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TOROIDAL SURGERIES AND THE GENUS OF A KNOT

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Abstract
In this paper we give an upper bound for the slopes yielding an incompressible torus by surgery on a hyperbolic knot in the 3-sphere in terms of its genus.

1. Introduction

One way to get closed 3-manifolds from a knot in S^3 is the so-called *Dehn surgery*, which consists of removing a regular neighborhood $\mathcal{N}(K)$ of the knot K and fill it with a solid torus glued differently.

The different ways of doing surgery on a knot K are parametrized by *slopes*, that is, isotopy classes of essential closed simple curves in $\partial\mathcal{N}(K)$, which can be identified with $\mathbb{Q} \cup \{1/0\}$ ([25]). Given a slope r on $\partial\mathcal{N}(K)$, we use $K(r)$ to denote the result of *r-Dehn surgery* on K, that is, the 3-manifold obtained by gluing a solid torus J to the exterior of K, $E(K) = S^3 - \mathrm{int}\mathcal{N}(K)$, in such a way that a meridian of J is identified with a curve of slope r.

It is known that any closed and orientable 3-manifold can be obtained by means of Dehn surgery in a link in S^3 ([33]) and ([18]).

By Thurston [32], a knot K in S^3 is either *hyperbolic* (i.e. its complement admits a complete Riemannian metric of constant sectional curvature -1), satellite (i.e. its exterior contains an incompressible and not ∂-parallel 2-torus), or it is a torus knot (i.e. its exterior is Seifert fibered). By Thurston [32], if K is hyperbolic then $K(r)$ will be hyperbolic for all but finitely many slopes r. The slopes that produce non-hyperbolic manifolds, whereas K is hyperbolic, are called *exceptional slopes*. According to Perelman [22, 23, 24] and Thurston [32], if a closed orientable 3-manifold is non-hyperbolic then it is either reducible (i.e. it contains an essential 2-sphere), toroidal (i.e. it contains an incompressible 2-torus), or it is Seifert fibered.

Many works in low dimensional topology are focused on exceptional slopes; see [6, 7, 8, 9]. In particular, if K is a hyperbolic knot, then $K(r)$ contains an incompressible torus for only finitely many slopes r ([10] or [32]). Such slopes r and the corresponding surgeries are said to be *toroidal*. Gordon and Luecke [11] proved that any toroidal slope $r = p/q$ is either

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integral or half-integral, that is, either $|q| = 1$ or 2. On the other hand, Eudave-Muñoz [2] constructed an infinite family of hyperbolic knots $K(l, m, n, p)$ having half-integral toroidal surgeries. Surprisingly, Gordon and Luecke proved that if K admits a half-integral toroidal surgery, then K is one of Eudave-Muñoz knots $K(l, m, n, p)$ [12].

Teragaito obtained several results concerning integral toroidal surgeries. He showed that every integer is a toroidal surgery slope for some hyperbolic knot [26]. He also showed that any two integral toroidal slopes r and s for a hyperbolic knot satisfy $|r - s| \leq 4$ unless K is the figure eight knot [28]. Furthermore, Gordon and Wu have determined all hyperbolic knots which have integral toroidal slopes r, s, with $|r - s| = 4$.

The hitting number of a toroidal surgery $K(r)$, denoted by t, is the minimal intersection number between a core of the attached solid torus and all incompressible tori in $K(r)$. For half-integral surgery, it is known that always $t = 2$. For integral surgery many examples have been constructed such that $t = 4$; see [3], [5], [29], [30]. It is also known that there are infinitely many knots with $t \geq 6$; indeed in [21], Osoinach gave an infinite family of hyperbolic knots, with a toroidal surgery, for which there is no upper bound for the hitting number t. A precise determination of the hitting number t for these examples is given in [31]

Let $g(K)$ denote the genus of the knot K. In 2003 Teragaito proposed the following conjecture in [27].

Conjecture 1. *If a hyperbolic knot K in S^3 has a toroidal slope r, then $|r| \leq 4g(K)$.*

It follows from the work of Ichihara [15] that $|r| \leq 3 \cdot 2^{7/4} g(K)$. Teragaito proved that his conjecture is true for genus one knots and alternating knots. On the other hand, S. Lee [17] proved that this conjecture is also true for genus two knots. For the case when K is hyperbolic and $K(r)$ contains a Klein bottle, which in many cases is toroidal, Ichihara and Teragaito [16] proved that $|r| \leq 4g(K)$.

The goal of this paper is to give upper bounds for toroidal slopes close to the conjectured by Teragaito, we prove the following

Theorem 1. *If a hyperbolic knot K in S^3 has a toroidal slope r, then $|r| \leq 4g(K) - \frac{3}{2}$ if r is half-integral, $|r| \leq 4g(K)$ if r is integral and $t \geq 6$, $|r| \leq 6g(K) - 3$ if r is integral and $t = 4$ and $|r| \leq 4g(K) + 8$ if r is integral and $t = 2$.*

EXAMPLES. The pretzel knot $K(-2, 3, 7)$ has genus five and its set of toroidal slopes is $\{16, 37/2, 20\}$. For $r = 37/2$, we may note that the upper bound is reached in this case.

If K is the figure eight knot, its genus is one and its set of toroidal-slopes is $\{-4, 0, 4\}$. But -4 and 4 are also the slopes yielding Klein bottles, and the complement of K contains a once-punctured Klein bottle.

To prove Theorem 1 we proceed as follows. By [11] we know that the toroidal slope r is integral or half-integral. In section 2 we prove the Theorem when the slope is half-integral; in this case we know all the knots with a toroidal surgery, and furthermore the genus and toroidal slopes of such knots can be calculated, so this gives a way to verify the given bound. In section 3 we develop the case when the toroidal slope is integral, this is done by means of graphs of intersection.

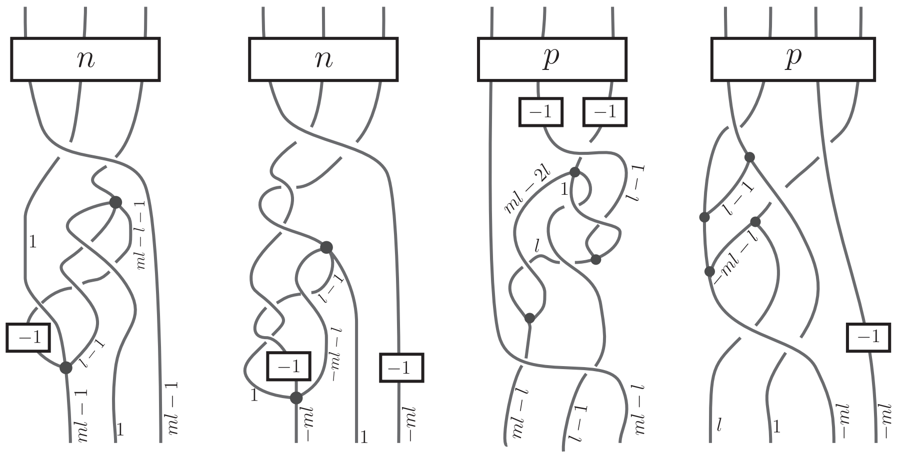
2. Half-integral slope.

Let K be a hyperbolic knot with a half-integral toroidal slope r. By [12], K is an Eudave-Muñoz knot $K(l, m, n, p)$, a family of knots parametrized by four integers l, m, n, p (where at least one of p and n is zero). Excluding the cases when $l = 0, \pm 1$, $m = 0$, $(l, m) = \{(2, 1), (-2, -1)\}$, $(m, n) = \{(1, 0), (-1, 1)\}$ and $(l, m, p) = (2, 2, 1)$, the knots result to be hyperbolic with only one half-integral toroidal slope. By [4], this slope is given by

$$r = \begin{cases} l(2m-1)(1-lm) + n(2lm-1)^2 - 1/2, & \text{for } K(l, m, n, 0) \text{ knots} \\ l(2m-1)(1-lm) + p(2lm-l-1)^2 - 1/2, & \text{for } K(l, m, 0, p) \text{ knots} \end{cases}$$

The mirror image of the knot $K(l, m, n, 0)$ is the knot $K(-l, -m, 1-n, 0)$, and the mirror image of $K(l, m, 0, p)$ is the knot $K(-l, 1-m, 0, 1-p)$ [4]. So, in what follows there is no loss of generality in assuming that $l > 0$.

In Figures 12, 13 of [4], an explicit presentation of the knots $K(l, m, n, p)$ as closed braids is given. However, Yi Ni observed that there is a discrepancy in the parameters in Figure 13 of [4]. Correct presentations of the knots are given in Figure 2 of [20]. In Fig. 1 we give a presentation of the knots in the case $l > 0$, $n \leq 0$, $p \leq 0$ (which is Figure 2 of [20]), and in Fig. 2 we give a presentation of the knots in the case $l > 0$, $n > 0$, $p > 0$.



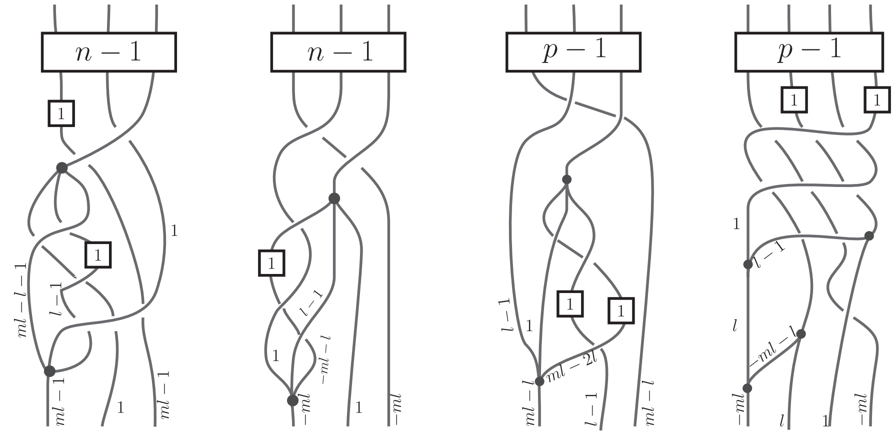
$p = 0, m > 0, n \leq 0$ $\quad p = 0, m < 0, n \leq 0$ $\quad n = 0, m > 0, p \leq 0$ $\quad n = 0, m < 0, p \leq 0$

Fig. 1

It follows from this presentation that $K(l, m, n, p)$ is the closure of a positive or negative braid, hence it is fibered, and the genus can be computed by the formula $(C - N + 1)/2$, where C is the crossing number in the positive or negative braid and N is the index [4].

The value of N is given by

$$N = \begin{cases} 2lm - 1, & \text{if } l > 0, m > 0 \text{ and } p = 0. \\ -2lm + 1, & \text{if } l > 0, m < 0 \text{ and } p = 0. \\ 2lm - l - 1, & \text{if } l > 0, m > 0 \text{ and } n = 0. \\ -2lm + l + 1, & \text{if } l > 0, m < 0 \text{ and } n = 0. \end{cases}$$



$p=0, m>0, n>0$ $p=0, m<0, n>0$ $n=0, m>0, p>0$ $n=0, m<0, p>0$

Fig. 2

The explicit value of C is given in the following table ("$>$" means >0, and the analogous for $<, \leq, \geq$).

l	m	n	p	Crossing number C
$>$	$>$	$\leq$	0	$2l^2m^2 - l^2m - 3lm + 2l - n(2lm-1)(2lm-2)$
$>$	$>$	$>$	0	$2l^2m^2 + l^2m - 3lm - 2l + 2 + (n-1)(2lm-1)(2lm-2)$
$>$	$<$	$\leq$	0	$2l^2m^2 - l^2m - lm - n(2lm)(2lm-1)$
$>$	$<$	$>$	0	$2l^2m^2 + l^2m - lm + (n-1)(2lm)(2lm-1)$
$>$	$>$	0	$\leq$	$2l^2m^2 - l^2m - 3lm + l - p(2lm-l-1)(2lm-l-2)$
$>$	$>$	0	$>$	$2l^2m^2 - 3l^2m - 3lm + l^2 + 2l + 2 + (p-1)(2lm-l-1)(2lm-l-2)$
$>$	$<$	0	$\leq$	$2l^2m^2 - l^2m - lm + l - p(2lm-l-1)(2lm-l)$
$>$	$<$	0	$>$	$2l^2m^2 - 3l^2m - lm + l^2 + (p-1)(2lm-l-1)(2lm-l)$

Computing the genus g of $K(l,m,n,p)$ by the formula $(C-N+1)/2$ we have,

l	m	n	p	Genus g
$>$	$>$	$\leq$	0	$1/2[2l^2m^2 - l^2m - 5lm + 2l + 2 - n(2lm-1)(2lm-2)]$
$>$	$>$	$>$	0	$1/2[2l^2m^2 + l^2m - 5lm - 2l + 4 + (n-1)(2lm-1)(2lm-2)]$
$>$	$<$	$\leq$	0	$1/2[2l^2m^2 - l^2m + lm - n(2lm)(2lm-1)]$
$>$	$<$	$>$	0	$1/2[2l^2m^2 + l^2m + lm + (n-1)(2lm)(2lm-1)]$
$>$	$>$	0	$\leq$	$1/2[2l^2m^2 - l^2m - 5lm + 2l + 2 - p(2lm-l-1)(2lm-l-2)]$
$>$	$>$	0	$>$	$\frac{1}{2}[2l^2m^2 - 3l^2m - 5lm + l^2 + 3l + 4 + (p-1)(2lm-l-1)(2lm-l-2)]$
$>$	$<$	0	$\leq$	$1/2[2l^2m^2 - l^2m + lm - p(2lm-l-1)(2lm-l)]$
$>$	$<$	0	$>$	$1/2[2l^2m^2 - 3l^2m + lm + l^2 - l + (p-1)(2lm-l-1)(2lm-l)]$

We will see that $4g - |r| \geq 3/2$ for each case.

CASE 1. $l, m > 0, n \leq 0$ and $p = 0$.

Note that if $l, m > 0$ and $n \leq 0$ then $|r| = -r$.

Therefore $4g - |r| = lm(l(2m-1) - 8) + 3l + n(-4l^2m^2 + 8lm - 3) + 7/2$.

Suposse $n = 0$, then $4g - |r| = lm(l(2m-1) - 8) + 3l + 7/2$. Since $(m, n) \neq (1, 0)$ and $l \neq \pm 1$, then $l, m \geq 2$. If $l = m = 2$, then $4g - |r| = lm(l(2m-1) - 8) + 3l + 7/2 = 2l(3l-8) + 3l + 7/2 = 3/2$. When $l \geq 3$ and $m \geq 2$, $4g - |r| = lm(l(2m-1) - 8) + 3l + 7/2 \geq lm + 3l + 7/2 \geq 37/2$. When $m \geq 3$ and $l \geq 2$, $4g - |r| = lm(l(2m-1) - 8) + 3l + 7/2 \geq 2lm + 3l + 7/2 \geq 43/2$.

Now we can assume that $n \leq -1$ and $l \geq 2$. Then $4g - |r| = lm(l(2m-1) - 8) + 3l + n(-4l^2m^2 + 8lm - 3) + 7/2 \geq lm(l(2m-1) - 8) + 3l + 4l^2m^2 - 8lm + 3 + 7/2 = 6l^2m^2 - l^2m - 16lm + 3l + 13/2 = lm(l(6m-1) - 16) + 3l + 13/2$. If $m = 1$, then $l \geq 3$ (since $(l, m) \neq (2, 1)$), then $4g - |r| \geq lm(l(6m-1) - 16) + 3l + 13/2 \geq -l + 3l + 13/2 \geq 25/2$. If $m \geq 2$, then $4g - |r| \geq lm(l(6m-1) - 16) + 3l + 13/2 \geq 12l + 3l + 13/2 \geq 43/2$.

In both cases, $4g - |r| \geq 3/2$.

CASE 2. $l, m, n > 0$ and $p = 0$.

Here $|r| = r$. Also, we have $4g - |r| = 2l^2m^2 + l^2m - 8lm - 3l + 15/2 + (n-1)(4l^2m^2 - 8lm + 3)$. Note that $lm \geq 3$ since $l \neq \pm 1$ and $(l, m) \neq (2, 1)$.

If $n = 1$, then $4g - |r| = 2l^2m^2 + l^2m - 8lm - 3l + 15/2 = 2(lm-2)^2 + l(lm-3) - 1/2 \geq 3/2$.

Now suposse that $n \geq 2$ and $l \geq 2$. Then $4g - |r| = 2l^2m^2 + l^2m - 8lm - 3l + 15/2 + (n-1)(4l^2m^2 - 8lm + 3) \geq 2l^2m^2 + l^2m - 8lm - 3l + 15/2 + 4l^2m^2 - 8lm + 3 = 6l^2m^2 + l^2m - 16lm - 3l + 21/2 = 2l^2m^2 + 4(lm-2)^2 + l(lm-3) - 11/2 \geq 18 + 4 - 11/2 = 33/2$.

In both cases, $4g - |r| \geq 3/2$.

CASE 3. $l > 0, m < 0, n \leq 0$ and $p = 0$.

Here $|r| = -r$. Also we have, $4g - |r| = 2l^2m^2 - l^2m + 4lm - l - n(4l^2m^2 - 1) - 1/2$. Note that $lm \leq -2$ since $l \geq 2$ and $m \leq -1$.

Suposse $n = 0$, then $4g - |r| = 2l^2m^2 - l^2m + 4lm - l - 1/2 = 2(lm+1)^2 + l(-lm-1) - 5/2 \geq 2 + 2 - 5/2 = 3/2$.

Now we can assume that $n \leq -1$, then $4g - |r| \geq 2l^2m^2 - l^2m + 4lm - l + 4l^2m^2 - 1 - 1/2 = 6l^2m^2 - l^2m + 4lm - l - 3/2 = 4l^2m^2 + 2(lm+1)^2 + l(-lm-1) - 7/2 \geq 16 + 2 + 2 - 7/2 = 33/2 > 3/2$.

In both cases, $4g - |r| \geq 3/2$.

CASE 4. $l > 0$, $m < 0$, $n > 0$ and $p = 0$.

Since $n \geq 1$, then $r \geq l(lm-1)(2m+1) + 1/2 \geq 0$, so $|r| = r$. Therefore $4g - |r| = 2l^2m^2 + l^2m + 4lm + l - 1/2 + (n-1)(4l^2m^2 - 1)$.

Suposse $n = 1$, then $m \leq -2$ (since $(m, n) \neq (-1, 1)$) and $4g - |r| = 2l^2m^2 + l^2m + 4lm + l - 1/2 \geq (3l-4)(2l) + l - 1/2 \geq 19/2$.

Now, if $n \geq 2$, then $4g - |r| \geq 2l^2m^2 + l^2m + 4lm + l - 1/2 + 4l^2m^2 - 1 = lm(l(6m+1) + 4) + l - 3/2 \geq 25/2$.

In both cases, $4g - |r| \geq 3/2$.

CASE 5. $l > 0$, $m > 0$, $n = 0$ and $p \leq 0$.

Here $|r| = -r$, then $4g - |r| = 2l^2m^2 - l^2m - 8lm + 3l + 7/2 - p(2lm - l - 1)(2lm - l - 3)$. Note that $l, m \geq 2$ since $l \neq 1$ and $(m, n) \neq (1, 0)$.

Suposse $p = 0$, then $4g - |r| = 2l^2m^2 - l^2m - 8lm + 3l + 7/2$. If $m = 2$, then $4g - |r| = (3l-2)(2l-3) - 5/2 \geq 4 - 5/2 = 3/2$. If $m \geq 3$, then $4g - |r| = lm(l(2m-1) - 8) + 3l + 7/2 \geq 12 + 6 + 7/2 = 43/2$.

Now we can assume that $p \leq -1$. Then $4g - |r| = 2l^2m^2 - l^2m - 8lm + 3l + 7/2 - p(2lm - l - 1)(2lm - l - 3) \geq 2l^2m^2 - l^2m - 8lm + 3l + 7/2 + 4l^2m^2 - 2l^2m - 6lm - 2l^2m + l^2 + 3l - 2lm + l + 3 = lm(l(6m-5) - 16) + l^2 + 7l + 13/2$. If $m = 2$, then $4g - |r| \geq 2l(7l - 16) + l^2 + 7l + 13/2 = 15l^2 - 25l + 13/2 = 5l(3l-5) + 13/2 \geq 33/2$. If $m \geq 3$, then $4g - |r| \geq lm(l(6m-5) - 16) + l^2 + 7l + 13/2 \geq 60 + 4 + 14 + 13/2 > 3/2$.

In both cases, $4g - |r| \geq 3/2$.

CASE 6. $l > 0$, $m > 0$, $n = 0$ and $p > 0$.

In this case $|r| = r$, then $4g - |r| = 2l^2m^2 - 3l^2m - 8lm + l^2 + 5l + 15/2 + (p-1)(2lm - l - 1)(2lm - l - 3)$. Note also that $l, m \geq 2$ since $l \neq 1$ and $(m, n) \neq (1, 0)$.

Suposse $p = 1$. Since $(l, m, p) \neq (2, 2, 1)$, then $l \geq 3$ or $m \geq 3$. If $m = 2$, then $l \geq 3$ and $4g - |r| = (3l-2)(l-3) + 3/2 \geq 3/2$. If $l = 2$, then $m \geq 3$ and $4g - |r| = 4m(2m-7) + 14 + 15/2 \geq 2 + 15/2 > 3/2$. Now, if $l \geq 3$ and $m \geq 3$, then $4g - |r| = lm(l(2m-3) - 8) + l^2 + 5l + 15/2 \geq 9 + 9 + 15 + 15/2 > 3/2$.

Now if $p \geq 2$, then $4g - |r| \geq 2l^2m^2 - 3l^2m - 8lm + l^2 + 5l + 15/2 + (2lm - l - 1)(2lm - l - 3) = 6l^2m^2 - 7l^2m - 16lm + 2l^2 + 9l + 21/2$. If $m = 2$, then $4g - |r| \geq (3l-2)(4l-5) + 1/2 \geq 12 + 1/2 > 3/2$. If $m \geq 3$, then $4g - |r| \geq lm(l(6m-7) - 16) + 2l^2 + 9l + 21/2 \geq 36 + 8 + 18 + 21/2 > 3/2$.

In both cases, $4g - |r| \geq 3/2$.

CASE 7. $l > 0$, $m < 0$, $n = 0$ and $p \leq 0$.

Here $|r| = -r$ and $4g - |r| = 2l^2m^2 - l^2m + 4lm - l - 1/2 - p(2lm - l - 1)(2lm - l + 1)$.

Suposse that $p = 0$, then $4g - |r| = 2l^2m^2 - l^2m + 4lm - l - 1/2 = l(m(l(2m-1)+4)-1)-1/2$. If $l = 2$ and $m = -1$, then $4g - |r| = 3/2$. If $l \geq 2$ and $m \leq -2$ then $l(2m-1)+4 \leq -6$ implies that $l(m(l(2m-1)+4)-1) \geq 22$, so $4g - |r| \geq 43/2$. If $l \geq 3$ and $m \leq -1$ then $l(2m-1)+4 \leq -5$ implies that $l(m(l(2m-1)+4)-1) \geq 12$, so $4g - |r| \geq 23/2$.

Now if $p \leq -1$, since $l \geq 2$ and $m \leq -1$, then $4g - |r| \geq 6l^2m^2 - 5l^2m + 4lm - l + l^2 - 3/2 = 6(lm + 1/3)^2 + (l - 1/2)^2 - 5l^2m - 29/12 \geq 26/3 + 9/4 + 20 - 29/12 > 3/2$.

In both cases, $4g - |r| \geq 3/2$.

CASE 8. $l > 0$, $m < 0$, $n = 0$ and $p > 0$.
Here $|r| = r$ and $4g - |r| = 2l^2m^2 - 3l^2m + 4lm + l^2 - 3l - 1/2 + (p-1)(2lm - l - 1)(2lm - l + 1)$.

First suposse that $p = 1$, then $4g - |r| = 2l^2m^2 - 3l^2m + 4lm + l^2 - 3l - 1/2 = 2(lm+1)^2 - 3l^2m + (l - 3/2)^2 - 1/2 - 9/4 - 2 \geq 2 + 12 + 1/4 - 1/2 - 9/4 - 2 = 19/2 > 3/2$.

Now assume that $p \geq 2$, then $4g - |r| \geq 2l^2m^2 - 3l^2m + 4lm + l^2 - 3l - 1/2 + (2lm - l - 1)(2lm - l + 1) = 6l^2m^2 - 7l^2m + 4lm + 2l^2 - 3l - 3/2 = 4l^2m^2 - 7l^2m + 2(lm+1)^2 + (2l-1)(l-1) - 9/2 \geq 16 + 28 + 2 + 3 - 9/2 > 3/2$.

In all cases, $4g - |r| \geq 3/2$.

□

3. Integral slope.

Let r be an integral slope and $\widehat{T}$ an incompressible torus in $K(r)$ that intersects the attached solid torus J in a disjoint union of meridional disks $v_1, v_2, \ldots, v_t$ numbered successively along J. We assume that $\widehat{T}$ is chosen so that t is minimal among all incompressible tori in $K(r)$. We also assume that r is not the longitudinal slope, then $\widehat{T}$ must be separating in $K(r)$ and hence t is even.

Let S be a minimal genus $g(K)$, Seifert surface for K. By shrinking S suitably, we may assume that S is properly embedded in $E(K) = S^3 - \mathrm{int}\mathcal{N}(K)$. We cap off ∂S with a disk u to obtain a closed surface $\widehat{S}$. Let $T = \widehat{T} \cap E(K)$. We isotope T so that $S \cap T$ consists of circles and arcs that are essential in both S and T.

The intersection $S \cap T$ defines two labeled graphs G_S on $\widehat{S}$ and G_T on $\widehat{T}$ as follows. The graph G_S has only one (fat) vertex u, while the graph G_T has t (fat) vertices $v_1, v_2, \ldots, v_t$. For each graph G_S and G_T, the edges are the arc components of $S \cap T$. For each $x = 1, 2, \ldots, t$, there are r points in $\partial u \cap \partial v_x$, which are endpoints of some edges in G_S (G_T). We label these r points by x in G_S. Then labels $1, 2, \ldots, t$ appear in order around the vertex of G_S

repeatedly r times. We number consecutively each of these r blocks of labels according to the orientation of ∂u, starting at some block (see Fig. 3). For each edge endpoint in G_S, we will make use of another label which is the number of the block that it belongs. Each edge of G_S has some labels x and y at its endpoints, and this edge connects v_x and v_y in G_T with labels i and j, where i and j are the labels of the blocks to which the edge endpoints belongs in G_S. Here x and y have opposite parities by the parity rule [1] and then $x \neq y$. Note that the number of the edges in G_S (or G_T) is $|r|t/2$.

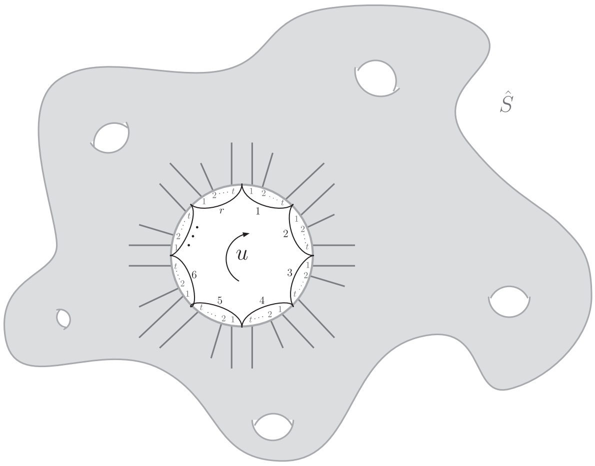


Fig. 3

Let $\widehat{B}, \widehat{W}$ be the two sides of $\widehat{T}$ in $K(r)$, also let $B = \widehat{B} \cap E(K)$ and $W = \widehat{W} \cap E(K)$. For each face f of G_S, we color f black or white according to whether a collar of ∂f lies in B or W.

An edge of G_S is called *x-edge* if it has label x at one endpoint, and it is called an (x, y)-*edge* if it has label y at the other endpoint. Note that the number of x-edges in G_S is $|r|$ for each $x = 1, 2, \ldots, t$.

If the subgraph of G_S consisting of all x-edges contains disk faces, we call them *x-faces*. We frequently regard an x-face as a configuration in G_S. If an x-face is a disk face of G_S, then all the edges in the boundary of the x-face have the same label pair $\{x, y\}$, where $|x - y| = 1$. The cycle of the edges of such an x-face is called a *Scharlemann cycle*. A Scharlemann cycle with only two edges is called an *S-cycle*. A cycle of G_S inmediatly surrounding a Scharlemann cycle is called an *extended Scharlemann cycle*.

Lemma 1. *If $|r| > 4g(K)$, $K(r)$ does not contain a Klein bottle.*

Proof. This follows from [16], Corollary 1.3. □

Lemma 2. *If $|r| > 2g(K) - 1$, $K(r)$ is irreducible.*

Proof. Lemma 2.3 of [17] or Theorem 1.3 of [19]. □

Lemma 3. *G_S cannot contain an extended Scharlemann cycle if $t \geq 4$.*

Proof. This is Theorem 3.2 of [11]. □

Lemma 4. *No two edges are parallel in both graphs G_S and G_T.*

Proof. Lemma 2.1 of [10]. □

Lemma 5. *If $|r| > 4g(K)$ and $t \geq 4$, then G_S cannot contain two S-cycles on disjoint label pairs.*

Proof. Lemma 2.6 of [17]. □

Lemma 6. *There are at most four labels of Scharlemann cycles in G_S.*

Proof. This is Lemma 2.3(4) of [14]. □

Lemma 7. *The edges of a Scharlemann cycle of G_S cannot lie in a disk in $\hat{T}$.*

Proof. This is Lemma 3.1 of [11]. □

Let x be a label of G_S. Define Γ_x be the subgraph of G_S consisting of all x-edges and $\overline{F}_x$ be the number of disk faces of Γ_x. Thus Γ_x has exactly $|r|$ edges.

Lemma 8. *If $|r| \geq 4g(K) - 1$, then Γ_x contains a disk face of length at most 3 for any label x.*

Proof. Assume that Γ_x has no disk face of lenght at most 3. Then $4\overline{F}_x \leq 2|r|$. By an Euler characteristic count in $\hat{S}$ we have

$$\frac{|r|}{2} \geq \overline{F}_x \geq 1 - 2g(K) + |r|$$

Thus $|r| \leq 4g(K) - 2$, a contradiction. □

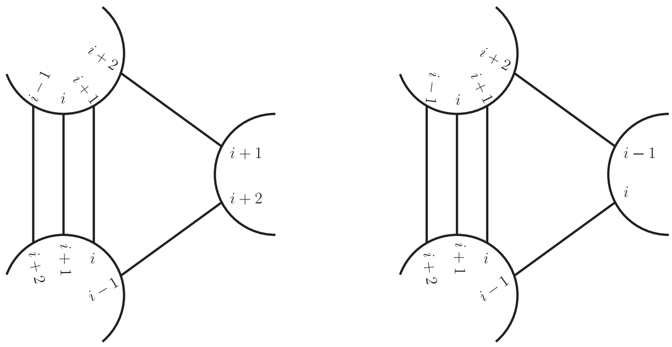
3.1. t ≥ 6. Suposse $|r| > 4g(K)$. By Lemma 1, $K(r)$ does not contain a Klein bottle. We mainly follow the argument in [[11], Section 5]. Let $C_{\pm}(i)$ be the configuration in G_S as illustrated in Fig. 4.

Let $\mathcal{S}$ denote the set of labels of Scharlemann cycles of lenght at most 3 in G_S, and let $\mathcal{L}_0 = \{1, 2, \dots, t\} \backslash \mathcal{S}$. Recall that Γ_x contains a disk face of lenght at most three for any label x by Lemma 8.

Lemma 9. *G_S contains a configuration $C_{\pm}(i)$ for just one label i.*

Proof. Let x be a label in $\mathcal{L}_0$. Then Γ_x contains a bigon or a trigon face f by Lemma 8. Since there is no extended Scharlemann cycle, Γ_x has a disk face as shown in Figure 5.1 of [11] (see Lemma 5.1 of [11]), and hence has a configuration $C_{\pm}(i)$. The uniqueness of such a configuration follows from [[11], Lemma 5.4]. □

Lemma 10. $|\mathcal{S}| \leq 4$ *and* $|\mathcal{L}_0| \leq 4$.



$C_+(i)$ $C_-(i)$

Fig.4

Proof. The first part follows from Lemma 6, and the second is Theorem 5.8 of [11]. □

Corollary 1. $t \leq 8$.

Proof. By Lemma 10, $t = |\mathcal{S}| + |\mathcal{L}_0| \leq 8$. □

Proposition 1. $t = 6$ *is impossible.*

Proof. By Lemma 9, we may assume that G_S contain an S-cycle with label pair $\{1, 2\}$, and $\mathcal{L}_0 \subset \{3, 4, 5, 6\}$. By the argument of [[11], page 626], we see that $\mathcal{L}_0 = \{3, 6\}$ and $\mathcal{S} = \{1, 2, 4, 5\}$. Hence G_S contains 12- and 45-Scharlemann cycles of lenght 2 or 3. This is impossible by [[11], Theorem 3.9]. □

Proposition 2. $t = 8$ *is impossible.*

Proof. If $t = 8$, then $|\mathcal{S}| = |\mathcal{L}_0| = 4$. By Lemma 9, G_S contains an S-cycle ρ with label pair $\{i, i+1\}$ for some i, and $\mathcal{L}_0 = \{i-2, i-1, i+2, i+3\}$ by [[11], Lemma 5.3(2)]. Since $|\mathcal{S}| > 2$, there is another $(j, j+1)$-Scharlemann cycle σ of length at most 3 with $j \neq i$, and with $j, j+1 \notin \mathcal{L}_0$. Hence $j, j+1 \notin \{i-2, i-1, i, i+1, i+2, i+3\}$. The rest of the proof is the same as those of [[11], pages 625-626]. □

3.2. t = 4. Suposse that $|r| \geq 6g(K) - 2$. By Lemmas 5 and 6, we can take a label x of G_S which is not a label of any S-cycle. Every disk face of Γ_x has at least 3 sides, since otherwise G_S would contain an extended Scharlemann cycle. An Euler characteristic count for Γ_x gives $\overline{F}_x \geq 1 - 2g(K) + |r|$. Also $2|r| \geq 3\overline{F}_x$, then $|r| \leq 6g(K) - 3$, a contradiction.

3.3. t = 2. Suposse that $|r| \geq 4g(K) + 1$. Since $t = 2$, G_T has exactly two vertices, then has at most four edge classes as shown in Fig. 5.

We label each edge e of G_S by the label of its class on G_T and we denote this label by $\mathcal{L}(e)$.

Note that when $t = 2$, all disk faces of G_S are Scharlemann cycles.

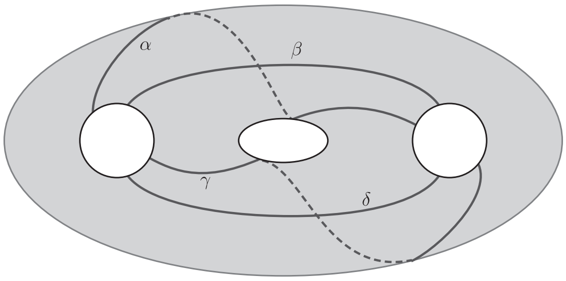


Fig. 5

Let f be a disk face of G_S and suposse that χ_1, χ_2 are two distinct edge classes in G_T. We say that f is a (χ_1, χ_2)-*face* if the edges of f belong to $\chi_1 \cup \chi_2$. When f is a (χ_1, χ_2)-face, f is said to be χ_i-*good* ($i = 1, 2$) if no two consecutive edges of f belong to class χ_i. If f is χ_i-good for some $i = 1, 2$, then f is said to be (χ_1, χ_2)-*good*. We denote by $|f|$ the number of edges of f.

Lemma 11. *For any two edge class labels χ_1, χ_2, G_S cannot have (χ_1, χ_2)-good faces on both sides of $\widehat{T}$.*

Proof. This is Lemma 4.2 of [17]. □

In what follows, let $H_{i,i+1}$ be the part of J between consecutive components i and $i+1$ of ∂T.

Lemma 12. *If a disk face of G_S have at most* 3 *edges, the edges are contained in an essential annulus on $\widehat{T}$.*

Proof. Let σ be a disk face of G_S, then σ is a Scharlemann cycle.

If $|\sigma| = 2$, the result is a direct consequence of Lemma 7.

If $|\sigma| = 3$. Let a, b, c be the edges of σ and suppose $\mathcal{L}(a) \neq \mathcal{L}(b) \neq \mathcal{L}(c)$ (otherwise we would have finished). Extending the edges a, b, c to the corners of σ in $T \cup H_{1,2}$, they look like Figure 6. According to such a figure, to complete the cycle σ we need to connect the ends of the arcs a, b, c in $H_{1,2}$. We have two options: (*i*) to connect an end of a with an end of c, an end of b with an end of a and an end of c with an end of b; or (*ii*) to connect an end of a with an end of b, an end of b with an end of c and an end of c with an end of a. However, it is not difficult to see that is not possible to realize such connections in $H_{1,2}$ without obtaining autointersections. □

Note that by Lemma 12, the disk faces of G_S of length at most three are good faces.

Lemma 13. *G_S does not contain two bigons of the same color on distinct edge class pairs.*

Proof. Otherwise, $K(r)$ would contain a Klein bottle. See the proof of Lemma 5.2 [13]. □

Lemma 14. *If a bigon and a trigon of G_S have the same color, then they have disjoint pairs of edge class labels.*

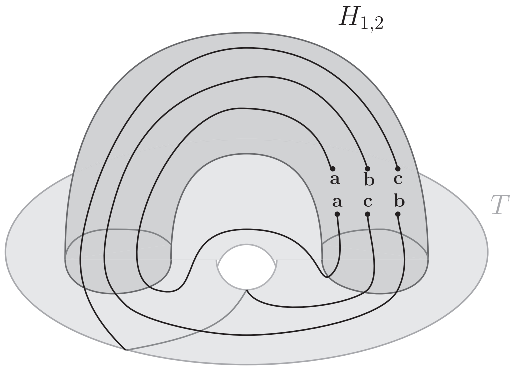


Fig. 6

Proof. This is Lemma 4.5 of [17]. □

Lemma 15. (1) *If two trigons of G_S of the same color have different pairs of class labels, then the pairs are disjoint.*

(2) *Suppose that two trigons of G_S of the same color have the same pair of edge class labels, say, $\{\chi_1, \chi_2\}$. If one has an edge in class χ_1 and two edges in class χ_2, then the other also has one edge in class χ_1 and two edges in class χ_2.*

(3) *If two trigons of G_S have opposite colors, then they cannot have the same pair of edge class labels.*

Proof. This is Lemma 4.6 of [17] □

Recall that u is the unique vertex of G_S. Let a, b be some two intersection points of ∂u and ∂v_x $(x = 1, 2)$. Then both points have label x in G_S. Since r is an integral slope, the points a and b are consecutive on ∂v_x of G_T if and only if there is exactly one edge endpoint in G_S between the points. See Fig. 7, where $x = 1$.

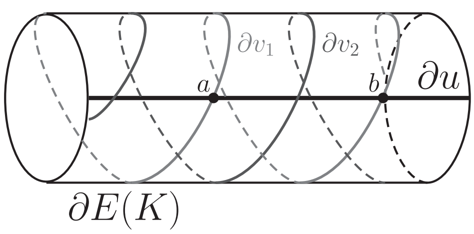


Fig. 7

We orient ∂v_1 counterclockwise around v_1, ∂v_2 clockwise around v_2 and ∂u counterclockwise around u. We may assume that the three curves ∂v_1, ∂v_2 and ∂u proceed in the same direction along the knot K when they proceed along their orientations.

Let $\chi \in \{\alpha, \beta, \gamma, \delta\}$. For two edges e, e' in class χ, we write $e < e'$ if the point $e \cap \partial v_1$ precedes the point $e' \cap \partial v_1$ with respect to the orientation of ∂v_1. Note that $e < e'$ if and only if the point $e \cap \partial v_2$ precedes the point $e' \cap \partial v_2$ with respect to the orientation of ∂v_2. We say that e is the *first edge* in the class χ if $e < e'$ for any other edge e' in class χ. Similarly, the last edge is defined.

Lemma 16. *Let a_1, a_2 be two edge endpoints of G_S such that there is exactly one endpoint betweeen them as in* Fig. 8. *Let e_i be the edge of G_S incident to a_i ($i = 1, 2$). Let $\chi_i = \mathcal{L}(e_i)$ and assume $\chi_1 \neq \chi_2$. Then on G_T, e_1 is the last edge in class χ_1, while e_2 is the first edge in class χ_2. Also, a_1, a_2 appear consecutively and in order on a vertex of G_T.*

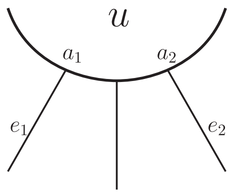


Fig. 8

Proof. This is Lemma 4.7 of [17]. □

Lemma 17. *Any two bigons of G_S cannot be adjacent.*

Proof. Lemma 4.8 of [17]. □

Lemma 18. *If a bigon and a trigon of G_S are adjacent, then G_S contains only one bigon.*

Proof. It follows from the first part of the proof of Lemma 4.9 of [17]. □

Recall that the graph G_S have $|r|$ edges. Let F be the number of disk faces on G_S.

An Euler characteristic calculus gives $1 - |r| + F \geq 2 - 2g$, then

$$F \geq |r| - 2g(K) + 1 \tag{1}$$

Lemma 19. *If $|r| \geq 6g(K) - 2$, then G_S cannot have a bigon and a trigon which are adjacent.*

Proof. Suposse there is a bigon adjacent to a trigon in G_S. By Lemma 18, G_S contains only one bigon. Applying (1), $2|r| \geq 3(F - 1) + 2 = 3F - 1 \geq 3(|r| + 1 - 2g(K)) - 1 = 3|r| - 6g(K) + 2$, hence $|r| \leq 6g(K) - 2$. Also $|r| \geq 6g(K) - 2$ by assumption, then $|r| = 6g(K) - 2$.

Let F^b, F^w be the number of black and white disk faces of G_S, respectively. Suposse that the unique bigon on G_S is black. Then $|r| \geq 3(F^b - 1) + 2 = 3F^b - 1$, so $F^b \leq (|r| + 1)/3$. Hence $(|r|+1)/3 + F^w \geq F^b + F^w = F \geq |r| + 1 - 2g(K)$, wich yields $F^w \geq (2|r|+2)/3 - 2g(K) = 2g(K) - 2/3$ and hence, $F^w \geq 2g$. Also $|r| \geq 3F^w$, then $|r| \geq 3F^w \geq 6g(K)$, contradicting that $|r| = 6g(K) - 2$. □

Lemma 20. *Let f be a disk face of G_S with $|f| \geq 4$. Then the following hold.*

- *f cannot be surrounded by bigons.*
- *If $|f|$ is odd, then f is adjacent to at most $|f| - 2$ bigons.*

Proof. Lemma 4.10 of [17]. □

Lemma 21. *G_S cannot contain a bigon, a trigon and a tetragon of the same color.*

Proof. Lemma 4.11 of [17]. □

Definition. We say that two disk faces in G_S are *of the same type* if they have the same color, the same length, the same set of class edges labels and the same number of edges in each edge class. This is an equivalence relation in the set of disk faces of G_S, and we call as *a type of n-faces* an equivalence class of a disk face with n sides.

Note that by Lemmas 12 and 13, there are at most two types of bigons in G_S (one for each color). Also, by Lemma 15, there are at most four types of trigons in G_S (two for each color).

Convention on the Figures. From now on, the same number and shape of big dots on the edges in the graph G_S will indicate that the edges are the same.

Definition. Two disk faces of the same type in G_S, are *consecutive with respect to one corner* if the corresponding labels of one of their corners are consecutive in G_T.

Remark. Note that if two disk faces are consecutive with respect to one corner and if in addition the corresponding edges that form the corners are parallel edges in G_T, then the faces are consecutive with respect to the other corners at the ends of the edges.

Definition. A finite set of disk faces in G_S, $\{\sigma_i\}_I$ is *consecutive* if the faces are of the same type and we can enumerate them as $\{\sigma_j\}_{j=1}^m$ for some $m \geq 1$, in such a way that for all $1 \leq j \leq m$, σ_j and σ_{j+1} are consecutive faces with respect to one corner (and then with respect to all their corners by the remark above).

Fig. 9 shows an example of a consecutive set of faces $\sigma_1, \sigma_2, \ldots, \sigma_m$ in G_S. The blue labels are the labels that they have in G_T and which are supossed to be consecutive.

We denote by F_i the set of faces with i sides in G_S and by $|F_i|$ its cardinality. In the same way, we denote by F_i^w and F_i^b be the set of white and black disk faces with i sides in G_S, respectively.

Lemma 22. *The set of bigons of the same color in G_S, is consecutive.*

Proof. Fix a color, say white. By Lemma 13, the bigons in F_2^w have the same two edge classes, say $\{\chi_1, \chi_2\}$, then they are all of the same type. In fact, the edges of the bigons in F_2^w lie in an annulus A on $\widehat{T}$. Let a_i^f and b_i^l be the first and last edge, respectively of class χ_i, $i = 1, 2$ for bigons in F_2^w. Suppose we have an edge c on a white face σ such that $a_1^f < c < b_1^l$ or $a_2^f < c < b_2^l$. Note that when going along the edges of σ on $\widehat{T} \cup H_{1,2}$ they cannot get out

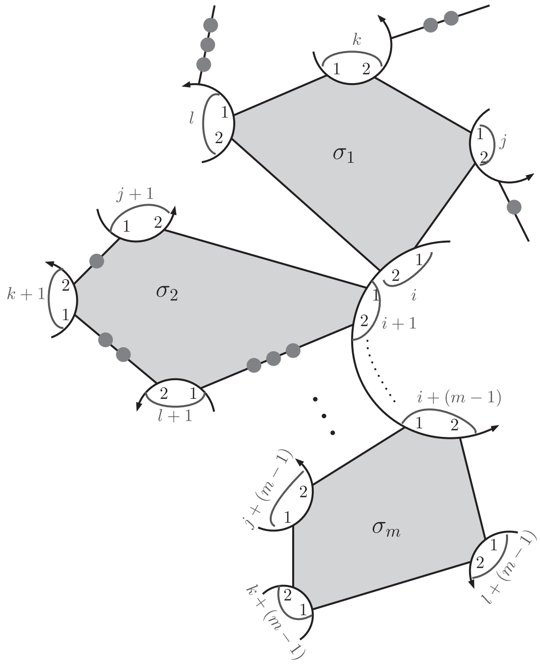


Fig. 9

of A and also they cannot give more than one lap around A because if not σ would not close. Then σ is a bigon inside A with edge classes $\{\chi_1, \chi_2\}$. Therefore the edges of the bigons in F_2^w of the same class, are consecutive on G_T and hence the set of white bigons is consecutive in G_S. □

Lemma 23. *The bigons of the same color are adjacent to at most two faces in G_S, each one having the same number of edges belonging to bigons.*

Proof. Fix a color, say black. By Lemma 22, F_2^b is a consecutive set in G_S. Then we can enumerate its elements according to the order they appear on the vertex u of G_S. Suppose that in such an order, the bigons are b_m, $m \in \{1, 2, \ldots n\}$ and have labels $i+(m-1)$, $j+(m-1)$, respectively on its corners on G_T for some fixed $i, j \in \{1, 2, \ldots, |r|\}$. Start with the first bigon, it its adjacent to at most two faces, f_1, f_2 (not necessarily different). By following the labels of the blocks in ∂u from i to $i+1$, we note that the second bigon must be adjacent to the face f_2 on one side and adjacent to the face f_1 on the other side (if we follow the labels from j to $j+1$), so they appear in G_S as in Fig. 10.

The third bigon has labels $i+2$, $j+2$ on its corners, then it must be adjacent to the face f_2 by one side (if we follow the labels from $j+1$ to $j+2$), and adjacent to the face f_1 on the other side (if we follow the labels from $i+1$ to $i+2$), as in Fig. 11. Continuing with this process, we get that all bigons are adjacent to the same two faces, and in fact f_1 and f_2 have the same number of edges belonging to bigons. □

We have similar properties for the trigons.

Lemma 24. *The set of trigons of the same type is consecutive in G_S.*

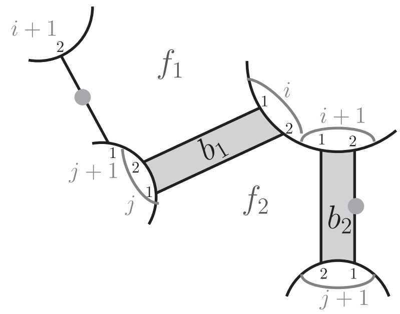


Fig. 10

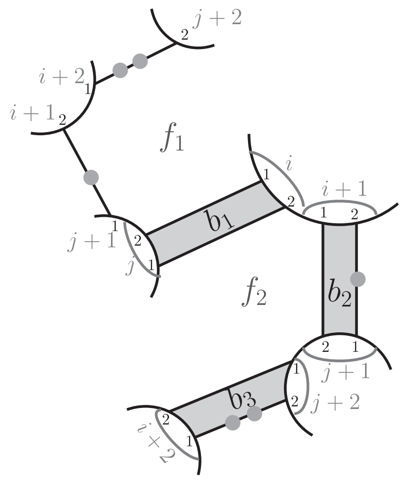


Fig. 11

Proof. Let C be the set of trigons of the same type in G_S. Without loss of generality, suppose that the trigons in C are white and have one edge of class χ_1 and two edges of class χ_2. By the remark above, we just need to prove that the trigons of the same type are consecutive with respect to the corner which contain the edge of class χ_1 on each trigon. By Lemma 12, the edges of the trigons of the same type lie in an essential annulus A on $\widehat{T}$. Let a^f and b^l be the first and last edge of class χ_1 for trigons in C, respectively. Suppose we have an edge c on a white face σ such that $a^f < c < b^l$. Note that when going along the edges of σ on $\widehat{T} \cup H_{1,2}$ they cannot get out of A and also they cannot take more than one lap around A because if not σ would not close (see Fig. 12). Then σ is a trigon inside A with edge classes $\{\chi_1, \chi_2\}$. Therefore the edges of the trigons must be consecutive on G_T and then the set of trigons is consecutive in G_S. □

Lemma 25. *The trigons of the same type are adjacent to at most three faces on G_S, each one having the same number of edges belonging to trigons.*

Proof. Let C be the set of trigons of the same type in G_S. By Lemma 24, the set C is consecutive. Suppose that the trigons in C have one edge of class χ_1 and two edges of class χ_2. We can enumerate the trigons in C according to the order that the edges of class χ_1 of the trigons appear on the vertex u. Suppose that in such an order, the trigons are c_m,

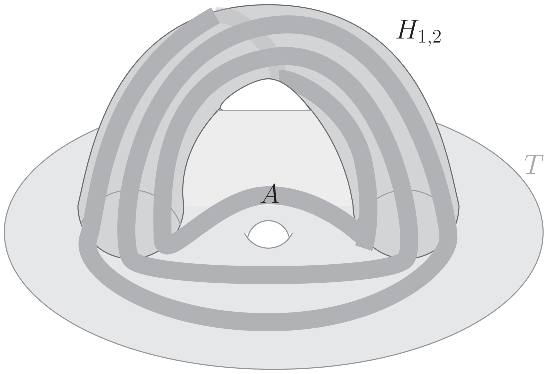


Fig. 12

$m \in \{1, 2, \ldots, n\}$ and have edge labels in G_T, $i + (m - 1)$, $j + (m - 1)$, $l + (m - 1)$ (mod $|r|$), respectively for some fixed numbers $i, j, l \in \{1, 2, \ldots, |r|\}$. Start with the first trigon, it is adjacent to at most three faces f_1, f_2, f_3 (not necessarily different). By following the labels from i to $i+1$ on the vertex u, we note that the second trigon is adjacent to the face f_2 by one side, adjacent to the face f_1 by other side and adjacent to f_3 on the third side (if we follow the labels from j to $j + 1$ and also from l to $l + 1$), so they appear in G_S like in Fig. 13.

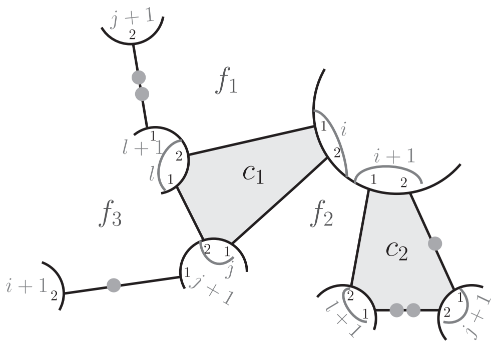


Fig. 13

The third trigon has edge labels $i + 2$, $j + 2$, $l + 2$ in G_T. Hence it must be adjacent to f_2 by one side (if we follow the labels from $l + 1$ to $l + 2$), and adjacent to f_1 (if we follow the labels from $j+1$ to $j+2$) and f_3 (if we follow the labels from $i+1$ to $i+2$) on its other sides, like in Fig. 14. Continuing with this process, we get that all trigons of the same type must glue to the same faces and in fact, f_1, f_2 and f_3 have the same number of edges belonging to trigons. □

Lemma 26. *If on G_S there are two adjacent trigons, they are unique with respect to their type.*

Proof. Suppose that we have two adjacent trigons c_1 and c_2 on G_S (say c_1 black and c_2 white). Without loss of generality, we can assume that c_1 is a $\{\chi_1, \chi_2\}$-good face and c_2 is a $\{\chi_2, \chi_3\}$-good face. By Lemma 25, c_1 is adjacent to all white trigons of type $\{\chi_2, \chi_3\}$, and in the same way c_2 is adjacent to all black trigons of type $\{\chi_1, \chi_2\}$. Since c_1 and c_2

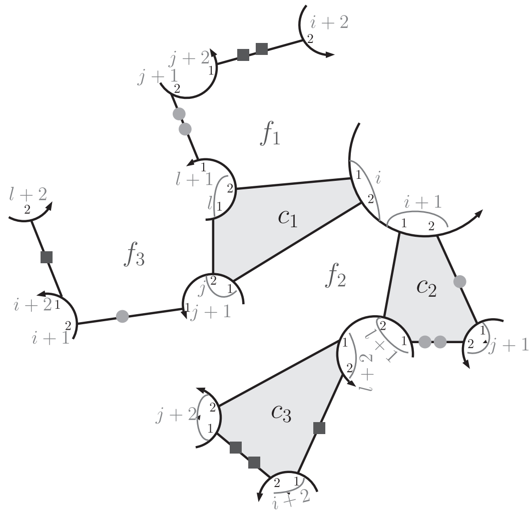


Fig. 14

are adjacent then there are at most two black trigons of type $\{\chi_1, \chi_2\}$ and two white trigons of type $\{\chi_2, \chi_3\}$. Suppose c_1 is adjacent to two white trigons of type $\{\chi_2, \chi_3\}$ (otherwise we would have finished). It means that c_1 have two edges of class χ_2 and one of class χ_1. Applying Lemma 16 on the corners of c_1 containing the edge of class χ_1, we can see that this is the only edge of its class. This implies that there are no more black trigons of type $\{\chi_1, \chi_2\}$.

Let a, b, c, d be the rest of edges of the trigons according to Fig. 15.

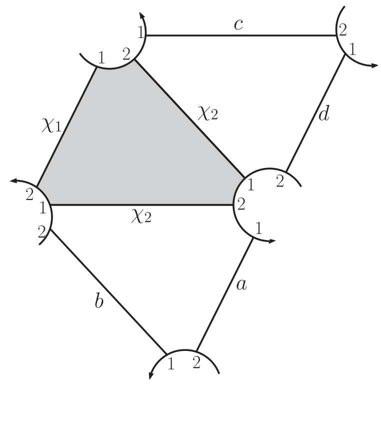


Fig. 15

Applying Lemma 16, we see that c is the first edge on its class, while b is the last edge on its class. For the edges a, b, c, d we have the following classes options. The superscripts f and l indicate that the edge is the first or the last of its class, respectively.

(1) $\mathcal{L}(a) = \chi_2; \mathcal{L}(b) = \chi_3^l; \mathcal{L}(c) = \chi_2^f; \mathcal{L}(d) = \chi_3$
(2) $\mathcal{L}(a) = \chi_2; \mathcal{L}(b) = \chi_3^l; \mathcal{L}(c) = \chi_3^f; \mathcal{L}(d) = \chi_2$
(3) $\mathcal{L}(a) = \chi_3; \mathcal{L}(b) = \chi_2^l; \mathcal{L}(c) = \chi_2^f; \mathcal{L}(d) = \chi_3$
(4) $\mathcal{L}(a) = \chi_3; \mathcal{L}(b) = \chi_2^l; \mathcal{L}(c) = \chi_3^f; \mathcal{L}(d) = \chi_2$

(5) $\mathcal{L}(a) = \chi_3; \mathcal{L}(b) = \chi_3^l; \mathcal{L}(c) = \chi_3^f; \mathcal{L}(d) = \chi_3$

Applying Lemma 16 on the first case, we have that d is the last edge of the class χ_3, which contradicts the fact that b is such an edge.

On the second case, since the edges of the class χ_2 are consecutive, we can label them with i, j, like in Fig. 16. However such configuration implies that b and c are the same, which is not possible since we are assuming that they lie in different faces.

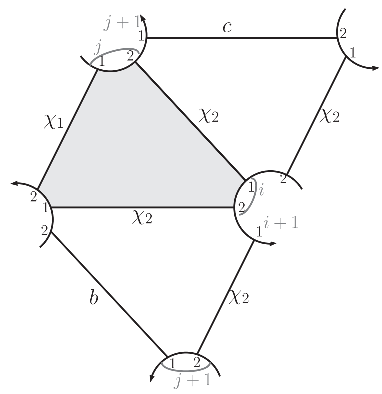


Fig. 16

Applying Lemma 16 on the third case, we have that the edges of class χ_2 in the black trigon are the first and the last edges of its class, contradicting that b and c are such edges.

Applying Lemma 16 on the fourth case, we see that one of the edges of class χ_2 is the last edge of this class, contradicting that b is such an edge.

Finally, applying Lemma 16 on the last case, we see that a and d are the first and last edge of class χ_3, respectively, contradicting that c and b are such edges. □

Note that Lemmas 15 and 26 imply that on G_S there are at most two pairs of adjacent trigons and in such case the trigons are unique.

Denote by n the number of faces to which the bigons (of both colors) in G_S are adjacent and denote by m the number of faces to which the trigons (of both colors) in G_S are adjacent. By Lemmas 23 and 25, $n \leq 4$ and $m \leq 12$, respectively.

We use Lemmas from 11 to 26 to separate the proof of the theorem on eleven cases. In Fig. 17 we see the tree of cases and the bound obtained for $|r|$ on each case.

I. Without bigons or trigons.

By counting the edges on G_S we have that $2|r| \geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3|) = 4F - 2|F_2| - |F_3|$. By (1), $2|r| \geq 4(|r| - 2g(K) + 1) - 2|F_2| - |F_3|$. Since $|F_2| = |F_3| = 0$, then $|r| \leq 4g(K) - 2$, which finishes this case.

II. Without trigons.

Since any two bigons cannot be adjacent by Lemma 17 and G_S does not have any trigon, then the faces to which the bigons are adjacent have at least four sides. On the other hand, a

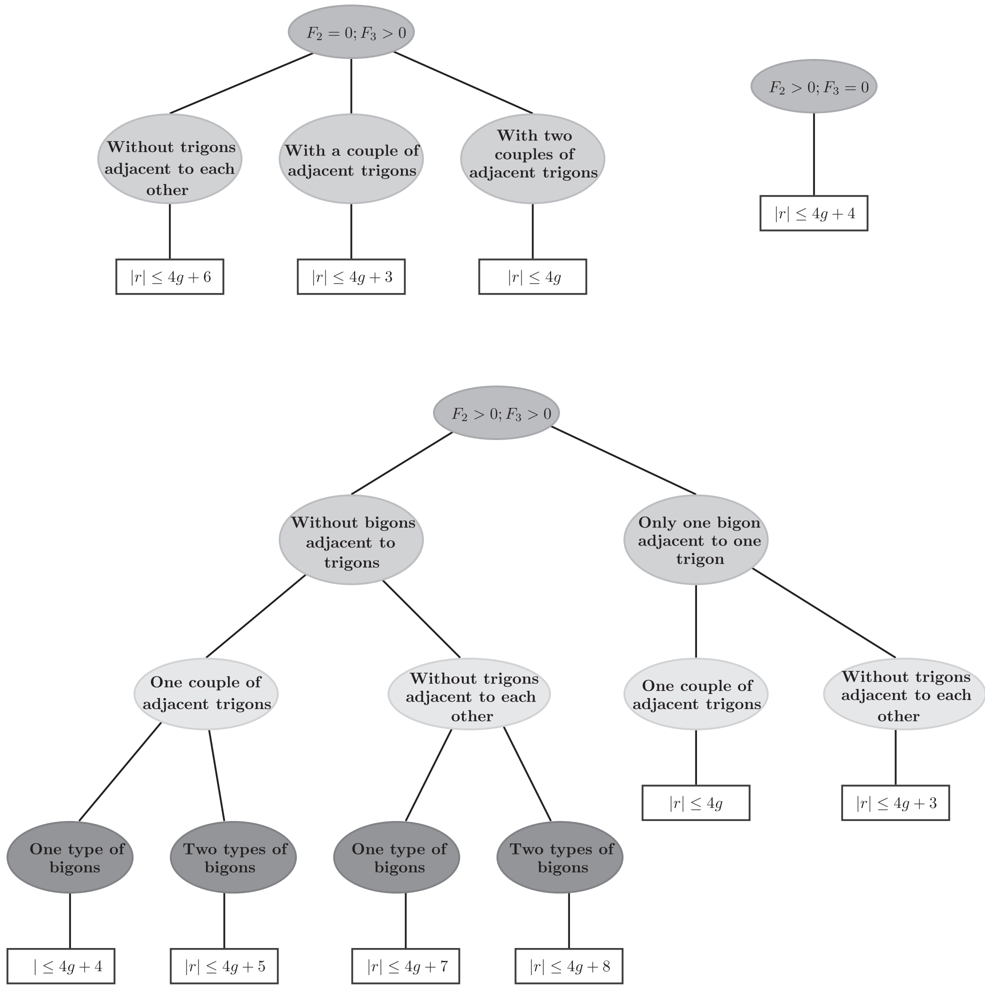


Fig. 17

disk face σ in G_S is adjacent to at most $|\sigma| - 1$ bigons by Lemma 20. By counting the edges on G_S and applying (1), we have

$$\begin{aligned}
2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n) + 2|F_2| + n \\
&= -|F_3| + 4F - 3n \\
&\geq 4(|r| - 2g(K) + 1) - 12
\end{aligned}$$

Then, $|r| \leq 4g(K) + 4$.

III. Without adjacent trigons and without bigons.

Since any two trigons are not adjacent and there is no bigons, the faces to which the trigons are adjacent have at least four sides. By counting the number of edges on G_S and applying (1), we have

$$\begin{aligned} 2|r| &\geq 3|F_3| + 4(F - |F_3| - m) + 3|F_3| \\ &= 2|F_3| + 4F - 4m \\ &\geq 2|F_3| + 4(|r| - 2g(K) + 1) - 48 \end{aligned}$$

Then

$$|F_3| \leq 4g(K) + 22 - |r| \tag{2}$$

Also we have

$$\begin{aligned} 2|r| &\geq 3|F_3| + 4(F - |F_3|) \\ &= 4F - |F_3| \\ &\geq 4(|r| - 2g(K) + 1) - |F_3| \end{aligned}$$

Then

$$|F_3| \geq -8g(K) + 4 + 2|r| \tag{3}$$

Comparing (2) and (3) gives $|r| \leq 4g(K) + 6$.

IV. Without bigons, with a couple of adjacent trigons.

By Lemma 26, if on G_S there is a couple of adjacent trigons, they are unique with respect to its type. Then G_S have this two trigons and other trigons of at most two types. Denote by F_3' the set of trigons that are not adjacent to any other trigon. Then $|F_3| = |F_3'| + 2$.

Let m' be the number of faces to which the trigons on F_3' are adjacent. By Lemmas 25 and 20, $m' \leq 6$ and such faces have at least four sides. By counting the number of edges on G_S and applying (1), we have

$$\begin{aligned} 2|r| &\geq 3|F_3| + 4(F - |F_3| - m') + 3|F_3'| \\ &= 2|F_3| + 4F - 4m' - 6 \\ &\geq 2|F_3| + 4(|r| - 2g(K) + 1) - 30 \end{aligned}$$

Then,

$$|F_3| \leq 4g(K) + 13 - |r| \tag{4}$$

Comparing (3) and (4) we get $|r| \leq 4g(K) + 3$.

V. Without bigons, with two couples of adjacent trigons.

By Lemma 26, G_S have only four trigons, i.e. $|F_3| = 4$. By counting the number of edges on G_S and applying (1), we get

$$\begin{aligned} 2|r| &\geq 3|F_3| + 4(F - |F_3|) \\ &= 4F - 4 \\ &\geq 4(|r| - 2g(K) + 1) - 4 \end{aligned}$$

Then, $|r| \leq 4g(K)$.

VI. One type of bigons, without bigons adjacent to trigons and with a couple of adjacent trigons.

By Lemma 14, G_S has at most three types of trigons. Since we do not have any bigon adjacent to any trigon, the faces to which the bigons are adjacent are at most two (Lemma 23) and have at least four sides.

On the other hand, we have a couple of adjacent trigons, then by Lemma 26, such trigons are the unique of its type. Since G_S has at most three types of trigons, then it has at most one type of trigons not adjacent to any trigon. Then we have $n \leq 2$ and $m \leq 3$. By counting the number of edges on G_S and applying (1), we have

$$\begin{aligned} 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n - m) + 2|F_2| + n + 3|F_3| - 6 \\ &= 2|F_3| + 4F - 3n - 4m - 6 \\ &\geq 2|F_3| + 4(|r| - 2g(K) + 1) - 24 \end{aligned}$$

Then

$$|F_3| \leq 4g(K) + 10 - |r| \tag{5}$$

Also

$$\begin{aligned} 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n) + 2|F_2| + n \\ &= 4F - 3n - |F_3| \\ &\geq 4(|r| - 2g(K) + 1) - 6 - |F_3| \end{aligned}$$

Then

$$|F_3| \geq -8g(K) - 2 + 2|r| \tag{6}$$

Comparing (5) and (6) we get $|r| \leq 4g(K) + 4$.

VII. Two types of bigons, one couple of adjacent trigons and without bigons adjacent to trigons.

Since any two bigons cannot be adjacent by Lemma 17, and we are assuming that any bigon is not adjacent to any trigon, then the faces to which the bigons are adjacent have at least four sides. On the other hand G_S has two types of bigons, then by Lemma 14 there are at most two types of trigons. We have also a couple of adjacent trigons, then by Lemma 26, they are the unique of its type and then $|F_3| = 2$. By counting the number of edges on G_S and applying (1), we have

$$\begin{aligned} 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n) + 2|F_2| + n \\ &= -|F_3| + 4F - 3n \\ &\geq 4(|r| - 2g(K) + 1) - 14 \end{aligned}$$

Then

$$|r| \leq 4g(K) + 5.$$

VIII. One type of bigons, without bigons adjacent to trigons and without trigons adjacent between them.

By Lemma 14, G_S has at most three types of trigons. We are assuming that any bigon is not adjacent to any trigon, then the faces to which the bigons are adjacent have at least four sides.

Then $n \leq 2$ and $m \leq 9$. By counting the number of edges on G_S and applying (1), we have

$$\begin{aligned}2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n - m) + 2|F_2| + n + 3|F_3| \\ &= 2|F_3| + 4F - 3n - 4m \\ &\geq 2|F_3| + 4(|r| - 2g(K) + 1) - 42\end{aligned}$$

Then

$$|F_3| \leq 4g(K) + 19 - |r| \tag{7}$$

Also

$$\begin{aligned}2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n) + 2|F_2| + n \\ &= 4F - 3n - |F_3| \\ &\geq 4(|r| - 2g(K) + 1) - 6 - |F_3|\end{aligned}$$

Then

$$|F_3| \geq -8g(K) - 2 + 2|r| \tag{8}$$

Comparing (7) and (8) we get that $|r| \leq 4g(K) + 7$.

IX. Two types of bigons, without bigons adjacent to trigons and without trigons adjacent between them.

By Lemma 14, on G_S there are at most two types of trigons. Then we have that $n \leq 4$ and $m \leq 6$.

Since any bigon is not adjacent to any bigon or trigon, the faces to which the bigons are adjacent have at least four sides. The same happens to the trigons since they are not adjacent to any trigon. We also apply Lemma 20 to obtain

$$\begin{aligned}2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n - m) + 2|F_2| + n + 3|F_3| \\ &= 2|F_3| + 4F - 3n - 4m \\ &\geq 2|F_3| + 4(|r| - 2g(K) + 1) - 12 - 24\end{aligned}$$

Then

$$2|F_3| \leq 8g(K) + 32 - 2|r| \tag{9}$$

Also

$$\begin{aligned}2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n) + 2|F_2| + n \\ &= 4F - 3n - |F_3| \\ &\geq 4(|r| - 2g(K) + 1) - 12 - |F_3|\end{aligned}$$

Then

$$|F_3| \geq -8g(K) - 8 + 2|r| \tag{10}$$

Comparing (9) and (10) we get $|r| \leq 4g(K) + 8$.

X. With a bigon adjacent to a trigon and a couple of adjacent trigons.

By Lemma 18, there is only one bigon on G_S, i.e. $|F_2| = 1$. Also by Lemma 24, the trigon adjacent to the only bigon is the only one on its class, let t_1 be such a trigon. We will see that $|F_3| \leq 3$.

Let t_2 and t_3 be two adjacent trigons on G_S. By Lemma 26, t_2 and t_3 are the unique with respect to its type. By Lemma 14, G_S has at most three types of trigons.

If t_1, t_2 and t_3 were all different, they would be the only trigons on G_S and then $|F_3| = 3$.

Now suppose that two of $\{t_1, t_2, t_3\}$ were equal, say $t_1 = t_3$. Let b be the only bigon on G_S. Suppose that b is black and it has class edges $\{\chi_1, \chi_2\}$. Then t_1 is a $\{\chi_2, \chi_3\}$-white trigon and t_2 is a $\{\chi_3, \chi_4\}$-black trigon (b and t_1 are adjacent on an edge of class χ_2, while t_1 and t_2 are adjacent on an edge of class χ_3). Applying repeatedly Lemma 16, we get a subgraph on G_S like Fig. 18.

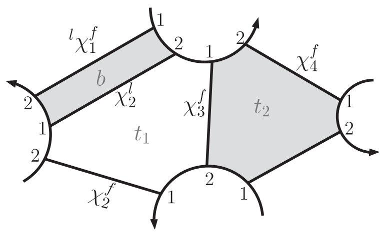


Fig. 18

If G_S had another type of trigons different than those already mentioned, it must be of type $\{\chi_1, \chi_4\}$-white trigon. But there is only one edge of class χ_1 on G_S, then there could be only one of such trigons. It follows that $|F_3| \leq 3$.

Therefore

$$\begin{aligned} 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3|) \\ &= 4F - |F_3| - 2|F_2| \geq 4(|r| - 2g(K) + 1) - 2 - 3 = 4|r| - 8g(K) - 1 \end{aligned}$$

Then

$$|r| \leq 4g(K).$$

XI. With a bigon adjacent to a trigon and without trigons adjacent between them.

Let b and t_1 be a bigon and a trigon adjacent on G_S, respectively. By Lemmas 18 and 24, b is the only bigon on G_S and t_1 is the only trigon of its type. By Lemma 14, G_S has at most three types of trigons. Then there is at most two types of trigons such that they are not adjacent to any trigon or bigon. Therefore $m \leq 6$. By counting the number of edges in G_S and applying (1) we have

$$\begin{aligned}2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - m) + 3|F_3| - 3\\ &= 2|F_3| + 4F - 4m - 5\\ &\geq 2|F_3| + 4(|r| - 2g(K) + 1) - 29\end{aligned}$$

Then

$$|F_3| \leq 4g(K) + 12 - |r| \tag{11}$$

Also

$$\begin{aligned}2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3|)\\ &= 4F - 2|F_2| - |F_3|\\ &\geq 4(|r| - 2g(K) + 1) - 2 - |F_3|\end{aligned}$$

Then

$$|F_3| \geq -8g(K) + 2 + 2|r| \tag{12}$$

Comparing (11) and (12) we get that $|r| \leq 4g(K) + 3$.

This concludes the proof. □

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