \mathbb{N}}$ satisfies the conditions in Section 2.1, π is called a *random partition*. In addition, if each τ_k is an $(\mathcal{F}_t)$-stopping time, we call π an *optional partition*. We denote the sequence $(\tau_k(\omega))_{k \in \mathbb{N}}$ by $\pi(\omega)$ for each $\omega \in \Omega$.

The name optional partition is borrowed from [10, p.4]. This terminology is reasonable because the term optional time is often used as another name for stopping time: see, for example, [11, Chapter IV. No. 49]. For a sequence of optional partitions $\Pi = (\pi_n)_{n \in \mathbb{N}}$, each partition is denoted by $\pi_n = (\tau_k^n)_{k \in \mathbb{N}}$, for example. Given an optional partition π and a càdlàg process X, the oscillation of paths of X along π is defined by

$$O_t(X(\omega), \pi(\omega)) = \sup_{t_i \in \pi} \sup\{\|X_s(\omega) - X_u(\omega)\| \mid s, u \in [t_i, t_{i+1}[\cap [0, t]\}.$$

Then, we can easily check that $(\omega, t) \mapsto O_t(X(\omega), \pi(\omega))$ is a càdlàg adapted process because X is càdlàg and adapted, and τ^n_k's are stopping times. Also, recall that if a sequence (g_n) in $L^p(P)$ satisfies $\sum_{n\in\mathbb{N}} \|g_n - g\|^p_{L^p(P)} < +\infty$, then (g_n) converges almost surely to g.

Suppose that X is a semimartingale under $P \in \mathcal{P}$. Let us consider the usual quadratic variation $[X, X]^P$ of a semimartingale X under P. It is a well known fact that if a sequence of optional partitions $\Pi = (\pi_n)$ satisfies $|\pi_n(\omega)| \to 0$ almost surely, the sequence $[X(\omega), X(\omega)]^{\pi_n(\omega)}$ converges to $[X, X]^P$ in the ucp topology. Hence, by taking a proper subsequence, P-almost all paths of X have quadratic variation along $\pi(\omega)$ and they are almost surely equal to $[X, X]^P$.

Our aim is to find a sufficient condition for Π under which P-almost every path of X has quadratic variation along Π without taking a subsequence. The following Proposition is due to [10, Proposition 2.4] for continuous semimartingales. Its proof can directly applied to càdlàg semimartingales. For the proof of the proposition, we use the notion of $\mathcal{S}^p$ spaces, semimartingale $\mathcal{H}^p$ spaces, and prelocalization of processes. See, [5] or [36] for these contents.

Proposition 2.23. *Suppose that, for all $P \in \mathcal{P}$, a càdlàg adapted process X is a P-semimartingale and $\Pi = (\pi_n)$ is a sequence of optional partitions such that $|\pi_n(\omega)| \to 0$ holds P-almost surely. Moreover, suppose that for every $P \in \mathcal{P}$ and every $T \in \mathbb{R}_{\geq 0}$, the series $\sum_n O_T(X(\omega), \pi_n(\omega))$ converges P-almost surely. Then, for all $P \in \mathcal{P}$, P-almost every path of X has quadratic variation along Π that coincides with the usual quadratic variation $[X, X]^P$, P-almost surely.*

Proof. We can assume $X_0 = 0$ without loss of generality. Fix $P \in \mathcal{P}$. Recall that X satisfies

$$[X, X]^P = X^2 - X_- \bullet X$$

where the last term of the right-hand side denotes the stochastic integral with respect to P. First we suppose $X \in \mathcal{H}^4(P)$. Fix an arbitrary T in $\mathbb{R}_{\geq 0}$ and define $K_T := \sum_n O_T(X, \pi_n)$. Then, K_T is a random variable since it is defined as the (convergent) series of a sequence of positive random variables. We define a probability measure Q equivalent to P by

$$Q(A) = \frac{1}{E[\exp(-K_T)]} \int_A e^{-K_T(\omega)} P(d\omega).$$

Then, X is again a semimartingale with respect to Q. Here recall that the stochastic integral is invariant under an equivalent change of probability measure. (See, for example, [23, 12.22 Theorem].) We have $K_T \in L^4(Q)$, and $X \in \mathcal{H}^4(Q)$ by definition of Q. Let $H^n := \sum_k X_{\tau^n_k} 1_{]\!]\tau^n_k, \tau^n_{k+1}]\!]}$ and let $K^n := H^n - X_-$. Then H^n and K^n are locally bounded predictable processes. By definition, we see that

$$\sum_{n\in\mathbb{N}} \sup_{t\in[0,T]} |K^n_t|^2 \leq \left(\sum_{n\in\mathbb{N}} \sup_{t\in[0,T]} |K^n_t| \right)^2 \leq K_T^2 \in L^2(Q). \tag{2.10}$$

Consider a decomposition $X = M + A$ such that $M \in \mathcal{M}_{\mathrm{loc},0}(Q)$ and $A \in \mathcal{V}$. Since K^n is locally bounded, the decomposition $K^n \bullet X = K^n \bullet M + K^n \bullet A$ satisfies $K^n \bullet M \in \mathcal{M}_{\mathrm{loc},0}(Q)$ and $K^n \bullet A \in \mathcal{V}$. Combining (2.10), the monotone convergence theorem, Schwarz's inequality, and Theorem 2 of [36, Chapter V], we get

$$\sum_{n\in\mathbb{N}} \left\|(K^n \bullet X)^T\right\|^2_{S^2(Q)} \le C\, \|K_T\|^2_{L^4(Q)} \left(\left\|[M,M]_T^{1/2}\right\|^2_{L^4(Q)} + \|V(A)_T\|^2_{L^4(Q)} \right)$$

where C is a positive constant. Since $X \in \mathcal{H}^4(Q)$, we can take a decomposition such that the right-hand side of the above inequality is finite. This proves that $(K^n \bullet X)^*_T \to 0$ holds Q-almost surely, and hence P-almost surely. Consequently, we have P-almost surely

$$\sup_{t\in[0,T]} \left| [X(\omega), X(\omega)]_t^{\pi_n(\omega)} - [X,X]^P_t(\omega) \right| = \sup_{t\in[0,T]} |(K^n \bullet X)_t(\omega)| \longrightarrow 0,$$

as $n \to \infty$. This completes the proof for $X \in \mathcal{H}^4(P)$. The general case is proved by the above argument and the usual prelocalization procedure. □

Remark 2.24. An example of optional partitions used in Proposition 2.23 can be given in the following manner. For a càdlàg semimartingale X satisfying the assumption of Proposition 2.23, we define a sequence of stopping times inductively as

$$\tau^n_0(\omega) = 0,$$

$$\tau^n_{k+1}(\omega) = \inf\left\{ t > \tau^n_k(\omega) \;\middle|\; |X_t(\omega) - X_{\tau^n_k}(\omega)| > \frac{1}{2^{n+1}} \right\} \wedge \left(\tau^n_k(\omega) + \frac{1}{n} \right).$$

Then $\Pi = ((\tau^n_k)_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ clearly satisfies the assumption of Proposition 2.23 for all $P \in \mathcal{P}$.

Remark 2.25. Because of Theorem 2.13 and Proposition 2.23, the Itô-Föllmer integral $\int_0^t g(X_{s-})\,dX_s$ for g of C^1-class converges almost surely along $\Pi = (\pi_n)$ provided that $\sum_n O_t(X,\pi_n) < \infty$ a.s.. Under this assumption, Π seems to control the oscillations of the paths of X, only. It is known that, in general, we have to control the oscillation of integrands to obtain the pathwise convergence of stochastic integrals: see [27]. The oscillation of the paths of $g \circ X_-$ is controlled by that of X in this particular case. Hence, our result is valid.

3. Integral equations with respect to Itô-Föllmer integrals and their applications to finance

We have seen that some fundamental formulas in stochastic integration theory are valid within the framework of Itô-Föllmer integrals. In this section, using these formulas, we compute explicit solutions to certain integral equations.

3.1. Linear equations. For a given $X \in QV(\Pi)$ and an $H \in D(\mathbb{R}_{\ge 0}, \mathbb{R})$, we consider the following equation for Y.

$$Y_t = H_t + \int_0^t Y_{s-} dX_s. \tag{3.1}$$

If H is (not) constant, the equation (3.1) is called an (in)homogeneous linear equation. A solution of (3.1) is an admissible integrand of X that satisfies (3.1).

We first consider the following proposition.

Proposition 3.1. *Let $X \in QV(\Pi)$ and let*

$$\mathcal{E}(X)_t = \exp\left(X_t - X_0 - \frac{1}{2}[X,X]^{\mathrm{c}}_t\right) \prod_{0<s\le t} (1 + \Delta X_s) e^{-\Delta X_s}.$$

Then $\mathcal{E}(X)$ is the unique solution to the homogeneous linear equation

$$Y_t = 1 + \int_0^t Y_{s-} dX_s. \tag{3.2}$$

Proof. Let

$$V(X)_t = \prod_{0<s\le t} (1 + \Delta X_s) e^{-\Delta X_s}.$$

It is well known that this infinite product converges absolutely and $V(X)$ is a càdlàg purely discontinuous function of locally finite variation. (See, for example, [25, Chapter I, 4.61 Theorem].) Here we define $f(x,a,b) = e^{x - X_0 - a/2} b$. It is easy to see that f is of C^2-class. Since $\frac{\partial}{\partial x} f(X_t, [X,X]^{\mathrm{c}}_t, V_t) = \mathcal{E}(X)_t$ holds, the function $\mathcal{E}(X)$ is an admissible integrand of X. Applying the Itô-Föllmer formula to $f(X_t, [X,X]^{\mathrm{c}}_t, V_t)$, we see that $Y = \mathcal{E}(X)$ satisfies (3.2).

For uniqueness, we will take an arbitrary solution Y of (3.2) and show that $Y = \mathcal{E}(X)$. Let Y be a solution of (3.2), and let $Z = X - X_0 - (1/2)[X,X]^{\mathrm{c}}$. Then we see that $\mathbf{X} := (Y, Z)$ belongs to $QV(\Pi; \mathbb{R}^2)$ by the representation $Y_t = 1 + \int_0^t Y_{s-} dX_s$ and by Corollary 2.9 and Proposition 2.17. The Itô-Föllmer formula implies the existence of the Itô-Föllmer integral $\int_0^t \langle \nabla g(Y_{s-}, Z_{s-}), d\mathbf{X}_s \rangle$ where $g(y,z) = ye^{-z}$. Since Y is represented as $Y = F(A, X)$ by some F of $C^{1,2}$-class and A of locally finite variation, the Itô-Föllmer integral $\int_0^t g(Y_{s-}, Z_{s-})\, dX_s$ exists. Moreover, by Remark 2.16

$$\int_0^t g(Y_{s-}, Z_{s-})\, dZ_s = \int_0^t g(Y_{s-}, Z_{s-})\, dX_s - \frac{1}{2} \int_0^t g(Y_{s-}, Z_{s-})\, d[X,X]^{\mathrm{c}}_s$$

also exists. Therefore, we can write

$$\int_0^t \langle \nabla g(Y_{s-}, Z_{s-}), d\mathbf{X}_s \rangle = -\int_0^t g(Y_{s-}, Z_{s-})\, dZ_s + \int_0^t e^{-Z_{s-}} dY_s. \tag{3.3}$$

Applying the Itô-Föllmer formula to $W = g(Y, Z) = Ye^{-Z}$ and using (3.3), we obtain

$$\begin{aligned} W_t - 1 &= \int_0^t -W_{s-} dZ_s + \int_0^t e^{-Z_{s-}} dY_s + \frac{1}{2} \int_0^t W_{s-} d[Z,Z]^{\mathrm{c}}_s \\ &\quad + \int_0^t -e^{-Z_{s-}} d[Y,Z]^{\mathrm{c}}_t + \sum_{0<s\le t} \left\{ \Delta W_s + W_{s-} \Delta Z_s - e^{-Z_{s-}} \Delta Y_s \right\} \\ &= \sum_{0<s\le t} \Delta W_s. \end{aligned}$$

Note that W satisfies

$$W_t = 1 + \int_0^t W_{s-} dR_s, \tag{3.4}$$

where R is a purely discontinuous function of locally finite variation defined by

$$R_t = \sum_{0<s\le t} \left\{ (1 + \Delta X_s) e^{-\Delta X_s} - 1 \right\}.$$

The function $U := \mathcal{E}(X)e^{-Z}$ clearly satisfies (3.4) because $\mathcal{E}(X)$ is a solution of (3.2). Hence

$$W_t - U_t = \int_0^t (W_{s-} - U_{s-})\, dR_s$$

holds for all t. Gronwall's lemma for Stieltjes integrals implies $W = U$, and so $Y = \mathcal{E}(X)$. This establishes uniqueness. □

Remark 3.2. If $X \in QV(\Pi)$ satisfies $\Delta X_t \neq -1$ for all $t \geq 0$, we have $\mathcal{E}(X)_t \neq 0$ and $\mathcal{E}(X)_{t-} \neq 0$ for all $t \geq 0$. Moreover, if $\Delta X_t > -1$ holds for all $t \geq 0$, then $\mathcal{E}(X)_t > 0$ and $\mathcal{E}(X)_{t-} > 0$ hold for all $t \geq 0$.

Remarks 2.14 and 3.2 allow us to apply the Itô-Föllmer formula (Theorem 2.13) to $1/\mathcal{E}(X)_t$. Consequently, we obtain the following lemma.

Lemma 3.3. *Let $X \in QV(\Pi)$ satisfy $\Delta X_t \neq -1$ for all $t \in \mathbb{R}_{\geq 0}$. Then $1/\mathcal{E}(X) \in QV(\Pi)$ and the following representation holds.*

$$\frac{1}{\mathcal{E}(X)_t} - 1 = -\int_0^t \frac{dX_s}{\mathcal{E}(X)_{s-}} + \int_0^t \frac{d[X,X]^c_s}{\mathcal{E}(X)_{s-}} + \sum_{0<s\leq t} \frac{1}{\mathcal{E}(X)_{s-}} \left[\frac{(\Delta X_s)^2}{1+\Delta X_s} \right].$$

With the help of Proposition 3.1 and Lemma 3.3, we obtain the following proposition. The first expression (3.5) was introduced by [26] for inhomogeneous linear SDEs.

Proposition 3.4. *Suppose that $X \in QV(\Pi)$ satisfies $\Delta X_t \neq -1$ for all $t \in \mathbb{R}_{\geq 0}$ and that H is an admissible integrand of X. Define*

$$Z_t = H_t - \mathcal{E}(X)_t \int_0^t H_{s-} d\left(\frac{1}{\mathcal{E}(X)}\right)_s, \quad t \in \mathbb{R}_{\geq 0}. \tag{3.5}$$

Then, it is the unique solution to the following inhomogeneous linear equation.

$$Z_t = H_t + \int_0^t Z_{s-} dX_s. \tag{3.6}$$

Moreover, if H is represented as

$$H_t = \int_0^t \xi_{s-} dX_s + A_t \tag{3.7}$$

by an admissible integrand ξ and $A \in FV_{\mathrm{loc}}$, then Z admits another expression as follows.

$$Z_t = \mathcal{E}(X)_t \left(H_0 + \int_0^t \frac{dH_s}{\mathcal{E}(X)_{s-}} - \int_0^t \frac{d[H,X]^c_s}{\mathcal{E}(X)_{s-}} - \sum_{0<s\leq t} \frac{\Delta H_s \Delta X_s}{\mathcal{E}(X)_{s-}(1+\Delta X_s)} \right). \tag{3.8}$$

Proof. We first prove that Z is a solution of (3.6). The existence of the Itô-Föllmer integral $Y_t := \int_0^t H_{s-} d(\mathcal{E}(X)^{-1})_s$ follows from the assumption about H, Theorem 2.19, and Lemma 3.3. Applying Corollary 2.21 and Proposition 2.17 to $Y\mathcal{E}(X)$, we get

$$\begin{aligned} Z_t - H_t = &\int_0^t H_{s-} dX_s - \int_0^t H_{s-} d[X,X]^c_s - \sum_{0<s\leq t} \frac{H_{s-}(\Delta X_s)^2}{1+\Delta X_s} \\ &- \int_0^t Y_{s-}\mathcal{E}(X)_{s-} dX_s + \int_0^t H_{s-} d[X,X]_s - \sum_{0<s\leq t} \frac{H_{s-}(\Delta X_s)^3}{1+\Delta X_s} \end{aligned}$$

$$= \int_0^t Z_{s-} dX_s.$$

Hence Z satisfies equation (3.6).

Next we show uniqueness. Let W be an arbitrary solution of (3.6). Corollary 2.21 yields

$$\frac{W_t - H_t}{\mathcal{E}(X)_t} = \int_0^t (W_{s-} - H_{s-})\, d\left(\frac{1}{\mathcal{E}(X)}\right)_s + \int_0^t \frac{W_{s-}}{\mathcal{E}(X)_{s-}} dX_s + \left[W - H, \frac{1}{\mathcal{E}(X)}\right]_t.$$

Using Proposition 2.17 and Corollary 2.9, the quadratic covariation part is given by

$$\left[W - H, \frac{1}{\mathcal{E}(X)}\right]_t = -\int_0^t \frac{W_{s-}}{\mathcal{E}(X)_{s-}} d[X,X]^{\mathrm{c}}_s - \sum_{0<s\le t} \frac{W_{s-}}{\mathcal{E}(X)_{s-}} \frac{(\Delta X_s)^2}{1+\Delta X_s}.$$

Therefore,

$$\frac{W_t - H_t}{\mathcal{E}(X)_t} = -\int_0^t H_{s-} d\left(\frac{1}{\mathcal{E}(X)}\right)_s.$$

This indicates $W = Z$ and establishes the uniqueness of solutions.

Now, suppose H is represented as (3.7). By Corollary 2.21, Proposition 2.17, and Lemma 3.3 we have

$$\frac{H_t}{\mathcal{E}(X)_t} - H_0 = \int_0^t \frac{dH_s}{\mathcal{E}(X)_{s-}} + \int_0^t H_{s-} d\left(\frac{1}{\mathcal{E}(X)}\right)_s - \int_0^t \frac{d\,[H,X]_s}{\mathcal{E}(X)_{s-}} - \sum_{0<s\le t} \frac{1}{\mathcal{E}(X)_{s-}} \frac{\Delta H_s \Delta X_s}{1+\Delta X_s}.$$

Combining this and (3.5), we obtain the expression of (3.8). □

3.2. Nonlinear equations. By combining Proposition 3.4 and results in [13, Section 3], we can solve a certain class of nonlinear equations.

Proposition 3.5. *Let $X \in QV(\Pi)$ satisfy $\Delta X_t \neq -1$ for all t, and let f be a continuous function on $\mathbb{R}_{\ge 0} \times \mathbb{R}$. Suppose that the integral equation*

$$Y_t = x + \int_0^t \frac{f(s, Y_s \mathcal{E}(X)_s)}{\mathcal{E}(X)_s} ds \tag{3.9}$$

with respect Y has a unique solution. Then, the path $Z = Y\mathcal{E}(X)$, where Y denotes the unique solution of (3.9)*, is the unique solution of*

$$Z_t = x + \int_0^t f(s, Z_s)\, ds + \int_0^t Z_{s-} dX_s. \tag{3.10}$$

Here, the terminology "solution of (3.10)" means an admissible integrand of X that satisfies (3.10).

Proof. We see that, by Corollary 2.21,

$$\begin{aligned} Y_t \mathcal{E}(X)_t &= \int_0^t \mathcal{E}(X)_s \frac{f(s, Y_s \mathcal{E}(X)_s)}{\mathcal{E}(X)_s} ds + \int_0^t Y_s \mathcal{E}(X)_{s-} dX_s \\ &= \int_0^t f(s, Y_s \mathcal{E}(X)_s)\, ds + \int_0^t Y_{s-} \mathcal{E}(X)_{s-} dX_s. \end{aligned}$$

Therefore, $Y\mathcal{E}(X)$ is a solution of (3.10). For uniqueness, let Z be an arbitrary solution of (3.10) and let

$$H_t = x + \int_0^t f(s, Z_s)\, ds.$$

Then, by Proposition 3.4, Z satisfies

$$Z_t = \mathcal{E}(X)_t \left(x + \int_0^t \frac{f(s, Z_s)}{\mathcal{E}(X)_s} ds \right).$$

Hence the path $W := Z/\mathcal{E}(X)$ satisfies (3.9). By the uniqueness of solutions of (3.9), we have $Z = W\mathcal{E}(X) = Y\mathcal{E}(X)$. □

Remark 3.6. Equation (3.9) has a unique solution if, for example, f satisfies the following conditions.

- (Local Lipschitz condition) For each $T, M > 0$, there is a constant $L_{T,M} > 0$ such that for all $x, y \in [-M, M]$ and all $t \in [0, T]$,

$$|f(t, x) - f(t, y)| \leq L_{T,M}|x - y|.$$

- (Linear growth condition) For each $T > 0$, there is a constant $K_T > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$,

$$|f(t, x)| \leq K_T(1 + |x|).$$

Under these assumptions, $g(t, y) = f(t, y\, \mathcal{E}(X)_t)/\mathcal{E}(X)_t$ again satisfies both local Lipschitz and linear growth conditions. Indeed, fix $T > 0$ and $M > 0$, and define $M' = \sup_{t\in[0,T]}|\mathcal{E}(X)_t|M$. Then g satisfies

$$|g(t, x) - g(t, y)| = L_{T,M'}|y - x|.$$

Moreover, it satisfies

$$|g(t, y)| \leq K_T \left(\frac{1}{\inf_{t\in[0,T]}|\mathcal{E}(X)_t|} + 1 \right) (1 + |y|)\, .$$

By the assumption that $\Delta X_t \neq -1$, Remark 3.2 and the càdlàg property of $\mathcal{E}(X)$, we see that $\inf_{[0,T]}|\mathcal{E}(X)_t| > 0$. Hence, g satisfies the linear growth condition.

Remark 3.7. There is another class of nonlinear equations that is solvable within our framework. [33] discuss an extension of the so called Doss–Sussmann method within the framework of Föllmer's calculus for continuous paths. The Doss–Sussmann method is a way to solve SDEs by using the solution of ODEs [12, 42].

3.3. Drawdown equations. In this section, we deal with integral equations which are called drawdown equations. We follow [4, Section 2–3], which studies Azéma–Yor processes and related drawdown equations. We interpret their results in our pathwise setting.

Given an $\mathbb{R}$-valued càdlàg path X, we define its running maximum, denoted by $\overline{X}$, through the formula $\overline{X}_t = \sup_{0\leq s\leq t} X_s$ for all $t \in \mathbb{R}_{\geq 0}$. Then the path $\overline{X}$ is càdlàg and increasing. In this subsection, we always assume that $\overline{X}$ is continuous. Recall that this continuity assumption implies

$$\int_0^t (\overline{X}_s - X_s)\, d\overline{X}_s = 0, \quad t \in \mathbb{R}_{\geq 0}. \tag{3.11}$$

Proposition 3.8. *Suppose that $X \in QV(\Pi)$ satisfies $X_0 = a$ and that the running maximum $\overline{X}$ is continuous. Let $U : [a, \infty[\to \mathbb{R}$ be a C^2 function such that $U(a) = a^*$. (We consider only right derivatives at a.) Then the function*

$$M_t^U(X) := U(\overline{X}_t) - U'(\overline{X}_t)(\overline{X}_t - X_t)$$

has quadratic variation along Π and satisfies

$$M_t^U(X) = a^* + \int_0^t U'(\overline{X}_s)\, dX_s.$$

Proof. The function $f(a, x) = U(a) - U'(a)(a - x)$ is clearly belongs to $C^{1,2}$-class. The existence of the quadratic variation of $M^U(X) = f(\overline{X}, X)$ follows from Proposition 2.7. Applying the Itô-Föllmer formula to $f(\overline{X}, X)$, we obtain

$$f(\overline{X}_t, X_t) - a^* = -\int_0^t U''(\overline{X}_s)(\overline{X}_s - X_s)\, d\overline{X}_s + \int_0^t U'(\overline{X}_s)\, dX_s.$$

By (3.11) we see that

$$\int_0^t \left| U''(\overline{X}_s)(\overline{X}_s - X_s) \right| d\overline{X}_s \leq \sup_{s \in [0,t]} |U''(\overline{X}_s)| \int_0^t (\overline{X}_s - X_s)\, d\overline{X}_s = 0.$$

Hence

$$M_t^U(X) = f(\overline{X}_t, X_t) = a^* + \int_0^t U'(\overline{X}_s)\, dX_s.$$

holds for all $t \in \mathbb{R}_{\geq 0}$. □

DEFINITION 3.9. We call the function $M^U(X)$ defined in Proposition 3.8 *the Azéma–Yor path associated with U and X*.

The following proposition, which was originally proved by [4], is still valid within the framework of Itô-Föllmer integration. It is easy to see that the proof of [4, Proposition 2.2] is pathwise and does not use Itô calculus. So we can replace the word "max-continuous semimartingale" with "path of $QV(\Pi)$ with continuous running maximum."

Proposition 3.10 ([4, Proposition 2.2])**.** (i) *Let X, U satisfy the same assumptions as in* Proposition 3.8. *Moreover, we suppose that U is increasing. Then $M^U(X)$ still has a continuous running maximum and satisfies $\overline{M_t^U(X)} = U(\overline{X}_t)$ for all $t \in \mathbb{R}_{\geq 0}$.*

(ii) *In addition to the assumptions of (i), let F be an increasing C^2 function such that $U \circ F$ is well defined. Then, $M_t^U(M^F(X)) = M_t^{U \circ F}(X)$.*

The following is a direct consequence of Propositions 3.10 and 3.8.

Corollary 3.11 ([4, Corollary 2.4])**.** *Consider a strictly increasing C^2 function U defined on $[a, \infty[$ such that $U(a) = a^*$. Let $V : [a^*, U(\infty)[\to [a, \infty[$ denotes the inverse of U where $U(\infty) = \lim_{x \to \infty} U(x)$. Moreover, suppose that $X \in QV(\Pi)$ has continuous running maximum and satisfies $X_0 = a$. Then, we have $X_t = M_t^V(M^U(X))$. If we define $Y_t = M_t^U(X)$, X and Y are expressed as follows.*

$$Y_t = a^* + \int_0^t U'(\overline{X}_s)\,dX_s, \quad X_t = a + \int_0^t V'(\overline{Y}_s)\,dY_s, \quad t \in \mathbb{R}_{\geq 0}.$$

Definition 3.12. Let X be a real-valued càdlàg path and w be a real-valued function defined on some subset of $\mathbb{R}$ including $[X_0, \infty[$. We say that X *satisfies the w-drawdown constraint* if $X_t \wedge X_{t-} > w(\overline{X}_t)$ holds for all $t \in \mathbb{R}_{\geq 0}$.

Let us consider the Azéma–Yor path $M^U(X)$ associated with a strictly positive valued X and a strictly increasing U. We can construct a function w such that $M^U(X)$ satisfies the w-drawdown constraint. Indeed, let $V = U^{-1}$, $h(x) = U(x) - xU'(x)$, and $w = h \circ V$. Then we have

$$\begin{aligned} M_t^U(X) \wedge M_{t-}^U(X) &= U(\overline{X}_t) - U'(\overline{X}_t)\overline{X}_t + U'(\overline{X}_t)(X_t \wedge X_{t-}) \\ &> U(\overline{X}_t) - U'(\overline{X}_t)\overline{X}_t = h(V(U(\overline{X}_t))) = w(\overline{M_t^U(X)}). \end{aligned}$$

Conversely, given a "nice" function w, we can construct a càdlàg path that satisfies the w-drawdown constraint as follows.

Proposition 3.13. (i) *Let $X \in QV(\Pi)$ satisfy $X_0 = a$ and $X_t \wedge X_{t-} > 0$. Suppose that its running maximum is continuous. We consider a function $w\colon [a^*, \infty[\to \mathbb{R}$ of C^1-class satisfying $y - w(y) > 0$ for all $y \in [a^*, \infty[$. Moreover, we define*

$$V(y) = a \exp\left(\int_{[a^*, y]} \frac{1}{s - w(s)}\,ds \right), \quad y \in [a^*, \infty[$$

and $U = V^{-1}$. Then $M^U(X)$ is the unique solution of

$$Y_t = a^* + \int_0^t \frac{Y_{s-} - w(\overline{Y}_s)}{X_{s-}}\,dX_s \tag{3.12}$$

that satisfies the w-drawdown constraint and has continuous running maximum.

(ii) *Let $Y \in QV(\Pi)$. Suppose that Y satisfies the w-drawdown constraint, $Y_0 = a^*$, and its running maximum is continuous. Then, $X_t = M_t^V(Y)$ is the unique càdlàg path of $QV(\Pi)$ such that $\overline{X}$ is continuous, $X_0 = a$ and X satisfies* (3.12).

The integral equation (3.12) is called a drawdown equation. The terminology "solution of (3.12)" means an admissible integrand of X that satisfies equation (3.12). To prove uniqueness, we use the following lemma.

Lemma 3.14. *Suppose X and Y of $QV(\Pi)$ satisfy X_t, X_{t-}, Y_t, $Y_{t-} > 0$, $X_0 = Y_0$ and*

$$\int_0^t \frac{dX_s}{X_{s-}} = \int_0^t \frac{dY_s}{Y_{s-}}$$

for all $t \in \mathbb{R}_{\geq 0}$. Then $X = Y$.

Proof. Let

$$Z_t = \int_0^t \frac{dX_s}{X_{s-}} = \int_0^t \frac{dY_s}{Y_{s-}}, \quad t \in \mathbb{R}_{\geq 0}.$$

Then we can easily check that both X and Y are admissible integrands of Z. Theorem 2.19 implies that X/X_0 and Y/Y_0 are solutions to the homogeneous linear equation driven by Z

with initial value 1. Hence, by Proposition 3.1, we obtain $X = X_0\mathcal{E}(Z) = Y_0\mathcal{E}(Z) = Y$. □

Proof of Proposition 3.13. (i) We first prove that $Y = M^U(X)$ satisfies (3.12). By definition, the derivative of V is $V'(y) = V(y)/(y-w(y))$ and the derivative of U is $U' = 1/V'$. Therefore, by Proposition 3.10 and Corollary 3.11, we have

$$Y_{t-} - w(\overline{Y}_t) = Y_{t-} - U(\overline{X}_t) + U'(\overline{X}_t)V(U(\overline{X}_t)) = U'(\overline{X}_t)X_{t-}.$$

This proves

$$Y_t = \int_0^t U'(\overline{X}_s)dX_s = \int_0^t \frac{Y_{s-} - w(\overline{Y}_s)}{X_{s-}}dX_s.$$

The Itô-Föllmer integral of this equation is well-defined since $U'(\overline{X})$ has locally finite variation.

Next, we show uniqueness. Suppose that Y is a solution of (3.12) satisfying the w-drawdown constraint and that the running maximum $\overline{Y}$ is continuous. Since Y satisfies the w-drawdown constraint, $M_t^V(Y)$ and $M_{t-}^V(Y)$ are strictly positive. It is easy to see that $\frac{Y-w(\overline{Y})}{X}$ is an admissible integrand of X, and $\frac{1}{Y-w(\overline{Y})}$ is an admissible integrand of X. Then we have

$$\int_0^t \frac{dY_s}{Y_{s-} - w(\overline{Y}_s)} = \int_0^t \frac{1}{Y_{s-} - w(\overline{Y}_s)} \frac{Y_{s-} - w(\overline{Y}_s)}{X_{s-}}dX_s = \int_0^t \frac{dX_s}{X_{s-}}$$

thanks to Theorem 2.19. Moreover, we see that

$$\int_0^t \frac{dY_s}{Y_{s-} - w(\overline{Y}_s)} = \int_0^t \frac{V'(\overline{Y}_s)}{M_{s-}^V(Y)}dY_s = \int_0^t \frac{dM_s^V(Y)}{M_{s-}^V(Y)}.$$

The second equality follows from Theorem 2.19. Hence,

$$\int_0^t \frac{dX_s}{X_{s-}} = \int_0^t \frac{dM_s^V(Y)}{M_{s-}^V(Y)}.$$

This proves $X = M^V(Y)$ because of Lemma 3.14. Then we can deduce that $Y_t = M_t^U(X)$ from Corollary 3.11.

(ii) Suppose that $Y \in QV(\Pi)$ satisfies the w-drawdown constraint and $Y_0 = a^*$, and that the running maximum $\overline{Y}$ is continuous. Let $X := M^V(Y)$. Then, Y is an admissible integrand of X due to Corollary 3.11 and the definition of an Azéma–Yor path. Hence $\frac{Y-w(\overline{Y})}{M^V(Y)}$ is also an admissible integrand of X. By the associativity of Itô-Föllmer integrals, we have

$$\int_0^t \frac{Y_{s-} - w(\overline{Y}_s)}{M_{s-}^V(Y)}dM_s^V(Y) = \int_0^t \frac{Y_{s-} - w(\overline{Y}_s)}{M_{s-}^V(Y)}V'(\overline{Y}_s)\,dY_s = \int_0^t \frac{M_{s-}^V(Y)}{M_{s-}^V(Y)}dY_s = Y_t - Y_0.$$

Therefore $X = M^V(Y)$ satisfies (3.12). Let Z be any càdlàg path of $QV(\Pi)$ satisfying (3.12) such that $Z_0 = a$ and $\overline{Z}$ is continuous. Then we can show

$$\int_0^t \frac{dZ_s}{Z_{s-}} = \int_0^t \frac{dM_s^V(Y)}{M_{s-}^V(Y)},$$

in the same way as the proof of the first part. Consequently, we obtain $M^V(Y) = Z$. □

3.4. Applications to Finance.

3.4.1. Model-free CPPI with jumps. As an application of the results for linear integral equations, we consider model-free constant proportion portfolio insurance (CPPI) and dynamic proportion portfolio insurance (DPPI) strategies with jumps, which extend the results of [38].

Throughout this section, we will fix a sequence of partitions $\Pi = (\pi_n)_{n\in\mathbb{N}}$ such that $|\pi_n| \to 0$ as $n \to \infty$. Assume that there is one riskless asset and one risky asset in the market. The price process of the riskless asset B is a càdlàg path of locally finite variation with $B_0 = 1$. The risky asset price process S belongs to $QV(\Pi)$. We suppose that both B and S are strictly positive valued and that B_{t-} and S_{t-} are also strictly positive. We consider a portfolio that consists of ξ_t units of the risky asset and η_t units of the riskless asset at time t. We will assume that ξ is an admissible integrand of S and η is a càdlàg path. The pair (ξ, η) is called a trading strategy. The value of the portfolio at time t is defined by $V_t = \xi_t S_t + \eta_t B_t$. A strategy (ξ, η) is self-financing if

$$V_t = V_0 + \int_0^t \xi_{s-} dS_s + \int_0^t \eta_{s-} dB_s$$

holds for all t.

Let K be a nonnegative càdlàg path of locally finite variation. A DPPI strategy, introduced by [38], is a trading strategy given by

$$\xi_t = \frac{m_t(V_t - K_t)}{S_t}, \quad \eta_t = \frac{V_t - \xi_t S_t}{B_t}, \tag{3.13}$$

where the multiplier $m \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is an admissible integrand of X. When m is constant, this is a so called CPPI strategy. Our aim is to construct a self-financing DPPI strategy satisfying $V_t \geq K_t$ for all $t \in \mathbb{R}_{\geq 0}$. From now on, we assume that K has the form $K = LB$, where L is a nonincreasing càdlàg function.

Proposition 3.15. *Let $S \in QV(\Pi)$ be a price process of a risky asset, $B \in FV_{\mathrm{loc}}$ be that of a riskless asset and $V_0 \geq 0$ be the initial wealth. Suppose that S, S_-, B, B_- are strictly positive and L is a nonincreasing path such that $V_0 \geq L_0$. For an admissible integrand $m : \mathbb{R}_{\geq 0} \to \mathbb{R}$ of S, we define a path X by*

$$X_t = \int_0^t \frac{m_{s-}}{S_{s-}} dS_s + \int_0^t \frac{(1 - m_{s-})}{B_{s-}} dB_s.$$

If m satisfies $\Delta X_t \neq -1$ for all t, the DPPI strategy given below is self-financing.

$$V_t = L_t B_t + \mathcal{E}(X)_t \left(V_0 - L_0 - \int_0^t \frac{B_{s-}}{\mathcal{E}(X)_{s-}} dL^c_s - \sum_{0<s\leq t} \frac{B_s}{\mathcal{E}(X)_{s-}} \frac{\Delta L_s}{1 + \Delta X_s} \right),$$

$$\xi_t = \frac{m_t(V_t - L_t B_t)}{S_t}, \quad \eta_t = \frac{V_t - \xi_t S_t}{B_t}.$$

Moreover, if the condition $\Delta X_t > -1$ holds for all $t \geq 0$, the value process of this portfolio satisfies $V_t \geq L_t B_t$ for all t.

Proof. Let $K_t = L_t B_t$. If the DPPI strategy (ξ, η) given by (3.13) is self-financing, the value process must satisfy

$$V_t - V_0 = \int_0^t \frac{m_{s-}(V_{s-} - K_{s-})}{S_{s-}} dS_s + \int_0^t \frac{V_{s-} - m_{s-}(V_{s-} - K_{s-})}{B_{s-}} dB_s. \tag{3.14}$$

Set

$$X_t = \int_0^t \frac{m_{s-}}{S_{s-}} dS_s + \int_0^t \frac{(1 - m_{s-})}{B_{s-}} dB_s, \quad H_t = V_0 - \int_0^t K_{s-} \left[dX_s - \frac{1}{B_{s-}} dB_s \right].$$

These paths are well-defined by assumption. Then (3.14) can be rewritten as

$$V_t = H_t + \int_0^t V_{s-} dX_s. \tag{3.15}$$

Because X satisfies $\Delta X_t \neq -1$ for all t, the equation (3.15) has the unique solution

$$V_t = \mathcal{E}(X)_t \left(H_0 + \int_0^t \frac{dH_s}{\mathcal{E}(X)_{s-}} - \int_0^t \frac{d[H, X]^c_s}{\mathcal{E}(X)_{s-}} - \sum_{0<s\leq t} \frac{1}{\mathcal{E}(X)_{s-}} \frac{\Delta H_s \Delta X_s}{1 + \Delta X_s} \right) \tag{3.16}$$

by Proposition 3.4. Consequently, if V is given by (3.16), the DPPI strategy of (3.13) is self-financing. Using Lemma 3.3, Corollaries 2.21 and 2.9, and (3.16) we have

$$\frac{V_t}{\mathcal{E}(X)_t} - \frac{K_t}{\mathcal{E}(X)_t} = V_0 - L_0 - \int_0^t \frac{B_{s-}}{\mathcal{E}(X)_{s-}} dL^c_s - \sum_{0<s\leq t} \frac{B_s}{\mathcal{E}(X)_{s-}} \cdot \frac{\Delta L_s}{1 + \Delta X_s}. \tag{3.17}$$

This proves the first part of the assertion.

Now suppose that $\Delta X_t > -1$ holds for all $t \in \mathbb{R}_{\geq 0}$. Then by Proposition 3.2 we see that $\mathcal{E}(X)_t$ is strictly positive. (3.17) implies that $V_t/\mathcal{E}(X)_t - K_t/\mathcal{E}(X)_t \geq 0$ holds for all t since L is nonincreasing and satisfies $V_0 \geq L_0$. Hence $V_t \geq K_t$ holds for all $t \in \mathbb{R}_{\geq 0}$. □

Let us consider a case where the multiple m is constant in Proposition 3.15. If $m_t = \overline{m}$ for all t, the process X in the proposition is

$$X_t = \overline{m} \int_0^t \frac{1}{S_{s-}} dS_s + (1 - \overline{m}) \int_0^t \frac{1}{B_{s-}} dB_s.$$

Then we have

$$\Delta X_t = \overline{m} \frac{\Delta S_t}{S_{t-}} + (1 - \overline{m}) \frac{\Delta B_t}{B_{t-}}.$$

By the assumptions $S, S_-, B, B_- > 0$, the conditions $\Delta S_t/S_{t-} > -1$ and $\Delta B_t/B_{t-} > -1$ are always satisfied. If we choose $\overline{m}$ such that $0 \leq \overline{m} \leq 1$, we have $\Delta X_{t-} > -1$. In this case, the value process V always satisfies $V_t \geq K_t$.

3.4.2. Portfolio strategies satisfying drawdown constraint. In this subsection, we will consider portfolio strategies satisfying the drawdown constraint as an application of observations about drawdown equations. Let S, B be the same paths as in Section 3.4.1, V_0 be initial wealth and w be a function satisfying the assumptions of Proposition 3.13. Our purpose is to find a self-financing trading strategy satisfying the w-drawdown constraint. First we show the following characterization of the self-financing condition. This equivalence is a well-known result in the classical Itô calculus framework. It also holds within our pathwise framework.

Proposition 3.16. *Let $S \in QV(\Pi)$ be a risky asset, $B \in FV_{\mathrm{loc}}$ be a riskless asset with initial price $B_0 = 1$ and $V_0 \geq 0$ be an initial wealth. Suppose that all of S, S_-, B, B_- are strictly positive valued. Define $\widetilde{V} = V/B$ and $\widetilde{S} = S/B$. Then for a trading strategy (ξ, η), the following two conditions are equivalent.*

(i) *The trading strategy (ξ, η) is self-financing.*

(ii) *For all $t \in \mathbb{R}_{\geq 0}$,*

$$\widetilde{V}_t = V_0 + \int_0^t \xi_{s-} d\widetilde{S}_s. \tag{3.18}$$

Proof. We first show that an admissible integrand ξ is also an admissible integrand of $\widetilde{S}$. Fix $T > 0$ and choose $A \in FV^m_{\mathrm{loc}}$ and a function F of $C^{1,2}$-class that satisfies $\xi_t = \frac{d}{dx}F(A_t, S_t)$ ($t \in [0, T]$). Then ξ can be represented as $\xi = \frac{d}{dx}g(A_t, B_t, \widetilde{S}_t)$, where $g(a, b, x) = F(a, bx)/b$. This means that ξ is an admissible integrand of $\widetilde{S}$.

Next we prove that (i) implies (ii). By Corollary 2.21, we have

$$\begin{aligned} \frac{V_t}{B_t} &= \frac{V_0}{B_0} + \int_0^t \xi_{s-}\left(S_{s-}d\left(\frac{1}{B}\right)_s + \frac{1}{B_{s-}}dS_s + d\left[S, \frac{1}{B}\right]_s\right) \\ &\quad + \int_0^t \eta_{s-}\left(B_{s-}d\left(\frac{1}{B}\right)_s + \frac{1}{B_{s-}}dB_s + d\left[B, \frac{1}{B}\right]_s\right), \end{aligned} \tag{3.19}$$

$$\widetilde{S}_t = \frac{S_0}{B_0} + \int_0^t S_{s-}d\left(\frac{1}{B}\right)_s + \int_0^t \frac{1}{B_{s-}}dS_s + \left[S, \frac{1}{B}\right]_t, \tag{3.20}$$

$$\int_0^t B_{s-}d\left(\frac{1}{B}\right)_s + \int_0^t \frac{1}{B_{s-}}dB_s + \left[B, \frac{1}{B}\right]_t = 0.$$

Combining these three equations, we obtain (3.18).

It remains to prove that (ii) implies (i). Using condition (ii), Corollary 2.21, Theorem 2.19, (3.20) and (3.19), we get

$$V_t = V_0 + \int_0^t \xi_{s-}dS_s + \int_0^t \eta_{s-}dB_s + \int_0^t \xi_{s-}\left(B_{s-}d\left[S, \frac{1}{B}\right]_s + \frac{1}{B_{s-}}d[S, B]_s + d\left[\left[S, \frac{1}{B}\right], B\right]_s\right).$$

The integrator of the last term satisfies

$$\begin{aligned} &\int_0^t B_{s-}d\left[S, \frac{1}{B}\right]_s + \int_0^t \frac{1}{B_{s-}}d[S, B]_s + \left[\left[S, \frac{1}{B}\right], B\right]_t \\ &= \sum_{0<s\leq t} \Delta S_s\left(B_{s-}\Delta\frac{1}{B_s} + \frac{1}{B_{s-}}\Delta B_s + \Delta\frac{1}{B_s}\Delta B_s\right) = 0. \end{aligned}$$

Therefore, V satisfies the self-financing condition. □

According to Proposition 3.16, we can assume that $B = 1$ identically when we consider a self-financing portfolio. Now we apply the results about Azéma–Yor paths and drawdown equations to trading strategies.

Proposition 3.17. *Suppose that the running maximum of the risky price process $S \in QV(\Pi)$ is continuous and that the riskless price process is identically 1. Let $V_0 \geq 0$ be an initial wealth, and suppose $w\colon [V_0, \infty[\to \mathbb{R}$ satisfies the same assumptions as in Proposition 3.13. Functions U and W are defined as*

$$W(y) = S_0 \exp\left(\int_{[V_0,y]} \frac{1}{s - w(s)} ds\right), \quad y \in [V_0, \infty[,$$

and $U = W^{-1}$. If we set

$$\xi_t = U'(\overline{S}_t), \quad \eta_t = M^U(S)_t - \xi_t S_t,$$

the trading strategy (ξ, η) is self-financing. Furthermore, the value of portfolio $V := \xi S + \eta$ satisfies the w-drawdown constraint in the sense of Definition 3.12.

Proof. The self-financing property of the strategy (ξ, η) follows from Proposition 3.8. Moreover by Proposition 3.13, we see that $V = M^U(S)$ satisfies w-drawdown constraint. □

Appendix A On the convergence of a sequence of discrete measures on $\mathbb{R}_{\geq 0}$

Let us consider a sequence of measures (μ_n) of the form

$$\mu_n = \sum_{i \geq 0} a_i^n \delta_{t_i^n}, \tag{A.1}$$

where $a_i^n \geq 0$ and $\pi_n = (t_i^n)_{i \in \mathbb{N}}$ is a partition of $\mathbb{R}_{\geq 0}$.

Lemma A.1. *Let $\Pi = (\pi_n)_{n \in \mathbb{N}}$ be a partition of $\mathbb{R}_{\geq 0}$ satisfying $|\pi_n| \to 0$. Suppose that, for each $n \in \mathbb{N}$, μ_n is a locally finite measure of the form* (A.1) *and the distribution functions of (μ_n) converge pointwise to that of a positive locally finite measure μ with $\mu(\{0\}) = 0$. Moreover, we assume the following condition.*

- ($*$) *For each $t \in \mathbb{R}_{\geq 0}$ take a sequence (i_n) such that $t \in]t_{i_n}^n, t_{i_n+1}^n]$. Then $(a_{i_n}^n)_n$ converges to $\mu(\{t\})$.*

Then for all $f \in D(\mathbb{R}_{\geq 0}, \mathbb{R})$ and all $t \in \mathbb{R}_{\geq 0}$, we have

$$\lim_{n \to \infty} \int_{[0,t]} f(s) \mu_n(ds) = \int_{[0,t]} f(s-) \mu(ds).$$

Remark A.2. Condition ($*$) is true if (μ_n) is given by

$$\mu_n = \sum_i \mu(]t_i^n, t_{i+1}^n]) \delta_{t_i^n}.$$

This condition is also satisfied if $\mu_n = \mu_X^{\pi_n}$ for $X \in QV(\Pi)$. In this case, Condition ($*$) corresponds to Condition (ii) of Definition 2.1.

Proof. Let $D_t = \{s \in [0, t] \mid \Delta f(s) \neq 0\}$ and $D_t^n = \{s \in D_t \mid |\Delta f(s)| > 1/n\}$. For $m \in \mathbb{N}_{\geq 1}$ define finite positive measures on $[0, t]$ by

$$\mu'_m(E) = \mu(D_t^m \cap E), \quad \mu''_m(E) = \mu(E) - \mu'_m(E) = \mu(E \setminus D_t^m),$$
$$\mu'_{m,n} = \sum_{t_i^n \leq t} a_i^n 1_{\{i \mid D_t^m \cap]t_i^n, t_{i+1}^n] \neq \emptyset\}}(i) \delta_{t_i^n}, \quad \mu''_{m,n} = \mu_n - \mu'_{m,n}.$$

We can deduce from the finiteness of D_t^m and the condition ($*$) that

$$\lim_{n \to \infty} \int_{[0,t]} f(s) \mu'_{m,n}(ds) = \int_{[0,t]} f(s-) \mu'_m(ds). \tag{A.2}$$

Moreover, we see the distribution functions of $(\mu'_{m,n})_n$ converge pointwise to that of μ'_m on $[0,t]$; similarly the distribution functions of $(\mu''_{m,n})_n$ converge to that of μ''_m.

Let $C = \sup_n \mu_n([0,t]) + \mu([0,t])$. Now we fix an arbitrary $\delta > 0$ and pick m such that $1/m < \delta/3C$. Take a function g of the form

$$g = f(0)1_{\{0\}} + \sum_{1\le i\le N} b_{i-1}1_{]s_{i-1},s_i]}, \quad 0 = s_0 < s_1 < \cdots < s_N \le t,$$

such that $|g(t) - f(t-)| \le \delta/3C$ on $[0,t]$. Then, we have

$$\begin{aligned}
&\left|\int_{[0,t]} f(s)\mu''_{m,n}(ds) - \int_{[0,t]} f(s-)\mu''_m(ds)\right| \\
&\quad\le \left|\int_{[0,t]} \{f(s) - f(s-)\}\,\mu''_{m,n}(ds)\right| + \left|\int_{[0,t]} \{f(s-) - g(s)\}\,\mu''_{m,n}(ds)\right| \\
&\qquad + \left|\int_{[0,t]} g(s)\mu''_{m,n}(ds) - \int_{[0,t]} g(s)\mu''_m(ds)\right| + \left|\int_{[0,t]} \{g(s) - f(s-)\}\,\mu''_m(ds)\right| \\
&\quad= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By assumption we see that, for sufficiently large n,

$$I_1 \le \frac{1}{m}\sup_n \mu_n([0,t]) \le \frac{\delta}{3}, \quad I_2 \le \frac{\delta}{3C}\sup_n \mu_n([0,t]) \le \frac{\delta}{3}, \quad I_4 \le \frac{\delta}{3C}\mu([0,t]) \le \frac{\delta}{3}.$$

Furthermore we have $\lim_{n\to\infty} I_3 = 0$ by assumption. Consequently,

$$\overline{\lim_{n\to\infty}}\left|\int_{[0,t]} f(s)\mu''_{m,n}(ds) - \int_{[0,t]} f(s-)\mu''_m(ds)\right| \le \delta. \tag{A.3}$$

Combining (A.2) and (A.3), we obtain

$$\overline{\lim_{n\to\infty}}\left|\int_{[0,t]} f(s)\mu_n(ds) - \int_{[0,t]} f(s-)\mu(ds)\right| \le \delta.$$

Because δ is chosen arbitrarily, we have the desired result. □

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