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INVARIANTS OF THE TRACE MAP AND UNIFORM SPECTRAL PROPERTIES FOR DISCRETE STURMIAN DIRAC OPERATORS

Roberto A. PRADO and Ruy C. CHARÃO

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Abstract

We establish invariants for the trace map associated to a family of 1D discrete Dirac operators with Sturmian potentials. Using these invariants we prove that the operators have purely singular continuous spectrum of zero Lebesgue measure, uniformly on the mass and parameters that define the potentials. For rotation numbers of bounded density we prove that these Dirac operators have purely α-continuous spectrum, as to the Schrödinger case, for some $\alpha \in (0,1)$. To the Sturmian Schrödinger and Dirac models we establish a comparison between invariants of the trace maps, which allows to compare the numbers α's and lower bounds on transport exponents.

1. Introduction

In this paper we study spectral properties for the family of discrete Dirac operators

$$\mathbb{D}_{\lambda,\theta,\rho}(m,c) = \mathbb{D}_0(m,c) + V_{\lambda,\theta,\rho} I_2 = \begin{pmatrix} mc^2 + V_{\lambda,\theta,\rho} & c\mathcal{D}^* \\ \\ c\mathcal{D} & -mc^2 + V_{\lambda,\theta,\rho} \end{pmatrix} \tag{1}$$

acting on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, with almost periodic Sturmian potentials $V_{\lambda,\theta,\rho}$. Here $m \geq 0$ is the mass of a particle in the lattice $\mathbb{Z}$, $c > 0$ represents the speed of light and $\mathcal{D}^*$ is the adjoint of the operator $\mathcal{D}$ with $(\mathcal{D}\varphi)(k) := \varphi(k+1) - \varphi(k)$, $k \in \mathbb{Z}$. The operators $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ act on $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \ell^2(\mathbb{Z}, \mathbb{C}^2)$ as follows

$$\left[\mathbb{D}_{\lambda,\theta,\rho}(m,c)\Psi\right](k) = \begin{pmatrix} c\,(\psi_2(k-1) - \psi_2(k)) + \left(mc^2 + V_{\lambda,\theta,\rho}(k)\right)\psi_1(k) \\ \\ c\,(\psi_1(k+1) - \psi_1(k)) + \left(-mc^2 + V_{\lambda,\theta,\rho}(k)\right)\psi_2(k) \end{pmatrix}$$

and the potentials $V_{\lambda,\theta,\rho} : \mathbb{Z} \longrightarrow \mathbb{R}$ are given by

$$V_{\lambda,\theta,\rho}(k) = \lambda\, \chi_{[1-\theta,1)}(k\theta + \rho \mod 1) \tag{2}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$ is the coupling constant, $\theta \in (0,1)$ is an irrational rotation number, $\rho \in [0,1)$ is the phase and χ_I denotes the characteristic function of an interval $I \subset [0,1)$. Important properties of the Sturmian potentials (2) can be found in [2, 10, 11]. The operators $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ are bounded self-adjoint operators on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$.

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The Sturmian Dirac model (1) can be interpreted as a relativistic version of the usual Schrödinger operator on $\ell^2(\mathbb{Z})$, given by

$$(3) \qquad (H_{\lambda,\theta,\rho}\,\phi)(k) = \frac{1}{2m}\left[\phi(k+1) + \phi(k-1)\right] + \tilde{V}_{\lambda,\theta,\rho}(k)\phi(k)$$

for mass $m > 0$ and potentials $\tilde{V}_{\lambda,\theta,\rho}(k) = V_{\lambda,\theta,\rho}(k) - 1/m$. In fact, by Theorem 1 in [15] the nonrelativistic limit ($c \to \infty$) of the resolvent of each Sturmian Dirac operator (1) is the resolvent of the corresponding Sturmian Schrödinger operator (3) (when projected on a proper subspace). The family of operators $H_{\lambda,\theta,\rho}$ given by (3) has been intensively studied and used to describe spectral properties of one-dimensional quasicrystals (see [2, 6, 10, 12]). It is well known [10] that each $H_{\lambda,\theta,\rho}$ has purely singular continuous spectrum of zero Lebesgue measure and for a number θ of bounded density (see the definition in (5)), $H_{\lambda,\theta,\rho}$ has purely α-continuous spectrum, for some $\alpha \in (0, 1)$. In this paper we show that the family of Sturmian Dirac operators $\mathbb{D}_{\lambda,\theta,\rho}(m, c)$ defined by (1)-(2) also present these spectral properties (see Theorems 1.1 and 1.2 below).

Spectral properties for discrete Dirac operators $\mathbb{D}_V(m, c) = \mathbb{D}_0(m, c) + VI_2$ of type (1) has been studied in [5, 15], but with sparse and random potentials V. In [5], it was considered sparse potentials with randomly distributed positions; the authors have determined the Hausdorff dimension of the spectral measure and they showed that there is a sharp transition between pure point and singular continuous spectra. In [15], it was considered random Bernoulli potentials taking two values; for almost all realizations and for all values of the mass, it is shown that its spectrum is pure point. For periodic potentials V_p it can be shown, adapting ideas from the Schrödinger context [25], that the Dirac operators $\mathbb{D}_{V_p}(m, c) = \mathbb{D}_0(m, c) + V_p I_2$ of type (1) has purely absolutely continuous spectrum. Along the aperiodic cases, several classes of potentials have been studied for discrete Schrödinger operators. These classes include potentials generated by circle maps [14], potentials generated by substitutions [1, 4, 7, 9] and in special the Sturmian potentials [2, 6, 10, 11], which lead to purely singular continuous spectrum. Here we are interested in studying this last class of potentials from a relativistic point of view (i.e. for the Dirac model), and we also consider the case of mass $m = 0$, which is not included in the Schrödinger context.

To study spectral properties for the Dirac operators (1)-(2), we follow the usual path of the context of Schrödinger operators, that is, the construction of the trace map and associated invariants (see Section 2). Although the trace map x_k for Sturmian Dirac model satisfies a recursive relation similar to the Sturmian Schrödinger case (Proposition 2.1-(i)), the different forms of the transfer matrices for discrete Schrödinger and Dirac models lead to different invariants for trace map. For the Sturmian Dirac model the invariants are functions that depend continuously on the energy E and are given by

$$\begin{aligned} \mathcal{I}_k^{(D)}(E) &= \frac{\lambda^2}{c^6}E^4 - \frac{2\lambda^3}{c^6}E^3 + \left(\frac{\lambda^4 - 2m^2c^4\lambda^2}{c^6}\right)E^2 + \frac{2m^2c^4\lambda^3}{c^6}E - \frac{m^2c^4}{c^6}\lambda^4 \\ &\quad + \left(m^4c^2 + 4m^2\right)\lambda^2 + 4, \end{aligned}$$

while in the case of Sturmian Schrödinger model these invariants are constants in the energy E' and they are given by (for more details see Proposition 2.1 below and their remarks)

$$\mathcal{I}_k^{(S)}(E') = 4m^2\lambda^2 + 4.$$

The different expressions of these invariants is one of the motivations for our spectral study.

Let $\theta \in (0,1)$ be irrational and consider on $\mathbb{T} \cong [0,1)$ the rotation $R_\theta : \mathbb{T} \to \mathbb{T}$ given by $R_\theta(\rho) = \rho + \theta \mod 1$. The dynamical system $(\mathbb{T}, R_\theta)$ is strictly ergodic (i.e. uniquely ergodic and minimal) with the Lebesgue measure on $\mathbb{T}$ as the ergodic measure. Note that the Sturmian potentials (2) can be written as $V_{\lambda,\theta,\rho}(k) = f(R_\theta^k(\rho))$, where $f : \mathbb{T} \to \mathbb{R}$ is the mensurable function given by $f(\rho) = \lambda\, \chi_{[1-\theta,1)}(\rho)$. Thus, for $m \geq 0$, λ and θ fixed, the family (in ρ) of Dirac operators $\left\{\mathbb{D}_{\lambda,\theta,\rho}(m,c)\right\}_{\rho\in[0,1)}$ is strictly ergodic. Therefore, the spectral properties of $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ are independent of ρ for Lebesgue almost every ρ. Due to minimality, there exists a set $\Sigma_{\lambda,\theta}(m) \subset \mathbb{R}$ such that the spectrum of $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ equals $\Sigma_{\lambda,\theta}(m)$ for all $\rho \in [0,1)$. Using the invariants $\mathcal{I}_k^{(D)}(E)$ described above, we show (Theorem 3.1) that $\Sigma_{\lambda,\theta}(m)$ coincides with the set of zeros of the Lyapunov exponent. By a result of Kotani in [19], extended to the Dirac model (Theorem 4.1), we conclude that $\Sigma_{\lambda,\theta}(m)$ has zero Lebesgue measure and therefore, for any λ, θ and $m \geq 0$, the absolutely continuous spectrum of $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ is empty (see Theorem 4.2). Due to specific properties of the Sturmian potentials $V_{\lambda,\theta,\rho}$ and using again the invariants $\mathcal{I}_k^{(D)}(E)$, we also obtain uniform absence of eigenvalues (see Theorem 5.1).

The following theorem is the first main result of this paper, which describes the spectral type of the Dirac operators $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ defined by (1)-(2). This result will be proven in the Section 5.

Theorem 1.1. *Fix $m \geq 0$. For every λ, θ, ρ, the operator $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ has purely singular continuous spectrum supported on a set of zero Lebesgue measure.*

This theorem says that we have a new class of almost periodic relativistic models with purely singular continuous spectrum of zero Lebesgue measure.

Our second goal is establishing Hausdorff-dimensional properties of spectral measures of the operators $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$. The definitions of continuity and singularity of a Borel measure with respect to Hausdorff measure appear in Section 6.

Given $\theta \in (0,1)$ irrational, we consider its expansion in continued fractions (see [2, 18]):

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [a_1, a_2, a_3, \ldots]$$

with uniquely determined $a_k \in \mathbb{Z}^+ = \{1, 2, \ldots\}$. The best rational approximations associated $\dfrac{p_k}{q_k} = [a_1, \ldots, a_k]$ are defined by

$$\begin{aligned} p_0 &= 0, \quad p_1 = 1, \quad p_k = a_k p_{k-1} + p_{k-2} \quad \text{for } k \geq 2, \\ q_0 &= 1, \quad q_1 = a_1, \quad q_k = a_k q_{k-1} + q_{k-2} \quad \text{for } k \geq 2. \end{aligned} \tag{4}$$

Recall that θ is called a *bounded density number* if the following condition holds [10]:

$$d(\theta) := \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} a_k < \infty. \tag{5}$$

Using properties of the Sturmian potentials and the invariants $\mathcal{I}_k^{(D)}(E)$, we establish upper and lower bounds on the growth of solutions of the eigenvalue problem $\mathbb{D}_{\lambda,\theta,\rho}(m,c)\Psi = E\Psi$ (see Propositions 7.1 and 7.2), which allows us to obtain purely α-continuous spectrum for the Dirac operators $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ by a general method described in Theorem 7.1. This method is well known for discrete Schrödinger operators (see Theorem 1 in [10]) and here is extended to Dirac operators. The second main result of this paper, which will be proved in Section 7, is the following.

Theorem 1.2. *Fix $m \geq 0$ and let θ be a bounded density number. Then, for any $\lambda \neq 0$ there exists $\alpha = \alpha(m,\lambda,\theta) \in (0,1)$ such that for all $\rho \in [0,1)$ and $\Phi \in \ell^2(\mathbb{Z},\mathbb{C}^2)$, the spectral measure for the pair $(\mathbb{D}_{\lambda,\theta,\rho}(m,c),\Phi)$ is purely α-continuous, that is, $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ has purely α-continuous spectrum.*

A consequence of Theorem 1.2 is that for the set of numbers θ of bounded density and for any $\lambda \neq 0$, $\rho \in [0,1)$ and $m \geq 0$, the spectrum $\sigma(\mathbb{D}_{\lambda,\theta,\rho}(m,c))$ is a set of positive Hausdorff dimension. In fact, the positive numbers α's obtained in Theorem 1.2 are lower bounds for the Hausdorff dimension of $\sigma(\mathbb{D}_{\lambda,\theta,\rho}(m,c))$. In Section 8 we compare these numbers α's obtained for the Schrödinger and Dirac models with Sturmian potentials (see relations (36) and (37)).

Although our spectral results (Theorems 1.1 and 1.2) are similar to the corresponding results obtained in [10] for Sturmian Schrödinger operators, the extension to the Dirac setting is not immediate and the proofs present important nontrivial parts. We highlight below the main points that motivated us to study similar properties for the Sturmian Dirac model.

- First of all, as previously mentioned, the different expressions of the invariants $\mathcal{I}_k^{(D)}(E)$ and $\mathcal{I}_k^{(S)}(E')$. It is important to point out that $\mathcal{I}_k^{(D)}(E)$ play a central role in the proofs of Theorem 3.1, Corollary 3.1, Lemma 5.1 and Propositions 7.1 and 7.2; these results are used to prove Theorems 1.1 and 1.2. Moreover, a comparison between $\mathcal{I}_k^{(D)}(E)$ and $\mathcal{I}_k^{(S)}(E')$ is developed in Section 8 (see relations (32) and (33)), which allows us to compare for such models the values of α's mentioned above and lower bounds on transport exponents.
- For Dirac operators with periodic potentials their spectra are purely absolutely continuous and can be characterized by boundedness of traces of transfer matrices, as in Schrödinger case; such result is used in periodic approximations in Proposition 3.1.
- The validity of a result of Kotani for Dirac operators (Theorem 4.1), whose long details are not reported here.
- The different representations of the m-functions (see Section 6) and the version of the Jitomirskaya-Last inequality for discrete Dirac operators (Lemma 6.1), which are used in the proof of Theorem 7.1.

The organization of this paper is as follows. In Section 2 we establish the trace map associated with Sturmian Dirac model and invariants for this map. In Section 3 it is shown that the spectrum of Sturmian Dirac operators coincides with the set of zeros of the Lyapunov exponent. In Section 4 we establish zero Lebesgue measure spectrum and empty absolutely continuous spectrum for all Sturmian Dirac operators. In Section 5 we establish absence of point spectrum for all Sturmian Dirac operators and we present the proof of Theorem 1.1. In Section 6 we introduce the m-functions for discrete Dirac operators, we present their relation with spectral measures, Jitomirskaya-Last inequality for discrete Dirac operators and recall the definitions of α-singular and α-continuous Borel measure. In Section 7 we extend to Dirac operators a criterion to establish α-continuity of spectral measures of a whole-line Dirac operator from power-law bounds on the solutions of a half-line and so we prove Theorem 1.2. Finally, Section 8 is dedicated to comparison of the invariants of the trace maps, of the dimension estimates and of lower bounds on exponents of transport, for Sturmian Schrödinger and Dirac models.

2. Sturmian Dirac Trace Map and Associated Invariants

In this section we discuss the trace map associated with the Sturmian Dirac model (1)-(2). In first place, we need to recall the local structure of Sturmian sequences. The Sturmian words S_k over the alphabet $\mathcal{A} = \{0, \lambda\}$ are defined by

$$S_0 = 0, \quad S_1 = 0^{a_1-1}\lambda, \quad S_{k+1} = S_k^{a_{k+1}} S_{k-1} \ \text{ for } k \geq 1, \tag{6}$$

where the numbers a_k's are the coefficients of the continued fraction expansion of θ. By definition, S_k is a prefix of S_{k+1} for each $k \geq 1$ and has length $|S_k| = q_k \to \infty$. It is known [2, 12] that the one-sided infinite sequence defined by $\omega_\theta = \lim_{k\to\infty} S_k$ coincides with the potential sequence $\{V_{\lambda,\theta,0}(k)\}_{k\in\mathbb{Z}^+}$ defined by (2).

Consider the Sturmian Dirac operators $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ defined by (1)-(2). If $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is a solution of the eigenvalue equation

$$\mathbb{D}_{\lambda,\theta,\rho}(m,c)\Psi = E\Psi \tag{7}$$

with $E \in \mathbb{R}$, then for $k \geq 1$,

$$\begin{pmatrix} \psi_1(k+1) \\ \psi_2(k) \end{pmatrix} = T\left(m, E, V_{\lambda,\theta,\rho}(k)\right) \cdots T\left(m, E, V_{\lambda,\theta,\rho}(1)\right) \begin{pmatrix} \psi_1(1) \\ \psi_2(0) \end{pmatrix} \tag{8}$$

where

$$T(m,E,V) = \begin{pmatrix} 1 + \dfrac{m^2c^4 - (E-V)^2}{c^2} & \dfrac{mc^2 + E - V}{c} \\ \dfrac{mc^2 - (E-V)}{c} & 1 \end{pmatrix}.$$

Fixed mass $m \geq 0$ and energy E, for each word $w = w_1 \cdots w_k \in \mathcal{A}^k$, we define the transfer matrix

$$M(m,E,w) := T(m,E,w_k) \cdots T(m,E,w_1).$$

Thus, if Ψ is solution of (7), then by (8) we have

$$\tilde{\Psi}(k+1) = M(m,E,V_{\lambda,\theta,\rho}(1) \cdots V_{\lambda,\theta,\rho}(k))\tilde{\Psi}(1)\,, \quad k \geq 1, \tag{9}$$

with $\tilde{\Psi}(k+1) = \begin{pmatrix} \psi_1(k+1) \\ \psi_2(k) \end{pmatrix}$.

Fixed m, E, λ, θ and taking $\rho = 0$, we consider the notation

$$\begin{aligned} M_k &:= M(m,E,V_{\lambda,\theta,0}(1) \cdots V_{\lambda,\theta,0}(q_k)) \\ &= T(m,E,V_{\lambda,\theta,0}(q_k)) \cdots T(m,E,V_{\lambda,\theta,0}(1)), \ \ k \geq 1, \end{aligned}$$

and we define the matrices

$$M_0 \;:=\; T(m,E,0) = \begin{pmatrix} 1 + \dfrac{m^2c^4 - E^2}{c^2} & \dfrac{mc^2+E}{c} \\ \\ \dfrac{mc^2-E}{c} & 1 \end{pmatrix}$$

and

$$\begin{aligned} M_{-1} \;&:=\; T(m,E,\lambda)\, T(m,E,0)^{-1} \\ &= \begin{pmatrix} 1 + \dfrac{\lambda(mc^2+E-\lambda)}{c^2} & -\dfrac{\lambda}{c}\left(1 + \dfrac{(mc^2+E)(mc^2+E-\lambda)}{c^2}\right) \\ \\ \dfrac{\lambda}{c} & 1 - \dfrac{\lambda(mc^2+E)}{c^2} \end{pmatrix}. \end{aligned}$$

We note here that

$$M_k \in SL(2,\mathbb{R}) := \{B \in M_{2\times 2}(\mathbb{R}) : \det(B) = 1\},\ \forall k \geq -1.$$

Using (6) one obtains the following recursive relation for the matrices M_k:

$$M_{k+1} = M_{k-1} M_k^{a_{k+1}}, \quad k \geq 0. \tag{10}$$

Now consider the Chebyshev polynomials $U_k(x)$, $x \in \mathbb{R}$, defined by

$$U_{-1}(x) = 0\,, \quad U_0(x) = 1\,, \quad U_k(x) = xU_{k-1}(x) - U_{k-2}(x) \ \text{ for } \ k \geq 1. \tag{11}$$

For these polynomials, the quantity $U_kU_{k-2} - U_{k-1}^2$ is constant in k:

$$U_kU_{k-2} - U_{k-1}^2 = U_1U_{-1} - U_0^2 = -1\,, \quad \forall k \in \mathbb{Z}^+. \tag{12}$$

Given a matrix $B \in SL(2,\mathbb{R})$ and using (11) one shows by induction on k that (see [2])

$$B^k = U_{k-1}(\mathrm{tr}\,(B))B - U_{k-2}(\mathrm{tr}\,(B))I_2, \tag{13}$$

where $\mathrm{tr}\,(B)$ denotes the trace of the matrix B.

The following result establishes a recursive relation and invariants for the traces of the matrices M_k defined above.

Proposition 2.1. *Let $\{x_k\}_{k\geq -1}$ be the sequence defined by $x_k := \mathrm{tr}\,(M_k)$.*

(i) *If $|x_{k-1}| > 2$ for $k \geq 1$, then*

$$x_{k+1} = U_{a_{k+1}-1}(x_k)\frac{U_{a_k}(x_{k-1})}{U_{a_k-1}(x_{k-1})}x_k - U_{a_{k+1}-2}(x_k)x_{k-1} - \frac{U_{a_{k+1}-1}(x_k)}{U_{a_k-1}(x_{k-1})}x_{k-2}\,. \tag{14}$$

(ii) *The quantity*

$$\mathcal{I}_k^{(D)} := x_{k+1}^2 + x_k^2 + [\mathrm{tr}\,(M_kM_{k+1})]^2 - x_{k+1}x_k\,\mathrm{tr}\,(M_kM_{k+1}) \tag{15}$$

is constant in k and

$$\begin{aligned} \mathcal{I}_k^{(D)} \;=\; \mathcal{I}_{-1}^{(D)} \;&=\; \frac{\lambda^2}{c^6}E^4 - \frac{2\lambda^3}{c^6}E^3 + \left(\frac{\lambda^4 - 2m^2c^4\lambda^2}{c^6}\right)E^2 + \frac{2m^2c^4\lambda^3}{c^6}E - \frac{m^2c^4}{c^6}\lambda^4 \\ &\quad + \left(m^4c^2 + 4m^2\right)\lambda^2 + 4 \end{aligned}$$

for all $k \geq -1$.

Remarks: **1.** We call the recursive relation (14) *Sturmian Dirac trace map*, which is analogous to Schrödinger case (Proposition 2 in [2]). In (i) the hypothesis $|x_{k-1}| > 2$ implies that $U_{a_k-1}(x_{k-1}) \neq 0$.
2. In the particular case of Fibonacci potential $V_{\lambda,\theta,\rho}$ where $\theta = (\sqrt{5}-1)/2$, one has $a_k = 1$ for all $k \in \mathbb{Z}^+$, and the relation (14) becomes $x_{k+1} = x_k x_{k-1} - x_{k-2}$.
3. The quantities $\mathcal{I}_k^{(D)} = \mathcal{I}_k^{(D)}(m, c, E, \lambda)$, defined in (15), are the *Dirac invariants for the trace map*. Fixed m, c and λ, $\mathcal{I}_k^{(D)}$ can be seen as a polynomial function of the energy E; in this case to emphasize the dependence on the energy E, we write $\mathcal{I}_k^{(D)}(E)$. For the Sturmian Schrödinger model (3), these invariants are constant in the energy E' and takes the values $\mathcal{I}_k^{(S)}(E') = 4m^2\lambda^2 + 4$ (see [2] for mass $m = 1/2$). In Section 8 we compare $\mathcal{I}_{-1}^{(D)}(E)$ with $\mathcal{I}_{-1}^{(S)}(E')$ for energies E, E' in the corresponding spectra (see relations (32) and (33)).
4. For fixed $m > 0, \lambda$ and E we can obtain, via nonrelativistic limit, the invariants of the Schrödinger trace map from the Dirac invariants:

$$\lim_{c\to\infty}\left(\mathcal{I}_{-1}^{(D)} - m^4c^2\lambda^2\right) = 4m^2\lambda^2 + 4 = \mathcal{I}_{-1}^{(S)}.$$

Proof of Proposition 2.1:
(i) The recursive relation (14) follows from relations (10)-(13). For more details see [2].
(ii) For matrices $A, B \in SL(2,\mathbb{R})$ the following properties are valid:

$$\mathrm{tr}\,(AB) = \mathrm{tr}\,(A)\mathrm{tr}\,(B) - tr\left(AB^{-1}\right), \quad \mathrm{tr}\,(AB) = \mathrm{tr}\,(BA) \quad \text{and} \quad tr\left(A^{-1}\right) = \mathrm{tr}\,(A).$$

Using these properties for the matrices M_k, we have

$$\begin{aligned}
\mathrm{tr}\,\left(M_{k+1}^{-1}M_k^{-1}M_{k+1}M_k\right) &= \mathrm{tr}\,\left((M_kM_{k+1})^{-1}\right)tr\,(M_{k+1}M_k) - tr\,(M_{k+1}M_kM_kM_{k+1})\\
&= [\mathrm{tr}\ (M_{k+1}M_k)]^2 - tr\left(M_k^2M_{k+1}^2\right)\\
&= [\mathrm{tr}\ (M_{k+1}M_k)]^2 - \mathrm{tr}\,(M_k)\ tr\left(M_kM_{k+1}^2\right) + tr\left(M_{k+1}^2\right)\\
&= [\mathrm{tr}\ (M_kM_{k+1})]^2 - \mathrm{tr}\,(M_k)\,[\mathrm{tr}\,(M_kM_{k+1})\mathrm{tr}\,(M_{k+1}) - \mathrm{tr}\,(M_k)]\\
&\quad + [\mathrm{tr}\,(M_{k+1})]^2 - 2\\
&= x_{k+1}^2 + x_k^2 + [\mathrm{tr}\,(M_kM_{k+1})]^2 - x_{k+1}x_k\ \mathrm{tr}\,(M_kM_{k+1}) - 2\\
&= \mathcal{I}_k^{(D)} - 2.
\end{aligned}$$

On the other hand, using the recursive relation (10), we can obtain that

$$\begin{aligned}
\mathrm{tr}\,\left(M_{k+1}^{-1}M_k^{-1}M_{k+1}M_k\right) &= \mathrm{tr}\,\left(M_{k+1}M_kM_{k+1}^{-1}M_k^{-1}\right)\\
&= \mathrm{tr}\,\left(M_{k-1}M_k^{a_{k+1}+1}\left(M_k^{a_{k+1}}\right)^{-1}M_{k-1}^{-1}M_k^{-1}\right)\\
&= \mathrm{tr}\,\left(M_{k-1}^{-1}M_k^{-1}M_{k-1}M_k\right)\\
&= \mathcal{I}_{k-1}^{(D)} - 2.
\end{aligned}$$

Therefore, $\mathcal{I}_k^{(D)} = \mathcal{I}_{k-1}^{(D)} = \mathcal{I}_{-1}^{(D)}$, $\forall k \geq 0$, that is, $\mathcal{I}_k^{(D)}$ is constant in k.

To conclude the proof, let us calculate this invariant:

$$
\begin{aligned}
\mathcal{I}_k^{(D)} &= \mathcal{I}_{-1}^{(D)} = x_0^2 + x_{-1}^2 + [\operatorname{tr}(M_{-1}M_0)]^2 - x_0 x_{-1} \operatorname{tr}(M_{-1}M_0) \\
&= \left(2 + \frac{m^2c^4 - E^2}{c^2}\right)^2 + \left(2 - \frac{\lambda^2}{c^2}\right)^2 + \left(2 + \frac{m^2c^4 - (E-\lambda)^2}{c^2}\right)^2 \\
&\quad - \left(2 + \frac{m^2c^4 - E^2}{c^2}\right)\left(2 - \frac{\lambda^2}{c^2}\right)\left(2 + \frac{m^2c^4 - (E-\lambda)^2}{c^2}\right) \\
&= \left[2 + \frac{m^2c^4 - (E-\lambda)^2}{c^2} - \left(2 + \frac{m^2c^4 - E^2}{c^2}\right)\right]^2 + \left(2 - \frac{\lambda^2}{c^2}\right)^2 \\
&\quad + \frac{\lambda^2}{c^2}\left(2 + \frac{m^2c^4 - E^2}{c^2}\right)\left(2 + \frac{m^2c^4 - (E-\lambda)^2}{c^2}\right) \\
&= \frac{(E-\lambda)^4}{c^4} - \frac{2E^2(E-\lambda)^2}{c^4} + \frac{E^4}{c^4} + 4 + \frac{\lambda^4}{c^4} + \frac{4m^2c^4\lambda^2 - 2(E-\lambda)^2\lambda^2 - 2E^2\lambda^2}{c^4} \\
&\quad + \frac{m^4c^8\lambda^2 - m^2c^4\lambda^2\left[(E-\lambda)^2 + E^2\right] + E^2(E-\lambda)^2\lambda^2}{c^6} \\
&= \frac{\lambda^2}{c^6}E^4 - \frac{2\lambda^3}{c^6}E^3 + \left(\frac{\lambda^4 - 2m^2c^4\lambda^2}{c^6}\right)E^2 + \frac{2m^2c^4\lambda^3}{c^6}E - \frac{m^2c^4}{c^6}\lambda^4 \\
&\quad + \left(m^4c^2 + 4m^2\right)\lambda^2 + 4.
\end{aligned}
$$

The next result is a version of the Proposition 4 in [2] for the Sturmian Dirac model (1). We omit the proof, since this can be done as in [2] using Chebyshev polynomials (11) and Proposition 2.1.

Proposition 2.2. *The sequence $\{x_k\}_{k\geq -1}$, with $x_k = \operatorname{tr}(M_k)$, is unbounded if and only if*

$$|x_{k_0-1}| \leq 2\,, \quad |x_{k_0}| > 2\,, \quad |x_{k_0+1}| > 2$$

for some $k_0 \geq 0$. This number k_0 is unique, it holds that

$$|x_{k+2}| > \frac{|x_{k+1}||x_k|}{2} > 2 \quad \text{for } k \geq k_0$$

and

$$\frac{|x_k|}{2} > C^{q_k} \quad \text{for some } C > 1,$$

with q_k the positive integers given in (4).

If $\{x_k\}_{k\geq -1}$ is bounded, then

$$|x_k| \leq 2 + \sqrt{4 + \mathcal{I}_{-1}^{(D)}} \quad \text{for } k \geq -1,$$

where $\mathcal{I}_{-1}^{(D)} = \mathcal{I}_{-1}^{(D)}(m, c, E, \lambda)$ is given by Proposition 2.1.

3. Spectrum and Vanishing Lyapunov Exponents

In this section our goal is to show that the spectrum $\sigma\left(\mathbb{D}_{\lambda,\theta,\rho}(m,c)\right) = \Sigma_{\lambda,\theta}(m)$ of the Sturmian Dirac operators (1)-(2) coincides with the set of zeros of the Lyapunov exponent.

Fixed $m \geq 0$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $\theta \in (0,1)$ irrational, we denote $\Sigma_{\lambda,\theta}(m)$ by Σ.

By the subadditive ergodic theorem [3], for each $E \in \mathbb{C}$ there exists a number $\Gamma(E) \geq 0$, called Lyapunov exponent, such that for almost every $\rho \in [0,1)$ with respect to Lebesgue measure,

$$\Gamma(E) := \lim_{k\to\infty} \frac{1}{k} \ln \left\| M(m, E, V_{\lambda,\theta,\rho}(1) \cdots V_{\lambda,\theta,\rho}(k)) \right\| ,$$

where $M(m, E, V_{\lambda,\theta,\rho}(1) \cdots V_{\lambda,\theta,\rho}(k))$ are the transfer matrices of the relation (9). We note that the Lyapunov exponent $\Gamma = \Gamma(E)$ is a function of $E \in \mathbb{C}$.

Now, we consider the set $\mathcal{Z}$ of real zeros of the Lyapunov exponent function, that is,

$$\mathcal{Z} = \{E \in \mathbb{R} : \Gamma(E) = 0\}.$$

The main result of this section is the following.

Theorem 3.1. *Let $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ be the Dirac operators defined by* (1)-(2). *For any λ, θ, ρ and $m \geq 0$, we have $\Sigma = \mathcal{Z}$.*

Consider the stable set

$$\mathcal{B} = \left\{ E \in \mathbb{R} : \ |x_k| \leq 2 + \sqrt{4 + \mathcal{I}_{-1}^{(D)}} \quad \text{for all } k \geq -1 \right\},$$

where $\mathcal{I}_{-1}^{(D)} = \mathcal{I}_{-1}^{(D)}(m, c, E, \lambda)$ is given by Proposition 2.1.

The proof of Theorem 3.1 use ideas of [2, 8] and will be obtained from the next three Propositions. In fact, combining these propositions we get the following chain of inclusions $\Sigma \subset \mathcal{B} \subset \mathcal{Z} \subset \Sigma$, which proves the theorem.

Proposition 3.1. $\Sigma \subset \mathcal{B}$.

Proof. For $m \geq 0$, $\lambda \in \mathbb{R}\setminus\{0\}$ and $\theta \in (0,1)$ irrational, consider the operator $\mathbb{D}_{\lambda,\theta,0}(m,c)$ on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, defined by (1). Let $\{\mathbb{D}_{\lambda,\theta_k,0}(m,c)\}_{k\geq 1}$ be the sequence of q_k-periodic Dirac operators on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, defined by (1), with potentials $V_{\lambda,\theta_k,0}$ defined by (2), where $\theta_k = p_k/q_k$ is the best approximation to θ given by (4). These potentials $V_{\lambda,\theta_k,0}$ are periodic with period q_k, taking the values $S_k = V_{\lambda,\theta_k,0}(1) \cdots V_{\lambda,\theta_k,0}(q_k)$ on its period, and $\sigma(\mathbb{D}_{\lambda,\theta_k,0}(m,c)) = \{E \in \mathbb{R} : |x_k| \leq 2\}$. We have that $\mathbb{D}_{\lambda,\theta,0}(m,c)$ is the strong limit of $\mathbb{D}_{\lambda,\theta_k,0}(m,c)$ as $k \to \infty$. Denoting $\Omega_k = \mathbb{R} \setminus \sigma(\mathbb{D}_{\lambda,\theta_k,0}(m,c)) = \{E \in \mathbb{R} : |x_k| > 2\}$, it follows from Theorem VIII.24 in [24] (see also [28]) that

$$\bigcup_{N\in\mathbb{N}} \mathrm{Int}\left(\bigcap_{k\geq N} \Omega_k \right) \subset \Sigma^c , \tag{16}$$

where $\mathrm{Int}(\Omega)$ denotes the interior of a set Ω and $\Sigma^c = \mathbb{R} \setminus \Sigma$.

Using Proposition 2.2 and (16), we obtain

$$\mathcal{B}^c \subset \bigcup_{N\in\mathbb{N}} \mathrm{Int}\left(\bigcap_{k\geq N} \Omega_k \right) \subset \Sigma^c ,$$

which implies the result. □

Proposition 3.2. $\mathcal{B} \subset \mathcal{Z}$.

Proof. Fix m, λ, θ and pick some $\rho \in [0, 1)$ for which $\Gamma(E)$ exists. Suppose there exists $E \in \mathcal{B}$ such that $\Gamma(E) > 0$. By Osceledec's theorem [21] there exists a solution Ψ of the eigenvalue equation (7) with

$$\|\tilde{\Psi}(n+1)\| \leq e^{-\Gamma(E)n} \quad \text{for large } n.$$

Since $E \in \mathcal{B}$ there is a constant $C = 2 + \sqrt{4 + \mathcal{I}_{-1}^{(D)}}$ such that

$$|x_k| = |trM(m, E, S_k)| \leq C \quad \forall k \geq 1, \tag{17}$$

where $S_k = V_{\lambda,\theta,0}(1) \cdots V_{\lambda,\theta,0}(q_k)$. Now, the word S_k occur in the sequence $V_{\lambda,\theta,\rho}$ for all ρ, as being $V_{\lambda,\theta,\rho}(n+1) \cdots V_{\lambda,\theta,\rho}(n+q_k)$ for $n \geq n_0$ (see [11, 12]). Thus, we can use (17) for each ρ. Pick n_0 such that, for every $n \geq n_0$ and every $j \in \mathbb{Z}^+$, the solution Ψ obeys

$$\|\tilde{\Psi}(n+j)\| \leq e^{-\frac{1}{2}\Gamma(E)j}\|\tilde{\Psi}(n)\|. \tag{18}$$

Now, we choose k such that $e^{-\frac{1}{2}\Gamma(E)q_k} < \frac{1}{2C}$. Considering the word

$$S_k S_k = V_{\lambda,\theta,\rho}(l+1) \cdots V_{\lambda,\theta,\rho}(l+q_k) \cdots V_{\lambda,\theta,\rho}(l+2q_k) \quad \text{for} \quad l \geq n_0,$$

and applying the Cayley-Hamilton theorem, we obtain

$$\tilde{\Psi}(l+2q_k) - \operatorname{tr} M(m, E, S_k)\tilde{\Psi}(l+q_k) + \tilde{\Psi}(l) = 0. \tag{19}$$

By (17) and (19) we have

$$2C \max\left\{\|\tilde{\Psi}(l+2q_k)\|, \|\tilde{\Psi}(l+q_k)\|\right\} \geq \|\tilde{\Psi}(l+2q_k)\| + C\|\tilde{\Psi}(l+q_k)\| \geq \|\tilde{\Psi}(l)\|. \tag{20}$$

Finally, using (18) with $n = l$ and $j = q_k$ or $j = 2q_k$, and then (20), one obtains

$$\begin{aligned} \max\left\{\|\tilde{\Psi}(l+2q_k)\|, \|\tilde{\Psi}(l+q_k)\|\right\} &\leq e^{-\frac{1}{2}\Gamma(E)q_k}\, \|\tilde{\Psi}(l)\| \\ &\leq e^{-\frac{1}{2}\Gamma(E)q_k}\, 2C \max\left\{\|\tilde{\Psi}(l+2q_k)\|, \|\tilde{\Psi}(l+q_k)\|\right\} \\ &< \max\left\{\|\tilde{\Psi}(l+2q_k)\|, \|\tilde{\Psi}(l+q_k)\|\right\}, \end{aligned}$$

which is a contradiction. □

Proposition 3.3. $\mathcal{Z} \subset \Sigma$.

Proof. Let $E \in \Sigma^c$. We introduce the two-components Green's function [5, 15]

$$\begin{pmatrix} G^{11}_{\lambda,\theta,\rho}(k, 1; E) \\ G^{21}_{\lambda,\theta,\rho}(k, 1; E) \end{pmatrix} := \begin{pmatrix} \left\langle \delta_{1,k}, \left(\mathbb{D}_{\lambda,\theta,\rho}(m, c) - EI\right)^{-1} \delta_{1,1} \right\rangle \\ \left\langle \delta_{2,k}, \left(\mathbb{D}_{\lambda,\theta,\rho}(m, c) - EI\right)^{-1} \delta_{1,1} \right\rangle \end{pmatrix},$$

so that

$$(\mathbb{D}_{\lambda,\theta,\rho}(m, c) - E) \begin{pmatrix} G^{11}_{\lambda,\theta,\rho}(k, 1; E) \\ G^{21}_{\lambda,\theta,\rho}(k, 1; E) \end{pmatrix} = \delta_{1,1}(k),$$

where $\{\delta_{1,k}, \delta_{2,k}\}_{k\in\mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z}, \mathbb{C}^2)$. By Combes-Thomas estimate for discrete Dirac operators (Proposition 1 in [23]), there exist constants $\Delta = \operatorname{dist}(E, \Sigma) > 0$ and

$b = b(m, c) > 0$ such that for $k \in \mathbb{Z}$ and $\beta \in \{1, 2\}$, it holds that

$$|G^{\beta 1}_{\lambda,\theta,\rho}(k, 1; E)| \ \leq \ \frac{2}{\Delta} e^{-b|k-1|} \, .$$

This implies that the solution $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ of the eigenvalue equation (7), with initial conditions $\psi_1(1) = 0$ and $\psi_2(0) = 1$, grows exponentially with a rate $r > 0$ (that is, $\|\tilde{\Psi}(k + 1)\| \geq \tilde{C} e^{rk}$ for $\tilde{C} > 0$ and large k), due to constancy (in k) of the Wronskian:

$$W\left[\begin{pmatrix} G^{11}_{\lambda,\theta,\rho}(k, 1; E) \\ G^{21}_{\lambda,\theta,\rho}(k, 1; E) \end{pmatrix}, \Psi(k)\right] := G^{11}_{\lambda,\theta,\rho}(k + 1, 1; E)\psi_2(k) - G^{21}_{\lambda,\theta,\rho}(k, 1; E)\psi_1(k + 1).$$

It follows from (9) that

$$\left\|M(m, E, V_{\lambda,\theta,\rho}(1) \cdots V_{\lambda,\theta,\rho}(k))\right\| \geq \|\tilde{\Psi}(k + 1)\| \geq \tilde{C} e^{rk},$$

which implies

$$\Gamma(E) = \lim_{k\to\infty} \frac{1}{k} \ln \left\|M(m, E, V_{\lambda,\theta,\rho}(1) \cdots V_{\lambda,\theta,\rho}(k))\right\| \geq \lim_{k\to\infty} \left(\frac{1}{k} \ln \tilde{C} + r\right) = r > 0.$$

Therefore $E \in \mathcal{Z}^c$. □

Since the spectrum $\Sigma = \sigma\left(\mathbb{D}_{\lambda,\theta,\rho}(m, c)\right)$ is compact and the bounds on the traces $x_k = \operatorname{tr}(M_k)$ for energies $E \in \Sigma$ depend continuously on E (see definition of stable set $\mathcal{B}$), we can find a global bound for these traces. We conclude this section with a result that will be used in Section 5.

Corollary 3.1. *For each $m \geq 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$, there exists a constant $C_\lambda(m) \in (2, \infty)$ such that for all irrational $\theta \in (0, 1)$, $E \in \Sigma$ and $k \in \mathbb{Z}^+$, we have*

$$\max\{|x_k|, |y_k|, |z_k|\} \leq C_\lambda(m) \, ,$$

where $x_k = \operatorname{tr} M(m, E, S_k)$, $y_k = x_{k-1}$ and $z_k = \operatorname{tr} M(m, E, S_k S_{k-1})$.

Proof. By Proposition 2.1(ii) the invariant

$$\mathcal{I}^{(D)}_{-1} = x_k^2 + y_k^2 + z_k^2 - x_k y_k z_k \tag{21}$$

is a polynomial function in E and is uniformly bounded on Σ (compact set) by a constant $C_{1,\lambda}(m) > 0$. For every $E \in \Sigma$, there exists the constant $C_{2,\lambda}(m) := 2 + \sqrt{4 + C_{1,\lambda}(m)}$ such that using the Proposition 3.1 and definition of the stable set $\mathcal{B}$, we obtain

$$|x_k| \leq C_{2,\lambda}(m) \quad \text{and} \quad |y_k| \leq C_{2,\lambda}(m) \, , \ \forall k \in \mathbb{Z}^+.$$

Now, solving the equation (21) in the variable z_k and using the boundedness of x_k and y_k, one obtains

$$|z_k| \leq \frac{|x_k||y_k| + \sqrt{x_k^2 y_k^2 + 4|\mathcal{I}^{(D)}_{-1}|}}{2} \leq C_\lambda(m) \, , \quad \forall k \in \mathbb{Z}^+,$$

where $C_\lambda(m) := \dfrac{C_{2,\lambda}(m)^2 + \sqrt{C_{2,\lambda}(m)^4 + 4C_{1,\lambda}(m)}}{2}$. Therefore, the result follows. □

4. Zero-Measure Spectrum

In this section we establish zero Lebesgue measure spectrum for all Sturmian Dirac operators $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$, with $m \geq 0$ (see Theorem 4.2). Consequently, these operators have empty absolutely continuous spectrum.

Let (Ω, T, μ) be an ergodic dynamical system and $f : \Omega \to \mathbb{R}$ a measurable bounded function. Define potentials

$$V_\omega(k) = f(T^k\omega)\,, \quad \omega \in \Omega,\ k \in \mathbb{Z},$$

and consider the ergodic family of Dirac operators $\{\mathbb{D}_\omega(m,c)\}_{\omega\in\Omega}$ on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, defined by

$$\left[\mathbb{D}_\omega(m,c)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\right](k) = \begin{pmatrix} c\,(\psi_2(k-1) - \psi_2(k)) + \left(mc^2 + V_\omega(k)\right)\psi_1(k) \\ \\ c\,(\psi_1(k+1) - \psi_1(k)) + \left(-mc^2 + V_\omega(k)\right)\psi_2(k)\end{pmatrix}. \tag{22}$$

Similarly to the Sturmian case, due to subadditive ergodic theorem [3], for each fixed $m \geq 0$ and $E \in \mathbb{C}$ there exists a number $\Gamma(E) = \Gamma(m,E) \in [0,\infty)$, called Lyapunov exponent, defined by

$$\Gamma(E) = \lim_{k\to\infty} \frac{1}{k} \ln \|M(m,E,V_\omega(1)\cdots V_\omega(k))\|$$

for μ-almost every $\omega \in \Omega$, where $M(m,E,V_\omega(1)\cdots V_\omega(k))$ are the transfer matrices associated with $\mathbb{D}_\omega(m,c)$. Let

$$\mathcal{Z} = \{E \in \mathbb{R} : \Gamma(E) = 0\}.$$

The following theorem is a version of a result of the Kotani theory [19] adapted for the Dirac operators (22). We omit the proof since this is very long and analogous to the case of Schrödinger operators.

Theorem 4.1. *Let $\{\mathbb{D}_\omega(m,c)\}_{\omega\in\Omega}$ be a ergodic family of Dirac operators defined by (22) with potentials $V_\omega(k) = f(T^k\omega)$ that are μ-almost surely not periodic and $f : \Omega \to \mathbb{R}$ is a function that takes a finite number of values. Then $\ell(\mathcal{Z}) = 0$, where ℓ denotes the Lebesgue measure.*

Remark. The hypothesis that the potentials $V_\omega(k) = f(T^k\omega)$ are μ-almost surely not periodic implies that f is not constant. In fact, if f is constant then V_ω is periodic for all $\omega \in \Omega$ and $\mu(\Omega) = 1 \neq 0$.

Since the family of Sturmian Dirac operators $\left\{\mathbb{D}_{\lambda,\theta,\rho}(m,c)\right\}_{\rho\in[0,1)}$ is strictly ergodic (see Introduction) with potentials $V_{\lambda,\theta,\rho}$ not periodic and taking two values 0 or $\lambda \neq 0$, then by using Theorems 3.1 and 4.1, we get the following result:

Theorem 4.2. *Let $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ be the Dirac operators defined by* (1)-(2). *For any λ, θ, ρ and $m \geq 0$, the spectrum $\Sigma = \sigma\left(\mathbb{D}_{\lambda,\theta,\rho}(m,c)\right)$ has zero Lebesgue measure and the absolutely continuous spectrum of $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ is empty.*

5. Absence of Point Spectrum

In this section we establish absence of point spectrum for all Sturmian Dirac operators $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ with $m \geq 0$ (Theorem 5.1), and we present the proof of Theorem 1.1.

For studying the behavior of solutions of the eigenvalue equation (7), we consider the norm $\|\cdot\|_L$ on a lattice interval of length $L \in \mathbb{R},\ L \geq 1$, defined on functions $\Psi : \mathbb{Z}^+ \to \mathbb{C}^2$, $\Psi(k) = \begin{pmatrix} \psi_1(k) \\ \psi_2(k) \end{pmatrix}$, by

$$\|\Psi\|_L = \left(\sum_{k=1}^{\lfloor L \rfloor} \|\Psi(k)\|^2 + (L - \lfloor L \rfloor)\|\Psi(\lfloor L \rfloor + 1)\|^2 \right)^{1/2},$$

where $\|\Psi(k)\|^2 = |\psi_1(k)|^2 + |\psi_2(k)|^2$ and $\lfloor L \rfloor$ denotes the integer part of L. The behavior of $\|\Psi\|_L$ can be investigated through behavior of $\|\tilde{\Psi}\|_L$, where $\tilde{\Psi}(k+1) = \begin{pmatrix} \psi_1(k+1) \\ \psi_2(k) \end{pmatrix}$ for $k \geq 1$, since there exists constants $D_1, D_2 > 0$ such that

$$D_1\|\tilde{\Psi}\|_L \leq \|\Psi\|_L \leq D_2\|\tilde{\Psi}\|_L\,.$$

We will assume that a solution Ψ of (7) has normalized initial condition (N.I.C.) in the sense that

$$\|\tilde{\Psi}(1)\|^2 = |\psi_1(1)|^2 + |\psi_2(0)|^2 = 1.$$

Now, due to partition Lemma (see [10, 11]), every sequence $V_{\lambda,\theta,\rho}$ may be partitioned into words S_k or S_{k-1}, defined by (6). Using this property, together with the uniform bounds on traces given in Corollary 3.1, we obtain the following result, similar to Lemma 4.1 in [10], for the Sturmian Dirac operators $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ defined by (1)-(2).

Lemma 5.1. *Fix $m \geq 0$. Let λ, θ, ρ be arbitrary, $E \in \Sigma$, and let Ψ be a solution of (7) with N.I.C.. Then, for every $k \geq 8$, the following inequality holds*

$$\|\tilde{\Psi}\|_{q_k} \geq B_\lambda(m)\|\tilde{\Psi}\|_{q_{k-8}}$$

with $B_\lambda(m) = \left(1 + \frac{1}{4C_\lambda(m)^2}\right)^{1/2}$, where $C_\lambda(m) \in (2,\infty)$ is the uniform constant given in Corollary 3.1.

Lemma 5.1 will be used in the proof of Theorem 5.1 below and also in Proposition 7.1 to obtain power-law lower bounds on solutions of (7) for certain rotation numbers.

Theorem 5.1. *Fix $m \geq 0$. For every λ, θ, ρ, the operator $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ has empty point spectrum.*

Proof. Fix $m \geq 0$. Let λ, θ, ρ be arbitrary, $E \in \Sigma$, and let Ψ be a solution of (7) with N.I.C.. Then, by Lemma 5.1 we have

$$\|\tilde{\Psi}\|_{q_{8k}} \geq B_\lambda(m)\|\tilde{\Psi}\|_{q_{8k-8}} \geq \cdots \geq B_\lambda(m)^k\|\tilde{\Psi}\|_{q_0} = B_\lambda(m)^k,$$

for all $k \geq 1$ and constant $B_\lambda(m) > 1$. This implies that

$$\sum_{p\in\mathbb{Z}} \|\Psi(p)\|^2 \geq \|\Psi\|_{q_{8k}} \geq D_1\|\tilde{\Psi}\|_{q_{8k}} \geq D_1 B_\lambda(m)^k, \quad \forall k \geq 1.$$

Thus $\Psi \notin \ell^2(\mathbb{Z}, \mathbb{C}^2)$. Therefore, $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ have no eigenvalues and its point spectrum is empty. □

Now are ready to complete the proof of the first main result of this paper.

Proof of Theorem 1.1. It follows directly from Theorems 4.2 and 5.1.

6. m-Functions and Decomposition of Borel Measures

Consider Dirac operators $\mathbb{D}(m,c) = \mathbb{D}_0(m,c) + VI_2$ on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ defined as in (22), associated with arbitrary potentials $V : \mathbb{Z} \to \mathbb{R}$. The study of spectral properties of an operator $\mathbb{D}(m,c)$ is related to the study of the Weyl m-function. In this section we introduce the m-functions for $\mathbb{D}(m,c)$ and we present its relation with spectral measures and a version of Jitomirskaya-Last inequality for the Dirac operators $\mathbb{D}(m,c)$; we also define α-singular and α-continuous Borel measure. These definitions and results will be used in Section 7.

Let $\mathbb{Z}^+ = \{1,2,3,\ldots\}$ and $\mathbb{Z}^- = \{\ldots,-2,-1,0\}$. To each whole-line operator $\mathbb{D}(m,c)$ we associate two half-line operators

$$\mathbb{D}_+(m,c) = \mathcal{P}_+^*\,\mathbb{D}(m,c)\mathcal{P}_+ \quad \text{and} \quad \mathbb{D}_-(m,c) = \mathcal{P}_-^*\,\mathbb{D}(m,c)\mathcal{P}_-\,,$$

where $\mathcal{P}_\pm$ denote the inclusions $\mathcal{P}_\pm : \ell^2(\mathbb{Z}^\pm, \mathbb{C}^2) \hookrightarrow \ell^2(\mathbb{Z}, \mathbb{C}^2)$.

For each $z \in \mathbb{C} \setminus \mathbb{R}$, the equation

$$\mathbb{D}(m,c)\Psi = z\Psi \tag{23}$$

has unique solutions $\Psi_z^\pm = \begin{pmatrix} \psi_{1,z}^\pm \\ \psi_{2,z}^\pm \end{pmatrix}$ with $\psi_{2,z}^\pm(0) = 1$ and $\sum_{k=0}^{\infty} \left\|\Psi_z^\pm(\pm k)\right\|^2 < \infty$. Let $\mathbf{u}_{\varphi,z}^\pm = \begin{pmatrix} u_{1,\varphi,z}^\pm \\ u_{2,\varphi,z}^\pm \end{pmatrix}$ and $\mathbf{v}_{\varphi,z}^\pm = \begin{pmatrix} v_{1,\varphi,z}^\pm \\ v_{2,\varphi,z}^\pm \end{pmatrix}$ solutions of (23), defined on $\mathbb{Z}^\pm$, satisfying the initial conditions

$$\begin{matrix} u_{1,\varphi,z}^\pm(1) = \cos\varphi & v_{1,\varphi,z}^\pm(1) = \sin\varphi \\ u_{2,\varphi,z}^\pm(0) = -\sin\varphi & v_{2,\varphi,z}^\pm(0) = \cos\varphi \end{matrix}\;, \quad \varphi \in (-\pi/2, \pi/2]\,. \tag{24}$$

Let $\Psi_{\varphi,z}^\pm$ be $\Psi_z^\pm$ normalized by $\psi_{2,\varphi,z}^\pm(0)\cos\varphi + \psi_{1,\varphi,z}^\pm(1)\sin\varphi = 1$. For $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$, the right and left Weyl m-functions, $\mathbf{m}_\varphi^\pm(z)$, are uniquely defined by

$$\Psi_{\varphi,z}^\pm = \mathbf{v}_{\varphi,z}^\pm \mp \mathbf{m}_\varphi^\pm(z)\mathbf{u}_{\varphi,z}^\pm\,.$$

For $\varphi = 0$ we should use the notation $\mathbf{m}^\pm(z) = \mathbf{m}_0^\pm(z)$. The functions $\mathbf{m}^\pm(z)$ and $\mathbf{m}_\varphi^\pm(z)$ are related of the following form:

$$\mathbf{m}^\pm(z) = \frac{\mathbf{m}_\varphi^\pm(z)\cos\varphi \mp \sin\varphi}{\cos\varphi \pm \mathbf{m}_\varphi^\pm(z)\sin\varphi}\,. \tag{25}$$

Moreover, we have that (see [5])

$$\mathbf{m}^+(z) = \langle \delta_{1,1}, (\mathbb{D}_+(m,c) - zI)^{-1}\,\delta_{1,1}\rangle = -\psi_{1,z}^+(1)\,,$$

$$\mathbf{m}^-(z) = \langle \delta_{2,0}, (\mathbb{D}_-(m,c) - zI)^{-1}\,\delta_{2,0}\rangle = \psi_{1,z}^-(1)\,,$$

where $\delta_{1,k}$ and $\delta_{2,k}$ denotes the vectors of the canonical basis of $\ell^2(\cdot\,,\mathbb{C}^2)$ supported at k with $\delta_{1,k}(k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\delta_{2,k}(k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Note that the pair of vectors $\{\delta_{1,1}, \delta_{2,0}\} \subset \ell^2(\mathbb{Z},\mathbb{C}^2)$ is cyclic for $\mathbb{D}(m,c)$. For the whole-line problem, the m-function $\mathbf{m}(z)$ is defined, for $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$, as the trace of the Weyl matrix $M_{2\times 2}(z)$:

$$\begin{bmatrix} a & b \end{bmatrix} M(z) \begin{bmatrix} a \\ b \end{bmatrix} = \left\langle a\delta_{2,0} + b\delta_{1,1}, (\mathbb{D}(m,c) - zI)^{-1}(a\delta_{2,0} + b\delta_{1,1})\right\rangle .$$

Developing this relation, one finds

$$\begin{aligned} M(z) &= \frac{1}{\psi^+_{1,z}(1)\psi^-_{2,z}(0) - \psi^+_{2,z}(0)\psi^-_{1,z}(1)} \begin{bmatrix} \psi^-_{2,z}(0)\psi^+_{2,z}(0) & \psi^-_{2,z}(0)\psi^+_{1,z}(1) \\ \psi^+_{1,z}(1)\psi^-_{2,z}(0) & \psi^-_{1,z}(1)\psi^+_{1,z}(1) \end{bmatrix} \\ &= \frac{1}{-\mathbf{m}^+(z) - \mathbf{m}^-(z)} \begin{bmatrix} 1 & -\mathbf{m}^+(z) \\ -\mathbf{m}^+(z) & -\mathbf{m}^-(z)\mathbf{m}^+(z) \end{bmatrix} \end{aligned}$$

and

$$\mathbf{m}(z) = \mathrm{tr}\,(M(z)) = \frac{\mathbf{m}^+(z)\mathbf{m}^-(z) - 1}{\mathbf{m}^+(z) + \mathbf{m}^-(z)} . \tag{26}$$

Due to spectral theorem (see also [5]), the m-functions can be written as Borel transform of spectral measures, that is,

$$\mathbf{m}^\pm(z) = \int_{-\infty}^{\infty} \frac{d\Lambda^\pm(t)}{t - z} \quad , \quad \mathbf{m}(z) = \int_{-\infty}^{\infty} \frac{d\Lambda(t)}{t - z} , \tag{27}$$

where Λ^+, Λ^- are the spectral measures for the pairs $(\mathbb{D}_+(m,c), \delta_{1,1})$, $(\mathbb{D}_-(m,c), \delta_{2,0})$, respectively, and Λ is the sum of the spectral measures for the pairs $(\mathbb{D}(m,c), \delta_{1,1})$ and $(\mathbb{D}(m,c), \delta_{2,0})$. Using (27) one shows that for $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$ one has $\mathrm{Im}(\mathbf{m}^\pm(z)) > 0$ and $\mathrm{Im}(\mathbf{m}(z)) > 0$.

Let $\mathbf{u}^+_{\varphi,E} = \begin{pmatrix} u^+_{1,\varphi,E} \\ u^+_{2,\varphi,E} \end{pmatrix}$ and $\mathbf{v}^+_{\varphi,E} = \begin{pmatrix} v^+_{1,\varphi,E} \\ v^+_{2,\varphi,E} \end{pmatrix}$ solutions of the eigenvalue equation

$$\mathbb{D}(m,c)\Psi = E\Psi \tag{28}$$

defined on $\mathbb{Z}^+$, satisfying initial conditions as in (24) with $z = E \in \mathbb{R}$. Given any $\epsilon > 0$, we define lengths $L^+_\varphi(\epsilon) \in [1,\infty)$ by requiring the equality

$$\|\mathbf{u}^+_{\varphi,E}\|_{L^+_\varphi(\epsilon)} \cdot \|\mathbf{v}^+_{\varphi,E}\|_{L^+_\varphi(\epsilon)} = \frac{c}{2\epsilon} . \tag{29}$$

The following result is the version of Jitomirskaya-Last inequality (well known in the context of Schrödinger operators [17]) for the discrete Dirac operators $\mathbb{D}_+(m,c)$. This inequality was obtained in Theorem 4.3 in [5] for Dirichlet boundary condition ($\varphi = 0$) and one adapts to any $\varphi \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Lemma 6.1. *Let $\mathbb{D}_+(m,c)$ be a Dirac operator on $\ell^2(\mathbb{Z}^+,\mathbb{C}^2)$ and let $E \in \mathbb{R}$, $\epsilon > 0$ be given. Then the following inequality holds*

$$\frac{5-\sqrt{24}}{\left|\mathbf{m}^{+}_{\varphi}(E+i\epsilon)\right|} < \frac{\|\mathbf{u}^{+}_{\varphi,E}\|_{L^{+}_{\varphi}(\epsilon)}}{\|\mathbf{v}^{+}_{\varphi,E}\|_{L^{+}_{\varphi}(\epsilon)}} < \frac{5+\sqrt{24}}{\left|\mathbf{m}^{+}_{\varphi}(E+i\epsilon)\right|} .$$

Now we recall some useful definitions and the decomposition of Borel measures with respect to Hausdorff measure, which can be found in [17, 20, 26, 27]. Given a Borel set $S \subset \mathbb{R}$ and $\alpha \in [0,1]$, consider the number

$$Q_{\alpha,\delta}(S) = \inf\left\{\sum_{\nu=1}^{\infty} |b_\nu|^{\alpha} : |b_\nu| < \delta; S \subset \bigcup_{\nu=1}^{\infty} b_\nu\right\} ,$$

with the infimum taken over all covers by intervals b_ν of size at most δ. The limit

$$h^{\alpha}(S) := \lim_{\delta\downarrow 0} Q_{\alpha,\delta}(S)$$

is called *α-dimensional Hausdorff measure*. Note that h^0 is the counting measure and h^1 coincides with the Lebesgue measure. For every non-empty Borel set S, there is a unique number $\alpha_S \in [0,1]$, called the *Hausdorff dimension* of S, such that $h^{\alpha}(S) = 0$ if $\alpha > \alpha_S$ and $h^{\alpha}(S) = \infty$ if $\alpha < \alpha_S$.

We recall the notions of continuity and singularity of a measure with respect to Hausdorff measure. Given $\alpha \in [0,1]$, a measure μ is called *α-continuous* if $\mu(S) = 0$ for every Borel set S with $h^{\alpha}(S) = 0$; it is called *α-singular* if it is supported on some Borel set S with $h^{\alpha}(S) = 0$.

Given a finite Borel measure μ on $\mathbb{R}$ and $\alpha \in [0,1]$, the upper α-derivative of μ at E is defined by

$$D^{\alpha}_{\mu}(E) := \limsup_{\epsilon\to 0} \frac{\mu\left((E-\epsilon, E+\epsilon)\right)}{(2\epsilon)^{\alpha}} .$$

Consider the sets

$$T^{\alpha}_{f} = \{E \in \mathbb{R} : D^{\alpha}_{\mu}(E) < \infty\} , \quad T^{\alpha}_{\infty} = \{E \in \mathbb{R} : D^{\alpha}_{\mu}(E) = \infty\}.$$

The measure μ can be decomposed uniquely with respect to Hausdorff measure h^{α} as

$$\mu = \mu_{\alpha c} + \mu_{\alpha s} ,$$

being $\mu_{\alpha c}(\cdot) = \mu(T^{\alpha}_{f} \cap \cdot)$ an α-continuous measure and $\mu_{\alpha s}(\cdot) = \mu(T^{\alpha}_{\infty} \cap \cdot)$ an α-singular measure. Therefore, if $D^{\alpha}_{\mu}(E) < \infty$ a.e. then μ is α-continuous and if $D^{\alpha}_{\mu}(E) = \infty$ a.e. then μ is α-singular.

Now, for each Dirac operator $\mathbb{D}(m,c)$ and each $\Phi \in \ell^2(\mathbb{Z},\mathbb{C}^2)$ we denote by μ^{m}_{Φ} the spectral measure for the pair $(\mathbb{D}(m,c),\Phi)$. The sets

$$\ell^2(\mathbb{Z},\mathbb{C}^2)_{\alpha c} = \{\Phi : \mu^{m}_{\Phi} \text{ is } \alpha-\text{continuous}\} , \quad \ell^2(\mathbb{Z},\mathbb{C}^2)_{\alpha s} = \{\Phi : \mu^{m}_{\Phi} \text{ is } \alpha-\text{singular}\}$$

are closed subspaces of $\ell^2(\mathbb{Z},\mathbb{C}^2)$, mutually orthogonal, invariants by $\mathbb{D}(m,c)$ and

$$\ell^2(\mathbb{Z},\mathbb{C}^2) = \ell^2(\mathbb{Z},\mathbb{C}^2)_{\alpha c} \oplus \ell^2(\mathbb{Z},\mathbb{C}^2)_{\alpha s} .$$

The α-continuous spectrum $\sigma_{\alpha c}(\mathbb{D}(m,c))$ and α-singular spectrum $\sigma_{\alpha s}(\mathbb{D}(m,c))$ of the operator $\mathbb{D}(m,c)$ are defined as the spectrum of the restriction of $\mathbb{D}(m,c)$ to corresponding subspaces. We have that $\sigma(\mathbb{D}(m,c)) = \sigma_{\alpha c}(\mathbb{D}(m,c)) \cup \sigma_{\alpha s}(\mathbb{D}(m,c))$.

7. α-Continuity of the Spectral Measures

In this section we extend to Dirac operators a criteria well known for discrete Schrödinger operators (Theorem 1 in [10]), which allows us to obtain α-continuous spectrum for a Dirac operator $\mathbb{D}(m,c)$ on $\ell^2(\mathbb{Z},\mathbb{C}^2)$. Such criterion (Theorem 7.1 below) establishes α-continuity of the spectral measures of $\mathbb{D}(m,c)$ from power-law upper and lower bounds of the form

$$C_1(E)L^{\gamma_1} \le \|\Psi\|_L \le C_2(E)L^{\gamma_2} \tag{30}$$

for all solutions of (28) with N.I.C. and for $L \ge 1$ sufficiently large, where $\alpha = \frac{2\gamma_1}{\gamma_1+\gamma_2}$. We will show that the bounds (30) can be established for every Sturmian Dirac operators $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ with rotation numbers θ of bounded density, proving so the Theorem 1.2.

Theorem 7.1. *Let Σ be a bounded set. Suppose that there are constants γ_1,γ_2 such that for each $E \in \Sigma$, every solution of* (28) *with N.I.C. obeys the estimate* (30) *for $L \ge 1$ sufficiently large and suitable constants $C_1(E), C_2(E) > 0$. Then for each $m \ge 0$, the operator $\mathbb{D}(m,c)$ has purely α-continuous spectrum on Σ with $\alpha = \frac{2\gamma_1}{\gamma_1+\gamma_2} \in (0,1)$, that is, for any $\Phi \in \ell^2(\mathbb{Z},\mathbb{C}^2)$ the spectral measure μ^m_Φ for the pair $(\mathbb{D}(m,c),\Phi)$ is purely α-continuous on Σ.*

Proof. The proof is based on ideas from [10] used in the context of discrete Schrödinger operators. Let $\alpha = \frac{2\gamma_1}{\gamma_1+\gamma_2}$. Using (30) for the solutions $\mathbf{u}^+_{\varphi,E}$ and $\mathbf{v}^+_{\varphi,E}$ of (28), we have

$$\frac{\|\mathbf{u}^+_{\varphi,E}\|_L}{\|\mathbf{v}^+_{\varphi,E}\|_L^{\frac{\alpha}{2-\alpha}}} \ge \frac{C_1(E)L^{\gamma_1}}{(C_2(E)L^{\gamma_2})^{\frac{\alpha}{2-\alpha}}} = \frac{C_1(E)}{C_2(E)^{\frac{\alpha}{2-\alpha}}}L^{\gamma_1-\gamma_2\frac{\alpha}{2-\alpha}} = \frac{C_1(E)}{C_2(E)^{\frac{\alpha}{2-\alpha}}} > 0$$

for all $\varphi \in (-\pi/2,\pi/2]$ and $L \ge 1$ sufficiently large.

By (29) and Lemma 6.1 we obtain

$$\frac{(5-\sqrt{24})c^{1-\alpha}}{(2\epsilon)^{1-\alpha}|\mathbf{m}^+_\varphi(E+i\epsilon)|} < \left(\frac{\|\mathbf{u}^+_{\varphi,E}\|_L}{\|\mathbf{v}^+_{\varphi,E}\|_L^{\frac{\alpha}{2-\alpha}}}\right)^{2-\alpha} < \frac{(5+\sqrt{24})c^{1-\alpha}}{(2\epsilon)^{1-\alpha}|\mathbf{m}^+_\varphi(E+i\epsilon)|}.$$

It follows from the two estimates above that

$$\limsup_{\epsilon\to 0} \epsilon^{1-\alpha}|\mathbf{m}^+_\varphi(E+i\epsilon)| < \infty, \quad \forall\varphi \in (-\pi/2,\pi/2].$$

Thus, there exists $0 < C_3(E) < \infty$ such that

$$\sup_\varphi |\mathbf{m}^+_\varphi(E+i\epsilon)| \le C_3(E)\epsilon^{\alpha-1}. \tag{31}$$

The next step is to transfer the estimate (31) for the m-function $\mathbf{m}(E+i\epsilon)$ given by (26). Fix $E \in \Sigma$ and $\epsilon > 0$. Introducing variables $\xi = e^{2i\varphi}$ and $\nu = \frac{\mathbf{m}^+-i}{\mathbf{m}^++i}$, we have

$$\begin{aligned}
\frac{1+\nu\xi}{1-\nu\xi} &= \frac{e^{i\varphi}\left(e^{-i\varphi} + (\frac{\mathbf{m}^+-i}{\mathbf{m}^++i})e^{i\varphi}\right)}{e^{i\varphi}\left(e^{-i\varphi} - \left(\frac{\mathbf{m}^+-i}{\mathbf{m}^++i}\right)e^{i\varphi}\right)} \\
&= \frac{(\cos\varphi - i\sin\varphi)(\mathbf{m}^+ + i) + (\mathbf{m}^+ - i)(\cos\varphi + i\sin\varphi)}{(\cos\varphi - i\sin\varphi)(\mathbf{m}^+ + i) - (\mathbf{m}^+ - i)(\cos\varphi + i\sin\varphi)} \\
&= \frac{\sin\varphi + \cos\varphi\,\mathbf{m}^+}{i(\cos\varphi - \sin\varphi\,\mathbf{m}^+)} = -i\mathbf{m}^+_\varphi,
\end{aligned}$$

where in last step we use the relation (25). Thus, we may rewrite (31) as

$$\sup_{|\xi|=1}\left|\frac{1+\nu\xi}{1-\nu\xi}\right| \le C_3(E)\epsilon^{\alpha-1}.$$

Note that $\mathrm{Im}(\mathbf{m}^+) > 0$ implies $|\nu| < 1$ and so $\frac{1+\nu\xi}{1-\nu\xi}$ defines an analytic function on unit disk $D_1(0) = \{\zeta : |\zeta| \le 1\}$. The point $\xi_1 = \frac{\mathbf{m}^- - i}{\mathbf{m}^- + i} \in D_1(0)$ since $\mathrm{Im}(\mathbf{m}^-) > 0$. By maximum modulus principle we have

$$\sup_{|\xi|\le 1}\left|\frac{1+\nu\xi}{1-\nu\xi}\right| = \sup_{|\xi|=1}\left|\frac{1+\nu\xi}{1-\nu\xi}\right| \le C_3(E)\epsilon^{\alpha-1}.$$

Applying this inequality to the point ξ_1 and using the expression (26), we obtain

$$|\mathbf{m}(E+i\epsilon)| = \left|\frac{1+\nu\xi_1}{1-\nu\xi_1}\right| \le C_3(E)\epsilon^{\alpha-1}.$$

This estimate and the representation (27) implies that

$$\Lambda((E-\epsilon, E+\epsilon)) \le 2\epsilon\mathrm{Im}(\mathbf{m}(E+i\epsilon)) \le 2\epsilon|\mathbf{m}(E+i\epsilon)| \le 2C_3(E)\epsilon^{\alpha}$$

for all $E \in \Sigma$ and $\epsilon > 0$. Therefore,

$$D^{\alpha}_{\Lambda}(E) = \limsup_{\epsilon\to 0}\frac{\Lambda((E-\epsilon, E+\epsilon))}{(2\epsilon)^{\alpha}} \le 2^{1-\alpha}C_3(E) < \infty,$$

from which Λ is α-continuous on Σ. Given any $\Phi \in \ell^2(\mathbb{Z}, \mathbb{C}^2)$, the spectral measure μ^m_{Φ} is absolutely continuous with respect to Λ and so must be α-continuous on Σ. This completes the proof of the theorem. □

Now our goal is to apply Theorem 7.1 to Sturmian Dirac operators $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$. For this, we will show the bounds (30) in Propositions 7.1 and 7.2 below. First, we establish a lower bound for solutions of (7), similar to Proposition 5.1 in [10].

Proposition 7.1. *Suppose that the sequence (q_k) associated with rotation number θ satisfies $q_k \le C_\theta^k$, for some $1 < C_\theta < \infty$. For every λ and $m \ge 0$, there exist $\gamma_1 = \gamma_1(m,\lambda,\theta) > 0$, $0 < C_1 < \infty$ such that for every $E \in \Sigma_{\lambda,\theta}(m)$ and all $\rho \in [0,1)$, every solution Ψ of (7) with N.I.C. obeys*

$$\|\Psi\|_L \ge C_1 L^{\gamma_1}$$

for L sufficiently large.

Proof. By hypothesis we have $C^k_{\theta,1} \le q_{8k} \le C^k_{\theta,2}$ for all $k \ge 1$, where $1 < C_{\theta,1} < C_{\theta,2} < \infty$, and by Lemma 5.1,

$$\|\tilde{\Psi}\|_{q_{8k}} \ge B_\lambda(m)^k \quad \forall k \ge 1,$$

with a constant $B_\lambda(m) > 1$. Choosing $\gamma = \gamma(m,\lambda,\theta) > 0$ such that $C^{\gamma}_{\theta,2} \le B_\lambda(m)$, follows that $\|\tilde{\Psi}\|_{q_{8k}} \ge q^{\gamma}_{8k}$.

Take $\epsilon \in (a_\theta\gamma, \gamma)$ where $a_\theta = \dfrac{\ln C_{\theta,2} - \ln C_{\theta,1}}{\ln C_{\theta,2}}$, and let $\gamma_1 := \gamma - \epsilon > 0$. We have $\dfrac{C^{\gamma_1}_{\theta,2}}{C^{\gamma}_{\theta,1}} < 1$. Choose $k \in \mathbb{N}$ such that

$$\left(\frac{C_{\theta,2}^{\gamma_1}}{C_{\theta,1}^{\gamma}}\right)^k \leq \frac{1}{C_{\theta,2}^{\gamma_1}}$$

and let L sufficiently large such that $q_{8k} \leq L < q_{8(k+1)}$. Thus, it follows that

$$\|\tilde{\Psi}\|_L \geq \|\tilde{\Psi}\|_{q_{8k}} \geq q_{8k}^{\gamma} \geq C_{\theta,1}^{k\gamma} \geq C_{\theta,2}^{(k+1)\gamma_1} \geq q_{8(k+1)}^{\gamma_1} \geq L^{\gamma_1}.$$

Therefore there exist a constant $C_1 = D_1 > 0$ such that

$$\|\Psi\|_L \geq C_1\|\tilde{\Psi}\|_L \geq C_1 L^{\gamma_1}$$

for every solution Ψ of (7) with N.I.C. and for L sufficiently large. □

The following result establishes a upper bound for solutions of (7), similar to Proposition 5.2 in [10].

Proposition 7.2. *Let θ be a bounded density number. For every λ and $m \geq 0$, there exist $\gamma_2 = \gamma_2(m, \lambda, \theta) > 0$, $0 < C_2 < \infty$ such that for every $E \in \Sigma_{\lambda,\theta}(m)$ and all $\rho \in [0, 1)$, every solution Ψ of (7) with N.I.C. obeys*

$$\|\Psi\|_L \leq C_2 L^{\gamma_2}$$

for all $L \geq 1$.

The main point of the proof of Proposition 7.2 is the Lemma 7.1 below. Since the upper boundedness of the transfer matrices $M(m, E, V_{\lambda,\theta,0}(1)\cdots V_{\lambda,\theta,0}(k))$ depends only on the structure of the Sturmian potentials, a direct adaptation of results of [16] in the Schrödinger setting shows that

Lemma 7.1. *Suppose that θ is a bounded density number. For every λ and $m \geq 0$, there is a constant $0 < C < \infty$ such that for all $E \in \Sigma_{\lambda,\theta}(m)$,*

$$\|M(m, E, V_{\lambda,\theta,0}(1)\cdots V_{\lambda,\theta,0}(k))\| \leq Ck^{\gamma} \quad \forall k \in \mathbb{Z}^+,$$

with $\gamma = Bd(\theta)\log\left(2 + \sqrt{4 + \max_{E\in\Sigma_{\lambda,\theta}(m)} \mathcal{I}_{-1}^{(D)}(E)}\right) > 0$, where B is some universal constant, $d(\theta)$ is as in (5) and $\mathcal{I}_{-1}^{(D)}$ is given by Proposition 2.1.

Proof of Proposition 7.2. If $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is solution of (7) with N.I.C., then follow from (9) and Lemma 7.1 that

$$\|\tilde{\Psi}(k+1)\| \leq \|M(m, E, V_{\lambda,\theta,0}(1)\cdots V_{\lambda,\theta,0}(k))\| \leq Ck^{\gamma} \quad \forall k \geq 1.$$

Hence, for all $L \geq 1$,

$$\|\tilde{\Psi}\|_L = \left(\sum_{k=1}^{\lfloor L\rfloor} \|\tilde{\Psi}(k)\|^2 + (L - \lfloor L\rfloor)\|\tilde{\Psi}(\lfloor L\rfloor + 1)\|^2\right)^{1/2} \leq \left(1 + C^2\right)^{1/2} L^{\gamma+1/2}.$$

Therefore there exist constants $C_2 = (1 + C^2)^{1/2} D_2 > 0$ and $\gamma_2 = \gamma + \frac{1}{2} > 0$ such that

$$\|\Psi\|_L \leq C_2 L^{\gamma_2} \quad \forall L \geq 1.$$

This shows the result for solutions of (7) corresponding to $\rho = 0$. Due to right continuity of the potential $V_{\lambda,\theta,\rho}$ in ρ, of the corresponding transfer matrices and the continuity of the norm $\|\cdot\|_L$, the result follows for all phase $\rho \in [0,1)$.

We are now ready to prove the second main result of this paper.

Proof of Theorem 1.2. By hypothesis θ is a number of bounded density, then by Lemma 2.3 in [6] there exists a constant $1 < C_\theta < \infty$ such that $q_k \leq C_\theta^k$. Thus, it follows from Propositions 7.1 and 7.2 that for $\lambda \neq 0$ and $m \geq 0$, there exist $\gamma_1, \gamma_2 > 0$ (depending on m, λ, θ), $0 < C_1, C_2 < \infty$, such that for each $E \in \Sigma_{\lambda,\theta}(m)$ and $\rho \in [0,1)$, every solution Ψ of (7) with N.I.C. obeys

$$C_1 L^{\gamma_1} \leq \|\Psi\|_L \leq C_2 L^{\gamma_2}$$

for $L \geq 1$ sufficiently large. Let $\alpha = \alpha(m,\lambda,\theta) := \frac{2\gamma_1}{\gamma_1+\gamma_2} \in (0,1)$. Therefore, by Theorem 7.1, for all $\rho \in [0,1)$ and $\Phi \in \ell^2(\mathbb{Z},\mathbb{C}^2)$, the spectral measure for the pair $(\mathbb{D}_{\lambda,\theta,\rho}(m,c),\Phi)$ is purely α-continuous, that is, $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ has purely α-continuous spectrum. This completes the proof.

8. Comparison of Invariants and of the Dimension Estimates

The goal of this section is to compare the numbers $\alpha^{(D)} = \frac{2\gamma_1^{(D)}}{\gamma_1^{(D)}+\gamma_2^{(D)}}$, obtained for the Sturmian Dirac operators $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$ in Theorem 1.2, with the corresponding numbers $\alpha^{(S)} = \frac{2\gamma_1^{(S)}}{\gamma_1^{(S)}+\gamma_2^{(S)}}$ obtained in Theorem 2 in [10] for the Schrödinger operators $H_{\lambda,\theta,\rho}$ given by (3). We consider the two models with same mass $m > 0$ and generated by the same Sturmian potentials. We also compare lower bounds for exponents of transport associated with these models.

To obtain a comparison between $\alpha^{(D)}$ and $\alpha^{(S)}$, the first step is to compare the Sturmian Dirac invariants (obtained in Proposition 2.1(ii))

$$\begin{aligned}\mathcal{I}_{-1}^{(D)}(E) &= \frac{\lambda^2}{c^6}E^4 - \frac{2\lambda^3}{c^6}E^3 + \left(\frac{\lambda^4 - 2m^2c^4\lambda^2}{c^6}\right)E^2 + \frac{2m^2c^4\lambda^3}{c^6}E - \frac{m^2c^4}{c^6}\lambda^4 \\ &\quad + \left(m^4c^2 + 4m^2\right)\lambda^2 + 4\end{aligned}$$

with the Sturmian Schrödinger invariants (for mass $m > 0$):

$$\begin{aligned}\mathcal{I}_{-1}^{(S)}(E') &= [\operatorname{tr}(\tilde{M}_0)]^2 + [\operatorname{tr}(\tilde{M}_{-1})]^2 + [\operatorname{tr}(\tilde{M}_{-1}\tilde{M}_0)]^2 - \operatorname{tr}(\tilde{M}_0)\operatorname{tr}(\tilde{M}_{-1})\operatorname{tr}(\tilde{M}_{-1}\tilde{M}_0) \\ &= (2mE'+2)^2 + 2^2 + (2m(E'-\lambda)+2)^2 - 2(2mE'+2)(2m(E'-\lambda)+2) \\ &= 4m^2\lambda^2 + 4\end{aligned}$$

where $\tilde{M}_{-1} = \begin{pmatrix} 1 & -2m\lambda \\ 0 & 1 \end{pmatrix}$ and $\tilde{M}_0 = \begin{pmatrix} 2mE'+2 & -1 \\ 1 & 0 \end{pmatrix}$, for all energies $E \in \Sigma^{(D)}$ and $E' \in \Sigma^{(S)}$, where $\Sigma^{(S)}$ and $\Sigma^{(D)}$ denote the spectra of the operators $H_{\lambda,\theta,\rho}$ and $\mathbb{D}_{\lambda,\theta,\rho}(m,c)$, respectively.

Since $\Sigma^{(D)}$ is singular continuous of zero Lebesgue measure (Theorem 1.1), it is not possible to calculate $\mathcal{I}_{-1}^{(D)}(E)$ directly for each $E \in \Sigma^{(D)}$ (analogous for $\mathcal{I}_{-1}^{(S)}(E')$); we will work on a larger set X, defined as follows. In [5, 15] it is shown that the spectrum of the free Dirac operator is given by

$$\sigma(\mathbb{D}_0(m,c)) = \left[-\sqrt{m^2c^4+4c^2}, -mc^2\right] \cup \left[mc^2, \sqrt{m^2c^4+4c^2}\right].$$

Thus, we have that

$$\Sigma^{(D)} = \sigma\left(\mathbb{D}_{\lambda,\theta,\rho}(m,c)\right) \subset \sigma\left(\mathbb{D}_0(m,c)\right) + [-|\lambda|, |\lambda|] = X$$

where $X := \left[-\sqrt{m^2c^4+4c^2} - |\lambda|, -mc^2+|\lambda|\right] \cup \left[mc^2 - |\lambda|, \sqrt{m^2c^4+4c^2} + |\lambda|\right]$. For simplicity, we will consider $\lambda > 0$; a similar analysis can be made for $\lambda < 0$. Note that if $0 < \lambda < mc^2$ then the two intervals of X are disjoint and so X has four boundary points; if $\lambda \geq mc^2$ then X has two boundary points. Fixed $m > 0$ and λ, the invariant $\mathcal{I}_{-1}^{(D)}(E)$ is a continuous (polynomial) function of the energy E, which assumes maximum and minimum values on the compact sets $\Sigma^{(D)}$ and X.

Let us determine $\max_{E\in X} \mathcal{I}_{-1}^{(D)}(E)$ and $\min_{E\in X} \mathcal{I}_{-1}^{(D)}(E)$. The critical points of $\mathcal{I}_{-1}^{(D)}$, which satisfies

$$\frac{d\mathcal{I}_{-1}^{(D)}}{dE}(E) = \frac{4\lambda^2}{c^6}E^3 - \frac{6\lambda^3}{c^6}E^2 + 2\left(\frac{\lambda^4 - 2m^2c^4\lambda^2}{c^6}\right)E + \frac{2m^2c^4\lambda^3}{c^6} = 0\,,$$

are given by $E \in \left\{\frac{\lambda}{2}, \frac{\lambda \pm \sqrt{\lambda^2+4m^2c^4}}{2}\right\}$. We have that $\frac{\lambda}{2} \in X$ if $\lambda > \frac{2}{3}mc^2$ and $\frac{\lambda \pm \sqrt{\lambda^2+4m^2c^4}}{2} \in X$ for all $\lambda > 0$. Calculating the value of $\mathcal{I}_{-1}^{(D)}(E)$ for each critical point, we obtain

$$\begin{aligned}
\mathcal{I}_{-1}^{(D)}\left(\frac{\lambda}{2}\right) &= \frac{1}{16c^6}\lambda^6 - \frac{m^2c^4}{2c^6}\lambda^4 + m^4c^2\lambda^2 + 4m^2\lambda^2 + 4 \\
&= \frac{\lambda^2}{16c^6}\left(\lambda^2 - 4m^2c^4\right)^2 + 4m^2\lambda^2 + 4 \;\geq\; 4m^2\lambda^2 + 4
\end{aligned}$$

and

$$\mathcal{I}_{-1}^{(D)}\left(\frac{\lambda \pm \sqrt{\lambda^2+4m^2c^4}}{2}\right) = -\frac{m^2c^4}{c^6}\lambda^4 + 4m^2\lambda^2 + 4 \;<\; 4m^2\lambda^2 + 4.$$

Now, calculating the value of $\mathcal{I}_{-1}^{(D)}(E)$ for each boundary point of the set X, we obtain for $\lambda > 0$,

$$\begin{aligned}
\mathcal{I}_{-1}^{(D)}\left(-\sqrt{m^2c^4+4c^2} - \lambda\right) &= \frac{4}{c^6}\lambda^6 + \frac{12\sqrt{m^2c^4+4c^2}}{c^6}\lambda^5 + \frac{(8m^2c^4+52c^2)}{c^6}\lambda^4 \\
&\quad + \frac{24\sqrt{m^2c^4+4c^2}}{c^4}\lambda^3 + \left(\frac{16}{c^2} + 4m^2\right)\lambda^2 + 4 \\
&> 4m^2\lambda^2 + 4\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{-1}^{(D)}\left(\sqrt{m^2c^4+4c^2} + \lambda\right) &= \frac{4}{c^4}\lambda^4 + \frac{8\sqrt{m^2c^4+4c^2}}{c^4}\lambda^3 + \left(\frac{16}{c^2} + 4m^2\right)\lambda^2 + 4 \\
&> 4m^2\lambda^2 + 4
\end{aligned}$$

and for $0 < \lambda < mc^2$,

$$\mathcal{I}_{-1}^{(D)}\left(-mc^2 + \lambda\right) = 4m^2\lambda^2 + 4$$

and

$$\begin{aligned}\mathcal{I}_{-1}^{(D)}\left(mc^2-\lambda\right) &= \frac{4}{c^6}\lambda^6-\frac{12mc^2}{c^6}\lambda^5+\frac{8m^2c^4}{c^6}\lambda^4+4m^2\lambda^2+4\\ &= \frac{4\lambda^4}{c^6}(\lambda-mc^2)(\lambda-2mc^2)+4m^2\lambda^2+4 \;>\; 4m^2\lambda^2+4.\end{aligned}$$

From the above calculations we conclude that for all $\lambda > 0$,

$$\min_{E\in X}\mathcal{I}_{-1}^{(D)}(E)=\mathcal{I}_{-1}^{(D)}\left(\frac{\lambda\pm\sqrt{\lambda^2+4m^2c^4}}{2}\right) \;<\; \mathcal{I}_{-1}^{(S)}(E')=4m^2\lambda^2+4$$

and

$$\max_{E\in X}\mathcal{I}_{-1}^{(D)}(E)=\mathcal{I}_{-1}^{(D)}\left(-\sqrt{m^2c^4+4c^2}-\lambda\right) \;>\; \mathcal{I}_{-1}^{(S)}(E').$$

Moreover, we have the following informations about the function $\mathcal{I}_{-1}^{(D)}(E)$:

1. $E=\dfrac{\lambda}{2}$ is local maximum point because it satisfies

$$\frac{d^2\mathcal{I}_{-1}^{(D)}}{dE^2}\left(\frac{\lambda}{2}\right)=-\frac{\lambda^4}{c^6}-\frac{4m^2c^4}{c^6}\lambda^2<0\;;$$

2. $E=\dfrac{\lambda\pm\sqrt{\lambda^2+4m^2c^4}}{2}$ are local (global) minimum points due to

$$\frac{d^2\mathcal{I}_{-1}^{(D)}}{dE^2}\left(\frac{\lambda\pm\sqrt{\lambda^2+4m^2c^4}}{2}\right)=\frac{2\lambda^4}{c^6}+\frac{8m^2c^4}{c^6}\lambda^2>0\;;$$

3. $\mathcal{I}_{-1}^{(D)}(E)$ is an decreasing function of E $\left(\dfrac{d\mathcal{I}_{-1}^{(D)}}{dE}(E)<0\right)$ in the energy intervals

$$\left(-\sqrt{m^2c^4+4c^2}-\lambda\;,\;\frac{\lambda-\sqrt{\lambda^2+4m^2c^4}}{2}\right)\cup\left(\frac{\lambda}{2}\;,\;\frac{\lambda+\sqrt{\lambda^2+4m^2c^4}}{2}\right)$$

and increasing $\left(\dfrac{d\mathcal{I}_{-1}^{(D)}}{dE}(E)>0\right)$ in the energy intervals

$$\left(\frac{\lambda-\sqrt{\lambda^2+4m^2c^4}}{2}\;,\;\frac{\lambda}{2}\right)\cup\left(\frac{\lambda+\sqrt{\lambda^2+4m^2c^4}}{2}\;,\;\sqrt{m^2c^4+4c^2}+\lambda\right);$$

4. $\mathcal{I}_{-1}^{(D)}(E)=\mathcal{I}_{-1}^{(S)}(E')=4m^2\lambda^2+4$ for energies $E\in\{-mc^2,-mc^2+\lambda,mc^2,mc^2+\lambda\}$.

Denoting for $\lambda>0$

$$J=\begin{cases}\left(-mc^2,-mc^2+\lambda\right)\cup\left(mc^2,mc^2+\lambda\right) & \text{if}\quad \lambda\le 2mc^2,\\ \left(-mc^2,mc^2\right)\cup\left(-mc^2+\lambda,mc^2+\lambda\right) & \text{if}\quad \lambda>2mc^2,\end{cases}$$

we obtain from the above calculations the following comparison of the invariants

$$\mathcal{I}_{-1}^{(D)}(E)\ge\mathcal{I}_{-1}^{(S)}(E')\quad\forall E\in(X\setminus J)\cap\Sigma^{(D)},\ \forall E'\in\Sigma^{(S)} \tag{32}$$

and

$$\mathcal{I}_{-1}^{(D)}(E) \le \mathcal{I}_{-1}^{(S)}(E') \quad \forall E \in \bar{J} \cap \Sigma^{(D)},\ \forall E' \in \Sigma^{(S)}. \tag{33}$$

It follows from (32) that

$$K^{(D)} := \max_{E \in (X \setminus J) \cap \Sigma^{(D)}} \mathcal{I}_{-1}^{(D)}(E) \ \ge\ \max_{E' \in \Sigma^{(S)}} \mathcal{I}_{-1}^{(S)}(E') = 4m^2\lambda^2 + 4.$$

Now, for all $E \in (X \setminus J) \cap \Sigma^{(D)}$ we have the bounds:

$$\begin{aligned}
|x_k| &\le 2 + \sqrt{4 + \mathcal{I}_{-1}^{(D)}(E)} \ \le\ 2 + \sqrt{4 + K^{(D)}} := C_0^{(D)}\,,\\
|y_k| &\le C_0^{(D)}\,,\\
|z_k| &\le \frac{|x_k||y_k| + \sqrt{x_k^2 y_k^2 + 4\mathcal{I}_{-1}^{(D)}(E)}}{2} \ \le\ \frac{(C_0^{(D)})^2 + \sqrt{(C_0^{(D)})^4 + 4K^{(D)}}}{2} := C^{(D)},
\end{aligned}$$

where $x_k = \operatorname{tr} M(m, E, S_k)$, $y_k = x_{k-1}$ and $z_k = \operatorname{tr} M(m, E, S_k S_{k-1})$. This implies that

$$\max\{|x_k|, |y_k|, |z_k|\} \le C^{(D)}.$$

In an analogous way to the above bounds, it is possible to show, for all $E' \in \Sigma^{(S)}$, that

$$\max\{|\tilde{x}_k|, |\tilde{y}_k|, |\tilde{z}_k|\} \le C^{(S)}$$

for some constant $C^{(S)}$, where $\tilde{x}_k$, $\tilde{y}_k$ and $\tilde{z}_k$ are the corresponding traces in the Schrödinger case. Since $K^{(D)} \ge 4m^2\lambda^2 + 4$, we have that $C^{(D)} \ge C^{(S)}$ and

$$B^{(D)} := \left(1 + \frac{1}{(2C^{(D)})^2}\right)^{1/2} \ \le\ B^{(S)} := \left(1 + \frac{1}{(2C^{(S)})^2}\right)^{1/2}.$$

Analysing the proofs of Proposition 7.1 and Proposition 5.1 in [10] (in the Schrödinger context) we have $C_{\theta,1}^k \le q_{8k} \le C_{\theta,2}^k$ for all $k \ge 1$, where $1 < C_{\theta,1} < C_{\theta,2} < \infty$. Let $\gamma^{(S)} > 0$ be such that $C_{\theta,2}^{\gamma^{(S)}} \le B^{(S)}$. Take $\gamma^{(D)} > 0$ such that $C_{\theta,2}^{\gamma^{(D)}} \le B^{(D)}$ and $a_\theta \gamma^{(S)} < \gamma^{(D)} \le \gamma^{(S)}$, where $a_\theta = \dfrac{\ln C_{\theta,2} - \ln C_{\theta,1}}{\ln C_{\theta,2}}$. Now choose $\epsilon \in \left(a_\theta \gamma^{(S)}, \gamma^{(D)}\right)$ and let $\gamma_1^{(S)} := \gamma^{(S)} - \epsilon > 0$, $\gamma_1^{(D)} := \gamma^{(D)} - \epsilon > 0$. Thus, referring to the energy intervals $(X \setminus J) \cap \Sigma^{(D)}$ and $\Sigma^{(S)}$ we obtain that

$$\gamma_1^{(D)} \le \gamma_1^{(S)}. \tag{34}$$

Note that with the above choices follow the proofs of Proposition 7.1 (in the energy interval $(X \setminus J) \cap \Sigma^{(D)}$) and Proposition 5.1 in [10].

On the other hand, the proofs of Proposition 7.2 and of Proposition 5.2 in [10] (in the Schrödinger context) are valid with

$$\gamma_2^{(D)} = Bd(\theta) \log\left(2 + \sqrt{4 + \max_{E \in (X \setminus J) \cap \Sigma^{(D)}} \mathcal{I}_{-1}^{(D)}(E)}\right) + \frac{1}{2}$$

and

$$\gamma_2^{(S)} = Bd(\theta) \log\left(2 + \sqrt{4 + \max_{E' \in \Sigma^{(S)}} \mathcal{I}_{-1}^{(S)}(E')}\right) + \frac{1}{2}\,,$$

respectively. The estimate (32) implies that

$$\gamma_2^{(D)} \geq \gamma_2^{(S)}, \tag{35}$$

referring to the energy intervals $(X \setminus J) \cap \Sigma^{(D)}$ and $\Sigma^{(S)}$.

By (34) and (35) we obtain, with respect to the energy intervals $(X \setminus J) \cap \Sigma^{(D)}$ and $\Sigma^{(S)}$, the following comparison between the dimension estimates for the models $H_{\lambda,\theta,\rho}$ and $\mathbb{D}_{\lambda,\theta,\rho}(m, c)$:

$$\alpha^{(D)} = \frac{2\gamma_1^{(D)}}{\gamma_1^{(D)} + \gamma_2^{(D)}} \leq \frac{2\gamma_1^{(S)}}{\gamma_1^{(S)} + \gamma_2^{(S)}} = \alpha^{(S)}, \quad \text{for all } \lambda > 0. \tag{36}$$

Similarly, by reproducing the above calculations by now using the estimate (33) instead of (32), we obtain with respect to the energy intervals $\bar{J} \cap \Sigma^{(D)}$ and $\Sigma^{(S)}$, the following comparison

$$\tilde{\alpha}^{(D)} := \frac{2\tilde{\gamma}_1{}^{(D)}}{\tilde{\gamma}_1{}^{(D)} + \tilde{\gamma}_2{}^{(D)}} \geq \frac{2\gamma_1^{(S)}}{\gamma_1^{(S)} + \gamma_2^{(S)}} = \alpha^{(S)}, \quad \text{for all } \lambda > 0, \tag{37}$$

where $\tilde{\gamma}_1{}^{(D)}$ and $\tilde{\gamma}_2{}^{(D)}$ are constructed as above in a similar way to $\gamma_1^{(D)}$ and $\gamma_2^{(D)}$.

Now we will apply (36) and (37) in the comparison of lower bounds for the exponents of transport associated with Sturmian Dirac and Schrödinger models. The standard quantities that are considered to measure the spreading of an initially localized wavepacket, under the dynamics governed by a Schrödinger operator H, are the time-averaged moments of the position operator

$$\mathcal{M}(p, T, \delta_1) := \frac{2}{T} \int_0^\infty e^{-2t/T} \sum_k |k|^p \left| \left\langle e^{-itH} \delta_1, \delta_k \right\rangle \right|^2 dt$$

with $p > 0$, $T > 0$ and $\{\delta_k\}$ the canonical basis of $\ell^2(\mathbb{Z})$. The faster $\mathcal{M}(p, T, \delta_1)$ grows, the faster $e^{-itH}\delta_1$ spreads out, at least averaged in time. It is also usual to consider the lower transport exponents

$$\beta^{(S)}(p, \delta_1) := \liminf_{T \to \infty} \frac{\log \mathcal{M}(p, T, \delta_1)}{\log T}.$$

By Theorem 2 in [10] the spectral measure for the Sturmian Schrödinger model $H_{\lambda,\theta,\rho}$, associated with δ_1, is $\alpha^{(S)}$-continuous with $\alpha^{(S)} = \frac{2\gamma_1^{(S)}}{\gamma_1^{(S)} + \gamma_2^{(S)}}$. It follows from Theorem 6.1 in [20] that

$$\beta^{(S)}(p, \delta_1) \geq p\alpha^{(S)} \quad \forall p > 0. \tag{38}$$

Similarly, we define the lower transport exponents $\beta^{(D)}(p, \delta_{1,1})$ associated with Sturmian Dirac model $\mathbb{D}_{\lambda,\theta,\rho}(m, c)$, where $\delta_{1,1}$ is the vector of the canonical basis of $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ supported at position $k = 1$ with $\delta_{1,1}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Theorem 6.1 in [20] is valid for Dirac operators $\mathbb{D}(m, c)$; using this result together with Theorem 1.2 we obtain

$$\beta^{(D)}(p, \delta_{1,1}) \geq p\alpha^{(D)} \quad \forall p > 0, \tag{39}$$

where $\alpha^{(D)} = \frac{2\gamma_1^{(D)}}{\gamma_1^{(D)} + \gamma_2^{(D)}}$ is given by Theorem 1.2. By (36) and (37) we conclude, by method above, that with respect to the energy intervals $(X \setminus J) \cap \Sigma^{(D)}$ and $\Sigma^{(S)}$ the lower bounds in (38) are greater than or equal to the corresponding lower bounds in (39), and with respect to the energy intervals $\bar{J} \cap \Sigma^{(D)}$ and $\Sigma^{(S)}$ the lower bounds in (39) are greater than or equal to the corresponding lower bounds in (38).

Finally, the papers [13, 22] have worked only on upper boundedness of transfer matrices, as in Lemma 7.1, and derived the following lower bounds

$$\beta^{(S)}(p,\delta_1) \geq \frac{p-3\gamma^{(S)}}{1+\gamma^{(S)}} \quad , \quad \beta^{(D)}(p,\delta_{1,1}) \geq \frac{p-3\gamma^{(D)}}{1+\gamma^{(D)}} \, , \tag{40}$$

for all $p > 0$, where $\gamma^{(D)} = \gamma$ is given by Lemma 7.1 and $\gamma^{(S)}$ by corresponding result in [16]. For large values of p, the bounds in (40) are better than in (38)-(39); for p small, the bounds in (38)-(39) are better.

Using (32) we obtain that with respect to the energy intervals $(X \setminus J) \cap \Sigma^{(D)}$ and $\Sigma^{(S)}$, $\gamma^{(S)} \leq \gamma^{(D)}$ which implies $\frac{p-3\gamma^{(S)}}{1+\gamma^{(S)}} \geq \frac{p-3\gamma^{(D)}}{1+\gamma^{(D)}}$. On the other hand, using (33) follows that with respect to the energy intervals $\bar{J} \cap \Sigma^{(D)}$ and $\Sigma^{(S)}$, $\frac{p-3\gamma^{(S)}}{1+\gamma^{(S)}} \leq \frac{p-3\tilde{\gamma}^{(D)}}{1+\tilde{\gamma}^{(D)}}$. We conclude again, now by this other method, that for the energy intervals $(X \setminus J) \cap \Sigma^{(D)}$ and $\Sigma^{(S)}$ the lower bounds for transport exponents $\beta^{(S)}(p,\delta_1)$ are greater than or equal to the corresponding lower bounds for $\beta^{(D)}(p,\delta_{1,1})$, and for the energy intervals $\bar{J} \cap \Sigma^{(D)}$ and $\Sigma^{(S)}$ the lower bounds for $\beta^{(D)}(p,\delta_{1,1})$ are greater than or equal to the corresponding lower bounds for $\beta^{(S)}(p,\delta_1)$.

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References

[1] J. Bellissard, A. Bovier and J.-M. Ghez: *Spectral Properties of a Tight Binding Hamiltonian with Period Doubling Potential*, Comm. Math. Phys. **135** (1991), 379–399.

[2] J. Bellissard, B. Iochum, E. Scoppola and D. Testard: *Spectral Properties of One-Dimensional Quasi-crystals*, Comm. Math. Phys. **125** (1989), 527–543.

[3] P. Bougerol, J. Lacroix: Products of Random Matrices with Applications to Schrödinger Operators, Birkhäuser, Boston, 1985.

[4] A. Bovier and J.-M. Ghez: *Spectral Properties of One-Dimensional Schrödinger Operators with Potentials Generated by Substitutions*, Comm. Math. Phys. **158** (1993), 45–66.

[5] S.L. Carvalho, C.R. de Oliveira and R.A. Prado: *Sparse One-dimensional Discrete Dirac Operators II: Spectral Properties*, J. Math. Phys. **52** (2011), 073501, 21pp.

[6] D. Damanik: α-continuity properties of one-dimensional quasicrystals, Comm. Math. Phys. **192** (1998), 169–182.

[7] D. Damanik: *Uniform singular continuous spectrum for the period doubling Hamiltonian*, Ann. Henri Poincaré, **2** (2001), 101–108.

[8] D. Damanik: *Substitution Hamiltonians with bounded trace map orbits*, J. Math. Anal. Appl. **249** (2000), 393–411.

[9] D. Damanik: *Singular continuous spectrum for a class of substitution Hamiltonians*, Lett. Math. Phys. **46** (1998), 303–311.

[10] D. Damanik, R. Killip and D. Lenz: *Uniform spectral properties of one-dimensional quasicrystals, III. α-continuity*, Comm. Math. Phys. **212** (2000), 191–204.
[11] D. Damanik and D. Lenz: *Uniform spectral properties of one-dimensional quasicrystals, I. Absence of eigenvalues*, Comm. Math. Phys. **207** (1999), 687–696.
[12] D. Damanik and D. Lenz: *Uniform spectral properties of one-dimensional quasicrystals, II. The Lyapunov exponent*, Lett. Math. Phys. **50** (1999), 245–257.
[13] D. Damanik, A. Sütő and S. Tcheremchantsev: *Power-law bounds on transfer matrices and quantum dynamics in one dimension. II*, J. Funct. Anal. **216** (2004), 362–387.
[14] F. Delyon and D. Petritis: *Absence of localization in a class of Schrödinger operators with quasiperiodic potential*, Comm. Math. Phys. **103** (1986), 441–444.
[15] C.R. de Oliveira and R.A. Prado: *Spectral and localization properties for the one-dimensional Bernoulli discrete Dirac operator*, J. Math. Phys. **46** (2005), 072105, 17pp.
[16] B. Iochum, L. Raymond and D. Testard: *Resistance of one-dimensional quasicrystals*, Phy A, **187** (1992), 353–368.
[17] S. Jitomirskaya and Y. Last: *Power-law subordinacy and singular spectra, II. Line operators*, Comm. Math. Phys. **211** (2000), 643–658.
[18] A.Ya. Khinchin: Continued Fractions, Dover Publications, Mineola, 1997.
[19] S. Kotani: *Jacobi matrices with random potentials taking finitely many values*, Rev. Math. Phys. **1** (1989), 129–133.
[20] Y. Last: *Quantum dynamics and decomposition of singular continuous spectra*, J. Funct. Anal. **142** (1996), 406–445.
[21] V. Oseledec: *A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197–231.
[22] R.A. Prado and C.R. de Oliveira: *Dynamical lower bounds for 1D Dirac operators*, Math. Z. **259** (2008), 45–60.
[23] R.A. Prado and C.R. de Oliveira: *Sparse 1D discrete Dirac operators I: quantum transport*, J. Math. Anal. Appl. **385** (2012), 947–960.
[24] M. Reed and B. Simon: Methods of Modern Mathematical Physics, Vol. I, Functional Analysis, Academic Press, New York, 1972.
[25] M. Reed and B. Simon: Methods of Modern Mathematical Physics, Vol. IV, Analysis of Operators, Academic Press, New York, 1977.
[26] C.A. Rogers: Hausdorff Measures, Cambridge Univ. Press, London, 1970.
[27] C.A. Rogers and S.J. Taylor: *The analysis of additive set functions in Euclidean space*, Acta Math. **101** (1959), 273–302.
[28] A. Sütő: *The spectrum of a quasiperiodic Schrödinger operator*, Comm. Math. Phys. **111** (1987), 409–415.

Roberto A. Prado
Department of Mathematics and Computer Science
Paulista State University, UNESP
19060-900, Presidente Prudente, SP
Brazil.
e-mail: robertoprado@fct.unesp.br

Ruy C. Charão
Department of Mathematics
Federal University of Santa Catarina, UFSC
88040-900, Florianópolis, SC
Brazil.
e-mail: ruy.charao@ufsc.br
ruycharao@gmail.com (corresponding author)