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HOMOTOPY GROUPS OF CERTAIN HIGHLY CONNECTED MANIFOLDS VIA LOOP SPACE HOMOLOGY

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Abstract

For $n \geq 2$ we consider $(n-1)$-connected closed manifolds of dimension at most $(3n-2)$. We prove that away from a finite set of primes, the p-local homotopy groups of M are determined by the dimension of the space of indecomposable elements in the cohomology ring $H^*(M;\mathbb{Q})$. Moreover, we show that these p-local homotopy groups can be expressed as a direct sum of p-local homotopy groups of spheres. This generalizes some of the results of our earlier work [1].

1. Introduction

In this document we compute homotopy groups of $(n-1)$-connected closed manifolds of dimension at most $(3n-2)$. The main techniques used arise from the theory of quadratic associative algebras and Lie algebras. These extend the arguments for $(n-1)$-connected $2n$-manifolds in [1]. Any $(n-1)$-connected $2n$-manifold has a cell structure obtained by attaching a single $2n$-cell to a wedge of n-spheres, a fact that is crucially used in [1]. For the more general manifolds in this paper one does not have such a cell structure, yet analogous results hold after inverting finitely many primes (cf. Theorem 3.6, Theorem 3.8).

Theorem 1.1. *Let M be a closed $(n-1)$-connected d-manifold with $n \geq 2$, $d \leq 3n-2$ and $\dim H^*(M;\mathbb{Q}) > 4$. Let r denote the dimension of the space of indecomposables in $H^*(M;\mathbb{Q})$. Then there is a finite set of primes Γ such that for $p \notin \Gamma$,*
(a) *the p-local homotopy groups of M can be expressed as a direct sum of p-local homotopy groups of spheres ;*
(b) *the number of summands $\pi_k(S^l)_{(p)}$ in $\pi_k(M)_{(p)}$ is a function that depends on r, but not on n or d.*

If a generator of $H^d(M;\mathbb{Q})$ is indecomposable then it follows from Poincaré duality that the rational cohomology of M is that of S^n. In this case $\dim H^*(M;\mathbb{Q}) = 2$ and $r = 1$. Conversely, suppose that $r = 1$ and M does not have the cohomology of a sphere. It follows that M is a manifold of dimension $2n$ with $H^*(M;\mathbb{Q}) = \mathbb{Q}[x]/(x^3)$ and n is even.

By the assumptions on M, the cup product of any three cohomology classes (of positive degree) is zero. Now we assume that $\dim H^*(M;\mathbb{Q}) > 4$. If any class $\alpha = a \cup b \in H^i(M;\mathbb{Q})$ (with $i < d$) is reducible then by Poincaré duality there exists $\beta \in H^{d-i}(M;\mathbb{Q})$ such that $\alpha \cup \beta$ is a generator of $H^d(M;\mathbb{Q})$. Thus, $a \cup b \cup \beta \neq 0$ and this violates our previous observation.

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Therefore, if $r > 1$ we deduce that

$$r = \dim(\oplus_{0<i<d} H^i(M;\mathbb{Q})) = \sum_{0<i<d} \dim H^i(M;\mathbb{Q}) = \dim H^*(M;\mathbb{Q}) - 2.$$

We note that the condition $\dim H^*(M;\mathbb{Q}) > 4$ is equivalent to $r \geq 3$. In terms of rational homotopy groups, M is rationally hyperbolic if and only if $r > 2$. In this case we have the following result (cf. Theorem 3.9).

Theorem 1.2. *Let M be a closed $(n-1)$-connected d-manifold with $n \geq 2$, $d \leq 3n-2$ and $\dim H^*(M;\mathbb{Q}) > 4$. Then the homotopy groups of M have unbounded p-exponents for all but finitely many primes.*

The above result verifies the Moore conjecture (see the discussion before Theorem 3.9 as well as [6] pp. 518) for such spaces. The Moore conjecture states that a simply connected finite CW complex is rationally elliptic if and only if it has a finite homotopy exponent at all primes. In the examples above the manifolds M are rationally hyperbolic, where the conjecture states that there exists primes p such that M does not have a finite homtopy exponent at p. The Theorem implies the stronger conclusion that M does not have finite homotopy exponent at almost all primes. The low rank cases, i.e., when $r = 1, 2$ are discussed in §3.3 (see Theorem 3.10).

2. Homology of the loop space

Let M be a closed $(n-1)$-connected d-manifold with $d \leq 3n-2$. The cohomology of M is finitely generated and has p-torsion only for a finite set of primes p. Let Σ be the set of primes such that the cohomology of M has p-torsion. Define

$$R_\Sigma = \mathbb{Z}[\tfrac{1}{p} \mid p \in \Sigma].$$

Then we may deduce the following facts.

(a) $H^*(M;R_\Sigma)$ is a free R_Σ-module.

(b) The natural map $H^*(M;R_\Sigma) \otimes_{R_\Sigma} \mathbb{Q} \to H^*(M;\mathbb{Q})$ is an isomorphism.

The first fact follows from Universal Coefficient Theorem for cohomology and the defining property of R_Σ. The second fact is clear.

As noted earlier, the only non-trivial products of positive dimensional classes are given by the intersection form. Therefore the module of indecomposables in $H^*(M;R_\Sigma)$ is given by $\mathcal{A}(M) = \oplus_{0<i<d} H^i(M;R_\Sigma)$. Let $x_1, \ldots, x_r$ be a basis of $\mathcal{A}(M)$. Fix a choice of an orientation class $[M] \in H^d(M;R_\Sigma)$ of M. Let $c_{ij} = \langle x_i x_j, [M] \rangle$.

In [3], the homology of ΩM is computed for M as above such that $\mathrm{Rank}(H^*(M;\mathbb{Z})) > 4$. Let us recall it. Consider the homology ring $H_*(\Omega M;\mathbb{Q})$ of the based loop space, equipped with the Pontrjagin product. This ring is freely generated as an associative algebra by classes $u_1, \ldots, u_r$ whose homology suspensions are dual to the classes $x_1, \ldots, x_r$ (in particular $|u_i| = |x_i| - 1$), modulo the single quadratic relation

$$\sum_{i,j} (-1)^{|u_i|+1} c_{ji} u_i u_j = 0.$$

The same argument works for a quotient field of R_Σ. We note this in the proposition below (cf. [3], Theorem 1.1). We use the following notation for a commutative ring A. If V is a free A-module then we shall denote by $T_A(V)$ (often abbreviated as $T(V)$) the tensor algebra generated by V. If the free module V has a basis $\{v_1, v_2, \ldots\}$, then we often write $T_A(V)$ as $T_A(v_1, v_2, \ldots)$.

Proposition 2.1. *For $k = \mathbb{Q}$ or a quotient field of R_Σ, there is an isomorphism of associative rings,*

$$H_*(\Omega M; k) \cong T_k(u_1, \ldots, u_r)/(\textstyle\sum(-1)^{|u_i|+1} c_{ji} u_i u_j).$$

This directly leads us to the following integral version.

Proposition 2.2. *As associative rings,*

$$H_*(\Omega M; R_\Sigma) \cong T_{R_\Sigma}(u_1, \ldots, u_r)/(\textstyle\sum(-1)^{|u_i|+1} c_{ji} u_i u_j).$$

Proof. Since M is an orientable manifold the homology $H_*(M - pt; R_\Sigma)$ matches $H_*(M; R_\Sigma)$ in all degrees up to $(d-1)$. From the conditions on M we deduce that $H^*(M - pt; R_\Sigma)$ is free on the classes x_i^* which are dual to the classes x_i and the products are all zero. It follows that $H_*(\Omega(M - pt); R_\Sigma)$ is a tensor algebra on the classes u_i. A way to see this is by deducing from [3, Theorem 4.1] that $M - pt$ is a formal space over R_Σ, and then from [3, Theorem 2.1] that for k a quotient field or a fraction field of R_Σ, $H_*(\Omega(M - pt); k)$ is the Koszul dual algebra of $H^*(M - pt; k)$. The rest follows from the fact that the Koszul dual algebra of an algebra with trivial products is the tensor algebra [14, Chapter 1, Section 2]. Therefore we have a map

$$\phi : T_{R_\Sigma}(u_1, \ldots, u_r) \xrightarrow{\cong} H_*(\Omega(M - pt); R_\Sigma) \longrightarrow H_*(\Omega M; R_\Sigma).$$

In the dimension range $0 \leq * \leq d - 1$, we may compute $H_*(\Omega M; R_\Sigma)$ using the Serre spectral sequence associated to the path-space fibration $\Omega M \to PM \to M$. This has the form

$$E^2_{p,q} = H_p(M) \otimes H_q(\Omega M) \implies H_*(pt)$$

with coefficients in R_Σ. From the multiplicative structure on the dual cohomology spectral sequence it follows that the indecomposable elements (with basis x_j^*) lie in the image of the transgression. Therefore the classes x_j are transgressive and they transgress onto the classes u_j, that is, $d(x_j) = u_j$.

The homology of M being torsion-free implies that the cohomology of M is just the dual. From the dual spectral sequence, we deduce that the classes $x_k \otimes u_j$ are mapped by differentials onto the classes $u_k \otimes u_j$ on the vertical 0-line. It follows that in degrees $\leq d - 1$, $H_*(\Omega M; R_\Sigma)$ are generated by the classes $u_i, u_i u_j$. The differential on the class $[M]$ hits a linear combination of $x_k \otimes u_j$. Hence, in this range of degrees $H_*(\Omega M; R_\Sigma) \cong T_{R_\Sigma}(u_1, \ldots, u_r)/(l)$ for some element l of homogeneous degree 2 in u_i.

Let $H_*(\Omega M; R_\Sigma)^{(2)}$ denote the free R_Σ-submodule generated by the homogeneous degree 2 elements which is isomorphic to $R_\Sigma\{u_i \otimes u_j\}/(l)$. The computations of [3] as quoted above imply that

$$R_\Sigma\{u_i \otimes u_j\}/(l) \otimes_{R_\Sigma} k \cong R_\Sigma\{u_i \otimes u_j\}/(\textstyle\sum(-1)^{|u_i|+1} c_{ji} u_i u_j) \otimes_{R_\Sigma} k$$

for k being either the fraction field of R_Σ, or $R_\Sigma/(\pi)$ for primes π in R_Σ. The first case implies that there are $a, b \in R_\Sigma$ such that

$$al = b(\sum(-1)^{|u_i|+1}c_{ji}u_iu_j)$$

and the second case implies that a and b are non-zero and differ by a unit modulo π for every prime π. Thus a and b are forced to be units after possible cancellations, and we may take $l = \sum(-1)^{|u_i|+1}c_{ji}u_iu_j$. Thus

$$H_*(\Omega M; R_\Sigma)^{(2)} \cong R_\Sigma\{u_i \otimes u_j\}/(\sum(-1)^{|u_i|+1}c_{ji}u_iu_j)$$

so that the element $\sum(-1)^{|u_i|+1}c_{ji}u_iu_j \in T_{R_\Sigma}(u_1, \ldots, u_r)$ goes to 0 under ϕ above. Thus we obtain a ring map

$$T_{R_\Sigma}(u_1, \ldots, u_r)/(\sum(-1)^{|u_i|+1}c_{ji}u_iu_j) \to H_*(\Omega M; R_\Sigma)$$

which is an isomorphism after tensoring with the fraction field of R_Σ or going modulo a prime from Proposition 2.1. The result now follows. □

3. Homotopy groups of certain $(n-1)$-connected manifolds

In this section we deduce results about the homotopy groups of $(n-1)$-connected manifolds of dimension $d \leq 3n-2$ after inverting finitely many primes. We use the computation of the homology of the loop space in Section 2. Note from Proposition 2.2 that $H_*(\Omega M; R_\Sigma)$ is a quadratic algebra. We prove that this possesses a nice basis and so does the corresponding quadratic Lie algebra. The basis of the Lie algebra is used to express $\pi_*(M)$ as a direct sum of homotopy groups of spheres after inverting finitely many primes.

3.1. Algebraic preliminaries. We start by recalling some algebraic preliminaries on quadratic algebras and quadratic Lie algebras. For further details we refer to [1, 14, 13]. Let A be a commutative ring (usually a principal ideal domain (PID)). If V is a free A-module then $T_A(V)$ denotes the tensor algebra generated by V, and the notation $Lie(V)$ (respectively $Lie^{gr}(V)$) denotes the free Lie algebra (respectively graded Lie algebra) on the A-module V.

Definition 3.1. For $R \subset V \otimes_A V$, the associative algebra $A(V, R) = T(V)/(R)$ is called a quadratic A-algebra.

If $R \subset V \otimes_A V$ lies in $Lie(V)$, the Lie algebra $L(V, R) = Lie(V)/((R))$ is called a quadratic Lie algebra over A. In the graded case this is denoted $L^{gr}(V, R)$.

It may be observed that the universal enveloping algebra of $L(V, R)$ is $A(V, R)$ and in the graded case the universal enveloping algebra of $L^{gr}(V, R)$ is $A(V, R)$ as graded modules. If in addition the modules $A(V, R)$ and $L(V, R)$ are free, there is a Poincaré-Birkhoff-Witt theorem which may be stated as $E_0(A(V, R)) \cong A[L(V, R)]$. The notation $A[L(V, R)]$ denotes the polynomial A-algebra on the module $L(V, R)$ and $E_0(A(V, R))$ denotes the associated graded for the filtration of $A(V, R)$ induced by the weight filtration on the tensor algebra. A similar statement holds for the graded case where one interprets the polynomial algebra as the polynomial algebra on even degree classes tensored with the exterior algebra on the odd degree classes. Finally from [5], [10] one may deduce that for a PID A, $L(V, R)$ is a free module if $A(V, R)$ is free and V has finite rank.

Next we recall the Diamond lemma from [4]. Let V be generated by a basis $\{x_1, \ldots, x_n\}$.

Fix an order of x_i, say $x_1 < x_2 < \cdots < x_n$, and obtain an induced order on monomials (ordered by degree, and lexicographically in each degree). Suppose that the free module R can be given a basis where each element is of the form $W_i - f_i$ where W_i is a monomial and f_i is a linear combination of monomials $< W_i$. Call a monomial R-indecomposable if it does not possess any submonomial which occurs as W_i in the above chosen basis. The Diamond lemma [4, Theorem 1.2] states certain sufficient conditions under which the R-indecomposable monomials form a basis of $A(V, R)$. The following implication of the Diamond lemma suffices for this paper [1, Theorem 2.8].

Proposition 3.2. *Suppose that R is generated by a single element of the form*

$$x_a \otimes x_\beta = \sum_{(i,j)\neq(\alpha,\beta)} a_{i,j}\ x_i \otimes x_j$$

with $\alpha \neq \beta$. Then the R-indecomposable elements form a basis for $A(V, R)$.

There is an analogous construction for Lie algebras $L(V, R)$ defined by generators and relations (see [9]). This is called a Lyndon basis. Start with a basis of V and an order on the basis set. We call a word in elements of V a Lyndon word if it is lexicographically smaller than its cyclic rearrangements. For a Lyndon word l there are unique Lyndon words l_1 and l_2 so that $l = l_1 l_2$ and l_2 is the largest possible Lyndon word occuring in the right in l. Inductively we may associate to the Lyndon word l the element $b(l) = [b(l_1), b(l_2)]$ of the free Lie algebra on V. One verifies that these elements form a basis of the free Lie algebra on V. In the case $R \neq 0$, we say a Lyndon word is R-standard if it cannot be further reduced using relations in R (with respect to the chosen order on V). From [9] we recall the following result (also see [1], Theorem 2.17).

Proposition 3.3. *Suppose that A is a localization of $\mathbb{Z}$ and R as in* Proposition 3.2. *Then, the R-standard Lyndon words give a basis of $L(V, R)$.*

3.2. Homotopy groups using loop space homology. Let us denote by $l(M)$ the sum $\sum(-1)^{|u_i|+1} c_{ji} u_i u_j$. It is clear that $l(M)$ lies in the free graded Lie algebra on the classes $u_1, \ldots, u_r$ which we denote by $Lie^{gr}(u_1, \ldots, u_r)$. For, if $i \neq j$ we have

$$c_{ij} = (-1)^{(|u_i|+1)(|u_j|+1)} c_{ji}$$

and hence

$$(-1)^{|u_i|+1} c_{ji} u_i u_j + (-1)^{|u_j|+1} c_{ij} u_j u_i = (-1)^{|u_i|+1} c_{ji} [u_i, u_j]^{gr},$$

where the graded Lie bracket is given by

$$[u_i, u_j]^{gr} = (u_i \otimes u_j - (-1)^{|u_i||u_j|} u_j \otimes u_i).$$

For the terms $c_{ii} u_i^2$, note that if $|u_i|$ is even, the formula $\langle x_i^2, [M]\rangle = c_{ii}$ implies that $c_{ii} = 0$, while if $|u_i|$ is odd, u_i^2 belongs to the free graded Lie algebra [13, Definition 8.1.1].

Consider the graded Lie Algebra $\mathcal{L}_r^{gr}(M)$ (over R_Σ) given by

$$\frac{Lie^{gr}(u_1, \ldots, u_r)}{(l(M))}$$

where $(l(M))$ denotes the graded Lie algebra ideal generated by $l(M)$. This is a quadratic graded Lie algebra. In this respect we denote

$$l(M) = \sum_{i<j} l_{i,j}[u_i, u_j]^{gr} + \sum_{|u_i|\ odd} l_{i,i}u_i^2 = \sum_{i<j} l_{i,j}(u_i \otimes u_j - (-1)^{|u_i||u_j|}u_j \otimes u_i) + \sum_{|u_i|\ odd} l_{i,i}u_i^2.$$

We make an analogous ungraded construction. Consider the element

$$l^u(M) = \sum_{i<j} l_{i,j}[u_i, u_j] = \sum_{i<j} l_{i,j}(u_i \otimes u_j - u_j \otimes u_i).$$

This element lies in $Lie(u_1, \ldots, u_r)$. We shall make use of the following notation:

$$A_r^u(M) = \frac{T_{R_\Sigma}(u_1, \ldots, u_r)}{(l^u(M))}, \ \mathcal{L}_r^u(M) = \frac{Lie(u_1, \ldots, u_r)}{(l^u(M))}.$$

The Lie algebra $\mathcal{L}_r^u(M)$ and the associative algebra $A_r^u(M)$ possess an induced grading.

Proposition 3.4. *$\mathcal{L}_r^u(M)$ and $\mathcal{L}_r^{gr}(M)$ are free over R_Σ. The Lyndon basis gives a basis of $\mathcal{L}_r^u(M)$.*

Proof. From the formulas in Proposition 2.2, it is clear that $H_*(\Omega M; R_\Sigma)$ is the universal enveloping algebra of the Lie algebra $\mathcal{L}^{gr}(M)$ in the graded sense. Analogously, $A_r^u(M)$ is the universal enveloping algebra of $\mathcal{L}^u(M)$. As R_Σ is a PID, for the first statement it suffices to show that $H_*(\Omega M; R_\Sigma)$ and $A_r^u(M)$ are free R_Σ-modules. We verify this last fact by proving $l(M)$ and $l^u(M)$ satisfy the hypothesis of Proposition 3.2. Now Proposition 3.3 implies the second statement as well.

Since the coefficients of u_iu_j for $i \neq j$ in $l(M)$ and $l^u(M)$ differ only by a sign, it suffices to write the element $l(M)$ as

$$u_iu_j = \text{combination of other terms not containing } u_iu_j.$$

This is equivalent to a change of basis of the x_i so that some c_{ji} equals 1. For $d > 2n$ note that n and $d-n$ are not the same. Now pick the basis x_i so that the dual classes in $H^n(M; R_\Sigma)$ are Poincaré dual to those in $H^{d-n}(M; R_\Sigma)$ (this is possible as $n \neq \frac{d}{2}$). For example we may start with a basis of $H^n(M; R_\Sigma)$ and then the dual basis of $H^{d-n}(M; R_\Sigma)$ and extend to a basis of $H^{0<*<d}(M; R_\Sigma)$. Order the basis so that $x_1^* \in H^n(M; R_\Sigma)$ and $x_r^* \in H^{d-n}(M; R_\Sigma)$ are dual to each other. Then $c_{1,r} = 1$ and thus

$$l(M) = \sum (-1)^{|u_i|+1} c_{ji}u_iu_j = u_1u_r + \text{combination of other terms.}$$

This completes the proof.

We have seen that the change of basis is possible for $d = 2n$ in [1, Proposition 2.9]. We repeat the proof here. In this case we need to change basis so that the matrix of the intersection form has a 1 at a position not on the diagonal. The proof for the n odd case is easier. The intersection form induces a symplectic inner product on $V = H^n(M; R_\Sigma)$ [12, Ch 1, Definition 1.3], and we have from [12, Ch 1, Corollary 3.5] that any symplectic inner product space over a Dedekind Domain possesses a symplectic basis. This implies the result in the n odd case.

For n even, the intersection form β is a symmetric bilinear form on $V = H^n(M; R_\Sigma)$. From the assumptions on M, we know that $\text{Rank}(V) = r \geq 3$. Choose some basis $\{a_1, \ldots, a_r\}$ of V over R_Σ. Due to dimensional constraints there exists a $\mathbb{Q}$-linear combination v of $a_2, \ldots, a_r$ such that $\beta(v, a_1) = 0$. Clear out denominators of v so that v is a primitive combination of

$a_2, \dots, a_r$ over R_Σ. Hence one may find a basis of $R_\Sigma\{a_2, \dots, a_r\}$ of the form $\{v, b_3, \dots, b_r\}$. Thus we may change basis and assume $\beta(a_1, a_2) = 0$. Now as the bilinear form is non-singular over R_Σ there exists w such that $\beta(a_1, w) = 1$. Note this computation does not change if we add multiples of a_2. Therefore in terms of the basis we may assume that the coefficient of a_2 in w is 1. We may now replace a_2 in the basis by w and obtain a new basis with the property $\beta(a_1, a_2) = 1$. This basis satisfies the required condition. □

Next we enlarge the set of primes so that the classes u_i are in the image of the Hurewicz homomorphism.

Proposition 3.5. *There exists a finite set of primes Γ containing Σ such that the classes u_i lie in the image of the Hurewiciz homomorphism $\pi_*(\Omega M) \otimes R_\Gamma \to H_*(\Omega M; R_\Gamma)$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \pi_*(\Omega M) \otimes R_\Sigma & \xrightarrow{\text{Hur}} & H_*(\Omega M; R_\Sigma) \\ \downarrow & & \downarrow \\ \pi_*(\Omega M) \otimes \mathbb{Q} & \xrightarrow{\text{Hur}} & H_*(\Omega M; \mathbb{Q}). \end{array}$$

Since $H_*(\Omega M; R_\Sigma)$ is a free R_Σ-module, the right vertical arrow is injective; it takes u_i to the corresponding element u_i. We know from the Milnor-Moore theorem that $H_*(\Omega M; \mathbb{Q})$ is the universal enveloping algebra on the rational homotopy Lie algebra $\pi_*(M) \otimes \mathbb{Q}$. It follows by standard methods [6, Theorem 15.11] that x_i's are dual to non-trivial elements in the rational homotopy groups in the appropriate degrees. This is because from [3], it follows that M is formal so that the x_i may be used in a minimal model for M.

As a consequence, the element u_i lies in the Hurewicz homomorphism for $\mathbb{Q}$ coefficients. It follows that for every i there is an integer d_i so that $d_i u_i$ lies in the image of the Hurewicz homorphism. Define Γ as Σ plus all the prime factors of d_i for $1 \leq i \leq r$. Consequently, over R_Γ, all the u_i are in the image of the Hurewicz homomorphism. □

Our goal is to compute the homotopy groups $\pi_*(M) \otimes R_\Gamma$. We work in the R_Γ-*local category*: that is, the category obtained from spaces by localizing with respect to $H_*(-; R_\Gamma)$-equivalences. Let S^n_Γ denote the R_Γ-local sphere. We know that if a map between simply connected spaces (or, more generally, simple spaces) is an $H_*(-; R_\Gamma)$-equivalence then it induces an isomorphism on $\pi_*(-) \otimes R_\Gamma$.

From Proposition 3.5, there are elements in $\pi_*(M) \otimes R_\Gamma$ whose adjoints have Hurewicz image u_i. By iterated Whitehead products we may map spheres into M corresponding to chosen elements of the Lie algebra $\mathcal{L}^u(M)$. We describe this in a precise fashion below.

We denote the degree of u_i by $|u_i|$. We fix a map $S^{|u_i|} \to \Omega M$ with Hurewicz image u_i. By adjunction we have a map $\alpha_i : S^{|u_i|+1} \to M$ with the property that after looping α_i the generator of the Pontrjagin ring maps to u_i.

There exists a Lyndon basis for $\mathcal{L}^u(M)$ by Proposition 3.4. List these elements in order as $l_1 < l_2 < \dots$ and define the height of a basis element by $h_i = h(l_i) = k + 1$ if $b(l_i) \in (\mathcal{L}^u(M))_k$ the k^{th}-graded piece. Then $h(l_i) \leq h(l_{i+1})$. Note that $b(l_i)$ represents an element of $Lie(u_1, \dots, u_r)$ and is thus represented by an iterated Lie bracket of u_i. Define $\lambda_i : S^{h_i}_\Gamma \to M$ as the Whitehead product replacing each u_i in the bracket by α_i.

Theorem 3.6. *There is an isomorphism*

$$\pi_*(M) \otimes R_\Gamma \cong \sum_{i \geq 1} \pi_* S^{h_i} \otimes R_\Gamma$$

and the inclusion of each summand is given by λ_i.

Observe that the right hand side is a finite sum in each degree.

Proof. The maps $\Omega\lambda_i : \Omega S^{h_i}_\Gamma \to \Omega M$ for $i = 1, \ldots, n$ can be multiplied using the H-space structure on ΩM to obtain a map from $S(n) = \prod_{i=1}^n \Omega S^{h_i}_\Gamma \to \Omega M$. Letting n vary $S(n)$ gives a directed system arising from the inclusion of subfactors using the basepoint. Fix an associative model for ΩM (for example using Moore loops) and observe that the various maps from $S(n)$ induces a map on the homotopy colimit

$$\Lambda : S := \mathrm{hocolim}_n\, S(n) \longrightarrow \Omega M.$$

Note that homotopy groups of S is the right hand side of the expression in the Theorem shifted in degree by 1. Hence it suffices to prove that Λ is a weak equivalence after inverting the primes in Γ. As both the domain and codomain are simple spaces, it suffices to show that this is an R_Γ-homology isomorphism.

The homology of S is a polynomial algebra with a generator for each copy of $\Omega S^{h_i}_\Gamma$

$$H_*(S; R_\Gamma) \cong T_{R_\Gamma}(c_{h_1-1}) \otimes T_{R_\Gamma}(c_{h_2-1}) \ldots \cong R_\Gamma[c_{h_1-1}, c_{h_2-1}, \ldots]$$

and $\Lambda_* c_{h_i-1} \in H_{h_i-1}(\Omega M; R_\Gamma)$ is the Hurewicz image of $\lambda_i \in \pi_{h_i-1}(\Omega M)$. Denote ρ as

$$\rho : \pi_n(X) \cong \pi_{n-1}(\Omega X) \xrightarrow{Hur} H_{n-1}(\Omega X; \mathbb{Z}) \to H_{n-1}(\Omega X; R_\Gamma).$$

We know from [8] that

$$\rho([a, b]) = \rho(a)\rho(b) - (-1)^{|a||b|}\rho(b)\rho(a). \tag{1}$$

Now from Proposition 2.2 we have the isomorphism $H_*(\Omega M; R_\Gamma) \cong T_{R_\Gamma}(u_1, \ldots, u_r)/(l(M))$ where the right hand side is the universal enveloping algebra of $\mathcal{L}^{gr}_r(M)$ (in the graded sense). From the Poincaré-Birkhoff-Witt theorem for graded Lie algebras we have

$$E_0 T_{R_\Gamma}(u_1, \ldots, u_r)/(l(M)) \cong E(\mathcal{L}^{gr}_r(M)^{odd}) \otimes P(\mathcal{L}^{gr}_r(M)^{even}).$$

The map ρ carries each α_i to u_i. The element $b(l_i)$ is mapped inside $H_*(\Omega M; R_\Gamma)$ to the element corresponding to the graded Lie algebra element by equation (1). We prove that $T_{R_\Gamma}(a_1, \ldots, a_r)/(l(M))$ has a basis given by monomials on $\rho(b(l_1)), \rho(b(l_2)), \cdots$.

Observe inductively that all the elements in $\mathcal{L}^{gr}_r(M)$ can be expressed as linear combinations of monomials in $\rho(b(l_i))$. It is clear for elements of weight 1. For the weight 2 elements note that they are generated by $[u_i, u_j]^{gr}$ for $i < j$, $(i, j) \neq (1, 2)$ and u_i^2 if $|u_i|$ is odd. The former are the Lyndon words and the latter is the square of a monomial. In the general case, a graded Lie algebra element is either a monomial or the square of a lower odd degree class; from one of the conditions in the definition of a graded Lie algebra the bracket with a square can be expressed as a bracket. Such a monomial may be obtained by applying ρ on the corresponding ungraded element. This is a linear combination of certain $b(l_i)$ and something in the ideal generated by $l^u(M)$. Applying ρ we obtain a combination of $\rho(b(l_i))$ and something in the ideal generated by $l(M)$ as $\rho(l^u(M)) = l(M)$ which verifies the induction step. As an application of the Poincaré-Birkhoff-Witt Theorem, we know that Λ_* is surjective.

Hence we have that the graded map

$$R_\Gamma[\rho(b(l_1)), \rho(b(l_2)), \ldots] \to T_{R_\Gamma}(u_1, \ldots, u_r)/(l(M)) \tag{2}$$

is surjective. We also know

$$R_\Gamma[b(l_1), b(l_2), \ldots] \to T_{R_\Gamma}(u_1, \ldots, u_r)/(l^u(M))$$

is an isomorphism. Now both $T_{R_\Gamma}(u_1, \ldots, u_r)/(l(M))$ and $T_{R_\Gamma}(u_1, \ldots, u_r)/(l^u(M))$ have bases given by the Diamond lemma and thus are of the same graded dimension. It follows that the graded pieces of $R_\Gamma[\rho(b(l_1)), \rho(b(l_2)), \ldots]$ and $T_{R_\Gamma}(u_1, \ldots, u_r)/(l(M))$ have the same rank which is finite. Thus on graded pieces one has a surjective map between free R_Γ-modules of the same rank which must be an isomorphism. □

The proof of Theorem 3.6 implies a stronger result about the loop space of the manifold M. We denote the R_Γ-localization of M by M_Γ. For a sequence of based spaces Y_i, we use the notation $\hat{\Pi}_{i\geq 0} Y_i$ for the homotopy colimit of finite products of Y_i.

Theorem 3.7. *With notations as above,*

$$\Omega M_\Gamma \simeq \hat{\Pi}_{i\geq 0} \Omega S_\Gamma^{h_i}.$$

We may now compute the number of copies of S^k in the expression of Theorem 3.6 from the rational cohomology groups of M. Let

$$q_M(t) = 1 - \sum_{n-1<i<d} b_i(M) t^i + t^d.$$

Then $\frac{1}{q_M(t)}$ is the generating series for ΩM (see [11], Theorem 3.5.1) from the fact that $H_*(\Omega M; R_\Gamma)$ is Koszul as an associative algebra (see [3]). Let

$$\eta_m := \text{coefficient of } t^m \text{ in } log(q_M(t)).$$

We may repeat the proof of Theorem 5.7 of [1] to deduce the following result.

Theorem 3.8. *The number of groups $\pi_s S^m \otimes R_\Gamma$ in $\pi_s(M) \otimes R_\Gamma$ is*

$$l_{m-1} = -\sum_{j|m-1} \mu(j) \frac{\eta_{(m-1)/j}}{j}$$

where μ is the Möbius function.

Proof. It is enough to compute the dimension l_j of the j^{th}-graded part of the Lie algebra $\mathcal{L}_r^u(M)$. We use the generating series to compute this from the universal enveloping algebra $H_*(\Omega M)$ as in [2].

The generating series for $H_*(\Omega M)$ is $p(t) = \frac{1}{q_M(t)}$. From (2), the symmetric algebra on $\mathcal{L}_r^u(M)$ is $H_*(\Omega M)$. Hence we have the equation

$$\frac{1}{\prod_j (1 - t^j)^{l_j}} = \frac{1}{1 - \sum_{n-1<i<d} b_i(M) t^i + t^d}.$$

Take log of both sides :

$$\begin{aligned}\log(1-\sum_{n-1<i<d} b_i(M)t^i + t^d) &= \sum_j l_j \log(1-t^j)\\ &= -\sum_j l_j\left(t^j + \frac{t^{2j}}{2} + \frac{t^{3j}}{3} + \cdots\right).\end{aligned}$$

Expanding this and equating coefficients, we see that

$$\eta_m := \text{coefficient of } t^m \text{ in } \log(1-\sum_{n-1<i<d} b_i(M)t^i + t^d) = -\frac{1}{m}\Big(\sum_{j|m} jl_j\Big).$$

We use the Möbius inversion formula; it gives us

$$l_m = -\sum_{j|m} \mu(j)\frac{\eta_{m/j}}{j}.$$

This completes the proof. □

Recall that simply connected, finite cell complexes either have finite dimensional rational homotopy groups or exponential growth of ranks of rational homotopy groups (cf. [6], §33). The former are called rationally elliptic while the latter are called rationally hyperbolic. From [3] we note that the $(n-1)$-connected manifolds of dimension at most $(3n-2)$ with $H^*(M)$ having rank at least 4 are all rationally hyperbolic. One may also verify this directly. Since the rank of $H^*(M)$ is at least 4 the number of generating u_i is at least 3. Then one observes that after switching the ordering appropriately the word

$$u_1u_2u_1u_2u_1u_3$$

is a Lyndon word in degree $> 2d$ as each u_i has degree $> \frac{d}{3}$. So these manifolds cannot be rationally elliptic. This forces by the Milnor-Moore Theorem, that $\mathcal{L}^{gr}(M)\otimes\mathbb{Q}$ has infinite rank. Hence, $\mathcal{L}^u(M)$ also has infinite rank.

There are many conjectures that lie in the dichotomy between rationally elliptic and hyperbolic spaces. We verify such a conjecture by Moore ([6], pp. 518) below. For a rationally hyperbolic space X the Moore conjecture states that there are primes p for which the homotopy groups do not have any exponent at p, that is, for any power p^r there is an element $\alpha \in \pi_*(X)$ of order p^r. We verify the following version.

Theorem 3.9. *If $p \notin \Gamma$, the homotopy groups of M do not have any exponent at p.*

Proof. We have noted above that $\mathcal{L}^u(M)$ has infinite rank. Thus there are elements of the Lyndon basis of arbitrarily large degree. Hence for arbitrarily large l, π_*S^l occurs as a summand of π_*M. The proof is complete by observing that any p^s may occur as the order of an element in π_*S^l for arbitrarily large l. This follows from [7]. This also follows from the fact that the same is true for the stable homotopy groups and these can be realized as $\pi_k^s \cong \pi_{k+l}S^l$ for $l > k+1$. Now torsion of order p^s for any s occurs in the image of the J-homomorphism (cf. [15], Theorem 1.1.13). □

3.3. The low rank cases. We end by demonstrating the above computations when the rank of $H_*(M;\mathbb{Q})$ is at most 4. Since H^0 and H^d are always $\mathbb{Q}$, we have to consider three possibilities: rank $2,3,4$. Our main techniques involve determining the rational homotopy type of M, of dimension d, and using it to compute the homotopy type at all but finitely many primes.

In the rank 2 case we know that rationally M is a sphere so that $M_{\mathbb{Q}} \simeq S^d_{\mathbb{Q}}$. Let Σ denote the finite set of primes which occur as torsion in the homology of M. With R_Σ-coefficients, the R_Σ localization M_Σ of M is a homology d-sphere. Thus $H^*(M; R) \cong H^*(S^d; R)$ for any ring R lying between R_Σ and $\mathbb{Q}$. Let α stand for the common notation for a generator of $H_*(M; R)$ for any such R. As $M_{\mathbb{Q}} \simeq S^d_{\mathbb{Q}}$, α lies in the image of the rational Hurewicz homomorphism. It follows that with R_Σ-coefficients, there is an integer k so that $k\alpha$ lies in the image of the Hurewicz homomorphism. Let Γ denote the union of Σ and the prime divisors of k. Then Γ is a finite set and α lies in the image of the R_Γ Hurewicz homomorphism. It is now clear that there is a map $S^d_\Gamma \to M$ which is an isomorphism with R_Γ-coefficients. Therefore, we have a homotopy equivalence $M_\Gamma \simeq S^d_\Gamma$. Note that the torsion in the homology can be quite varied. So this is precisely the sort of result we are looking for.

For the next case, let J_2S^n be the second stage of the James construction which is obtained as the mapping cone of the Whitehead product $[id, id]$. If $H^*(M; \mathbb{Q})$ has rank 3, by Poincaré duality the cohomology ring is forced to be $\mathbb{Q}[x_s]/(x_s^3)$ where $d = 2s$. By graded commutativity s is forced to be even. The rational homotopy of such a space may be computed directly from the cohomology ring structure as the ring structure forces the space to be formal. The minimal model is given by $\Lambda(x_s, y_{3s-1})$ with $d(x_s) = 0$ and $d(y_{3s-1}) = x_s^3$. Thus the rational homotopy groups of M are given by

$$\pi_k^{\mathbb{Q}}(M) = \begin{cases} \mathbb{Q} & \text{if } k = s, 3s-1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that on rationalizations we have a map $S^s_{\mathbb{Q}} \to M_{\mathbb{Q}}$ which is an isomorphism on $\pi_*^{\mathbb{Q}}$ for $* \leq 2s-2$. In degree $2s-1$ the homotopy groups are $\pi_{2s-1}^{\mathbb{Q}} S^s \cong \mathbb{Q}\{[id, id]\}$ and $\pi_{2s-1}^{\mathbb{Q}}(M) = 0$. Therefore the composite

$$S^{2s-1}_{\mathbb{Q}} \xrightarrow{[id,id]} S^s_{\mathbb{Q}} \longrightarrow M_{\mathbb{Q}}$$

is null-homotopic and thus factors through the cofibre $(J_2S^s)_{\mathbb{Q}}$. Therefore we obtain a map $(J_2S^s)_{\mathbb{Q}} \to M_{\mathbb{Q}}$ which is an isomorphism on H^s and by cup products also on H^d. As a result, one obtains that $M_{\mathbb{Q}} \simeq J_2S^s_{\mathbb{Q}}$.

Next we upgrade the rational homotopy result to one which is valid after inverting finitely many primes, that is, over a set Γ of finitely many primes that the Γ-localizations of the above two spaces are weakly equivalent. Let Σ denote all the primes which appear as torsion in the homology of M. As M is a simply connected compact CW-complex, the homotopy groups of M are finitely generated in each degree. Let Γ denote the primes in Σ together with the finite list of primes which appear as torsion in $\pi_{2s-1}(M)$ and those which need to be inverted so that x_s lies in the image of the Hurewicz homomorphism. We consider the following commutative diagram:

$$\begin{array}{ccccc} S^{2s-1}_\Gamma & \xrightarrow{[id,id]} & S^s_\Gamma & \longrightarrow & M_\Gamma \\ \downarrow & & \downarrow & & \downarrow \\ S^{2s-1}_{\mathbb{Q}} & \xrightarrow{[id,id]} & S^s_{\mathbb{Q}} & \longrightarrow & M_{\mathbb{Q}}. \end{array}$$

The composite in the bottom row is 0. The composite in the top row gives an element in $\pi_{2s-1}(M)\otimes R_\Gamma$ which injects into $\pi_{2s-1}(M)\otimes\mathbb{Q}$ by our choice of Γ. The latter group is 0 from our choices. It follows that the composite of the top row is 0 and thus we obtain a map from the mapping cone $J_2S^s_\Gamma \to M_\Gamma$ which is an isomorphism in cohomology with R_Γ coefficients by the same argument as that for $\mathbb{Q}$. Thus we deduce that

$$M_\Gamma \simeq (J_2S^s)_\Gamma.$$

It remains to consider the last case when total rank is 4. Let $\#^2J_2(n)$ denote the mapping cone of

$$[id^1, id^1] + [id^2, id^2] : S^{2n-1} \to S^n \vee S^n.$$

Then,

$$H^*(\#^2J_2(n);\mathbb{Q}) \cong \mathbb{Q}[x_n, y_n]/(x_n^3, y_n^3, x_ny_n, x_n^2 = y_n^2).$$

If $H^*(M;\mathbb{Q})$ has rank 4, then by Poincaré duality the rational cohomology ring is forced to be one of the following:
(a) $\{1, x_s, y_s, x_s^2 = y_s^2\}$ (where $d = 2s$ with s even),
(b) $\{1, x_k, y_{d-k}, x_k \cdot y_{d-k}\}$.
Notice that (a) is the rational cohomology ring of $\#^2J_2(s)$ while (b) is the rational cohomology ring of $S^k\times S^{d-k}$. We now deduce that the rational homotopy type of M must indeed be one of these.

The rational homotopy groups may be computed directly as the ring structure forces the space to be formal. The minimal model for type (a) is given by

$$\Lambda(x_s, y_s, u_{2s-1}, v_{2s-1}), d(x_s) = d(y_s) = 0, d(u_{2s-1}) = x_s^2 - y_s^2, d(v_{2s-1}) = x_sy_s.$$

Thus the rational homotopy groups of M are given by

$$\pi_k^{\mathbb{Q}}(M) = \begin{cases} \mathbb{Q} & \text{if } k = 0,\\ \mathbb{Q}^2 & \text{if } k = s, 2s-1,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, on rationalizations we have a map $S^s_{\mathbb{Q}} \vee S^s_{\mathbb{Q}} \to M_{\mathbb{Q}}$ representing x_s and y_s which is an isomorphism on $\pi_*^{\mathbb{Q}}$ for $* \leq 2s-2$. In degree $2s-1$ the homotopy group

$$\pi_{2s-1}^{\mathbb{Q}}(S^s \vee S^s) \cong \mathbb{Q}\{[id^1, id^1], [id^1, id^2], [id^2, id^2]\}.$$

In the computation of homotopy groups using minimal models one knows that the quadratic part of the differential represents the Whitehead product, and so it follows that the element $[id^1, id^1] + [id^2, id^2]$ goes to 0 in M. Therefore the composite

$$S^{2s-1}_{\mathbb{Q}} \xrightarrow{[id^1,id^1]+[id^2,id^2]} S^s_{\mathbb{Q}} \longrightarrow M_{\mathbb{Q}}$$

is null-homotopic and thus factors through the cofibre $(\#^2J_2(s))_{\mathbb{Q}}$. Therefore we obtain a map $(\#^2J_2(s))_{\mathbb{Q}} \to M_{\mathbb{Q}}$ which is an isomorphism on H^s and by cup products also on H^d. It follows that $M_{\mathbb{Q}} \simeq \#^2J_2(s)_{\mathbb{Q}}$.

If the cohomology algebra is of type (b), the minimal model matches that for the product $S^k \times S^{d-k}$. Thus, on rationalizations we have a map $S^k_{\mathbb{Q}} \vee S^{d-k}_{\mathbb{Q}} \to M_{\mathbb{Q}}$ which is an isomorphism on $\pi_*^{\mathbb{Q}}$ for $* \leq 2s-2$. In degree $2s-1$ the class $[id_k, id_{d-k}]$ generates a copy of $\mathbb{Q}$ in $\pi_{2s-1}^{\mathbb{Q}}(S^k \vee S^{d-k})$. As in the argument above, one shows that $[id_k, id_{d-k}]$ goes to 0 in M. Therefore we obtain a map $(S^k \times S^{d-k})_{\mathbb{Q}} \to M_{\mathbb{Q}}$ which is an isomorphism on $H^{\leq d-1}$ and by cup products also on H^d. It follows that $M_{\mathbb{Q}} \simeq (S^k \times S^{d-k})_{\mathbb{Q}}$.

As in the $r = 3$ case we upgrade the rational homotopy result to one which is valid after inverting finitely many primes. Let Σ denote all the primes which appear as torsion in the homology of M and Γ denote the primes in Σ together with the finite primes which appear as torsion in $\pi_{2s-1}(M)$, and so that the generators of $H_{\leq d-1}$ lie in the image of the R_Γ-Hurewicz homomorphism. Let ϕ denote $[id^1, id^1] + [id^2, id^2]$ or $[id^1, id^2]$ accordingly as $H^*(M;\mathbb{Q})$ is of type (a) or (b). We consider the following commutative diagram

$$\begin{array}{ccccc} S_\Gamma^{2s-1} & \xrightarrow{\phi} & S_\Gamma^s \vee S_\Gamma^s & \longrightarrow & M_\Gamma \\ \downarrow & & \downarrow & & \downarrow \\ S_{\mathbb{Q}}^{2s-1} & \xrightarrow{\phi} & S_{\mathbb{Q}}^s \vee S_{\mathbb{Q}}^s & \longrightarrow & M_{\mathbb{Q}}. \end{array}$$

The composite in the bottom row is 0. The composite in the top row gives an element in $\pi_{2s-1}(M) \otimes R_\Gamma$ which injects into $\pi_{2s-1}(M) \otimes \mathbb{Q}$ by our choice of Γ. The class ϕ maps to 0 in the latter group as proved above. It follows that the composite of the top row is 0 and thus we obtain a map from the mapping cone

$$\mathrm{Cone}(\phi)_\Gamma \to M_\Gamma$$

which is an isomorphism in cohomology with R_Γ coefficients by the same argument as that for $\mathbb{Q}$. Thus we deduce that

$$M_\Gamma \simeq \mathrm{Cone}(\phi)_\Gamma.$$

We summarize all the above computations and observations in the result below.

Theorem 3.10. *Let M be a $(n-1)$-connected d-manifold with $d \leq 3n-2$. Suppose that the total rank of $H^*(M;\mathbb{Q})$ is at most* 4.
(i) *If the rank is* 2*, then there is a finite set of primes Γ such that $M_\Gamma \simeq S_\Gamma^d$.*
(ii) *If the rank is* 3*, then there is a finite set of primes Γ such that $M_\Gamma \simeq J_2 S_\Gamma^{d/2}$.*
(iii) *If the rank is* 4*, then there is a finite set of primes Γ such that $M_\Gamma \simeq (\#^2 J_2(\frac{d}{2}))_\Gamma$ or $M_\Gamma \simeq (S^k \times S^{d-k})_\Gamma$.*

References

[1] S. Basu and S. Basu: *Homotopy groups of highly connected manifolds*, Adv. Math. **337** (2018), 363–416.

[2] S. Basu and S. Basu: *Homotopy groups and periodic geodesics of closed 4-manifolds*, Internat. J. Math. **26** (2015), 1550059, 34pp.

[3] A. Berglund and K. Börjeson: *Free loop space homology of highly connected manifolds*, Forum Math. **29** (2017), 201–228.

[4] G.M. Bergman: *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), 178–218.

[5] P. Cartier: *Remarques sur le théorème de Birkhoff-Witt*, Ann. Scuola Norm. Sup. Pisa (3), **12** (1958), 1–4.
[6] Y. Félix, S. Halperin and J.-C. Thomas: Rational homotopy theory, Graduate Texts in Mathematics **205**, Springer-Verlag, New York, 2001.
[7] B. Gray: *On the sphere of origin of infinite families in the homotopy groups of spheres*, Topology **8** (1969), 219–232.
[8] P.J. Hilton: *On the homotopy groups of the union of spheres*, J. London Math. Soc. **30** (1955), 154–172.
[9] P. Lalonde and A. Ram: *Standard Lyndon bases of Lie algebras and enveloping algebras*, Trans. Amer. Math. Soc. **347** (1995), 1821–1830.
[10] M. Lazard: *Sur les algèbres enveloppantes universelles de certaines algèbres de Lie*, Publ. Sci. Univ. Alger. Sér. A. **1** (1954), 281–294 (1955).
[11] J.-L. Loday and B. Vallette: Algebraic operads, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **346**, Springer, Heidelberg, 2012.
[12] J. Milnor and D. Husemoller: Symmetric bilinear forms, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73, Springer-Verlag, New York-Heidelberg, 1973.
[13] J. Neisendorfer: Algebraic methods in unstable homotopy theory, New Mathematical Monographs **12**, Cambridge University Press, Cambridge, 2010.
[14] A. Polishchuk and L. Positselski: Quadratic algebras, University Lecture Series **37**, American Mathematical Society, Providence, RI, 2005.
[15] D.C. Ravenel: Complex cobordism and stable homotopy groups of spheres, Pure and Applied Mathematics **121**, Academic Press, Inc., Orlando, FL, 1986.

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