Chen, Q.-G. and Yang, B.
Osaka J. Math.
56 (2019), 301–322

CLEFT COEXTENSION FOR SYMMETRIC TWISTED PARTIAL COACTIONS ON COALGEBRAS

Q.-G. CHEN and B. YANG

(Received July 10, 2017, revised November 29, 2017)

Abstract

In this paper, we will introduce the concepts of symmetric twisted partial Hopf coactions, and discuss under which conditions a given symmetric twisted partial Hopf coaction is globalizable. Then we will introduce the notion of partial cleft coextensions which are dual to partial cleft extensions introduced by M. M. S. Alves et.al., and discuss its relation with partial crossed coproducts introduced by the first author of this paper, which covers the classical results in classical Hopf algebra theory.

1. Introduction

Hopf-Galois extension was introduced by Kreimer and Takeuchi [13], which has its origin in the approach to Galois theory of groups acting on commutative rings developed by Chase, Harrison and Rosenberg and in the extension of this theory to coactions of a Hopf algebra acting on a commutative algebra developed by Chase and Sweedler [6]. Cleft extension of algebra by a Hopf algebra, or cleft Hopf comodule algebras, are one of the simplest and best known examples of Hopf-Galois extensions. Indeed, by [11] a Hopf-Galois extension with the normal basis property is necessary a cleft extension. In the standard Hopf algebra theory, cleft extension of Hopf algebras are examples of crossed products with Hopf algebras: indeed a cleft extension is the same as a crossed product with an invertible cocycle [11]. As a dual version of a cleft extension, a cleft coextension associated with a Hopf algebra was introduced by Masuoka and Doi [14], and it is showed in [10] that cleft coextension of Hopf algebras are exactly crossed coproduct.

In order to generalize the cleft extension of Hopf algebras to the case of partial actions of Hopf algebras, the notion of a twisted partial Hopf action and the associated partial crossed products was introduced by Alves et al. [4]. They also introduced the related notion of partial cleft extension and worked out its relation with partial crossed products. In this context, they proved that partial cleft extensions over the coinvariants A was exactly the crossed products by so-called symmetric twisted partial Hopf actions on A. Dually, the first author of this paper with the another coauthor introduced the notion of twisted partial Hopf coactions and constructed the partial crossed coproducts [9].

The aim of the present paper is to extend the theory of cleft coextensions to the case of partial coactions. To do it, we will need to introduce the concepts of symmetric twisted partial Hopf coactions, it is a interesting problem to discover under which conditions a given symmetric twisted partial Hopf coaction is globalizable.

2010 Mathematics Subject Classification. 16T05.

The paper is organized as follows.

In Section 2, we recall some basic concepts related to twisted partial Hopf coactions. We shall introduce the symmetric twisted partial coactions which are dual to [4], and discuss some properties in Section 3, and give the conditions to make a given twisted partial Hopf coaction be globalizable in Section 4. As the key content, we will introduce the notion of partial cleft coextensions and discuss its relation with partial crossed coproducts, and show that a crossed coproduct with some additional assumptions is a partial cleft coextension, and vice versa in Section 5.

2. Twisted Partial coactions and Partial Crossed Coproducts

Throughout k is a fixed field. All coalgebras, algebras, vector spaces and unadorned $\otimes$, Hom, etc. are over k. Given a vector space V, $id(or\ \iota) : V \to V$ means the identity map.

Now, we shall recall from [9] that the twisted partial coaction and the partial crossed coproduct. More knowledge about partial (co)actions of Hopf algebras can found in ([4]-[9]).

Definition 2.1. Let H be a Hopf algebra, A a coalgebra. Let $\rho_A : A \to H \otimes A$ and $\omega : A \to H \otimes H$ be two linear maps. We will write $\rho_A(a) = a_{[-1]} \otimes a_{[0]}$ and $\omega(a) = \omega(a)^1 \otimes \omega(a)^2$, for any $a \in A$. The pair (ρ, ω) is called a twisted partial coaction of H on A, if the following conditions holds:

$$\varepsilon_H(a_{[-1]})a_{[0]} = a, \tag{2.1}$$

$$a_{(1)[-1]}a_{(2)[-1]} \otimes a_{(1)[0]} \otimes a_{(2)[0]} = a_{[-1]} \otimes a_{[0](1)} \otimes a_{[0](2)}, \tag{2.2}$$

$$\begin{aligned} &a_{(1)[-1]}\omega(a_{(2)})^1 \otimes a_{(1)[0][-1]}\omega(a_{(2)})^2 \otimes a_{(1)[0][0]} \\ &\quad = \omega(a_{(1)})^1 a_{(2)[-1](1)} \otimes \omega(a_{(1)})^2 a_{(2)[-1](2)} \otimes a_{(2)[0]}, \end{aligned} \tag{2.3}$$

$$\omega(a_{(1)})^1 a_{(2)[-1](1)} \otimes \omega(a_{(1)})^2 a_{(2)[-1](2)}\varepsilon_A(a_{(2)[0]}) = \omega(a)^1 \otimes \omega(a)^2. \tag{2.4}$$

If H, A and (ρ_A, ω) satisfy Definition 2.1, we call (A, ρ_A, ω) a *twisted partial H-comodule coalgebra*.

Given a twisted partial coaction (ρ_A, ω), we have the following useful equations:

$$\begin{aligned} \omega(a)^1 \otimes \omega(a)^2 &= a_{(1)[-1]}\omega(a_{(2)})^1 \otimes a_{(1)[0][-1]}\omega(a_{(2)})^2\varepsilon_A(a_{(1)[0][0]}) \\ &= \varepsilon_A(a_{(1)[0]})a_{(1)[-1]}\omega(a_{(2)})^1 \otimes \omega(a_{(2)})^2. \end{aligned} \tag{2.5}$$

Given any two linear maps $\rho_A : A \to H \otimes A$ and $\omega : A \to H \otimes H$, we can define on the vector space $A \otimes H$ a coproduct given by the comultiplication

$$\Delta(a \otimes h) = a_{(1)} \otimes a_{(2)[-1]}\omega(a_{(3)})^1 h_{(1)} \otimes a_{(2)[0]} \otimes \omega(a_{(3)})^2 h_{(2)},$$

for all $a \in A$ and $h \in H$. Let $A\natural_{(\rho_A,\omega)}H$ be the subspace of $A \otimes H$ generated by the elements of the form $a\natural h = a_{(1)} \otimes a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})h$ for all $a \in A$ and $h \in H$.

The following proposition gives the necessary and sufficient conditions for $A \otimes H$ (so also $A\natural_{(\rho_A,\omega)}H$) to be coassociative and counital with the counit $\varepsilon_A \otimes \varepsilon_H$ in [9].

Proposition 2.2. *Let H a Hopf algebra, A a coalgebra, $\rho_A : A \to H \otimes A$ and $\omega : A \to H \otimes H$ two linear maps satisfying the conditions* (2.1), (2.2) *and* (2.4).

(i) *$\varepsilon = \varepsilon_A \otimes \varepsilon_H$ is the counit of $A\natural_{(\rho_A,\omega)}H$ if and only if, for all $a \in A$,*

$$\omega(a)^1\varepsilon_H(\omega(a)^2) = a_{[-1]}\varepsilon_A(a_{[0]}) = \varepsilon_H(\omega(a)^1)\omega(a)^2. \tag{2.6}$$

(ii) *Suppose that $\omega(a)^1\varepsilon_H(\omega(a)^2) = a_{[-1]}\varepsilon_A(a_{[0]})$ for all $a \in A$. The comultiplication Δ on $A \otimes H$ is coassociative if and only if the condition* (2.3) *holds and, for all $a \in A$,*

$$\begin{aligned} &a_{(1)[-1]}\omega(a_{(2)})^1 \otimes \omega(a_{(1)[0]})^1\omega(a_{(2)})^2_{(1)} \otimes \omega(a_{(1)[0]})^2\omega(a_{(2)})^2_{(2)} \\ &\quad = \omega(a_{(1)})^1\omega(a_{(2)})^1_{(1)} \otimes \omega(a_{(1)})^2\omega(a_{(2)})^1_{(2)} \otimes \omega(a_{(2)})^2. \end{aligned} \tag{2.7}$$

Given a twisted partial coaction (ρ_A, ω) of a Hopf algebra H on a coalgebra A, the coalgebra $A\natural_{(\rho_A,\omega)}H$ is called a crossed coproduct by a twisted partial coaction (in brief, a *partial crossed coproduct*), if the additional conditions (2.6) and (2.7) hold.

3. Symmetric Twisted Partial Coactions

In this section, we always assume that there exists a twisted partial coaction (ρ_A, ω) of a Hopf algebra H on a coalgebra A such that the map $\Lambda \in \mathrm{Hom}(A, H)$ given by $\Lambda(a) = a_{[-1]}\varepsilon_A(a_{[0]})$, is central with respect to the convolution product.

Now, we shall introduce the symmetric twisted partial coactions which are dual to [4, Definition 2]. Define two linear maps

$$f_1 : A \to H \otimes H, a \mapsto a_{[-1]}\varepsilon_A(a_{[0]}) \otimes 1_H,$$
$$f_2 : A \to H \otimes H, a \mapsto a_{[-1](1)} \otimes a_{[-1](2)}\varepsilon_A(a_{[0]}).$$

DEFINITION 3.1. Let (ρ_A, ω) is a twisted partial coaction. We will say that the twisted partial coaction is symmetric, if

(1) f_1 and f_2 are central in $\mathrm{Hom}(A, H \otimes H)$,
(2) ω satisfies conditions (2.6)and (2.7), and has a convolution inverse ω' in $\langle f_1 * f_2\rangle$,
(3) For all $a \in A$,

$$a_{[-1]} \otimes a_{[0][-1]}\varepsilon_A(a_{[0][0]}) = a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})a_{(2)[-1](1)} \otimes a_{(2)[-1](2)}\varepsilon_A(a_{(2)[0]}). \tag{3.1}$$

We remark more that to say that $\omega' : A \to H \otimes H$ lies in $\langle f_1 * f_2\rangle$ is equivalent to requiring the equalities

$$\begin{aligned} \omega'(a) &= \omega'(a_{(1)})^1 a_{(2)[-1](1)} \otimes \omega'(a_{(1)})^2 a_{(2)[-1](2)}\varepsilon_A(a_{(2)[0]}) \\ &= \omega'(a_{(1)})^1 a_{(2)[-1]}\varepsilon_A(a_{(2)[0]}) \otimes \omega'(a_{(1)})^2, \end{aligned} \tag{3.2}$$

and ω' is the inverse of ω in $\langle f_1 * f_2\rangle$ if and only if

$$(\omega' * \omega)(a) = (\omega * \omega')(a) = a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})a_{(2)[-1](1)} \otimes a_{(2)[-1](2)}\varepsilon_A(a_{(2)[0]}). \tag{3.3}$$

It follows easily from (3.2) and (3.3) that

$$\omega'(a)^1\varepsilon_H(\omega'(a)^2) = a_{[-1]}\varepsilon_A(a_{[0]}) = \varepsilon_H(\omega'(a)^1)\omega(a)'^2 \tag{3.4}$$

for all $a \in A$. Multiplying equality (2.3) on the right by ω' and using (3.1) of Definition 3.1, we obtain

$$(3.5)\qquad a_{[-1]}\otimes a_{[0][-1]}\otimes a_{[0][0]} = \omega(a_{(1)})^1 a_{(2)[-1](1)}\omega'(a_{(3)})^1\otimes\omega(a_{(1)})^2 a_{(2)[-1](2)}\omega'(a_{(3)})^2\otimes a_{(2)[0]}.$$

for all $a\in A$. Furthermore, multiplying (3.5) by ω' on the left and using the centrality of Λ, we have

$$\begin{aligned}
&\omega'(a_{(1)})^1\underline{a_{(2)[-1]}}\otimes\omega'(a_{(1)})^2\underline{a_{(2)[0][-1]}}\otimes\underline{a_{(2)[0][0]}}\\
(3.5)\ =&\underline{\omega'(a_{(1)})^1\omega(a_{(2)})^1}a_{(3)[-1](1)}\omega'(a_4)^1\\
&\otimes\underline{\omega'(a_{(1)})^2\omega(a_{(2)})^2}a_{(3)[-1](2)}\omega'(a_4)^2\otimes a_{(3)[0]}\\
(3.3)\ =&a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})\underline{a_{(2)[-1](1)}a_{(3)[-1](1)}}\omega'(a_4)^1\\
&\otimes\underline{a_{(2)[-1](2)}}\varepsilon_A(\underline{a_{(2)[0]}})\underline{a_{(3)[-1](2)}}\omega'(a_4)^2\otimes\underline{a_{(3)[0]}}\\
(2.2)\ =&a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})a_{(2)[-1](1)}\omega'(a_3)^1\otimes a_{(2)[-1](2)}\omega'(a_3)^2\otimes a_{(2)[0]}\\
=&a_{(1)[-1](1)}\omega'(a_2)^1a_{(3)[-1]}\varepsilon_A(a_{(3)[0]})\otimes a_{(1)[-1](2)}\omega'(a_2)^2\otimes a_{(1)[0]}\\
=&a_{(1)[-1](1)}\omega'(a_2)^1\otimes a_{(1)[-1](2)}\omega'(a_2)^2\otimes a_{(1)[0]}.
\end{aligned}$$

Therefore, ω' satisfies

$$(3.6)\qquad \omega'(a_{(1)})^1 a_{(2)[-1]}\otimes\omega'(a_{(1)})^2 a_{(2)[0][-1]}\otimes a_{(2)[0][0]} = a_{(1)[-1](1)}\omega'(a_2)^1\otimes a_{(1)[-1](2)}\omega'(a_2)^2\otimes a_{(1)[0]}.$$

In order to proceed the further discussion, we need to recall an interesting result about semigroups from [4, Lemma2] that, let $\mathfrak{S}$ be a semigroup and let ν, e, e' be elements of $\mathfrak{S}$. If there is an element $\nu'\in\mathfrak{S}$ such that

$$\nu\nu' = e,\ \nu'\nu = e',\ \nu'e = \nu',$$

then ν' satisfying the above relations is unique.

Lemma 3.2. *Let $(A,\rho_A,(\omega,\omega'))$ be a symmetric twisted partial H-comodule coalgebra. Then*

$$(3.7)\qquad a_{[-1]}\otimes\omega(a_{[0]})^1\otimes\omega(a_{[0]})^2 = \omega(a_{(1)})^1\omega(a_{(2)})^1_{(1)}\omega'(a_{(3)})^1\otimes\omega(a_{(1)})^2\omega(a_{(2)})^1_{(2)}\omega'(a_{(3)})^2_{(1)}\otimes\omega(a_{(2)})^2\omega'(a_{(3)})^2_{(2)},$$

$$(3.8)\qquad a_{[-1]}\otimes\omega'(a_{[0]})^1\otimes\omega'(a_{[0]})^2 = \omega(a_{(1)})^1\omega'(a_{(2)})^1_{(1)}\omega'(a_{(3)})^1\otimes\omega(a_{(1)})^2_{(1)}\omega'(a_{(2)})^1_{(2)}\omega'(a_{(3)})^2\otimes\omega(a_{(1)})^2_{(2)}\omega'(a_{(2)})^2,$$

for all $a\in A$.

Proof. Multiplying (2.7) by ω' on the right yields

$$a_{(1)[-1]}\omega(a_{(2)})^1\omega'(a_{(3)})^1\otimes\omega(a_{(1)[0]})^1\omega(a_{(2)})^2_{(1)}\omega'(a_{(3)})^2_{(1)}\otimes\omega(a_{(1)[0]})^2\omega(a_{(2)})^2_{(2)}\omega'(a_{(3)})^2_{(2)} = \omega(a_{(1)})^1\omega(a_{(2)})^1_{(1)}\omega'(a_{(3)})^1\otimes\omega(a_{(1)})^2\omega(a_{(2)})^1_{(2)}\omega'(a_{(3)})^2_{(1)}\otimes\omega(a_{(2)})^2\omega'(a_{(3)})^2_{(2)}.$$

Since the left-hand side equals

$$a_{(1)[-1]}\underline{\omega(a_{(2)})^1\omega'(a_{(3)})^1}\otimes\omega(a_{(1)[0]})^1\underline{\omega(a_{(2)})^2_{(1)}\omega'(a_{(3)})^2_{(1)}}\otimes\omega(a_{(1)[0]})^2\underline{\omega(a_{(2)})^2_{(2)}\omega'(a_{(3)})^2_{(2)}}$$

$$(3.3) = a_{(1)[-1]}a_{(2)[-1]} \otimes \omega(a_{(1)[0]})^1 a_{(2)[0][-1](1)}\varepsilon_A(a_{(2)[0][0]}) \otimes \omega(a_{(1)[0]})^2 a_{(2)[0][-1](2)}$$

$$(2.2) = a_{[-1]} \otimes \underline{\omega(a_{[0](1)})^1 a_{[0](2)[-1](1)}\varepsilon_A(a_{[0](2)[0]})} \otimes \underline{\omega(a_{[0](1)})^2 a_{[0](2)[-1](2)}}$$

$$(2.4) = a_{[-1]} \otimes \omega(a_{[0]})^1 \otimes \omega(a_{[0]})^2.$$

Thus (3.7) follows. The proof of (3.8) is a bit more involved. Consider $\mathrm{Hom}(A, H^{\otimes^3})$ as a multiplicative semigroup, and let

$$\nu(a) = a_{[-1]} \otimes \omega(a_{[0]})^1 \otimes \omega(a_{[0]})^2$$
$$\varsigma(a) = a_{[-1]} \otimes a_{[0][-1]} \otimes a_{[0][0][-1]}\varepsilon_A(a_{[0][0][0]}).$$

We shall show that

$$\kappa(a) = a_{[-1]} \otimes \omega'(a_{[0]})^1 \otimes \omega'(a_{[0]})^2$$
$$\kappa'(a) = \omega(a_{(1)})^1\omega'(a_{(2)})^1_{(1)}\omega'(a_{(3)})^1 \otimes \omega(a_{(1)})^2_{(1)}\omega'(a_{(2)})^1_{(2)}\omega'(a_{(3)})^2 \otimes \omega(a_{(1)})^2_{(2)}\omega'(a_{(2)})^2.$$

satisfy $\nu * \kappa = \varsigma, \kappa * \nu = \varsigma, \kappa * \varsigma = \kappa$ and $\nu * \kappa' = \varsigma, \kappa' * \nu = \varsigma, \kappa' * \varsigma = \kappa'$. It is easily checked that the first group of relations hold. Here we will prove the second group. We calculate $\nu * \kappa'$:

$$\begin{aligned}
&(\nu*\kappa')(a)\\
&= \underline{a_{(1)[-1]}\omega(a_{(2)})^1}\omega'(a_{(3)})^1_{(1)}\omega'(a_{(4)})^1\\
&\quad \otimes \underline{\omega(a_{(1)[0]})^1\omega(a_{(2)})^2_{(1)}}\omega'(a_{(3)})^1_{(2)}\omega'(a_{(4)})^2 \otimes \underline{\omega(a_{(1)[0]})^2\omega(a_{(2)})^2_{(2)}}\omega'(a_{(3)})^2\\
(2.7) &= \omega(a_{(1)})^1\underline{\omega(a_{(2)})^1_{(1)}\omega'(a_{(3)})^1_{(1)}}\omega'(a_{(4)})^1\\
&\quad \otimes \omega(a_{(1)})^2\underline{\omega(a_{(2)})^1_{(2)}\omega'(a_{(3)})^1_{(2)}}\omega'(a_{(4)})^2 \otimes \underline{\omega(a_{(2)})^2\omega'(a_{(3)})^2}\\
(3.3) &= \omega(a_{(1)})^1 a_{(2)[-1](1)}\omega'(a_{(3)})^1 \otimes \omega(a_{(1)})^2 a_{(2)[-1](2)}\omega'(a_{(3)})^2 \otimes a_{(2)[0][-1]}\varepsilon_A(a_{(2)[0][0]})\\
&= a_{(1)[-1]}\underline{\omega(a_{(2)})^1\omega'(a_{(3)})}^1 \otimes a_{(1)[0][-1]}\underline{\omega(a_{(2)})^2\omega'(a_{(3)})}^2 \otimes a_{(1)[0][0][-1]}\varepsilon_A(a_{(1)[0][0][0]})\\
(3.3) &= a_{(1)[-1]}a_{(2)[-1]} \otimes a_{(1)[0][-1]}a_{(2)[0][-1]}\varepsilon_A(a_{(2)[0][0]}) \otimes a_{(1)[0][0][-1]}\varepsilon_A(a_{(1)[0][0][0]})\\
(2.2) &= a_{[-1]} \otimes a_{[0][-1]} \otimes a_{[0][0][-1]}\varepsilon_A(a_{[0][0][0]})\\
&= \varsigma(a).
\end{aligned}$$

As for $\kappa' * \nu$, we have

$$\begin{aligned}
(\kappa' * \nu)(a) &= \omega(a_{(1)})^1\omega'(a_{(2)})^1_{(1)}\omega'(a_{(3)})^1\underline{a_{(4)[-1]}}\\
&\quad \otimes \omega(a_{(1)})^2_{(1)}\omega'(a_{(2)})^1_{(2)}\omega'(a_{(3)})^2\underline{\omega(a_{(4)[0]})^1} \otimes \omega(a_{(1)})^2_{(2)}\omega'(a_{(2)})^2\underline{\omega(a_{(4)[0]})^1}\\
(3.7) &= \omega(a_{(1)})^1\omega'(a_{(2)})^1_{(1)}\underline{\omega'(a_{(3)})^1\omega(a_{(4)})^1}\omega(a_{(5)})^1_{(1)}\omega'(a_{(6)})^1\\
&\quad \otimes \omega(a_{(1)})^2_{(1)}\omega'(a_{(2)})^1_{(2)}\underline{\omega'(a_{(3)})^2\omega(a_{(4)})^2}\omega(a_{(5)})^1_{(2)}\omega'(a_{(6)})^2_{(1)}\\
&\quad \otimes \omega(a_{(1)})^2_{(2)}\omega'(a_{(2)})^2\omega(a_{(5)})^2\omega'(a_{(6)})^2_{(2)}\\
(3.3) &= \omega(a_{(1)})^1\omega'(a_{(2)})^1_{(1)}a_{(3)[-1]}\varepsilon_A(a_{(3)[0]})a_{(4)[-1](1)}\omega(a_{(5)})^1_{(1)}\omega'(a_{(6)})^1\\
&\quad \otimes \omega(a_{(1)})^2_{(1)}\omega'(a_{(2)})^1_{(2)}a_{(4)[-1](2)}\omega(a_{(5)})^1_{(2)}\omega'(a_{(6)})^2_{(1)}\\
&\quad \otimes \omega(a_{(1)})^2_{(2)}\omega'(a_{(2)})^2\omega(a_{(5)})^2\omega'(a_{(6)})^2_{(2)}\varepsilon_A(a_{(4)[0]})\\
&= \omega(a_{(1)})^1 a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})\omega'(a_{(3)})^1_{(1)}a_{(4)[-1](1)}\omega(a_{(5)})^1_{(1)}\omega'(a_{(6)})^1
\end{aligned}$$

$$
\begin{aligned}
&\otimes \omega(a_{(1)})^2_{(1)}\omega'(a_{(3)})^1_{(2)}a_{(4)[-1](2)}\omega(a_{(5)})^1_{(2)}\omega'(a_{(6)})^2_{(1)}\\
&\otimes \omega(a_{(1)})^2_{(2)}\omega'(a_{(3)})^2\omega(a_{(5)})^2\omega'(a_{(6)})^2_{(2)}\varepsilon_A(a_{(4)[0]})\\
(3.5)\ =&\omega(a_{(1)})^1\omega'(a_{(2)})^1_{(1)}\underline{a_{(3)[-1](1)}\omega(a_{(4)})^1_{(1)}}\omega'(a_{(5)})^1\\
&\otimes \omega(a_{(1)})^2_{(1)}\omega'(a_{(2)})^1_{(2)}\underline{a_{(3)[-1](2)}\varepsilon_A(a_{(3)[0]})\omega(a_{(4)})^1_{(2)}}\omega'(a_{(5)})^2_{(1)}\\
&\otimes \omega(a_{(1)})^2_{(2)}\omega'(a_{(2)})^2\underline{\omega(a_{(4)})^2}\omega'(a_{(5)})^2_{(2)}\\
(2.5)\ =&\omega(a_{(1)})^1\underline{\omega'(a_{(2)})^1_{(1)}\omega(a_{(3)})^1_{(1)}}\omega'(a_{(4)})^1\\
&\otimes \omega(a_{(1)})^2_{(1)}\underline{\omega'(a_{(2)})^1_{(2)}\omega(a_{(3)})^1_{(2)}}\omega'(a_{(4)})^2_{(1)}\otimes \omega(a_{(1)})^2_{(2)}\underline{\omega'(a_{(2)})^2\omega(a_{(3)})^2}\omega'(a_{(4)})^2_{(2)}\\
(3.3)\ =&\omega(a_{(1)})^1a_{(2)[-1](1)}a_{(3)[-1](1)}\omega'(a_{(4)})^1\otimes \omega(a_{(1)})^2_{(1)}a_{(2)[-1](2)}a_{(3)[-1](2)}\omega'(a_{(4)})^2_{(1)}\\
&\otimes \omega(a_{(1)})^2_{(2)}\varepsilon_A(a_{(2)[0]})a_{(3)[-1](3)}\varepsilon_A(a_{(3)[0]})\omega'(a_{(4)})^2_{(2)}\\
(2.2)\ =&\omega(a_{(1)})^1a_{(2)[-1](1)}\omega'(a_{(3)})^1\\
&\otimes \omega(a_{(1)})^2_{(1)}a_{(2)[-1](2)}\omega'(a_{(3)})^2_{(1)}\otimes \omega(a_{(1)})^2_{(2)}\varepsilon_A(a_{(2)[0]})\omega'(a_{(3)})^2_{(2)}\\
=&a_{(1)[-1](1)}\underline{\omega(a_{(2)})^1\omega'(a_{(3)})^1}\\
&\otimes a_{(1)[-1](2)}\underline{\omega(a_{(2)})^2_{(1)}\omega'(a_{(3)})^2_{(1)}}\otimes \varepsilon_A(a_{(1)[0]})\underline{\omega(a_{(2)})^2_{(2)}\omega'(a_{(3)})^2_{(2)}}\\
(3.3)\ =&a_{(1)[-1](1)}a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})a_{(3)[-1](1)}\\
&\otimes a_{(1)[-1](2)}a_{(3)[-1](2)}\otimes \varepsilon_A(a_{(1)[0]})a_{(3)[-1](3)}\varepsilon_A(a_{(3)[0]})\\
(2.2)\ =&a_{(1)[-1](1)}a_{(2)[-1]}\otimes a_{(1)[-1](2)}a_{(2)[0][-1](1)}\otimes \varepsilon_A(a_{(1)[0]})a_{(2)[0][-1](2)}\varepsilon_A(a_{(2)[0][0]})\\
=&a_{(1)[-1](1)}a_{(2)[-1]}\varepsilon_A(a_{(2)[0](1)})\\
&\otimes a_{(1)[-1](2)}\varepsilon_A(a_{(1)[0]})a_{(2)[0](2)[-1](1)}\otimes a_{(2)[0](2)[-1](2)}\varepsilon_A(a_{(2)[0](2)[0]})\\
(2.2)\ =&a_{(1)[-1](1)}a_{(2)[-1]}a_{(3)[-1]}\varepsilon_A(a_{(2)[0]})\\
&\otimes a_{(1)[-1](2)}\varepsilon_A(a_{(1)[0]})a_{(3)[0][-1](1)}\otimes a_{(3)[0][-1](2)}\varepsilon_A(a_{(3)[0][0]})\\
=&a_{(1)[-1]}a_{(2)[-1](1)}\varepsilon_A(a_{(1)[0]})a_{(3)[-1]}\\
&\otimes a_{(2)[-1](2)}\varepsilon_A(a_{(2)[0]})a_{(3)[0][-1](1)}\otimes a_{(3)[0][-1](2)}\varepsilon_A(a_{(3)[0][0]})\\
=&a_{(1)[-1]}a_{(2)[-1]}\otimes a_{(1)[0][-1]}a_{(2)[0][-1](1)}\otimes a_{(2)[0][-1](2)}\varepsilon_A(a_{(1)[0][0]})\varepsilon_A(a_{(2)[0][0]})\\
(2.2)\ =&a_{[-1]}\otimes a_{[0](1)[-1]}a_{[0](2)[-1](1)}\otimes a_{[0](2)[-1](2)}\varepsilon_A(a_{[0](1)[0]})\varepsilon_A(a_{[0](2)[0]})\\
=&a_{[-1]}\otimes a_{[0][-1]}\otimes a_{[0][0][-1]}\varepsilon_A(a_{[0][0][0]}) = \varsigma(a),
\end{aligned}
$$

as desired. It is a good exercise for readers to check the relation $\kappa' * \varsigma = \kappa'$.

This ends the proof. □

EXAMPLE 3.3. Let $H = k < 1_H, g >$ be the group Hopf algebra with $g^2 = 1_H$ and $\Delta(g) = g \otimes g$, $S_H(g) = g = g^{-1}$. Now, we can define the partial coaction of H on the field k via $\rho_k : k \to H\otimes k, 1 \mapsto \frac{1}{2}(1_H+g)\otimes 1$, and $\omega : k \to H\otimes H, 1 \mapsto \frac{1}{4}(1_H\otimes 1_H+1_H\otimes g+g\otimes 1_H+g\otimes g)$. Notice easily that $f_1 : k \to H\otimes H, 1 \mapsto \frac{1}{2}(1_H+g)\otimes 1_H$ and $f_2 : k \to H\otimes H, 1 \mapsto \frac{1}{2}1_H\otimes 1_H+\frac{1}{2}g\otimes g$ are central in $\mathrm{Hom}(k, H\otimes H)$. The convolution inverse ω' in $< f_1 * f_2 >$ is ω. Thus $(k, \rho_k, (\omega, \omega'))$ be a symmetric twisted partial H-comodule coalgebra.

4. Globalization for twisted partial comodule coalgebras

Our main goal now is to introduce the concept of globalization for twisted partial comodule coaglebras. We need previously introduce the notion of induced twisted partial coaction. Thus, we start discussing about the conditions for the existence of an induced twisted partial coaction.

Let D be a (left) twisted H-comodule coalgebra via $\rho_D : d \to d_{[-1]} \otimes d_{[0]} \in H \otimes D$, a convolution invertible map $\Omega : D \to H \otimes H$ and C a subcoalgebra of D. In order to induce a coaction of H on C, we can restrict the coaction of D to C, but, in general, we do not have that $\rho_D(C) \subseteq H \otimes C$. Thus, we need to take a linear map $P : D \to C$ and then consider the following composite map

$$\rho_C : C \to H \otimes C,\ \rho_C(c) = c_{[\underline{-1}]} \otimes c_{[\underline{0}]} = c_{[-1]} \otimes P(c_{[0]})$$

and the other two linear maps $\varpi : C \to H \otimes H$, for $c \in C$,

$$\begin{aligned}\varpi(c)^1 \otimes \varpi(c)^2 &= c_{(1)[\underline{-1}]}\Omega(c_{(2)})^1 \otimes c_{(1)[\underline{0}][\underline{-1}]}\varepsilon_C(c_{(1)[\underline{0}][\underline{0}]})\Omega(c_{(2)})^2 \\ &= c_{(1)[\underline{-1}]}\Omega(c_{(2)})^1 c_{(3)[\underline{-1}](1)} \otimes \Omega(c_{(2)})^2 c_{(3)[\underline{-1}](2)}\varepsilon_C(c_{(1)[\underline{0}]})\varepsilon_C(c_{(3)[\underline{0}]}),\end{aligned}$$

and

$$\begin{aligned}\varpi'(c)^1 \otimes \varpi'(c)^2 &= \Omega^{-1}(c_{(1)})^1 c_{(2)[\underline{-1}]} \otimes \Omega^{-1}(c_{(1)})^2 c_{(2)[\underline{0}][\underline{-1}]}\varepsilon_C(c_{(2)[\underline{0}][\underline{0}]}) \\ &= c_{(1)[\underline{-1}](1)}\Omega(c_{(2)})^1 c_{(3)[\underline{-1}]} \otimes c_{(1)[\underline{-1}](2)}\Omega(c_{(2)})^2\varepsilon_C(c_{(1)[\underline{0}]})\varepsilon_C(c_{(3)[\underline{0}]}).\end{aligned}$$

Now we need to find sufficient conditions on P in order to (C, ϖ, ϖ') be a (left) symmetric partial H-comodule coalgebra structure via ρ_C. To do this, we study the conditions given in Definition 2.1.

First of all, observe that if P is a linear projection from D onto C, then the condition (2.1) holds. In fact, for $c \in C$, we have

$$\varepsilon_C(c_{[\underline{-1}]})c_{[\underline{0}]} = \varepsilon_C(c_{[-1]})P(c_{[0]}) = P(c) = c.$$

Supposing that P is a comultiplicative map (i.e. $\Delta \circ P = (P \otimes P) \circ \Delta$), then it follows that the condition (2.2) holds. In fact, given $c \in C$, we have

$$\begin{aligned}c_{[\underline{-1}]} \otimes c_{[\underline{0}](1)} \otimes c_{[\underline{0}](2)} &= c_{[-1]} \otimes P(c_{[0]})_{(1)} \otimes P(c_{[0]})_{(2)} \\ &= c_{[-1]} \otimes P(c_{[0](1)}) \otimes P(c_{[0](2)}) \\ &= c_{(1)[-1]}c_{(2)[-1]} \otimes P(c_{(1)[0]}) \otimes P(c_{(2)[0]}) \\ &= c_{(1)[\underline{-1}]}c_{(2)[\underline{-1}]} \otimes c_{(1)[\underline{0}]} \otimes c_{(2)[\underline{0}]}.\end{aligned}$$

Furthermore, we can check that the condition (2.4) holds. In fact,

$$\begin{aligned}&\varpi(c_{(1)})^1 c_{(2)[\underline{-1}](1)} \otimes \varpi(c_{(1)})^2 c_{(2)[\underline{-1}](2)}\varepsilon_C(c_{(2)[\underline{0}]}) \\ &= c_{(1)[\underline{-1}]}\Omega(c_{(2)})^1 c_{(3)[\underline{-1}](1)}c_{(4)[\underline{-1}](1)} \otimes \Omega(c_{(2)})^2 c_{(3)[\underline{-1}](2)}c_{(4)[\underline{-1}](2)}\varepsilon_C(c_{(4)[\underline{0}]})\varepsilon_C(c_{(1)[\underline{0}]})\varepsilon_C(c_{(3)[\underline{0}]}) \\ &= c_{(1)[\underline{-1}]}\Omega(c_{(2)})^1 c_{(3)[\underline{-1}](1)} \otimes \Omega(c_{(2)})^2 c_{(3)[\underline{-1}](2)}\varepsilon_C(c_{(1)[\underline{0}]})\varepsilon_C(c_{(3)[\underline{0}]}) \\ &= \varpi(c)^1 \otimes \varpi(c)^2.\end{aligned}$$

In order to ensure that the condition (2.3) is satisfied, we need to assume the following condition on P, for all $d \in D$,

$$P(d)_{[-1]} \otimes P(P(d)_{[0]}) = d_{(2)[-1]} \otimes \varepsilon_C(P(d_{(1)}))P(d_{(2)[0]}). \tag{4.1}$$

Under the assumption of the above condition (4.1), we can prove that the condition (2.3) holds. In fact, given $c \in C$, we have

$$\begin{aligned}
& c_{(1)[\underline{-1}]}\varpi(c_{(2)})^1 \otimes c_{(1)[\underline{0}][\underline{-1}]}\varpi(c_{(2)})^1 \otimes c_{(1)[\underline{0}][\underline{0}]} \\
&= c_{(1)[\underline{-1}]}c_{(2)[\underline{-1}]}\Omega(c_{(3)})^1 \otimes c_{(1)[\underline{0}][\underline{-1}]}c_{(2)[\underline{0}][\underline{-1}]}\varepsilon_C(c_{(2)[\underline{0}][\underline{0}]})\Omega(c_{(3)})^2 \otimes c_{(1)[\underline{0}][\underline{0}]} \\
&= c_{(1)[\underline{-1}]}\Omega(c_{(2)})^1 \otimes c_{(1)[\underline{0}](1)[\underline{-1}]}c_{(1)[\underline{0}](2)[\underline{-1}]}\varepsilon_C(c_{(1)[\underline{0}](2)[\underline{0}]})\Omega(c_{(2)})^2 \otimes c_{(1)[\underline{0}](1)[\underline{0}]} \\
&= c_{(1)[\underline{-1}]}\Omega(c_{(2)})^1 \otimes c_{(1)[\underline{0}][\underline{-1}]}\varepsilon_C(c_{(1)[\underline{0}][\underline{0}](2)})\Omega(c_{(2)})^2 \otimes c_{(1)[\underline{0}][\underline{0}](1)} \\
&= c_{(1)[\underline{-1}]}\Omega(c_{(2)})^1 \otimes c_{(1)[\underline{0}][\underline{-1}]}\Omega(c_{(2)})^2 \otimes c_{(1)[\underline{0}][\underline{0}]} \\
&= c_{(1)[-1]}\Omega(c_{(2)})^1 \otimes P(c_{(1)[0]})_{[-1]}\Omega(c_{(2)})^2 \otimes P(P(c_{(1)[0]})_{[0]}) \\
&= c_{(1)[-1]}\Omega(c_{(2)})^1 \otimes c_{(1)[0](2)[-1]}\Omega(c_{(2)})^2 \otimes P(c_{(1)[0](2)[0]})\varepsilon_C(P(c_{(1)[0](1)})) \\
&= c_{(1)[-1]}c_{(2)[-1]}\Omega(c_{(3)})^1 \otimes c_{(2)[0][-1]}\Omega(c_{(3)})^2 \otimes P(c_{(2)[0][0]})\varepsilon_C(P(c_{(1)[0]})) \\
&= c_{(1)[-1]}\Omega(c_{(2)})^1 c_{(3)[-1](1)} \otimes \Omega(c_{(2)})^2 c_{(3)[-1](2)} \otimes P(c_{(3)[0]})\varepsilon_C(P(c_{(1)[0]})) \\
&= c_{(1)[\underline{-1}]}\Omega(c_{(2)})^1 c_{(3)[\underline{-1}](1)} \otimes \Omega(c_{(2)})^2 c_{(3)[\underline{-1}](2)} \otimes c_{(3)[\underline{0}]}\varepsilon_C(c_{(1)[\underline{0}]}) \\
&= c_{(1)[\underline{-1}]}\Omega(c_{(2)})^1 c_{(3)[\underline{-1}](1)}c_{(4)[\underline{-1}](1)} \otimes \Omega(c_{(2)})^2 c_{(3)[\underline{-1}](2)}c_{(4)[\underline{-1}](2)}\varepsilon_C(c_{(1)[\underline{0}]})\varepsilon_C(c_{(3)[\underline{0}]}) \otimes c_{(4)[\underline{0}]} \\
&= \varpi(c_{(1)})^1 c_{(2)[\underline{-1}](1)} \otimes \varpi(c_{(1)})^2 c_{(2)[\underline{-1}](2)} \otimes c_{(2)[\underline{0}]}.
\end{aligned}$$

We are now able to define a globalization for a twisted partial comodule coalgebra as follows.

Definition 4.1. Let (A, ω, ω') and (A, ϖ, ϖ') be two symmetric twisted partial H-comodule coalgebras. $\Theta : A \to A'$ is an isomorphism of symmetric twisted partial H-comodule coalgebra, if

(1) Θ is a coalgebra isomorphism,
(2) $\Theta(a)_{[-1]} \otimes P(\Theta(a)_{[0]}) = a_{[-1]} \otimes \Theta(a_{[0]})$, for all $a \in A$,
(3) $\varpi \circ \Theta = \omega,\ \varpi' \circ \Theta = \omega'$.

Definition 4.2. Let A be a (left) symmetric twisted partial H-comodule coalgebra. Then a triple (D, Θ, P) is a globalization for A, where D is a twisted H-comodule coalgebra with invertible map $\Omega : D \to H \otimes H$ and Θ is a coalgebra monomorphism from A into D and P is a comultiplicative projection from D to $\Theta(A)$, if the following conditions hold:

(G1) $\Theta(A)$ is a subcoalgebra of D,
(G2) $P(d)_{[-1]} \otimes P(P(d)_{[0]}) = d_{(2)[-1]} \otimes \varepsilon(P(d_{(1)}))P(d_{(2)[0]})$, for all $d \in D$,
(G3) $\Theta : A \to \Theta(A)$ is an isomorphism of symmetric twisted partial partial H-comodule coalgebras, where $\Theta(A)$ has the structure induced by D as above.

The main theorem in this section will be given as follows.

Theorem 4.3. *Let A be a (left) symmetric twisted partial H-comodule coalgebra with the pair (ω, ω'). If there exists a convolution invertible linear map $\widetilde{\omega} : A \to H \otimes H$ satisfying*

$$\widetilde{\omega}(a)^1 \varepsilon_H(\widetilde{\omega}(a)^2) = \widetilde{\omega}(a)^2 \varepsilon_H(\widetilde{\omega}(a)^1) = \varepsilon_A(a)$$

and

$$\omega(a)^1 \otimes \omega(a)^2 = a_{(1)[-1]}\widetilde{\omega}(a_{(2)})^2 \otimes a_{(1)[0][-1]}\varepsilon_A(a_{(1)[0][0]})\widetilde{\omega}(a_{(2)})^1, \tag{4.2}$$

$$\omega'(a)^1 \otimes \omega'(a)^2 = a_{(1)[-1]}\widetilde{\omega}^{-1}(a_{(2)})^2 \otimes a_{(1)[0][-1]}\varepsilon_A(a_{(1)[0][0]})\widetilde{\omega}^{-1}(a_{(2)})^1. \tag{4.3}$$

Then A has a globalization.

Proof. Given a symmetric twisted partial H-comodule coalgebra (A, ω, ω'), we will prove that the triple $(A \otimes H^*, \Theta, P)$ is a globalization for A, where $\Theta : A \to A \otimes H^*, a \mapsto a \otimes 1_{H^*}$ ($A \otimes H^*$ with the coalgebra structure given by the tensor coalgebra) and

$$P : A \otimes H^* \to \Theta(A), a \otimes f \mapsto f(a_{[-1]})a_{[0]} \otimes 1_{H^*}.$$

Notice easily that Θ is injective and $\Theta(A)$ is a subcoalgebra of $A \otimes H^*$.

Let us prove that $A \otimes H^*$ is a twisted H-comodule coalgebra. First, The coaction of H on $A \otimes H^*$ is given by

$$\varrho : a \otimes f \mapsto \widetilde{\omega}(a_{(1)})^1 h_i \widetilde{\omega}^{-1}(a_{(3)})^1 \otimes a_{(2)} \otimes (\widetilde{\omega}^{-1}(a_{(3)})^2 \rightharpoonup f \leftharpoonup \widetilde{\omega}(a_{(1)})^2) * h_i^*,$$

where $\{h_i, h_i^*\}_{i=1}^n$ is a dual basis for H and H^*. The condition (2.1) can be checked easily. Now, we will check the condition (2.2). For all $a \in A, f \in H^*$, we have, on one hand,

$$\begin{aligned}&(a\otimes f)_{[-1]} \otimes (a \otimes f)_{[0](1)} \otimes (a \otimes f)_{[0](2)}\\ &=\widetilde{\omega}(a_{(1)})^1 h_i \widetilde{\omega}^{-1}(a_{(4)})^1 \otimes a_{(2)} \otimes (\widetilde{\omega}^{-1}(a_{(4)})^2 \rightharpoonup f \leftharpoonup \widetilde{\omega}(a_{(1)})^2)_{(1)} * h^*_{(1)}\\ &\quad \otimes a_{(3)} \otimes (\widetilde{\omega}^{-1}(a_{(4)})^2 \rightharpoonup f \leftharpoonup \widetilde{\omega}(a_{(1)})^2)_{(2)} * h^*_{(2)},\end{aligned}$$

on the other hand,

$$\begin{aligned}&(a_{(1)} \otimes f_{(1)})_{[-1]}(a_{(2)} \otimes f_{(2)})_{[-1]} \otimes (a_{(1)} \otimes f_{(1)})_{[0]} \otimes (a_{(2)} \otimes f_{(2)})_{[0]}\\ &= \widetilde{\omega}(a_{(1)})^1 h_i \widetilde{\omega}^{-1}(a_{(3)})^1 \widetilde{\omega}(a_{(4)})^1 \tilde{h}_i \widetilde{\omega}^{-1}(a_{(6)})^1\\ &\otimes a_{(2)} \otimes (\widetilde{\omega}^{-1}(a_{(3)})^2 \rightharpoonup f_{(1)} \leftharpoonup \widetilde{\omega}(a_{(1)})^2) * h_i^* \otimes a_{(5)} \otimes (\widetilde{\omega}^{-1}(a_{(6)})^2 \rightharpoonup f_{(2)} \leftharpoonup \widetilde{\omega}(a_{(4)})^2 * \tilde{h}_i^*.\end{aligned}$$

For all $l, l' \in H$, we have

$$\begin{aligned}&\widetilde{\omega}(a_{(1)})^1 h_i \widetilde{\omega}^{-1}(a_{(4)})^1 \otimes a_{(2)} \otimes a_{(3)} f(\widetilde{\omega}(a_{(1)})^2 l_{(1)} l'_{(1)} \widetilde{\omega}^{-1}(a_{(4)})^2) h^*(l_{(2)} l'_{(2)})\\ &\quad = \widetilde{\omega}(a_{(1)})^1 l_{(2)} l'_{(2)} \widetilde{\omega}^{-1}(a_{(4)})^1 \otimes a_{(2)} \otimes a_{(3)} f(\widetilde{\omega}(a_{(1)})^2 l_{(1)} l'_{(1)} \widetilde{\omega}^{-1}(a_{(4)})^2),\end{aligned}$$

and

$$\begin{aligned}&\widetilde{\omega}(a_{(1)})^1 h_i \widetilde{\omega}^{-1}(a_{(3)})^1 \widetilde{\omega}(a_{(4)})^1 \tilde{h}_i \widetilde{\omega}^{-1}(a_{(6)})^1\\ &\quad \otimes a_{(2)} f_{(1)}(\widetilde{\omega}(a_{(1)})^2 l_{(1)} \widetilde{\omega}^{-1}(a_{(3)})^2) h_i^*(l_{(2)}) \otimes a_{(5)} f_{(2)}(\widetilde{\omega}(a_{(4)})^2 l'_{(1)} \widetilde{\omega}^{-1}(a_{(6)})^2) \tilde{h}_i^*(l'_{(2)})\\ &\quad = \widetilde{\omega}(a_{(1)})^1 l_{(2)} \widetilde{\omega}^{-1}(a_{(3)})^1 \widetilde{\omega}(a_{(4)})^1 l'_{(2)} \widetilde{\omega}^{-1}(a_{(6)})^1\\ &\qquad \otimes a_{(2)} \otimes a_{(5)} f(\widetilde{\omega}(a_{(1)})^2 l_{(1)} \widetilde{\omega}^{-1}(a_{(3)})^2 \widetilde{\omega}(a_{(4)})^2 l'_{(1)} \widetilde{\omega}^{-1}(a_{(6)})^2)\\ &\quad = \widetilde{\omega}(a_{(1)})^1 l_{(2)} l'_{(2)} \widetilde{\omega}^{-1}(a_{(4)})^1 \otimes a_{(2)} \otimes a_{(3)} f(\widetilde{\omega}(a_{(1)})^2 l_{(1)} l'_{(1)} \widetilde{\omega}^{-1}(a_{(4)})^2).\end{aligned}$$

Thus, it follows that the condition (2.2) holds. If we define $\Omega : A \otimes H^* \to H \otimes H$ as the map

$$\begin{aligned}\Omega : a \otimes f \mapsto&\widetilde{\omega}(a_{(1)})^1 \widetilde{\omega}(a_{(2)})^2{}_{(2)} \widetilde{\omega}^{-1}(a_{(3)})^1{}_{(1)}\\ &\otimes \widetilde{\omega}(a_{(2)})^1 \widetilde{\omega}^{-1}(a_{(3)})^1{}_{(2)} f(\widetilde{\omega}(a_{(1)})^2 \widetilde{\omega}(a_{(2)})^2{}_{(1)} \widetilde{\omega}^{-1}(a_{(3)})^2),\end{aligned}$$

then its convolution inverse is

$$\Omega^{-1}: a\otimes f \mapsto \widetilde{\omega}(a_{(1)})^1{}_{(1)}\widetilde{\omega}^{-1}(a_{(2)})^2{}_{(2)}\widetilde{\omega}^{-1}(a_{(3)})^1 \otimes \widetilde{\omega}(a_{(1)})^1{}_{(2)}\widetilde{\omega}^{-1}(a_{(2)})^1 f(\widetilde{\omega}(a_{(1)})^2\widetilde{\omega}^{-1}(a_{(2)})^2{}_{(1)}\widetilde{\omega}^{-1}(a_{(3)})^2).$$

For all $a\in A$ and $f\in H^*$, we have, on one hand,

$$\begin{aligned}
&(a\otimes f)_{(1)[-1]}\Omega((a\otimes f)_{(2)})^1\otimes(a\otimes f)_{(1)[0][-1]}\Omega((a\otimes f)_{(2)})^2\otimes(a\otimes f)_{(1)[0][0]}\\
&= (a_{(1)}\otimes f_{(1)})_{[-1]}\Omega(a_{(2)}\otimes f_{(2)})^1\otimes(a_{(1)}\otimes f_{(1)})_{[0][-1]}\Omega(a_{(2)}\otimes f_{(2)})^2\otimes(a_{(1)}\otimes f_{(1)})_{[0][0]}\\
&= \widetilde{\omega}(a_{(1)})^1 h_i\widetilde{\omega}^{-1}(a_{(3)})^1\Omega(a_{(4)}\otimes f_{(2)})^1\\
&\quad\otimes(a_{(2)}\otimes(\widetilde{\omega}^{-1}(a_{(3)})^2\rightharpoonup f_{(1)}\leftharpoonup\widetilde{\omega}(a_{(1)})^2)*h_i^*)_{[-1]}\Omega(a_{(4)}\otimes f_{(2)})^2\\
&\quad\otimes(a_{(2)}\otimes(\widetilde{\omega}^{-1}(a_{(3)})^2\rightharpoonup f_{(1)}\leftharpoonup\widetilde{\omega}(a_{(1)})^2)*h_i^*)_{[0]}\\
&= \widetilde{\omega}(a_{(1)})^1 h_i\widetilde{\omega}^{-1}(a_{(5)})^1\Omega(a_{(6)}\otimes f_{(2)})^1\\
&\quad\otimes\widetilde{\omega}(a_{(2)})^1\tilde{h}_i\widetilde{\omega}^{-1}(a_{(4)})^1\Omega(a_{(6)}\otimes f_{(2)})^2\\
&\quad\otimes a_{(3)}\otimes[\widetilde{\omega}^{-1}(a_{(4)})^2\rightharpoonup((\widetilde{\omega}^{-1}(a_{(5)})^2\rightharpoonup f_{(1)}\leftharpoonup\widetilde{\omega}(a_{(1)})^2)*h_i^*\leftharpoonup\widetilde{\omega}(a_{(2)})^2]*\tilde{h}_i^*\\
&= \widetilde{\omega}(a_{(1)})^1 h_i\widetilde{\omega}^{-1}(a_{(5)})^1\widetilde{\omega}(a_{(6)})^1\widetilde{\omega}(a_{(7)})^2{}_{(2)}\widetilde{\omega}^{-1}(a_{(8)})^1{}_{(1)}\\
&\quad\otimes\widetilde{\omega}(a_{(2)})^1\tilde{h}_i\widetilde{\omega}^{-1}(a_{(4)})^1\widetilde{\omega}(a_{(7)})^1\widetilde{\omega}^{-1}(a_{(8)})^1{}_{(2)}f_{(2)}(\widetilde{\omega}(a_{(6)})^2\widetilde{\omega}(a_{(7)})^2{}_{(1)}\widetilde{\omega}^{-1}(a_{(8)})^2)\\
&\quad\otimes a_{(3)}\otimes[\widetilde{\omega}^{-1}(a_{(4)})^2\rightharpoonup((\widetilde{\omega}^{-1}(a_{(5)})^2\rightharpoonup f_{(1)}\leftharpoonup\widetilde{\omega}(a_{(1)})^2)*h_i^*\leftharpoonup\widetilde{\omega}(a_{(2)})^2]*\tilde{h}_i^*,
\end{aligned}$$

on the other hand,

$$\begin{aligned}
&\Omega(a_{(1)}\otimes f_{(1)})^1(a_{(2)}\otimes f_{(2)})_{[-1](1)}\otimes\Omega(a_{(1)}\otimes f_{(1)})^2(a_{(2)}\otimes f_{(2)})_{[-1](2)}\otimes(a_{(2)}\otimes f_{(2)})_{[0]}\\
&= \Omega(a_{(1)}\otimes f_{(1)})^1(\widetilde{\omega}(a_{(2)})^1 h_i\widetilde{\omega}^{-1}(a_{(4)})^1)_{(1)}\\
&\quad\otimes\Omega(a_{(1)}\otimes f_{(1)})^2(\widetilde{\omega}(a_{(2)})^1 h_i\widetilde{\omega}^{-1}(a_{(4)})^1)_{(2)}\otimes a_{(3)}\otimes(\widetilde{\omega}^{-1}(a_{(4)})^2\rightharpoonup f_{(2)}\leftharpoonup\widetilde{\omega}(a_{(2)})^2)*h_i^*\\
&= \Omega(a_{(1)}\otimes f_{(1)})^1\widetilde{\omega}(a_{(2)})^1{}_{(1)}h_{i(1)}\widetilde{\omega}^{-1}(a_{(4)})^1{}_{(1)}\\
&\quad\otimes\Omega(a_{(1)}\otimes f_{(1)})^2\widetilde{\omega}(a_{(2)})^1{}_{(2)}h_{i(2)}\widetilde{\omega}^{-1}(a_{(4)})^1{}_{(2)}\\
&\quad\otimes a_{(3)}\otimes(\widetilde{\omega}^{-1}(a_{(4)})^2\rightharpoonup f_{(2)}\leftharpoonup\widetilde{\omega}(a_{(2)})^2)*h_i^*\\
&= \widetilde{\omega}(a_{(1)})^1\widetilde{\omega}(a_{(2)})^2{}_{(2)}\widetilde{\omega}^{-1}(a_{(3)})^1{}_{(1)}\widetilde{\omega}(a_{(4)})^1{}_{(1)}h_{i(1)}\widetilde{\omega}^{-1}(a_{(6)})^1{}_{(1)}\\
&\quad\otimes\widetilde{\omega}(a_{(2)})^1\widetilde{\omega}^{-1}(a_{(3)})^1{}_{(2)}f_{(1)}(\widetilde{\omega}(a_{(1)})^2\widetilde{\omega}(a_{(2)})^2{}_{(1)}\widetilde{\omega}^{-1}(a_{(3)})^2)\widetilde{\omega}(a_{(4)})^1{}_{(2)}h_{i(2)}\widetilde{\omega}^{-1}(a_{(6)})^1{}_{(2)}\\
&\quad\otimes a_{(5)}\otimes(\widetilde{\omega}^{-1}(a_{(6)})^2\rightharpoonup f_{(2)}\leftharpoonup\widetilde{\omega}(a_{(4)})^2)*h_i^*.
\end{aligned}$$

For all $l\in H$, we compute as follows

$$\begin{aligned}
&\widetilde{\omega}(a_{(1)})^1 h_i\widetilde{\omega}^{-1}(a_{(5)})^1 h_i^*(\widetilde{\omega}(a_{(2)})^2{}_{(2)}l_{(2)}\widetilde{\omega}^{-1}(a_{(4)})^2{}_{(2)})\tilde{h}_i^*(l_{(3)})\widetilde{\omega}(a_{(6)})^1\widetilde{\omega}(a_{(7)})^2{}_{(2)}\widetilde{\omega}^{-1}(a_{(8)})^1{}_{(1)}\\
&\quad\otimes\widetilde{\omega}(a_{(2)})^1\tilde{h}_i\widetilde{\omega}^{-1}(a_{(4)})^1\widetilde{\omega}(a_{(7)})^1\widetilde{\omega}^{-1}(a_{(8)})^1{}_{(2)}f_{(2)}(\widetilde{\omega}(a_{(6)})^2\widetilde{\omega}(a_{(7)})^2{}_{(1)}\widetilde{\omega}^{-1}(a_{(8)})^2)\\
&\quad\otimes a_{(3)}f_{(1)}(\widetilde{\omega}(a_{(1)})^2\widetilde{\omega}(a_{(2)})^2{}_{(1)}l_{(1)}\widetilde{\omega}^{-1}(a_{(4)})^2{}_{(1)}\widetilde{\omega}^{-1}(a_{(5)})^2)\\
&\qquad= \widetilde{\omega}(a_{(1)})^1 h_i\widetilde{\omega}^{-1}(a_{(5)})^1\widetilde{\omega}(a_{(6)})^1\widetilde{\omega}(a_{(7)})^2{}_{(2)}\widetilde{\omega}^{-1}(a_{(8)})^1{}_{(1)}\\
&\qquad\quad\otimes\widetilde{\omega}(a_{(2)})^1\tilde{h}_i\widetilde{\omega}^{-1}(a_{(4)})^1\widetilde{\omega}(a_{(7)})^1\widetilde{\omega}^{-1}(a_{(8)})^1{}_{(2)}\\
&\qquad\quad\otimes a_{(3)}h_i^*(\widetilde{\omega}(a_{(2)})^2{}_{(2)}l_{(2)}\widetilde{\omega}^{-1}(a_{(4)})^2{}_{(2)})\tilde{h}_i^*(l_{(3)})
\end{aligned}$$

$$f(\widetilde{\omega}(a_{(1)})^2\widetilde{\omega}(a_{(2)})^2{}_{(1)}l_{(1)}\widetilde{\omega}^{-1}(a_{(4)})^2{}_{(1)}\widetilde{\omega}^{-1}(a_{(5)})^2\widetilde{\omega}(a_{(6)})^2\widetilde{\omega}(a_{(7)})^2{}_{(1)}\widetilde{\omega}^{-1}(a_{(8)})^2)$$
$$= \widetilde{\omega}(a_{(1)})^1 h_i \widetilde{\omega}(a_{(5)})^2{}_{(2)}\widetilde{\omega}^{-1}(a_{(6)})^1{}_{(1)}$$
$$\otimes \widetilde{\omega}(a_{(2)})^1 \tilde{h}_i \widetilde{\omega}^{-1}(a_{(4)})^1\widetilde{\omega}(a_{(5)})^1\widetilde{\omega}^{-1}(a_{(6)})^1{}_{(2)}$$
$$\otimes a_{(3)} h_i^*(\widetilde{\omega}(a_{(2)})^1{}_{(2)} l_{(2)}\widetilde{\omega}^{-1}(a_{(4)})^2{}_{(2)})\tilde{h}_i^*(l_{(3)})$$

$$f(\widetilde{\omega}(a_{(1)})^2\widetilde{\omega}(a_{(2)})^2{}_{(1)}l_{(1)}\widetilde{\omega}^{-1}(a_{(4)})^2{}_{(1)}\widetilde{\omega}(a_{(5)})^2{}_{(1)}\widetilde{\omega}^{-1}(a_{(6)})^2)$$
$$= \widetilde{\omega}(a_{(1)})^1\widetilde{\omega}(a_{(2)})^2{}_{(2)}l_{(2)}\widetilde{\omega}^{-1}(a_{(4)})^2{}_{(2)}\widetilde{\omega}(a_{(5)})^2{}_{(2)}\widetilde{\omega}^{-1}(a_{(6)})^1{}_{(1)}$$
$$\otimes \widetilde{\omega}(a_{(2)})^1 l_{(3)}\widetilde{\omega}^{-1}(a_{(4)})^1\widetilde{\omega}(a_{(5)})^1\widetilde{\omega}^{-1}(a_{(6)})^1{}_{(2)} \otimes a_{(3)}$$
$$f(\widetilde{\omega}(a_{(1)})^2\widetilde{\omega}(a_{(2)})^2{}_{(1)}l_{(1)}\widetilde{\omega}^{-1}(a_{(4)})^2{}_{(1)}\widetilde{\omega}(a_{(5)})^2{}_{(1)}\widetilde{\omega}^{-1}(a_{(6)})^2)$$
$$= \widetilde{\omega}(a_{(1)})^1\widetilde{\omega}(a_{(2)})^2{}_{(2)}l_{(2)}\widetilde{\omega}^{-1}(a_{(4)})^1{}_{(1)} \otimes \widetilde{\omega}(a_{(2)})^1 l_{(3)}\widetilde{\omega}^{-1}(a_{(4)})^1{}_{(2)} \otimes a_{(3)}$$
$$f(\widetilde{\omega}(a_{(1)})^2\widetilde{\omega}(a_{(2)})^2{}_{(1)}l_{(1)}\widetilde{\omega}^{-1}(a_{(4)})^2),$$

and

$$\widetilde{\omega}(a_{(1)})^1\widetilde{\omega}(a_{(2)})^2{}_{(2)}\widetilde{\omega}^{-1}(a_{(3)})^1{}_{(1)}\widetilde{\omega}(a_{(4)})^1{}_{(1)}h_{i(1)}\widetilde{\omega}^{-1}(a_{(6)})^1{}_{(1)}$$
$$\otimes \widetilde{\omega}(a_{(2)})^1\widetilde{\omega}^{-1}(a_{(3)})^1{}_{(2)}f_{(1)}(\widetilde{\omega}(a_{(1)})^2\widetilde{\omega}(a_{(2)})^2{}_{(1)}\widetilde{\omega}^{-1}(a_{(3)})^2)\widetilde{\omega}(a_{(4)})^1{}_{(2)}h_{i(2)}\widetilde{\omega}^{-1}(a_{(6)})^1{}_{(2)}$$
$$\otimes a_{(5)}f_{(2)}(\widetilde{\omega}(a_{(4)})^2 l_{(1)}\widetilde{\omega}^{-1}(a_{(6)})^2)h_i^*(l_{(2)})$$
$$= \widetilde{\omega}(a_{(1)})^1\widetilde{\omega}(a_{(2)})^2{}_{(2)}\widetilde{\omega}^{-1}(a_{(3)})^1{}_{(1)}\widetilde{\omega}(a_{(4)})^1{}_{(1)}l_{(2)}\widetilde{\omega}^{-1}(a_{(6)})^1{}_{(1)}$$
$$\otimes \widetilde{\omega}(a_{(2)})^1\widetilde{\omega}^{-1}(a_{(3)})^1{}_{(2)}\widetilde{\omega}(a_{(4)})^1{}_{(2)}l_{(3)}\widetilde{\omega}^{-1}(a_{(6)})^1{}_{(2)}$$
$$\otimes a_{(5)}f(\widetilde{\omega}(a_{(1)})^2\widetilde{\omega}(a_{(2)})^2{}_{(1)}\widetilde{\omega}^{-1}(a_{(3)})^2\widetilde{\omega}(a_{(4)})^2 l_{(1)}\widetilde{\omega}^{-1}(a_{(6)})^2)h_i^*(l_{(2)})$$
$$= \widetilde{\omega}(a_{(1)})^1\widetilde{\omega}(a_{(2)})^2{}_{(2)}l_{(2)}\widetilde{\omega}^{-1}(a_{(4)})^1{}_{(1)} \otimes \widetilde{\omega}(a_{(2)})^1 l_{(3)}\widetilde{\omega}^{-1}(a_{(4)})^1{}_{(2)}$$
$$\otimes a_{(3)}f(\widetilde{\omega}(a_{(1)})^2\widetilde{\omega}(a_{(2)})^2{}_{(1)}l_{(1)}\widetilde{\omega}^{-1}(a_{(4)})^2)h_i^*(l_{(2)}).$$

Thus, it follows that the condition (2.3) is satisfied. Observe that

$$\Omega(a\otimes f)^1\varepsilon_H(\Omega(a\otimes f)^2)$$
$$=\widetilde{\omega}(a_{(1)})^1\widetilde{\omega}(a_{(2)})^2{}_{(2)}\widetilde{\omega}^{-1}(a_{(3)})^1{}_{(1)}\varepsilon_H(\widetilde{\omega}(a_{(2)})^1\widetilde{\omega}^{-1}(a_{(3)})^1{}_{(2)})f(\widetilde{\omega}(a_{(1)})^2\widetilde{\omega}(a_{(2)})^2{}_{(1)}\widetilde{\omega}^{-1}(a_{(3)})^2)$$
$$=\widetilde{\omega}(a_{(1)})^1\widetilde{\omega}^{-1}(a_{(2)})^1 f(\widetilde{\omega}(a_{(1)})^2\widetilde{\omega}^{-1}(a_{(2)})^2)$$
$$=\varepsilon_A(a)f(1_H)1_H = \varepsilon(a\otimes f)1_H.$$

Similarly, we have $\Omega(a\otimes f)^2\varepsilon_H(\Omega(a\otimes f)^1) = \varepsilon(a\otimes f)1_H$. We also check that the condition (2.7) holds in a straightforward way. Thus, we have proved that $(A\otimes H^*, \varrho, \Omega)$ is a twisted H-comodule coalgebra.

Using the definition of P, we can check that the condition (2) of Definition (4.1) holds. In fact, for all $a\in A$, it follows that

$$\begin{aligned}\Theta(a)_{[-1]}\otimes P(\Theta(a)_{[0]}) &= (a\otimes 1_{H^*})_{[-1]}\otimes P((a\otimes 1_{H^*})_{[0]})\\ &= h_i\otimes P(a\otimes h_i^*)\\ &= a_{[-1]}\otimes a_{[0]}\otimes 1_{H^*}\\ &= a_{[-1]}\otimes \Theta(a_{[0]}),\end{aligned}$$

as desired. Also, we can check that the condition $(G2)$ holds in a similar way. It is left to us to check that the condition (3) of Definition (4.1) is satisfied. In fact, for all $a \in A$, we have

$$
\begin{aligned}
&\varpi(\Theta(a))^1 \otimes \varpi(\Theta(a))^2 \\
&= (a_{(1)} \otimes 1_{H^*})_{[\underline{-1}]}\Omega(a_{(2)} \otimes 1_{H^*})^1 \otimes (a_{(1)} \otimes 1_{H^*})_{[\underline{0}][\underline{-1}]}\varepsilon((a_{(1)} \otimes 1_{H^*})_{[\underline{0}][\underline{0}]})\Omega(a_{(2)} \otimes 1_{H^*})^2 \\
&= (a_{(1)} \otimes 1_{H^*})_{[-1]}\Omega(a_{(2)} \otimes 1_{H^*})^1 \\
&\quad \otimes P((a_{(1)} \otimes 1_{H^*})_{[0]})_{[\underline{-1}]}\varepsilon(P((a_{(1)} \otimes 1_{H^*})_{[0]})_{[\underline{0}]})\Omega(a_{(2)} \otimes 1_{H^*})^2 \\
&= (a_{(1)} \otimes 1_{H^*})_{[-1]}\Omega(a_{(2)} \otimes 1_{H^*})^1 \\
&\quad \otimes P((a_{(1)} \otimes 1_{H^*})_{[0]})_{[-1]}\varepsilon(P(P((a_{(1)} \otimes 1_{H^*})_{[0]})_{[0]}))\Omega(a_{(2)} \otimes 1_{H^*})^2 \\
&= h_i\widetilde{\omega}(a_{(2)})^2 \otimes P(a_{(1)} \otimes h_i^*)_{[-1]}\varepsilon(P(P(a_{(1)} \otimes h_i^*)_{[0]}))\widetilde{\omega}(a_{(2)})^1 \\
&= a_{(1)[-1]}\widetilde{\omega}(a_{(2)})^2 \otimes (a_{(1)[0]} \otimes 1_{H^*})_{[-1]}\varepsilon(P((a_{(1)[0]} \otimes 1_{H^*})_{[0]}))\widetilde{\omega}(a_{(1)})^1 \\
&= a_{(1)[-1]}\widetilde{\omega}(a_{(2)})^2 \otimes h_i\varepsilon(P(a_{(1)[0]} \otimes h_i^*))\widetilde{\omega}(a_{(2)})^1 \\
&= a_{(1)[-1]}\widetilde{\omega}(a_{(2)})^2 \otimes a_{(1)[0][-1]}\varepsilon(a_{(1)[0][0]} \otimes 1_{H^*})\widetilde{\omega}(a_{(2)})^1 \\
&= a_{(1)[-1]}\widetilde{\omega}(a_{(2)})^2 \otimes a_{(1)[0][-1]}\varepsilon_A(a_{(1)[0][0]})\widetilde{\omega}(a_{(2)})^1 \\
&= \omega(a)^1 \otimes \omega(a)^2
\end{aligned}
$$

Thus we have $\varpi \circ \Theta = \omega$. Similarly, we can prove that $\varpi' \circ \Theta = \omega'$. □

If $\widetilde{\omega}$ is trivial, i.e., $\widetilde{\omega}(a) = \varepsilon_A(a)1_H \otimes 1_H$, for all $a \in A$, then the map ω is trivial, and the twisted partial comodule coalgebras reduce to the partial comodule coalgebras. From Theorem 4.3, we have the following conclusion.

Corollary 4.4. *Every partial comodule coalgebra has a globalization.*

Example 4.5. Consider the symmetric twisted partial comodule coalgebra $(k, \rho_k, \omega, \omega')$ in Example 3.3. We define a linear map

$$\widetilde{\omega} : k \to H \otimes H, 1 \mapsto \frac{1}{2}(1_H \otimes 1_H + g \otimes 1_H + 1_H \otimes g - g \otimes g)$$

Notice easily that $\widetilde{\omega}^{-1} = \widetilde{\omega}$. Now we can check that the conditions 4.2 and 4.3 are satisfied. In fact, we compute

$$
\begin{aligned}
&[\frac{1}{2}(1_H + g) \otimes \frac{1}{2}(1_H + g)]\frac{1}{2}(1_H \otimes 1_H + g \otimes 1_H + 1_H \otimes g - g \otimes g) \\
&\quad = \frac{1}{2}(1_H + g) \otimes \frac{1}{2}(1_H + g),
\end{aligned}
$$

as desired. From Theorem 4.3, it follows that H^* is a globalization of $(k, \rho_k, \omega, \omega')$.

5. Partial Cleft Coextensions

In the section, we will introduce the concept of a partially cleft coextension and discuss the relations between partially cleft coextensions and partial crossed coproducts.

Definition 5.1. Let H is a Hopf algebra and D a right H-module coalgebra. Let us denote by H^+ the augmentation ideal $\mathrm{Ker}\varepsilon_H$ which is a Hopf ideal. Then $D \cdot H^+$ is a coideal of D, and $D/D \cdot H^+$ is a coalgebra with a trivial right H-module structure. Let C be the quotient coalgebra $D/D \cdot H^+$. Then we call D/C an H-coextension. The H-coextension D/C is partially cleft, if there is a pair of linear maps $\Phi, \Psi : D \to H$ such that

(a) $\varepsilon_H \circ \Phi = \varepsilon_D$,

(b) The diagrams below are communtative:

$$\begin{array}{ccc} D\otimes H & \xrightarrow{\Phi\otimes H} & H\otimes H \\ \downarrow{\cdot} & & \downarrow{m_H} \\ D & \xrightarrow{\Phi} & H \end{array} \qquad \begin{array}{ccc} D\otimes H & \xrightarrow{\Psi\otimes S} & H\otimes H \\ \downarrow{\cdot} & & \downarrow{m_H^{op}} \\ D & \xrightarrow{\Psi} & H \end{array}$$

(c) $\Delta_H \circ (\Phi * \Psi)$ is a central element in the convolution algebra $\mathrm{Hom}(C, H \otimes H)$,

(d) $F_{d_{(1)}} \otimes \overline{d}_{(2)} = F_{d_{(2)}} \otimes \overline{d}_{(1)}$,

(e) $d_{(1)} \cdot \Psi(d_{(2)})\Phi(d_{(3)}) = d$,

(f) $\Phi(d_{(1)}) \otimes E_{d_{(2)}} = (E_{d_{(1)}})_{(1)}\Phi(d_{(2)}) \otimes (E_{d_{(1)}})_{(2)}$,

(g) $\Psi(d_{(1)}) \otimes F_{d_{(2)}} = (F_{d_{(1)}})_{(2)}\Psi(d_{(2)}) \otimes (F_{d_{(1)}})_{(1)}$,

(h) $\Phi(d_{(1)})_{(1)} \otimes \Phi(d_{(1)})_{(2)}F_{d_{(2)}} = E_{d_{(1)}}\Phi(d_{(2)})_{(1)} \otimes \Phi(d_{(2)})_{(2)}$, where $E_d = (\Phi * \Psi)(d)$, $F_d = (\Psi * \Phi)(d)$, for all $d \in D$.

Note that in the case of a cleft coextension, with a convolution invertible map Φ, the axioms for partially cleft coextensions are automatically satisfied, if we take Ψ to be the convolution inverse of Φ [10, Lemma 2.3].

Given a partial cleft coextension, we also have $\varepsilon_H \circ \Psi = \varepsilon_D$, which follows from (e) of Definition 5.1. Applying Φ to both sides of (e) again, we conclude that

$$\Phi * \Psi * \Phi = \Phi.$$

In particular, multiplying the equality above by Ψ (resp. Φ) on the right, we obtain that $\Phi * \Psi$(resp. $\Psi * \Phi$) is an idempotent. For any linear map $\tau \in \mathrm{Hom}(C, H)$, we define a linear map

$$\nu : C \to H \otimes H,\ \overline{d} \mapsto \tau(\overline{d}) \otimes 1_H,$$

from (c) of Definition 5.1, we have

$$(\Phi * \Psi) * \tau = \tau * (\Phi * \Psi).$$

Thus $\Phi * \Psi$ is a central idempotent in the convolution algebra $\mathrm{Hom}(C, H)$.

Just unlike Φ, the map Ψ may not satisfy an equality $\Psi * \Phi * \Psi = \Psi$, but we can use $\Psi' = \Psi*\Phi*\Psi$ to replace Ψ. From $\Psi*\Phi$ being an idempotent, it follows that $\Psi' = \Psi'*\Phi*\Psi'$, and the pair (Φ, Ψ') still satisfies the properties (a)-(h) in Definition 5.1. Thus we always assume that Ψ satisfies the relation $\Psi = \Psi * \Phi * \Psi$.

Proposition 5.2. *If $(A, \rho_A, (\omega, \omega'))$ is a symmetric partial twisted H-comodule coalgebra, then $A\natural_{(\rho_A,\omega)}H$ is a partially cleft H-coextension.*

Proof. We see that $A\natural_{(\rho_A,\omega)}H$ is a right H-module coalgebra via

$$(a\natural h) \cdot g = a\natural hg$$

for all $a \in A$ and $h, g \in H$. Consider the maps $\Phi, \Psi : A\natural_{(\rho_A,\omega)}H \to H$ given by

$$\Phi(a\natural h) = a_{[-1]}\varepsilon_A(a_{[0]})h,$$

$$\Psi(a\natural h) = S(a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})h)S(\omega'(a_{(1)})^1)\omega'(a_{(1)})^2.$$

From the definition of Φ, we have $\varepsilon_H \circ \Phi = \varepsilon_A \otimes \varepsilon_H$, which satisfies the condition (a) of Definition 5.1. With respect to item (b), the first diagram is communicative by the definition of $\cdot$. As for the second diagram in (b), we have

$$\begin{aligned}(m_H^{op} \circ (\Psi \otimes S))(a\natural h \otimes g) =& S(g)S(a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})h)S(\omega'(a_{(1)})^1)\omega'(a_{(1)})^2\\ =& S(a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})hg)S(\omega'(a_{(1)})^1)\omega'(a_{(1)})^2\\ =& \Psi(a\natural hg),\end{aligned}$$

which finishes the proof of (b) of the definition of partial cleft coextension. Now,

$$\begin{aligned}(\Phi * \Psi)(a\natural h) =& \Phi(a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1 h_{(1)})\Psi(a_{(2)[0]}\natural\omega(a_{(3)})^2 h_{(2)})\\ =& a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})a_{(2)[-1]}\omega(a_{(3)})^1 h_{(1)}S(h_{(2)})\\ &\times S(a_{(2)[0](2)[-1]}\varepsilon_A(a_{(2)[0](2)[0]})\omega(a_{(3)})^2)S(\omega'(a_{(2)[0](1)})^1)\omega'(a_{(2)[0](1)})^2\\ (2.2) =& \varepsilon_H(h)a_{(1)[-1]}\omega(a_{(2)})^1\\ &\times S(a_{(1)[0](2)[-1]}\varepsilon_A(a_{(1)[0](2)[0]})\omega(a_{(2)})^2)S(\omega'(a_{(1)[0](1)})^1)\omega'(a_{(1)[0](1)})^2\\ (2.2) =& \varepsilon_H(h)a_{(1)[-1]}a_{(2)[-1]}\omega(a_{(3)})^1\\ &\times S(a_{(2)[0][-1]}\varepsilon_A(a_{(2)[0][0]})\omega(a_{(3)})^2)S(\omega'(a_{(1)[0]})^1)\omega'(a_{(1)[0]})^2\\ (2.2) =& \varepsilon_H(h)a_{(1)[-1]}\omega(a_{(2)})^1 S(\omega(a_{(2)})^2)S(\omega'(a_{(1)[0]})^1)\omega'(a_{(1)[0]})^2\\ =& \varepsilon_H(h)\omega(a_{(1)})^1\omega'(a_{(2)})^1_{(1)}\omega'(a_{(3)})^1\omega(a_{(4)})^1\\ &\times S(\omega'(a_{(2)})^1_{(2)}\omega'(a_{(3)})^2\omega(a_{(4)})^2)S(\omega(a_{(1)})^2_{(1)})\omega(a_{(1)})^2_{(2)}\omega'(a_{(2)})^2\\ =& \varepsilon_H(h)a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})\omega'(a_{(2)})^1_{(1)}\\ &\times \underline{\omega'(a_{(3)})^1\omega(a_{(4)})^1}S(\omega'(a_{(2)})^1_{(2)}\underline{\omega'(a_{(3)})^2\omega(a_{(4)})^2})\omega'(a_{(2)})^2\\ (3.3) =& \varepsilon_H(h)a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})\omega'(a_{(2)})^1_{(1)}a_{(3)[-1]}\varepsilon_A(a_{(3)[0]})\\ &\times a_{(4)[-1](1)}S(a_{(4)[-1](2)})S(\omega'(a_{(2)})^1_{(2)}\varepsilon_A(a_{(4)[0]}))\omega'(a_{(2)})^2\\ =& \varepsilon_H(h)a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})\omega'(a_{(2)})^1_{(1)}a_{(3)[-1]}\varepsilon_A(a_{(3)[0]})S(\omega'(a_{(2)})^1_{(2)})\omega'(a_{(2)})^2\\ =& \varepsilon_H(h)a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})\omega'(a_{(3)})^1_{(1)}S(\omega'(a_{(3)})^1_{(2)})\omega'(a_{(3)})^2\\ =& \varepsilon_H(h)a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})\omega'(a_{(2)})^1_{(1)}S(\omega'(a_{(2)})^1_{(2)})\omega'(a_{(2)})^2\\ =& \varepsilon_H(h)a_{[-1]}\varepsilon_A(a_{[0]}).\end{aligned}$$

Hence it follows that

$$\Delta_H \circ (\Phi * \Psi) = f_2 \circ (A \otimes \varepsilon_H),$$

and this implies that $\Delta_H \circ (\Phi * \Psi)$ is central in $\mathrm{Hom}(A, H \otimes H)$ in virtue of the convolution centrality of f_2. For each $h \in H$, we define a linear map

$$\tau_h : A \to H,\ \tau_h(a) = \varepsilon_A(a)h,$$

since $\Lambda(a) = a_{[-1]}\varepsilon_A(a_{[0]})$ is central in $\mathrm{Hom}(A, H)$ by assumption, we have

$$\begin{aligned} a_{[-1]}\varepsilon_A(a_{[0]})h =& a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})\varepsilon_A(a_{(2)})h \\ =&(\Lambda * \tau_h)(a) = (\tau_h * \Lambda)(a) \\ =& h a_{[-1]}\varepsilon_A(a_{[0]}). \end{aligned}$$

With respect to $(\Psi * \Phi)$,

$$\begin{aligned} (\Psi * \Phi)(a\natural h) =& \Psi(a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1 h_{(1)})\Phi(a_{(2)[0]}\natural\omega(a_{(3)})^2 h_{(2)}) \\ =& S(a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})a_{(3)[-1]}\omega(a_{(4)})^1 h_{(1)}) \\ & \times S(\omega'(a_{(1)})^1)\omega'(a_{(1)})^2 a_{(3)[0][-1]}\varepsilon_A(a_{(3)[0][0]})\omega(a_{(4)})^2 h_{(2)} \\ (2.2) =& S(a_{(2)[-1]}\omega(a_{(3)})^1 h_{(1)}) \\ & \times S(\omega'(a_{(1)})^1)\omega'(a_{(1)})^2 a_{(2)[0][-1]}\varepsilon_A(a_{(2)[0][0]})\omega(a_{(3)})^2 h_{(2)} \\ =& S(\omega'(a_{(1)})^1\omega(a_{(2)})^1 h_{(1)})\omega'(a_{(1)})^2\omega(a_{(2)})^2 h_{(2)} \\ (3.3) =& S(a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})a_{(2)[-1](1)}h_{(1)})a_{(2)[-1](2)}\varepsilon_A(a_{(2)[0]})h_{(2)} \\ (2.2) =& S(a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})h_{(1)})S(a_{(1)[-1](1)})a_{(1)[-1](2)}\varepsilon_A(a_{(1)[0]})h_{(2)} \\ =& S(a_{[-1]}\varepsilon_A(a_{[0]})h_{(1)})h_{(2)} \\ =& S(h_{(1)}a_{[-1]}\varepsilon_A(a_{[0]}))h_{(2)} \\ =& \varepsilon_H(h)S(a_{[-1]}\varepsilon_A(a_{[0]})). \end{aligned}$$

Thus it follows that

$$(\Psi * \Phi)(a\natural h) = \varepsilon_H(h)S(a_{[-1]}\varepsilon_A(a_{[0]})).$$

Using the equality above, we have

$$\begin{aligned} &(a\natural h)_{(1)} \cdot \Psi((a\natural h)_{(2)})\Phi((a\natural h)_{(3)}) \\ &= a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1 h_{(1)}\Psi(a_{(2)[0](1)}\natural a_{(2)[0](2)[-1]}\omega(a_{(2)[0](3)})^1\omega(a_{(3)})^2_{(1)}h_{(2)}) \\ &\quad \times \Phi(a_{(2)[0](2)[0]}\natural\omega(a_{(2)[0](3)})^2\omega(a_{(3)})^2_{(2)}h_{(3)}) \\ &= a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1 S(a_{(2)[0](2)[-1]}\varepsilon_A(a_{(2)[0](2)[0]})a_{(2)[0](3)[-1]}\omega(a_{(2)[0](4)})^1\omega(a_{(3)})^2_{(1)}) \\ &\quad \times S(\omega'(a_{(2)[0](1)})^1)\omega'(a_{(2)[0](1)})^2 a_{(2)[0](3)[0][-1]}\varepsilon_A(a_{(2)[0](3)[0][0]})\omega(a_{(2)[0](4)})^2\omega(a_{(3)})^2_{(2)}h \\ &= a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1 S(a_{(2)[0](2)[-1]}\varepsilon_A(a_{(2)[0](2)[0]})\omega(a_{(2)[0](3)})^1\omega(a_{(3)})^2_{(1)}) \\ &\quad \times S(\omega'(a_{(2)[0](1)})^1)\omega'(a_{(2)[0](1)})^2\omega(a_{(2)[0](3)})^2\omega(a_{(3)})^2_{(2)}h \\ &= a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1 S(\omega(a_{(2)[0](2)})^1\omega(a_{(3)})^2_{(1)}) \\ &\quad \times S(\omega'(a_{(2)[0](1)})^1)\omega'(a_{(2)[0](1)})^2\omega(a_{(2)[0](2)})^2\omega(a_{(3)})^2_{(2)}h \\ &= a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1 S(\omega(a_{(3)})^2_{(1)})S(a_{(2)[0](1)[-1]}\varepsilon_A(a_{(2)[0](1)[0]})a_{(2)[0](2)[-1](1)}) \\ &\quad \times a_{(2)[0](2)[-1](2)}\varepsilon_A(a_{(2)[0](2)[0]})\omega(a_{(3)})^2_{(2)}h \\ &= a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1 S(\omega(a_{(3)})^2_{(1)})S(a_{(2)[0](1)[-1](1)}a_{(2)[0](2)[-1]}\varepsilon_A(a_{(2)[0](2)[0]})) \\ &\quad \times a_{(2)[0](1)[-1](2)}\varepsilon_A(a_{(2)[0](1)[0]})\omega(a_{(3)})^2_{(2)}h \\ &= a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1 S(\omega(a_{(3)})^2_{(1)})S(a_{(2)[0][-1]}\varepsilon_A(a_{(2)[0][0]}))\omega(a_{(3)})^2_{(2)}h \end{aligned}$$

$$= a_{(1)}\natural a_{(2)[-1]}a_{(3)[-1]}\varepsilon_A(a_{(3)[0]})S(a_{(2)[0][-1]}\varepsilon_A(a_{(2)[0][0]}))h$$
$$= a_{(1)}\natural a_{(2)[-1]}S(a_{(2)[0][-1]}\varepsilon_A(a_{(2)[0][0]}))h$$
$$= a_{(1)}\natural a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})h = a\natural h,$$

which shows the condition (e) of Definition 5.1. For any $a \in A$ and $h \in H$, we have

$$\begin{aligned}
&(E_{(a\natural h)_{(1)}})_{(1)}\Phi((a\natural h)_{(2)}) \otimes (E_{(a\natural h)_{(1)}})_{(2)}\\
&\quad=\varepsilon_H(\omega(a_{(3)})^1)a_{(1)[-1](1)}\varepsilon_A(a_{(1)[0]})\Phi(a_{(2)}\natural\omega(a_{(3)})^2h) \otimes a_{(1)[-1](2)}\\
&\quad=a_{(1)[-1](1)}\varepsilon_A(a_{(1)[0]})a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})h \otimes a_{(1)[-1](2)}\\
&\quad=a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})a_{(2)[-1](1)}\varepsilon_A(a_{(2)[0]})h \otimes a_{(2)[-1](2)}\\
&\quad=a_{[-1]}h \otimes a_{[0][-1]}\varepsilon_A(a_{[0][0]})
\end{aligned}$$

and

$$\begin{aligned}
\Phi((a\natural h)_{(1)}) \otimes E_{(a\natural h)_{(2)}} &=\Phi(a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1h_{(1)}) \otimes \varepsilon_H(\omega(a_{(3)})^2h_{(2)})a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})\\
&=a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})a_{(2)[-1]}\\
&\quad\times \omega(a_{(3)})^1h_{(1)} \otimes \varepsilon_H(\omega(a_{(3)})^2h_{(2)})a_{(2)[0][-1]}\varepsilon_A(a_{(2)[0][0]})\\
&=a_{(1)[-1]}a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})h \otimes a_{(1)[0][-1]}\varepsilon_A(a_{(1)[0][0]})\\
&=a_{[-1]}h \otimes a_{[0][-1]}\varepsilon_A(a_{[0][0]}),
\end{aligned}$$

it follows that (f) holds. As for (h), we compute as follows:

$$\begin{aligned}
&\Phi((a\natural h)_{(1)})_{(1)} \otimes \Phi((a\natural h)_{(1)})_{(2)}F_{(a\natural h)_{(2)}}\\
&\quad=\Phi(a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1h_{(1)})_{(1)}\\
&\qquad\otimes \Phi(a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1h_{(1)})_{(2)}(\Psi * \Phi)(a_{(2)[0]}\natural\omega(a_{(3)})^2h_{(2)})\\
&\quad=(a_{(1)[-1]}\omega(a_{(2)})^1h)_{(1)}\\
&\qquad\otimes (a_{(1)[-1]}\omega(a_{(2)})^1h)_{(2)}\varepsilon_H(\omega(a_{(2)})^2)S(a_{(1)[0][-1]}\varepsilon_A(a_{(1)[0][0]}))\\
&\quad=(a_{[-1]})h)_{(1)} \otimes (a_{[-1]}h)_{(2)}S(a_{[0][-1]}\varepsilon_A(a_{[0][0]}))\\
&\quad=a_{[-1](1)}h_{(1)} \otimes a_{[-1](2)}h_{(2)}S(a_{[0][-1]}\varepsilon_A(a_{[0][0]}))\\
&\quad=a_{[-1](1)}h_{(1)} \otimes a_{[-1](2)}S(a_{[0][-1]}\varepsilon_A(a_{[0][0]}))h_{(2)}\\
&\quad=\varepsilon_A(a_{(1)[0]})a_{(1)[-1](1)}a_{(2)[-1](1)}h_{(1)} \otimes a_{(1)[-1](2)}a_{(2)[-1](2)}S(a_{(2)[-1](3)}\varepsilon_A(a_{(2)[0]}))h_{(2)}\\
&\quad=\varepsilon_A(a_{(1)[0]})a_{(1)[-1](1)}a_{(2)[-1]}\varepsilon_A(a_{(2)[0]})h_{(1)} \otimes a_{(1)[-1](2)}h_{(2)}\\
&\quad=a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})\varepsilon_A(a_{(2)[0]})a_{(2)[-1](1)}h_{(1)} \otimes a_{(2)[-1](2)}h_{(2)}
\end{aligned}$$

and

$$\begin{aligned}
&E_{(a\natural h)_{(1)}}\Phi((a\natural h)_{(2)})_{(1)} \otimes \Phi((a\natural h)_{(2)})_{(2)}\\
&\quad= (\Phi * \Psi)(a_{(1)}\natural a_{(2)[-1]}\omega(a_{(3)})^1h_{(1)})\\
&\qquad\times \Phi(a_{(2)[0]}\natural\omega(a_{(3)})^2h_{(2)})_{(1)} \otimes \Phi(a_{(2)[0]}\natural\omega(a_{(3)})^2h_{(2)})_{(2)}\\
&\quad= \varepsilon_H(a_{(2)[-1]}\omega(a_{(3)})^1)a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})\\
&\qquad\times a_{(2)[0][-1](1)}\varepsilon_A(a_{(2)[0][0]})\omega(a_{(3)})^2_{(1)}h_{(1)} \otimes a_{(2)[0][-1](2)}\omega(a_{(3)})^2_{(2)}h_{(2)}\\
&\quad= \varepsilon_H(\omega(a_{(3)})^1)a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})\\
&\qquad\times a_{(2)[-1](1)}\varepsilon_A(a_{(2)[0]})\omega(a_{(3)})^2_{(1)}h_{(1)} \otimes a_{(2)[-1](2)}\omega(a_{(3)})^2_{(2)}h_{(2)}
\end{aligned}$$

$$= a_{(1)[-1]}\varepsilon_A(a_{(1)[0]})a_{(2)[-1](1)}\varepsilon_A(a_{(2)[0]})h_{(1)} \otimes a_{(2)[-1](2)}h_{(2)},$$

as desired.

This ends the proof. □

We are now to prove the main result of this section. Our result shows that partially cleft coextensions are exactly partially crossed coproducts, and thus it is the dual version of [4, Theorem2].

Theorem 5.3. *Let D be a right H-comodule coalgebra and D/C an H-coextension. Then the following are equivalent:*

(1) *The H-coextension D/C is partially cleft,*
(2) *D is isomorphic to a partial crossed coproduct $C\natural_{(\rho_A,\omega)}H$ with respect to a symmetric twisted partial H-comodule structure on A.*

Proof. We have proved in Proposition 5.2 that (2) ⇒ (1). Conversely, assume that D is partially cleft by the pair of maps $\Phi, \Psi : D \to H$. The pair (Φ, Ψ) allows us to define a twisted partial coaction of H on $C = D/D\cdot H^+$ by

$$\rho^C : C \to H \otimes C,\ \rho^C(\overline{d}) = \Phi(d_{(1)})\Psi(d_{(3)}) \otimes \overline{d}_{(2)}.$$

The map ρ^C is well-defined by the condition (b) in Definition 5.1. Define two linear maps $\omega, \omega' : C \to H \otimes H$ by

$$\omega(\overline{d}) = \Phi(d_{(1)})\Psi(d_{(3)})_{(1)} \otimes \Phi(d_{(2)})\Psi(d_{(3)})_{(2)},$$

$$\omega'(\overline{d}) = \Phi(d_{(1)})_{(1)}\Psi(d_{(3)}) \otimes \Phi(d_{(1)})_{(2)}\Psi(d_{(2)}).$$

It follows easily that the maps ω and ω' are both well-defined.

Note that

$$\begin{aligned}
\varepsilon_H(\overline{d}_{[-1]})\overline{d}_{[0]} &= \varepsilon_H(\Phi(d_{(1)}))\varepsilon_H(\Psi(d_{(1)}))\overline{d_{(2)}} \\
&= \varepsilon_D(d_{(1)})\varepsilon_D(d_{(1)})\overline{d_{(2)}} \\
&= \overline{d}.
\end{aligned}$$

Next, given $d \in D$, we see that

$$\begin{aligned}
&\overline{d}_{[-1]} \otimes \overline{d}_{[0](1)} \otimes \overline{d}_{[0](2)} \\
&= \Phi(d_{(1)})\Psi(d_{(4)}) \otimes \overline{d}_{(2)} \otimes \overline{d}_{(3)} \\
&= \Phi(d_{(1)})\Psi(d_{(3)})\Phi(d_{(4)})\Psi(d_{(6)}) \otimes \overline{d}_{(2)} \otimes \overline{d}_{(5)} \\
&= \overline{d}_{(1)[-1]}\overline{d}_{(2)[-1]} \otimes \overline{d}_{(1)[0]} \otimes \overline{d}_{(2)[0]}.
\end{aligned}$$

The partial coaction is twisted by (ω, ω'), since

$$\begin{aligned}
&\overline{d}_{[-1]} \otimes \overline{d}_{[0][-1]} \otimes \overline{d}_{[0][0]} \\
&= \Phi(d_{(1)})\Psi(d_{(3)}) \otimes \overline{d}_{(2)[-1]} \otimes \overline{d}_{(2)[0]} \\
&= \Phi(d_{(1)})\Psi(d_{(5)}) \otimes \Phi(d_{(2)})\Psi(d_{(4)}) \otimes \overline{d}_{(3)} \\
&= \Phi(d_{(1)})\Psi(d_{(4)})_{(1)}\Phi(d_{(5)})_{(1)}\Psi(d_{(7)})
\end{aligned}$$

$$\otimes\Phi(d_{(2)})\Psi(d_{(4)})_{(2)}\Phi(d_{(5)})_{(2)}\Psi(d_{(6)})\otimes\overline{d}_{(3)}$$
$$=\Phi(d_{(1)})\Psi(d_{(3)})_{(1)}\Phi(d_{(4)})_{(1)}\Psi(d_{(6)})_{(1)}\Phi(d_{(7)})_{(1)}\Psi(d_{(9)})$$
$$\otimes\Phi(d_{(2)})\Psi(d_{(3)})_{(2)}\Phi(d_{(4)})_{(2)}\Psi(d_{(6)})_{(2)}\Phi(d_{(7)})_{(2)}\Psi(d_{(8)})\otimes\overline{d}_{(5)}$$
$$=\omega(\overline{d}_{(1)})^1\overline{d}_{(2)[-1](1)}\omega'(\overline{d}_{(3)})^1\otimes\omega(\overline{d}_{(1)})^2\overline{d}_{(2)[-1](2)}\omega'(\overline{d}_{(3)})^2\otimes\overline{d}_{(2)[0]}$$

for all $d\in D$.

With respect to ω and ω', first we observe that

$$\begin{aligned}\omega(\overline{d})^1\varepsilon_H(\omega(\overline{d})^2) &= \Phi(d_{(1)})\Psi(d_{(3)})_{(1)}\varepsilon_H(\Phi(d_{(2)})\Psi(d_{(3)})_{(2)})\\ &= \Phi(d_{(1)})\Psi(d_{(3)})\varepsilon_H(\Phi(d_{(2)}))\\ &= \Phi(d_{(1)})\Psi(d_{(2)})\\ &= \overline{d}_{[-1]}\varepsilon_C(\overline{d}_{[0]}),\end{aligned}$$

and also

$$\varepsilon_H(\omega(\overline{d})^1)\omega(\overline{d})^2=\overline{d}_{[-1]}\varepsilon_C(\overline{d}_{[0]}),$$

showing that ω is normalized. Note, furthermore, that

$$\varepsilon_C(\overline{d}_{(1)[0]})\overline{d}_{(1)[-1]}\omega(\overline{d}_{(2)})^1\otimes\omega(\overline{d}_{(2)})^2=\omega(\overline{d}_{(1)})^1\varepsilon_C(\overline{d}_{(2)[0]})\overline{d}_{(2)[-1]}\otimes\omega(\overline{d}_{(1)})^2=\omega(\overline{d}).$$

Indeed, because $(\Phi*\Psi)(d)=(\Phi*\Psi)(\overline{d})=\overline{d}_{[-1]}\varepsilon_C(\overline{d}_{[0]})$ and $\Phi*\Psi$ is central in $\mathrm{Hom}(C,H)$, we have

$$\begin{aligned}&\varepsilon_C(\overline{d}_{(1)[0]})\overline{d}_{(1)[-1]}\omega(\overline{d}_{(2)})^1\otimes\omega(\overline{d}_{(2)})^2\\ &=\varepsilon_C(\overline{d}_{(1)[0]})\overline{d}_{(1)[-1]}\omega(\overline{d}_{(2)})^1\otimes\omega(\overline{d}_{(2)})^2\\ &=\varepsilon_C(\overline{d}_{(1)[0]})\overline{d}_{(1)[-1]}\Phi(d_{(2)})\Psi(d_{(4)})_{(1)}\otimes\Phi(d_{(3)})\Psi(d_{(4)})_{(2)}\\ &=\Phi(d_{(1)})\Psi(d_{(2)})\Phi(d_{(3)})\Psi(d_{(5)})_{(1)}\otimes\Phi(d_{(4)})\Psi(d_{(5)})_{(2)}\\ &=\Phi(d_{(1)})\Psi(d_{(3)})_{(1)}\otimes\Phi(d_{(2)})\Psi(d_{(3)})_{(2)}\\ &=\omega(\overline{d})\end{aligned}$$

Analogously, as for ω', one shows that

$$\varepsilon_C(\overline{d}_{(1)[0]})\overline{d}_{(1)[-1]}\omega'(\overline{d}_{(2)})^1\otimes\omega'(\overline{d}_{(2)})^2=\omega'(\overline{d}_{(1)})^1\varepsilon_C(\overline{d}_{(2)[0]})\overline{d}_{(2)[-1]}\otimes\omega'(\overline{d}_{(1)})^2=\omega'(\overline{d}).$$

For $\omega*\omega'$, we have

$$\begin{aligned}&(\omega*\omega')(\overline{d})\\ &=\Phi(d_{(1)})\Psi(d_{(3)})_{(1)}\Phi(d_{(4)})_{(1)}\Psi(d_{(6)})\otimes\Phi(d_{(2)})\Psi(d_{(3)})_{(2)}\Phi(d_{(4)})_{(2)}\Psi(d_{(5)})\\ &=\Phi(d_{(1)})\Psi(d_{(4)})\otimes\Phi(d_{(2)})\Psi(d_{(3)})\\ &=\overline{d}_{[-1]}\otimes\overline{d}_{[0][-1]}\varepsilon_C(\overline{d}_{[0][0]}).\end{aligned}$$

For $\omega'*\omega$, using (g) and (h), we compute

$$\begin{aligned}&(\omega'*\omega)(\overline{d})\\ &=\Phi(d_{(1)})_{(1)}\Psi(d_{(3)})\Phi(d_{(4)})\Psi(d_{(6)})_{(1)}\otimes\Phi(d_{(1)})_{(2)}\Psi(d_{(2)})\Phi(d_{(5)})\Psi(d_{(6)})_{(2)}\\ &=\Phi(d_{(1)})_{(1)}F_{d_{(3)}}\Psi(d_{(5)})_{(1)}\otimes\Phi(d_{(1)})_{(2)}\Psi(d_{(2)})\Phi(d_{(4)})\Psi(d_{(5)})_{(2)}\end{aligned}$$

$$= \Phi(d_{(1)})_{(1)}(F_{d_{(2)}})_{(1)}\Psi(d_{(5)})_{(1)} \otimes \Phi(d_{(1)})_{(2)}(F_{d_{(2)}})_{(2)}\Psi(d_{(3)})\Phi(d_{(4)})\Psi(d_{(5)})_{(2)}$$
$$= \Phi(d_{(1)})_{(1)}\Psi(d_{(4)})_{(1)} \otimes \Phi(d_{(1)})_{(2)}\Psi(d_{(2)})\Phi(d_{(3)})\Psi(d_{(4)})_{(2)}$$
$$= \Phi(d_{(1)})_{(1)}\Psi(d_{(3)})_{(1)} \otimes \Phi(d_{(1)})_{(2)}F_{d_{(2)}}\Psi(d_{(3)})_{(2)}$$
$$= E_{d_{(1)}}\Phi(d_{(2)})_{(1)}\Psi(d_{(3)})_{(1)} \otimes \Phi(d_{(2)})_{(2)}\Psi(d_{(3)})_{(2)}$$
$$= \Phi(d_{(1)})\Psi(d_{(2)})\Phi(d_{(3)})_{(1)}\Psi(d_{(4)})_{(1)} \otimes \Phi(d_{(3)})_{(2)}\Psi(d_{(4)})_{(2)}$$
$$= \overline{d}_{(1)[-1]}\overline{d}_{(2)[-1](1)}\varepsilon_C(\overline{d}_{(1)[0]}) \otimes \overline{d}_{(2)[-1](2)}\varepsilon_C(\overline{d}_{(2)[0]})$$

Using the equalities presented above, we can obtain the initial form of the twisting condition (2.3) of Definition 2.1:

$$(\overline{d})_{(1)[-1]}\omega((\overline{d})_{(2)})^1 \otimes (\overline{d})_{(1)[0][-1]}\omega((\overline{d})_{(2)})^2 \otimes (\overline{d})_{(1)[0][0]}$$
$$= \omega(\overline{d}_{(1)})^1\overline{d}_{(2)[-1](1)}\omega'(\overline{d}_{(3)})^1\omega((\overline{d}_{(4)})^1$$
$$\otimes\omega(\overline{d}_{(1)})^2\overline{d}_{(2)[-1](2)}\omega'(\overline{d}_{(3)})^2\omega((\overline{d}_{(4)})^2 \otimes \overline{d}_{(2)[0]}$$
$$= \omega(\overline{d}_{(1)})^1\overline{d}_{(2)[-1](1)}\overline{d}_{(3)[-1]}\overline{d}_{(4)[-1](1)}\varepsilon_C(\overline{d}_{(3)[0]})$$
$$\otimes\omega(\overline{d}_{(1)})^2\overline{d}_{(2)[-1](2)}\overline{d}_{(4)[-1](2)}\varepsilon_C(\overline{d}_{(4)[0]}) \otimes \overline{d}_{(2)[0]}$$
$$= \omega(\overline{d}_{(1)})^1\overline{d}_{(2)[-1]}\varepsilon_C(\overline{d}_{(2)[0]})\overline{d}_{(3)[-1](1)}\overline{d}_{(4)[-1](1)}$$
$$\otimes\omega(\overline{d}_{(1)})^2\overline{d}_{(3)[-1](2)}\overline{d}_{(4)[-1](2)}\varepsilon_C(\overline{d}_{(4)[0]}) \otimes \overline{d}_{(3)[0]}$$
$$= \omega(\overline{d}_{(1)})^1\overline{d}_{(2)[-1](1)} \otimes \omega(\overline{d}_{(1)})^2\overline{d}_{(2)[-1](2)} \otimes \overline{d}_{(2)[0]}.$$

Using again $\omega' * \omega$, we obtain the similar twisting equality for ω'

$$\omega'((\overline{d})_{(1)})^1(\overline{d})_{(2)[-1]} \otimes \omega'((\overline{d})_{(1)})^2(\overline{d})_{(2)[0][-1]} \otimes (\overline{d})_{(2)[0][0]}$$
$$= \overline{d}_{(1)[-1](1)}\omega'(\overline{d}_{(2)})^1 \otimes \overline{d}_{(1)[-1](2)}\omega'(\overline{d}_{(2)})^2 \otimes \overline{d}_{(1)[0]}.$$

Note now that by (c) and (f) of Definition 5.1, we have

$$\overline{d}_{[-1]} \otimes \overline{d}_{[0][-1]}\varepsilon_C(\overline{d}_{[0][0]})$$
$$= \Phi(d_{(1)})\Psi(d_{(4)}) \otimes \Phi(d_{(2)})\Psi(d_{(3)})$$
$$= \Phi(d_{(1)})\Psi(d_{(3)}) \otimes E_{d_{(2)}}$$
$$= (E_{d_{(1)}})_{(1)}\Phi(d_{(2)})\Psi(d_{(3)}) \otimes (E_{d_{(1)}})_{(2)}$$
$$= \overline{d}_{(1)[-1](1)}\varepsilon_C(\overline{d}_{(1)[0]})\overline{d}_{(2)[-1]}\varepsilon_C(\overline{d}_{(2)[0]}) \otimes \overline{d}_{(1)[-1](2)}$$
$$= \overline{d}_{(1)[-1]}\varepsilon_C(\overline{d}_{(1)[0]})\overline{d}_{(2)[-1](1)}\varepsilon_C(\overline{d}_{(2)[0]}) \otimes \overline{d}_{(2)[-1](2)},$$

which gives (2.2) of Definition 2.1. Next, we see that ω absorbs $(\overline{d})_{[-1](1)}\otimes (\overline{d})_{[-1](2)}\, \varepsilon_A((\overline{d})_{[0]})$ on the right:

$$\omega((\overline{d})_{(1)})^1(\overline{d})_{(2)[-1](1)} \otimes \omega((\overline{d})_{(1)})^2(\overline{d})_{(2)[-1](2)}\varepsilon_A((\overline{d})_{(2)[0]})$$
$$= \Phi(d_{(1)})\Psi(d_{(3)})_{(1)}\Phi(d_{(4)})_{(1)}\Psi(d_{(5)})_{(1)} \otimes \Phi(d_{(2)})\Psi(d_{(3)})_{(2)}\Phi(d_{(4)})_{(2)}\Psi(d_{(5)})_{(2)}$$
$$= \Phi(d_{(1)})\Psi(d_{(3)})_{(1)} \otimes \Phi(d_{(2)})\Psi(d_{(3)})_{(2)}$$
$$= \omega(\overline{d}),$$

showing that (2.4) holds. Then using the twisting condition (2.3), we see that

$$
\begin{aligned}
&(\overline{d})_{(1)[-1]}\omega((\overline{d})_{(2)})^1 \otimes (\overline{d})_{(1)[0][-1]}\omega((\overline{d})_{(2)})^2\varepsilon_C((\overline{d})_{(1)[0][0]})\\
&\quad = \omega(\overline{d}_{(1)})^1\overline{d}_{(2)[-1](1)} \otimes \omega(\overline{d}_{(1)})^2\overline{d}_{(2)[-1](2)}\varepsilon_C(\overline{d}_{(2)[0]})\\
&\quad = \omega(\overline{d}).
\end{aligned}
$$

Thus we have that $\omega(\overline{d})$ absorbs $(\overline{d})_{[-1]} \otimes (\overline{d})_{[0][-1]}\varepsilon_C((\overline{d})_{[0][0]})$, $(\overline{d})_{[-1](1)}\otimes (\overline{d})_{[-1](2)}\ \varepsilon_A((\overline{d})_{[0]})$ and $(\overline{d})_{[-1]}\ \varepsilon_A((\overline{d})_{[0]})$ from both sides. In particular, ω is contained in the ideal $\langle f_1 * f_2\rangle$. Similarly, we can check that ω' belongs to $\langle f_1 * f_2\rangle$.

Now, we check the cocycle equality (2.7) for ω as follows:

$$
\begin{aligned}
&\overline{d}_{(1)[-1]}\omega(\overline{d}_{(2)})^1 \otimes \omega(\overline{d}_{(1)[0]})^1\omega(\overline{d}_{(2)})^2_{(1)} \otimes \omega(\overline{d}_{(1)[0]})^2\omega(\overline{d}_{(2)})^2_{(2)}\\
&\quad = \Phi(d_{(1)})\Psi(d_{(3)})\Phi(d_{(4)})\Psi(d_{(6)})_{(1)}\\
&\qquad \otimes\omega(\overline{d}_{(2)})^1\Phi(d_{(5)})_{(1)}\Psi(d_{(6)})_{(2)} \otimes \omega(\overline{d}_{(2)})^2\Phi(d_{(5)})_{(2)}\Psi(d_{(6)})_{(3)}\\
&\quad = \Phi(d_{(1)})(\Psi * \Phi)(d_{(3)})\Psi(d_{(5)})_{(1)}\\
&\qquad \otimes\omega(\overline{d}_{(2)})^1\Phi(d_{(4)})_{(1)}\Psi(d_{(5)})_{(2)} \otimes \omega(\overline{d}_{(2)})^2\Phi(d_{(4)})_{(2)}\Psi(d_{(5)})_{(3)}\\
&\quad = \Phi(d_{(1)})(\Psi * \Phi)(d_{(2)})\Psi(d_{(5)})_{(1)}\\
&\qquad \otimes\omega(\overline{d}_{(3)})^1\Phi(d_{(4)})_{(1)}\Psi(d_{(5)})_{(2)} \otimes \omega(\overline{d}_{(3)})^2\Phi(d_{(4)})_{(2)}\Psi(d_{(5)})_{(3)}\\
&\quad = \Phi(d_{(1)})\Psi(d_{(4)})_{(1)}\\
&\qquad \otimes\omega(\overline{d}_{(2)})^1\Phi(d_{(3)})_{(1)}\Psi(d_{(4)})_{(2)} \otimes \omega(\overline{d}_{(2)})^2\Phi(d_{(3)})_{(2)}\Psi(d_{(4)})_{(3)}\\
&\quad = \Phi(d_{(1)})\Psi(d_{(6)})_{(1)} \otimes \Phi(d_{(2)})\Psi(d_{(4)})_{(1)}\Phi(d_{(5)})_{(1)}\Psi(d_{(6)})_{(2)}\\
&\qquad \otimes\Phi(d_{(3)})\Psi(d_{(4)})_{(2)}\Phi(d_{(5)})_{(2)}\Psi(d_{(6)})_{(3)}\\
&\quad = \Phi(d_{(1)})\Psi(d_{(4)})_{(1)} \otimes \Phi(d_{(2)})\Psi(d_{(4)})_{(2)} \otimes \Phi(d_{(3)})\Psi(d_{(4)})_{(3)}\\
&\quad = \Phi(d_{(1)})\Psi(d_{(3)})_{(1)}\Phi(d_{(4)})_{(1)}\Psi(d_{(6)})_{(1)}\\
&\qquad \otimes\Phi(d_{(2)})\Psi(d_{(3)})_{(2)}\Phi(d_{(4)})_{(2)}\Psi(d_{(6)})_{(2)} \otimes \Phi(d_{(5)})\Psi(d_{(6)})_{(3)}\\
&\quad = \omega(\overline{d}_{(1)})^1\omega(\overline{d}_{(2)})^1_{(1)} \otimes \omega(\overline{d}_{(1)})^2\omega(\overline{d}_{(2)})^1_{(2)} \otimes \omega(\overline{d}_{(2)})^2.
\end{aligned}
$$

This completes the proof of the fact that $(C,\rho^C,(\omega,\omega'))$ is a symmetric twisted partial H-comodule coalgebra.

Finally, we claim that

$$
\begin{aligned}
\Gamma : D &\rightarrow (D/D\cdot H^+)\natural H,\\
d &\mapsto \overline{d}_{(1)}\natural\Phi(d_{(2)})
\end{aligned}
$$

is a coalgebra isomorphism with inverse given by

$$
\begin{aligned}
\Upsilon : (D/D\cdot H^+)\natural H &\rightarrow D,\\
\overline{d}\natural h &\mapsto d_{(1)}\cdot\Psi(d_{(2)})h.
\end{aligned}
$$

In order to show that Γ is a coalgebra homomorphism, it suffices to check that the following diagram is communicative:

$$
\begin{array}{ccc}
D & \xrightarrow{\quad\Gamma\quad} & (D/D\cdot H^+)\otimes H\\
\downarrow{\scriptstyle \Delta_D} & & \downarrow{\scriptstyle \Delta}\\
D\otimes D & \xrightarrow{\Gamma\otimes\Gamma} & (D/D\cdot H^+)\otimes H\otimes(D/D\cdot H^+)\otimes H
\end{array}
$$

Indeed, for all $d \in D$, we have

$$
\begin{aligned}
&(\Gamma \otimes \Gamma) \circ \Delta(d)\\
&= \overline{d}_{(1)} \otimes \Phi(d_{(2)}) \otimes \overline{d}_{(3)} \otimes \Phi(d_{(4)})\\
&= \overline{d}_{(1)} \otimes \Phi(d_{(2)})\Psi(d_{(4)})\Phi(d_{(5)}) \otimes \overline{d}_{(3)} \otimes \Phi(d_{(6)})\\
&= \overline{d}_{(1)} \otimes \Phi(d_{(2)})\Psi(d_{(4)})\Phi(d_{(5)})\\
&\quad \times\Psi(d_{(7)})_{(1)}\Phi(d_{(8)})_{(1)} \otimes \overline{d}_{(3)} \otimes \Phi(d_{(6)})\Psi(d_{(7)})_{(2)}\Phi(d_{(8)})_{(2)}\\
&= \overline{d}_{(1)} \otimes \overline{d}_{(2)[-1]}\Phi(d_{(3)})\Psi(d_{(5)})_{(1)}\\
&\quad \times\Phi(d_{(6)})_{(1)} \otimes \overline{d}_{(2)[0]} \otimes \Phi(d_{(4)})\Psi(d_{(5)})_{(2)}\Phi(d_{(6)})_{(2)}\\
&= \overline{d}_{(1)} \otimes \overline{d}_{(2)[-1]}\omega(\overline{d}_{(3)})^1\Phi(d_{(4)})_{(1)} \otimes \overline{d}_{(2)[0]} \otimes \omega(\overline{d}_{(3)})^2\Phi(d_{(4)})_{(2)}\\
&= (\Delta \circ \Gamma)(d),
\end{aligned}
$$

as desired. For all $d \in D$, we have

$$
\begin{aligned}
(\Upsilon \circ \Gamma)(d) \quad &= d_{(1)} \cdot \Psi(d_{(2)})\Phi(d_{(3)})\\
&= d.
\end{aligned}
$$

Thus it follows that $\Upsilon \circ \Gamma = \iota$. On the other hand, for all $\overline{d} \otimes h \in D/D \cdot H^+$, since

$$
\begin{aligned}
(\Gamma \circ \Upsilon)(\overline{d} \otimes h) \quad &= \Gamma(d_{(1)} \cdot \Psi(d_{(2)})h)\\
&= \overline{d}_{(1)} \otimes \Phi(d_{(2)} \cdot \Psi(d_{(3)})h)\\
&= \overline{d}_{(1)} \otimes \overline{d}_{(2)[-1]}\varepsilon_C(\overline{d}_{(2)[0]})h\\
&= \overline{d}_{(1)}\natural h.
\end{aligned}
$$

we have $\Gamma \circ \Upsilon = \iota$.

This ends the proof. □

Remark 5.4. If $a_{[-1]}\varepsilon_A(a_{[0]}) = \varepsilon_A(a)1_H$, then partial cleft H-coextension just is cleft H-coextension in sense of [14], and partial crossed coproducts just is the ordinary crossed coproducts, and Theorem 5.3 reduces to the main result of [10].

Acknowledgements. We thank the referee for the helpful comments. The work was supported by the National Natural Science Foundation of China (Nos. 1187301, 11471186, 11501317 and 11801304), the Shandong Provincial Natural Science Foundation of China (Nos. ZR2018MA012 and ZR2016AQ03).

References

[1] M.M.S. Alves and E. Batista: *Enveloping actions for partial Hopf actions*, Comm. Algebra, **38** (2010), 2872–2902.
[2] M.M.S. Alves and E. Batista: *Globalization theorems for partial Hopf (co)actions and some of their applications*; in Contemporary Mathematics, **537** (2011), 13–30 .

[3] M.M.S. Alves and E. Batista: *Partial Hopf actions, partial invariants and a Morita context*, Algebra Discrete Math. **3** (2009), 1–19.

[4] M.M.S. Alves, E. Batista, M. Dokuchaev and A. Paques: *Twisted Partial actions of Hopf algebras*, Israel J. Math. **197** (2013), 263–308.

[5] S. Caenepeel and K. Janssen: *Partial (co)actions of Hopf algebras and partial Hopf-Galois theory*, Comm. Algebra **36** (2008), 2923–2946.

[6] S.U. Chase and M.E. Sweedler: Hopf algebras and Galois theory, Springer-Verlag, Berlin-Heilderberg-New York, 1969.

[7] Q.G. Chen and D.G. Wang: *A class of coquasitriangular Hopf group algebras*, Comm. Algebra **44** (2016), 310–335.

[8] Q.G. Chen and D.G. Wang: *Constructing quasitriangular Hopf algebras*, Comm. Algebra **43** (2015), 1698–1722.

[9] Q.G. Chen and D.G. Wang: *Twisted partial coactions of Hopf algebras*, Front Math. China **12** (2017), 63–86.

[10] S. Dascalescu, G. Militaru and S. Raianu: *Crossed coproducts and cleft coextensions*, Comm. Algebra **24** (1996), 1229–1243.

[11] Y. Doi and M. Takeuchi: *Cleft comodule algebras for a bialgebra*, Comm. Algebra **14** (1986), 801–817.

[12] Y. Doi: *Equivalent Crossed Products for a Hopf Algebra*, Comm. Algebra **17** (1989), 3053–3085.

[13] H.F. Kreimer and M. Takeuchi: *Hopf algebras and Galois extension of an algebra*, Indiana Univ. Math. J. **30** (1981), 675–691.

[14] A. Masuoka and Y. Doi: *Generalization of cleft comodule algebras*, Comm. Algebra **20** (1992), 3703–3721.

Q.-G. Chen
School of Mathematical Sciences
Qufu Normal University
Shandong, Qufu 273165
China
e-mail: cqg211@163.com

B. Yang
School of Mathematical Sciences
Qufu Normal University
Shandong, Qufu 273165
China