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Osaka J. Math.
56 (2019), 33–50

ON HOCHSTER'S FORMULA FOR A CLASS OF QUOTIENT SPACES OF MOMENT-ANGLE COMPLEXES

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(Received October 13, 2016, revised August 9, 2017)

Abstract

Any finite simplicial complex $\mathcal{K}$ and a partition of the vertex set of $\mathcal{K}$ determines a canonical quotient space of the moment-angle complex of $\mathcal{K}$. We prove that the cohomology groups of such a space can be computed via some Hochster's type formula, which generalizes the usual Hochster's formula for the cohomology groups of moment-angle complexes. In addition, we show that the stable decomposition of moment-angle complexes can also be extended to such spaces. This type of spaces include all the quasitoric manifolds that are pullback from the linear models. And we prove that the moment-angle complex associated to a finite simplicial poset is always homotopy equivalent to one of such spaces.

1. Introduction

An *abstract simplicial complex* on a set $[m] = \{v_1, \cdots, v_m\}$ is a collection $\mathcal{K}$ of subsets $\sigma \subseteq [m]$ such that if $\sigma \in \mathcal{K}$, then any subset of σ also belongs to $\mathcal{K}$. We always assume that the empty set belongs to $\mathcal{K}$ and refer to $\sigma \in \mathcal{K}$ as an abstract *simplex* of $\mathcal{K}$. The simplex corresponding to the empty set is denoted by $\hat{\mathbf{0}}$. In particular, any element of $[m]$ is called a *vertex* of $\mathcal{K}$. We call the number of vertices of a simplex σ the *rank* of σ, denoted by $\mathrm{rank}(\sigma)$. Let $\dim(\sigma)$ denote the dimension of a simplex σ. So $\mathrm{rank}(\sigma) = \dim(\sigma) + 1$.

Any finite abstract simplicial complex $\mathcal{K}$ admits a *geometric realization* in some Euclidean space. But sometimes we also use $\mathcal{K}$ to denote its geometric realization when the meaning is clear in the context.

Given a finite abstract simplicial complex $\mathcal{K}$ on a set $[m]$ and a pair of spaces (X, A) with $A \subset X$, we can construct of a topological space $(X, A)^{\mathcal{K}}$ by:

$$(1) \qquad (X,A)^{\mathcal{K}} = \bigcup_{\sigma\in\mathcal{K}} (X,A)^{\sigma}, \text{ where } (X,A)^{\sigma} = \prod_{v_j\in\sigma} X \times \prod_{v_j\notin\sigma} A.$$

The symbol $\prod$ here and in the rest of this paper means Cartesian product. So $(X, A)^{\mathcal{K}}$ is a subspace of the Cartesian product of m copies of X. It is called the *polyhedral product* or the *generalized moment-angle complex* of $\mathcal{K}$ and (X, A). In particular, $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$ and $\mathbb{R}\mathcal{Z}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$ are called the *moment-angle complex* and *real moment-angle complex* of $\mathcal{K}$, respectively (see [5]). Moreover, we can define the polyhedral product $(\underline{\mathbb{X}}, \underline{\mathbb{A}})^{\mathcal{K}}$ of $\mathcal{K}$ with m pairs of spaces $(\underline{\mathbb{X}}, \underline{\mathbb{A}}) = \{(X_1, A_1), \cdots, (X_m, A_m)\}$ (see [2] or [6, Sec 4.2]).

2010 Mathematics Subject Classification. 57S15, 14M25.

The author is partially supported by Natural Science Foundation of China (grant no.11371188) and the PAPD (priority academic program development) of Jiangsu higher education institutions.

Originally, $\mathcal{Z}_{\mathcal{K}}$ and $\mathbb{R}\mathcal{Z}_{\mathcal{K}}$ were constructed by Davis and Januszkiewicz [9] in a different way. We will only explain the construction of $\mathcal{Z}_{\mathcal{K}}$ below (the $\mathbb{R}\mathcal{Z}_{\mathcal{K}}$ case is completely parallel). Let $\mathcal{K}'$ denote the barycentric subdivision of $\mathcal{K}$. We can consider $\mathcal{K}'$ as the set of chains of simplices in $\mathcal{K}$ ordered by inclusions. For each simplex $\sigma \in \mathcal{K}$, let F_σ denote the geometric realization of the poset $\mathcal{K}_{\geq\sigma} = \{\tau \in \mathcal{K} \,|\, \sigma \subseteq \tau\}$. Thus, F_σ is the subcomplex of $\mathcal{K}'$ consisting of all simplices of the form $\sigma = \sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_l$. Let $P_{\mathcal{K}}$ denote the cone on $\mathcal{K}'$. If σ is a $(k-1)$-simplex, then we say that $F_\sigma \subset P_{\mathcal{K}}$ is a face of codimension k in $P_{\mathcal{K}}$. The polyhedron $P_{\mathcal{K}}$ together with its decomposition into "faces" $\{F_\sigma\}_{\sigma\in\mathcal{K}}$ is called a *simple polyhedral complex* (see [9, p.428]).

Let $V(\mathcal{K})$ denote the vertex set of $\mathcal{K}$. Any map $\lambda : V(\mathcal{K}) \to \mathbb{Z}^r$ is called a $\mathbb{Z}^r$*-coloring of* $\mathcal{K}$, and any element of $\mathbb{Z}^r$ is called a *color*. For any simplex $\sigma \in \mathcal{K}$,

- let $V(\sigma)$ denote the vertex set of σ and,
- let $G_\lambda(\sigma)$ denote the toral subgroup of $T^r = (S^1)^r$ corresponding to the subgroup of $\mathbb{Z}^r$ generated by $\{\lambda(v) \,|\, v \in V(\sigma)\}$.

Given a $\mathbb{Z}^r$-coloring λ of $\mathcal{K}$, we obtain a space $X(\mathcal{K}, \lambda)$ defined by

$$X(\mathcal{K}, \lambda) := P_{\mathcal{K}} \times T^r / \sim \tag{2}$$

where $(p, g) \sim (p', g')$ whenever $p' = p \in F_\sigma$ and $g'g^{-1} \in G_\lambda(\sigma)$ for some $\sigma \in \mathcal{K}$.

In particular, if $r = |V(\mathcal{K})| = m$ and $\{\lambda(v_j)\,;\, 1 \leq i \leq m\}$ is a basis of $\mathbb{Z}^m$, $X(\mathcal{K}, \lambda)$ is homeomorphic to $\mathcal{Z}_{\mathcal{K}}$. Let $\pi_{\mathcal{K}} : P_{\mathcal{K}} \times T^m \to \mathcal{Z}_{\mathcal{K}}$ be the corresponding quotient map in (2). There is a *canonical action* of T^m on $\mathcal{Z}_{\mathcal{K}}$ defined by:

$$g' \cdot \pi_{\mathcal{K}}(p, g) = \pi_{\mathcal{K}}(p, gg'),\ p \in P_{\mathcal{K}},\ g, g' \in T^m. \tag{3}$$

Then any subgroup of T^m acts canonically on $\mathcal{Z}_{\mathcal{K}}$ through this action.

The following is another way to view the canonical T^m-action on $\mathcal{Z}_{\mathcal{K}}$. Recall

$$\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}} = \bigcup_{\sigma\in\mathcal{K}} \Big(\prod_{v_j\in\sigma} D^2_{(j)} \times \prod_{v_j\notin\sigma} S^1_{(j)}\Big) \subset \prod_{v_j\in[m]} D^2_{(j)} \tag{4}$$

where $D^2_{(j)}$ and $S^1_{(j)}$ are the copy of D^2 and S^1 associated to v_j. Notice that $D^1_{(j)} = S^1_{(j)} * v_j$ (the *join* of $S^1_{(j)}$ with v_j). So we can write

$$\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}} = \bigcup_{\sigma\in\mathcal{K}} \Big(\prod_{v_j\in\sigma} S^1_{(j)} * v_j \times \prod_{v_j\notin\sigma} S^1_{(j)}\Big). \tag{5}$$

We can identify $S^1_{(j)}$ with the j-th S^1-factor in $T^m = (S^1)^m$. Then for any $(g_1, \cdots, g_m) \in T^m$, let g_j act on $S^1_{(j)}$ through left translations. This is equivalent to the canonical T^m-action on $\mathcal{Z}_{\mathcal{K}}$ defined by (3).

For any map $\lambda : V(\mathcal{K}) \to \mathbb{Z}^r$ whose image spans the whole $\mathbb{Z}^r$, we can view the space $X(\mathcal{K}, \lambda)$ in (2) as a quotient space of $\mathcal{Z}_{\mathcal{K}}$ by a toral sbugroup of T^m. Indeed, let $\{e_1, \cdots, e_m\}$ be a unimodular basis of $\mathbb{Z}^m$ and define a group homomorphism

$$\rho_\lambda : \mathbb{Z}^m \longrightarrow \mathbb{Z}^r,\ \rho_\lambda(e_j) = \lambda(v_j),\ 1 \leq i \leq m.$$

The kernel of ρ_λ is a subgroup of $\mathbb{Z}^m$ which determines an $(m-r)$-dimensional toral subgroup $H_\lambda \subset T^m$. It is easy to see that $X(\mathcal{K}, \lambda)$ is homeomorphic to the quotient space $\mathcal{Z}_{\mathcal{K}}/H_\lambda$, where H_λ acts on $\mathcal{Z}_\mathcal{K}$ via the canonical T^m-action. Note that the action of H_λ on $\mathcal{Z}_\mathcal{K}$ is not necessarily free.

The cohomology groups of $\mathcal{Z}_\mathcal{K}$ can be computed via a Hochster's type formula as follows (see [4] or [5]). For any subset $J \subset [m]$, let $\mathcal{K}_J$ denote the full subcomplex of $\mathcal{K}$ obtained by restricting to J. Let $\mathbf{k}$ denote a field or $\mathbb{Z}$ below. We have

$$H^q(\mathcal{Z}_\mathcal{K};\mathbf{k}) = \bigoplus_{J\subset[m]} \widetilde{H}^{q-|J|-1}(\mathcal{K}_J;\mathbf{k}),\ q \geq 0. \tag{6}$$

where $\widetilde{H}^*(\mathcal{K}_J;\mathbf{k})$ is the reduced cohomology groups of $\mathcal{K}_J$ and $|J|$ denotes the number of elements in J. Here we adopt the convention that

$$\widetilde{H}^{-1}(\mathcal{K}_\varnothing;\mathbf{k}) = \mathbf{k}.$$

Moreover, it is shown in [4] and [5] that there is a natural bigrading on $H^*(\mathcal{Z}_\mathcal{K};\mathbf{k})$ so that it is isomorphic to $\mathrm{Tor}_{\mathbf{k}[v_1,\cdots,v_m]}(\mathbf{k}[\mathcal{K}];\mathbf{k})$ as bigraded algebras, where $\mathbf{k}[\mathcal{K}]$ is the face ring (or Stanley-Reisner ring) of $\mathcal{K}$ over $\mathbf{k}$ (also see [10]).

The cohomology rings of free quotient spaces of $\mathcal{Z}_\mathcal{K}$ can be reprsented in a similar way. Suppose a subtorus $H \subset T^m$ acts freely on $\mathcal{Z}_\mathcal{K}$ through the canonical action. It is shown in [15] (also see [5, Theorem 7.37]) that there is a graded algebra isomorphism

$$H^*(\mathcal{Z}_\mathcal{K}/H;\mathbf{k}) \cong \mathrm{Tor}_{H^*(B(T^m/H);\mathbf{k})}(\mathbf{k}[\mathcal{K}];\mathbf{k}) \tag{7}$$

where $B(T^m/H)$ is the classifying space for the principal T^m/H-bundle. However, $\mathrm{Tor}_{H^*(B(T^m/H);\mathbf{k}}(\mathbf{k}[\mathcal{K}];\mathbf{k})$ is not so easy to compute in practice and it is not clear whether there exists a Hochster's type formula for $H^*(\mathcal{Z}_P/H;\mathbf{k})$ in general as we have for $H^*(\mathcal{Z}_P;\mathbf{k})$ in (6).

Remark 1.1. For the calculation of the cohomology ring structure of general polyhedral products, the reader is referred to [2, 3, 19].

An important class of quotient spaces of moment-angle complexes are quasitoric manifolds. Let $\mathcal{K}_P$ be the simplicial sphere that is dual to an n-dimensional simple convex polytope P with m facets. Then $\mathcal{Z}_P = \mathcal{Z}_{\mathcal{K}_P}$ is an $(m+n)$-dimensional closed connected manifolds, called the *moment-angle manifold* of P. Suppose $H \cong T^{m-n}$ is a subgroup of T^m that acts freely on $\mathcal{Z}_P$ through the canonical action, the quotient space $\mathcal{Z}_P/H$ is called a *quasitoric manifold* over P. Quasitoric manifolds are introduced by Davis and Januszkiewicz in [9].

In this paper, we study a special class of quotient spaces of $\mathcal{Z}_\mathcal{K}$ and show that their cohomology groups can indeed be computed via some Hochster's type formula. These spaces are defined as follows.

- Let $\alpha = \{\alpha_1,\cdots,\alpha_k\}$ be a *partition* of the vertex set $V(\mathcal{K})$ of a simplicial complex $\mathcal{K}$, i.e. α_i's are disjoint nonempty subsets of $V(\mathcal{K})$ with $\alpha_1 \cup \cdots \cup \alpha_k = V(\mathcal{K})$.
- Let $\{\tilde{e}_1,\cdots,\tilde{e}_k\}$ be a basis of $\mathbb{Z}^k$. We define a $\mathbb{Z}^k$-coloring of $\mathcal{K}$, denoted by λ_α, which assigns $\tilde{e}_i$ to all the vertices in α_i $(1 \leq i \leq k)$.

Then we obtain a space $X(\mathcal{K},\lambda_\alpha)$ via construction (2), which can be thought of as a quotient space of $\mathcal{Z}_\mathcal{K}$ by the action of a rank $m-k$ subtorus H_{λ_α} of T^m. Note that it is

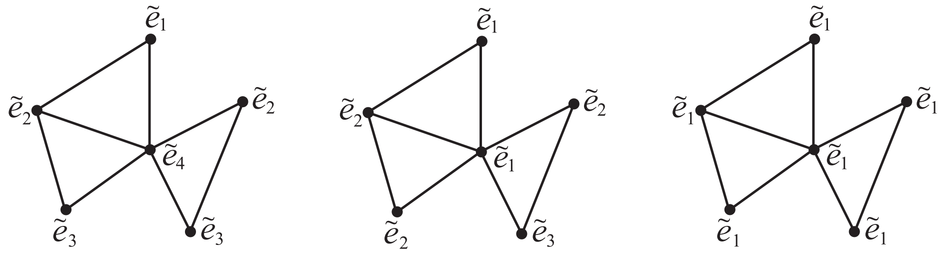

Fig. 1. Examples of $X(\mathcal{K}, \lambda_\alpha)$

possible that the two vertices of a 1-simplex in $\mathcal{K}$ are assigned the same color (see Figure 1 for example).

Let α^* denote the *trivial partition* of $V(\mathcal{K})$, i.e. $\alpha^* = (\alpha_1, \cdots, \alpha_m)$ where each $\alpha_j = \{v_j\}$ consists of only one vertex of $\mathcal{K}$. Then according to our definition,

$$X(\mathcal{K}, \lambda_{\alpha^*}) = \mathcal{Z}_{\mathcal{K}}.$$

For a non-trivial partition α of $V(\mathcal{K})$, the space $X(\mathcal{K}, \lambda_\alpha)$ is a priori not a moment-angle complex of any kind. But we will see that some topological properties of $X(\mathcal{K}, \lambda_\alpha)$ are very similar to moment-angle complexes. In particular, the cohomology groups of these spaces can also be computed by some Hochster's type formula as we do for moment-angle complexes.

Let $[k] = \{1, \cdots, k\}$. One should keep in mind the difference between $[k]$ and the vertex set $[m]$ of $\mathcal{K}$. For any simplex $\sigma \in \mathcal{K}$, let

$$\mathrm{I}_\alpha(\sigma) := \{i \in [k]\,;\, V(\sigma) \cap \alpha_i \neq \varnothing\} \subset [k],$$

which just tells us the set of colors on the vertices of σ defined by λ_α. Obviously we have $0 \leq |\mathrm{I}_\alpha(\sigma)| \leq \mathrm{rank}(\sigma)$. For any subset $\mathrm{L} \subset [k]$, define

$$\mathcal{K}_{\alpha,\mathrm{L}} := \text{the subcomplex of } \mathcal{K} \text{ consisting of } \{\sigma \in \mathcal{K}\,;\, \mathrm{I}_\alpha(\sigma) \subset \mathrm{L}\}. \tag{8}$$

The main results of this paper are the following two theorems.

Theorem 1.2. *Let $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ be a partition of the vertex set of a finite simplicial complex $\mathcal{K}$. Then we have group isomorphisms:*

$$H^q(X(\mathcal{K}, \lambda_\alpha); \mathbf{k}) \cong \bigoplus_{\mathrm{L} \subset [k]} \widetilde{H}^{q-|\mathrm{L}|-1}(\mathcal{K}_{\alpha,\mathrm{L}}; \mathbf{k}),\ \forall q \geq 0.$$

Note that the above formula for α^* gives (6). If the action of $H_{\lambda_\alpha} \subset T^m$ on $\mathcal{Z}_{\mathcal{K}}$ is free, we obtain from (7) that $H^*(X(\mathcal{K}, \lambda_\alpha); \mathbf{k}) \cong \mathrm{Tor}_{H^*(B(T^m/H_{\lambda_\alpha});\mathbf{k})}(\mathbf{k}[\mathcal{K}]; \mathbf{k})$. In this case the Tor-module splits by the refined $\mathbb{Z}^k$-graded components and then Theorem 1.2 gives the Hochster's formula for it. But generally $X(\mathcal{K}, \lambda_\alpha)$ may not be a free quotient of $\mathcal{Z}_{\mathcal{K}}$. For example let $\mathcal{K} = \partial\Delta^2$ be the boundary of a 2-simplex and $k = 1$, then λ_α assigns the same color $\tilde{e}_1$ to all the vertices of $\partial\Delta^2$. It is easy to see that $X(\partial\Delta^2, \lambda_\alpha)$ is not even a closed manifold while $\mathcal{Z}_{\partial\Delta^2} \cong S^5$. So in general we can not directly apply (7) to compute $H^*(X(\mathcal{K}, \lambda_\alpha); \mathbf{k})$.

In addition, it was shown in [2, Corollary 2.23] that the Hochster's formula for the cohomology groups of $\mathcal{Z}_{\mathcal{K}}$ follows from a stable decomposition of $\mathcal{Z}_{\mathcal{K}}$. We have parallel results for $X(\mathcal{K}, \lambda_\alpha)$ as well.

Theorem 1.3. *Let $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ be a partition of the vertex set of a finite simplicial complex $\mathcal{K}$. There are homotopy equivalences:*

$$\mathbf{\Sigma}(X(\mathcal{K}, \lambda_\alpha)) \simeq \bigvee_{\mathrm{L}\subset[k]} \mathbf{\Sigma}^{|\mathrm{L}|+2}(\mathcal{K}_{\alpha,\mathrm{L}})$$

where the bold $\mathbf{\Sigma}$ denotes the reduced suspension.

The paper is organized as follows. In section 2, we construct some natural cell decomposition of $X(\mathcal{K}, \lambda_\alpha)$ and use it to compute the the cohomology groups of $X(\mathcal{K}, \lambda_\alpha)$, which leads to a proof of Theorem 1.2. In section 3, we use the same strategy in [2] to study the stable decompositions of $X(\mathcal{K}, \lambda_\alpha)$ and give a proof of Theorem 1.3. In section 4, we show that the moment-angle complex of any finite simplicial poset $\mathcal{S}$ is homotopy equivalent to $X(\mathcal{K}, \lambda_\alpha)$ for some finite simplicial complex $\mathcal{K}$ and a partition α of $V(\mathcal{K})$. In section 5, we generalize our results on $X(\mathcal{K}, \lambda_\alpha)$ to a wider range of spaces.

2. Cohomology groups of $X(\mathcal{K}, \lambda_\alpha)$

Suppose the vertex set of $\mathcal{K}$ is $[m] = \{v_1, \cdots, v_m\}$. Let $\Delta^{[m]}$ be the simplex with vertex set $[m]$. For a partition $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ of $[m]$, let Δ^{α_i} denote the face of $\Delta^{[m]}$ whose vertex set is α_i. Then $\mathcal{K}$ can be thought of as a simplicial subcomplex of $\Delta^{[m]}$. Next, we construct a cell decomposition of $X(\mathcal{K}, \lambda_\alpha)$.

2.1. The cell decomposition of $X(\mathcal{K}, \lambda_\alpha)$.

According to the construction of $X(\mathcal{K}, \lambda_\alpha)$ in (2), it is easy to see that $X(\mathcal{K}, \lambda_\alpha)$ is homeomorphic to the quotient space of $\mathcal{Z}_{\mathcal{K}}$ by the canonical action of the toral subgroup H_{λ_α} of T^m corresponding to the subgroup of $\mathbb{Z}^m = \langle e_1, \cdots, e_m\rangle$ generated by the set

$$\{e_j - e_{j'} \,|\, v_j, v_{j'} \in \alpha_i \text{ for some } 1 \le i \le k\} \subset \mathbb{Z}^m.$$

In other words, the action H_{λ_α} on $\mathcal{Z}_{\mathcal{K}} = P_{\mathcal{K}} \times T^m/\sim$ identifies the $S^1_{(j)}$ and $S^1_{(j')}$ in $T^m = S^1_{(1)} \times \cdots \times S^1_{(m)}$ whenever v_j and $v_{j'}$ belong to the same α_i. Considering the partition α of the vertex set of $\mathcal{K}$, we can rewrite the decomposition of $\mathcal{Z}_{\mathcal{K}}$ in (5) as:

$$\mathcal{Z}_{\mathcal{K}} = \bigcup_{\sigma\in\mathcal{K}} \Big(\big(\prod_{i\in \mathrm{I}_\alpha(\sigma)} \prod_{v_j\in V(\sigma)\cap\alpha_i} S^1_{(j)} * v_j \big) \times \prod_{v_j\notin\sigma} S^1_{(j)} \Big). \tag{9}$$

Then with respect to this decomposition of $\mathcal{Z}_{\mathcal{K}}$, we obtain a decomposition of $X(\mathcal{K}, \lambda_\alpha)$ by Lemma 2.1 below.

$$\begin{aligned} X(\mathcal{K}, \lambda_\alpha) &= \bigcup_{\sigma\in\mathcal{K}} \Big(\prod_{i\in\mathrm{I}_\alpha(\sigma)} \big(S^1_{(i)} * \mathop{\ast}_{v\in V(\sigma)\cap\alpha_i} v\big) \times \prod_{i\in[k]\backslash \mathrm{I}_\alpha(\sigma)} S^1_{(i)} \Big) \\ &= \bigcup_{\sigma\in\mathcal{K}} \Big(\prod_{i\in\mathrm{I}_\alpha(\sigma)} S^1_{(i)} * (\sigma\cap\Delta^{\alpha_i}) \times \prod_{i\in[k]\backslash \mathrm{I}_\alpha(\sigma)} S^1_{(i)} \Big) \subset \prod_{i\in[k]} S^1_{(i)} * \Delta^{\alpha_i} \end{aligned} \tag{10}$$

where $S^1_{(i)}$ is a copy of S^1 corresponding to $i \in [k]$, which can be considered as the join of S^1 with the empty face of Δ^{α_i}.

Lemma 2.1. *If we identify all the S^1 factors in a product $(S^1 * v_1) \times \cdots \times (S^1 * v_s)$, the quotient space $(S^1 * v_1) \times \cdots \times (S^1 * v_s)/\sim$ is homeomorphic to $S^1 * (v_1 * \cdots * v_s)$. Note that $v_1 * \cdots * v_s$ can be identified with a simplex whose vertex set is $\{v_1, \cdots, v_s\}$.*

Proof. The points in $(S^1 * v_1) \times \cdots \times (S^1 * v_s)$ can be written as

$$((t_1 v_1 + (1 - t_1)x_1), \cdots, (t_s v_s + (1 - t_s)x_s)),\ x_i \in S^1, 0 \leq t_i \leq 1, 1 \leq i \leq s.$$

In we identify all the S^1 factors in the above product, the points in the quotient space can be written as

$$P_{t_1,\cdots,t_s,x} = ((t_1 v_1 + (1 - t_1)x), \cdots, (t_s v_s + (1 - t_s)x)),\ x \in S^1, 0 \leq t_i \leq 1, 1 \leq i \leq s.$$

Then mapping any $P_{t_1,\cdots,t_s,x}$ to the point $\hat{P}_{t_1,\cdots,t_s,x}$ in $S^1 * (v_1 * \cdots * v_s)$ below

$$\hat{P}_{t_1,\cdots,t_s,x} = \frac{t_1}{s} v_1 + \cdots + \frac{t_s}{s} v_s + \frac{1 - t_1 - \cdots - t_s}{s} x$$

defines a homeomorphism from $(S^1 * v_1) \times \cdots \times (S^1 * v_s)/\sim$ to $S^1 * (v_1 * \cdots * v_s)$. $\square$

REMARK 2.2. We see from (10) that the building blocks of $X(\mathcal{K}, \lambda_\alpha)$ are spaces obtained by mixtures of Cartesian products and joins of some simple spaces (i.e. points and S^1). The building blocks of polyhedral products $(X, A)^{\mathcal{K}}$, however, only involve Cartesian products of spaces. In addition, we have *polyhedral join* (see [1]) and *polyhedral smash product* (see [2]) whose building blocks only involve joins and smash products, respectively. It should be interesting to study spaces whose building blocks involve mixtures of Cartesian products, joins and smash products.

To obtain a cell decomposition of $X(\mathcal{K}, \lambda_\alpha)$ from (10), we need to choose a cell decomposition of the torus T^k. First of all, a circle $S^1 = \{z \in \mathbb{C}\,; |z| = 1\}$ has a natural cell decomposition $\{e^0, e^1\}$ where $e^0 = \{1\} \in S^1$ and $e^1 = S^1 \backslash e^0$. We consider T^k as the product $\prod_{i=1}^k S^1_{(i)}$ and equip T^k with the product cell structure (see [11, 3.B]). Then the cells in T^k can be indexed by subsets of $[k] = \{1, \cdots, k\}$. More specifically, any subset $\mathrm{L} \subset [k]$ determines a unique cell U_{L} in T^k where

$$U_{\mathrm{L}} = \prod_{i \in \mathrm{L}} e^1_{(i)} \times \prod_{i \in [k]\backslash \mathrm{L}} e^0_{(i)},\ \dim(U_{\mathrm{L}}) = |\mathrm{L}|.$$

Here $e^0_{(i)}, e^1_{(i)}$ denote the cells in $S^1_{(i)}$ for each $i \in [k]$.

Observe that for any $\sigma \in \mathcal{K}$, $\prod_{i \in \mathrm{I}_\alpha(\sigma)} S^1_{(i)} * (\sigma \cap \Delta^{\alpha_i})$ is homeomorphic to a closed ball of dimension $\mathrm{rank}(\sigma) + |\mathrm{I}_\alpha(\sigma)|$. Now for each $\mathrm{L} \subset [k] \backslash \mathrm{I}_\alpha(\sigma)$, define

$$B_{(\sigma,\mathrm{L})} := \text{the relative interior of} \Big(\prod_{i \in \mathrm{I}_\alpha(\sigma)} S^1_{(i)} * (\sigma \cap \Delta^{\alpha_i}) \Big) \times U_{\mathrm{L}}.$$

Then from (10), a cell decomposition of $X(\mathcal{K}, \lambda_\alpha)$ is given by:

$$\mathscr{B}_\alpha(\mathcal{K}) := \{B_{(\sigma,\mathrm{L})} \,|\, \sigma \in \mathcal{K},\ \mathrm{L} \subset [k] \backslash \mathrm{I}_\alpha(\sigma)\}. \tag{11}$$

Note that $B_{(\sigma,\mathrm{L})}$ is an open cell of dimension $\mathrm{rank}(\sigma) + |\mathrm{I}_\alpha(\sigma)| + |\mathrm{L}|$.

2.2. The cochain complex of $X(\mathcal{K},\lambda_\alpha)$. For any coefficient ring $\mathbf{k}$, let $C^*(X(\mathcal{K},\lambda_\alpha);\mathbf{k})$ be the cellular cochain complex corresponding to the cell decomposition $\mathscr{B}_\alpha(\mathcal{K})$. If we want to write the boundary maps of the cochains in $C^*(X(\mathcal{K},\lambda_\alpha);\mathbf{k})$, we need to put orientations on the base elements. To do this, we need to first assign orientations to all the simplicies of $\mathcal{K}$. For convenience, we put a total ordering $\prec$ on the vertex set $\{v_1,\cdots,v_m\}$ of $\Delta^{[m]}$ so that they appear in the increasing order in α_1 until α_k. In other words, for any $1\le i<k$ all the vertices of Δ^{α_i} have less order than the vertices in $\Delta^{\alpha_{i+1}}$. Moreover, the vertex-ordering of $\Delta^{[m]}$ induces a vertex-ordering of any simplex $\omega\in\Delta^{[m]}$, which determines an orientation of ω. Then the boundary of ω is

$$\partial\omega = \sum_{\substack{\sigma\subset\omega\\ \dim(\sigma)=\dim(\omega)-1}} \varepsilon(\sigma,\omega)\,\sigma. \tag{12}$$

Here if $V(\omega)=V(\sigma)\cup\{v\}$, then $\varepsilon(\sigma,\omega)$ is equal to $(-1)^{l(v,\omega)}$ where $l(v,\omega)$ is the number vertices of ω that are less than v with respect to the vertex-ordering $\prec$.

The following definition is very useful for us to simplify our argument later.

DEFINITION 2.3(SIMPLEX WITH A GHOST FACE). For any $m\ge 1$, let $\hat{0}$ denote the empty face of $\Delta^{[m]}$ (distinguished from the empty simplex $\hat{\mathbf{0}}$ in $\mathcal{K}$). In addition, we attach a *ghost face* $-\hat{1}$ to $\Delta^{[m]}$ with the following conventions.

- $\dim(\hat{0})=\dim(-\hat{1})=-1$, $\mathrm{rank}(\hat{0})=\mathrm{rank}(-\hat{1})=0$.
- The interiors of $\hat{0}$ and $-\hat{1}$ are themselves.
- The boundary of any vertex of $\Delta^{[m]}$ is $\hat{0}$.
- The boundaries of $\hat{0}$ and $-\hat{1}$ are empty.

In the rest of the paper we use $\widehat{\Delta}^{[m]}$ to denote $\Delta^{[m]}$ with the ghost face $-\hat{1}$.

Let $\{e^0,e^1\}$ be the cell decomposition of S^1 where $\dim(e^0)=0$, $\dim(e^1)=1$, and e^0, e^1 are both oriented. Then given an orientation of each face of $\Delta^{[m]}$, we obtain an oriented cell decomposition of $S^1*\Delta^{[m]}$ by

$$\{e^0\},\ \{e^1\},\ \{S^1*\sigma^\circ\,|\,\sigma \text{ is a nonempty simplex in } \Delta^{[m]}\}.$$

If we formally define $S^1*\hat{0}=e^1$ and $S^1*-\hat{1}=e^0$, we can write a basis of the cellular chain complex $C_*(S^1*\Delta^{[m]};\mathbf{k})$ as $\{S^1*\sigma^\circ\,|\,\sigma\in\widehat{\Delta}^{[m]}\}$ where the orientation of $S^1*\sigma^\circ$ is determined canonically by the orientations of S^1 and σ.

Let $\{y^\sigma\,|\,\sigma\in\widehat{\Delta}^{[m]}\}$ be a basis for the cellular cochain complex $C^*(S^1*\Delta^{[m]};\mathbf{k})$, where y^σ is the dual of $S^1*\sigma^\circ$. For any nonempty simplex σ in $\Delta^{[m]}$,

$$d(y^\sigma) = \sum_{\substack{\sigma\subset\tau\\ \dim(\tau)=\dim(\sigma)+1}} \varepsilon(\sigma,\tau)\cdot y^\tau. \tag{13}$$

In addition, we have

$$d(y^{-\hat{1}})=0,\ \ d(y^{\hat{0}})=\sum_{v\in[m]} y^v. \tag{14}$$

Note that $y^{-\hat{1}}$ and $y^{\hat{0}}$ are cochains in dimension 0 and 1, respectively.

By setting the σ in (10) to the big simplex $\Delta^{[m]}$, we obtain a homeomorphism

$$X(\Delta^{[m]},\lambda_\alpha)\cong\prod_{i\in[k]}S^1*\Delta^{\alpha_i}.$$

Let $X(\Delta^{[m]},\lambda_\alpha)$ be equipped with the product cell structure of each $S^1*\Delta^{\alpha_i}$. The corresponding cellular cochain complex $C^*(X(\Delta^{[m]},\lambda_\alpha);\mathbf{k})$ has a basis

$$\{\mathbf{y}^\Phi=y^{\sigma_1}\times\cdots\times y^{\sigma_k};\ \Phi=(\sigma_1,\cdots,\sigma_k),\ \text{where } \sigma_i\in\widehat{\Delta}^{\alpha_i},\ 1\le i\le k\}.$$

We need to introduce two more notations for our argument below.

- For any vertex v of $\mathcal{K}$ and a partition $\alpha=\{\alpha_1,\cdots,\alpha_k\}$ of $V(\mathcal{K})$, let $i_\alpha(v)\in[k]$ denote the index so that v belongs to the subset $\alpha_{i_\alpha(v)}$.
- For any $i\in\mathrm{L}\subset[k]$, define $\kappa(i,\mathrm{L})=(-1)^{r(i,\mathrm{L})}$, where $r(i,\mathrm{L})$ is the number of elements in L less than i. Moreover, for any simplex $\sigma\in\mathcal{K}_{\alpha,\mathrm{L}}$ we define

$$\kappa(\sigma,\mathrm{L}):=\prod_{v\in V(\sigma)}\kappa(i_\alpha(v),\mathrm{L}). \tag{15}$$

$$\text{So if } V(\omega)=V(\sigma)\cup\{v\}, \text{ we have } \kappa(\omega,\mathrm{L})=\kappa(\sigma,\mathrm{L})\cdot\kappa(i_\alpha(v),\mathrm{L}). \tag{16}$$

The differential of $\mathbf{y}^\Phi$ in $C^*(X(\Delta^{[m]},\lambda_\alpha);\mathbf{k})$ is given by:

$$d(\mathbf{y}^\Phi):=\sum_{1\le i\le k}\iota(\Phi,\sigma_i)\,y^{\sigma_1}\times\cdots\times dy^{\sigma_i}\times\cdots\times y^{\sigma_k}, \tag{17}$$

where $\iota(\Phi,\sigma_i)=(-1)^{\sum_{l=1}^{i-1}\dim(y^{\sigma_l})}$. For a simplex $\sigma\in\Delta^{[m]}$ and $\mathrm{J}\subset[k]\backslash\mathrm{I}_\alpha(\sigma)$, let

$$\Phi_\sigma^{\mathrm{J}}=(\sigma_1^{\mathrm{J}},\cdots,\sigma_k^{\mathrm{J}})$$

$$\text{where }\ \sigma_i^{\mathrm{J}}=\begin{cases}\sigma\cap\Delta^{\alpha_i}, & i\in\mathrm{I}_\alpha(\sigma);\\ \hat{0}\in\widehat{\Delta}^{\alpha_i}, & i\in\mathrm{J};\\ -\hat{1}\in\widehat{\Delta}^{\alpha_i}, & i\in[k]\backslash(\mathrm{I}_\alpha(\sigma)\cup\mathrm{J});\end{cases} \tag{18}$$

So by our definition when $\Phi=\Phi_\sigma^{\mathrm{J}}=(\sigma_1^{\mathrm{J}},\cdots,\sigma_k^{\mathrm{J}})$, we have

$$\dim(y^{\sigma_i^{\mathrm{J}}})=\begin{cases}\mathrm{rank}(\sigma_i^{\mathrm{J}})+1, & i\in\mathrm{I}_\alpha(\sigma);\\ 1, & i\in\mathrm{J};\\ 0, & i\in[k]\backslash(\mathrm{I}_\alpha(\sigma)\cup\mathrm{J}).\end{cases}$$

For $\Phi=\Phi_\sigma^{\mathrm{J}}$, the formula (17) reads:

$$\begin{aligned}d(\mathbf{y}^{\Phi_\sigma^{\mathrm{J}}}) &= \sum_{1\le i\le k}\iota(\Phi_\sigma^{\mathrm{J}},\sigma_i^{\mathrm{J}})\,y^{\sigma_1^{\mathrm{J}}}\times\cdots\times dy^{\sigma_i^{\mathrm{J}}}\times\cdots\times y^{\sigma_k^{\mathrm{J}}}\\ &\overset{(13)}{=} \sum_{1\le i\le k}\iota(\Phi_\sigma^{\mathrm{J}},\sigma_i^{\mathrm{J}})\,y^{\sigma_1^{\mathrm{J}}}\times\cdots\times\Big(\sum_{\substack{\sigma_i^{\mathrm{J}}\subset\tau\subset\Delta^{\alpha_i}\\ \dim(\tau)=\dim(\sigma)+1}}\varepsilon(\sigma_i^{\mathrm{J}},\tau)\cdot y^\tau\Big)\times\cdots\times y^{\sigma_k^{\mathrm{J}}}.\end{aligned}$$

Note that if $V(\omega)=V(\sigma)\cup\{v\}$ where $v\in\alpha_i$, then $\omega_i^{\mathrm{J}}=\sigma_i^{\mathrm{J}}\cup\{v\}$ and we have

$$\kappa(i,\mathrm{I}_\alpha(\sigma)\cup\mathrm{J})\cdot\varepsilon(\sigma,\omega)=\iota(\Phi_\sigma^{\mathrm{J}},\sigma_i^{\mathrm{J}})\cdot\varepsilon(\sigma_i^{\mathrm{J}},\tau).$$

This is because for each $i \in [k]\backslash(\mathrm{I}_\alpha(\sigma)\cup \mathrm{J})$, $\dim(y^{\sigma_i^{\mathrm{J}}}) = 0$. So we obtain

$$d(\mathbf{y}^{\Phi_\sigma^{\mathrm{J}}}) = \sum_{\substack{\sigma\subset\omega,\, \mathrm{I}_\alpha(\omega)\subset \mathrm{I}_\alpha(\sigma)\cup \mathrm{J}\\ \dim(\omega)=\dim(\sigma)+1}} \kappa(i_\alpha(\omega\backslash\sigma), \mathrm{I}_\alpha(\sigma)\cup \mathrm{J})\cdot \varepsilon(\sigma,\omega)\, \mathbf{y}^{\Phi_\omega^{\mathrm{I}_\alpha(\sigma)\cup \mathrm{J}\backslash \mathrm{I}_\alpha(\omega)}} \tag{19}$$

where $\omega\backslash\sigma$ denotes the only vertex of ω that is not in σ.

For the simplicial complex $\mathcal{K}\subset\Delta^{[m]}$ and any simplex $\sigma\in\mathcal{K}$, $\mathbf{y}^{\Phi_\sigma^{\mathrm{J}}}$ is a cochain of dimension $\mathrm{rank}(\sigma)+|\mathrm{I}_\alpha(\sigma)|+|\mathrm{J}|$ in $C^*(X(\mathcal{K},\lambda_\alpha);\mathbf{k})$ that is dual to the cell $B_{(\sigma,\mathrm{J})}$ (see (11)). So $C^*(X(\mathcal{K},\lambda_\alpha);\mathbb{Z})$ has a basis

$$\{\mathbf{y}^{\Phi_\sigma^{\mathrm{J}}}\,;\ \sigma\in\mathcal{K},\ \mathrm{J}\subset[k]\backslash\mathrm{I}_\alpha(\sigma)\}.$$

Note that for any $\sigma\in\mathcal{K}$, the differential $d(\mathbf{y}^{\Phi_\sigma^{\mathrm{J}}})$ in $C^*(X(\mathcal{K},\lambda_\alpha);\mathbf{k})$ is only the sum of those terms $\mathbf{y}^{\Phi_\omega^{\mathrm{I}_\alpha(\sigma)\cup \mathrm{J}\backslash \mathrm{I}_\alpha(\omega)}}$ on the right hand side of (19) with $\omega\in\mathcal{K}$.

Proof of Theorem 1.2. For any subset $\mathrm{L}\subset[k]$, let $C^{*,\mathrm{L}}(X(\mathcal{K},\lambda_\alpha);\mathbf{k})$ denote the $\mathbf{k}$-submodule of $C^*(X(\mathcal{K},\lambda_\alpha);\mathbf{k})$ generated by the following set

$$\{B_{(\sigma,\mathrm{J})}\,|\,\mathrm{I}_\alpha(\sigma)\cup\mathrm{J}=\mathrm{L},\ \sigma\in\mathcal{K},\ \mathrm{J}\subset[k]\backslash\mathrm{I}_\alpha(\sigma)\}.$$

From the differential of $C^*(X(\mathcal{K},\lambda_\alpha);\mathbf{k})$ in (19), we see that $C^{*,\mathrm{L}}(X(\mathcal{K},\lambda_\alpha);\mathbf{k})$ is actually a cochain subcomplex of $C^*(X(\mathcal{K},\lambda_\alpha);\mathbf{k})$. We denote its cohomology groups by $H^{*,\mathrm{L}}(X(\mathcal{K},\lambda_\alpha);\mathbf{k})$. Then we have the following decompositions:

$$H^*(X(\mathcal{K},\lambda_\alpha);\mathbf{k}) = \bigoplus_{\mathrm{L}\subset[k]} H^{*,\mathrm{L}}(X(\mathcal{K},\lambda_\alpha);\mathbf{k}). \tag{20}$$

For any subset $\mathrm{L}\subset[k]$, let $C^*(\mathcal{K}_{\alpha,\mathrm{L}};\mathbf{k})$ denote the simplicial cochain complex of $\mathcal{K}_{\alpha,\mathrm{L}}$. For any simplex $\sigma\in\mathcal{K}$, let σ^* be the cochain dual to σ in $C^*(\mathcal{K}_{\alpha,\mathrm{L}};\mathbf{k})$. Then we have the following isomorphism of $\mathbf{k}$-modules:

$$\begin{aligned}\varphi_\alpha^{\mathrm{L}} : C^*(\mathcal{K}_{\alpha,\mathrm{L}};\mathbf{k}) &\longrightarrow C^{*,\mathrm{L}}(X(\mathcal{K},\lambda_\alpha);\mathbf{k})\\ \sigma^* &\longmapsto \kappa(\sigma,\mathrm{L})\,\mathbf{y}^{\Phi_\sigma^{\mathrm{L}\backslash\mathrm{I}_\alpha(\sigma)}}\end{aligned}$$

Note $\dim(\mathbf{y}^{\Phi_\sigma^{\mathrm{L}\backslash\mathrm{I}_\alpha(\sigma)}}) = \mathrm{rank}(\sigma)+|\mathrm{L}|$. Moreover, $\varphi_\alpha^{\mathrm{L}}$ is actually a chain complex isomorphism. Indeed by the differential of $C^*(X(\mathcal{K},\lambda_\alpha);\mathbf{k})$ shown in (19),

$$\begin{aligned} d(\varphi_\alpha^{\mathrm{L}}(\sigma^*)) &= d\big(\kappa(\sigma,\mathrm{L})\,\mathbf{y}^{\Phi_\sigma^{\mathrm{L}\backslash\mathrm{I}_\alpha(\sigma)}}\big)\\ &= \kappa(\sigma,\mathrm{L}) \sum_{\substack{\sigma\subset\omega\in\mathcal{K},\,\mathrm{I}_\alpha(\omega)\subset\mathrm{L}\\ \dim(\omega)=\dim(\sigma)+1}} \kappa(i_\alpha(\omega\backslash\sigma),\mathrm{L})\cdot\varepsilon(\sigma,\omega)\,\mathbf{y}^{\Phi_\omega^{\mathrm{L}\backslash\mathrm{I}_\alpha(\omega)}}\\ &= \sum_{\substack{\sigma\subset\omega\in\mathcal{K},\,\mathrm{I}_\alpha(\omega)\subset\mathrm{L}\\ \dim(\omega)=\dim(\sigma)+1}} \kappa(\omega,\mathrm{L})\cdot\varepsilon(\sigma,\omega)\,\mathbf{y}^{\Phi_\omega^{\mathrm{L}\backslash\mathrm{I}_\alpha(\omega)}} = \varphi_\alpha^{\mathrm{L}}(d\sigma^*).\end{aligned}$$

The third "=" uses the relation $\kappa(\omega,\mathrm{L}) = \kappa(\sigma,\mathrm{L})\cdot\kappa(i_\alpha(\omega\backslash\sigma),\mathrm{L})$ (see (16)). So we have an additive isomorphism of cohomology groups

$$\widetilde{H}^q(\mathcal{K}_{\alpha,\mathrm{L}};\mathbf{k}) \cong H^{q+|\mathrm{L}|+1,\mathrm{L}}(X(\mathcal{K},\lambda_\alpha);\mathbf{k}).$$

$$\text{Let } \varphi_\alpha = \bigoplus_{\mathrm{L}\subset[k]} \varphi_\alpha^{\mathrm{L}} : \bigoplus_{\mathrm{L}\subset[k]} \widetilde{H}^{q-|\mathrm{L}|-1}(\mathcal{K}_{\alpha,\mathrm{L}};\mathbf{k}) \longrightarrow H^q(X(\mathcal{K},\lambda_\alpha);\mathbf{k}).$$

Then φ_α is an isomorphism that satisfies our requirement. □

Example 1(Balanced Simplicial Complex and Pullbacks from the Linear Model). An $(n-1)$-dimensional simplicial complex $\mathcal{K}$ is called *balanced* if there exists a map $\phi : V(\mathcal{K}) \to [n] = \{1, \cdots, n\}$ such that if $\{v, v'\}$ is an edge of $\mathcal{K}$, then $\phi(x) \neq \phi(y)$ (see [18]). We call ϕ an *n-coloring* on $\mathcal{K}$. It is easy to see that $\mathcal{K}$ is balanced if and only if $\mathcal{K}$ admits a non-degenerate simplicial map onto Δ^{n-1}. In fact, if we identify the vertex set of Δ^{n-1} with $[n]$, any n-coloring ϕ on $\mathcal{K}$ induces a non-degenerate simplicial map from $\mathcal{K}$ to Δ^{n-1} which sends a simplex $\sigma \in \mathcal{K}$ with $V(\sigma) = \{v_{i_1}, \cdots, v_{i_s}\}$ to the face of Δ^{n-1} spanned by $\{\phi(v_{i_1}), \cdots, \phi(v_{i_s})\}$.

Suppose $\phi : V(\mathcal{K}) \to [n]$ is an n-coloring of an $(n-1)$-dimensional simplicial complex $\mathcal{K}$. Let $\{e_1, \cdots, e_n\}$ be a basis of $\mathbb{Z}^n$. Then ϕ uniquely determines a $\mathbb{Z}^n$-coloring $\lambda^\phi : V(\mathcal{K}) \to \mathbb{Z}^n$ where $\lambda^\phi(v) = e_{\phi(v)}$. The space $X(\mathcal{K}, \lambda^\phi)$ is called a *pullback from the linear model* in [9, Example 1.15]. On the other hand, we have a partition of $V(\mathcal{K})$ defined by $\alpha_\phi := \{\phi^{-1}(1), \cdots \phi^{-1}(n)\}$. By our notation in section 1, we have $X(\mathcal{K}, \lambda^\phi) = X(\mathcal{K}, \lambda_{\alpha_\phi})$. Then by Theorem 1.2, the cohomology groups of $X(\mathcal{K}, \lambda^\phi)$ can be computed by

$$H^q(X(\mathcal{K},\lambda^\phi);\mathbf{k}) \cong \bigoplus_{\mathrm{L}\subset[n]} \widetilde{H}^{q-|\mathrm{L}|-1}(\mathcal{K}_{\alpha_\phi,\mathrm{L}};\mathbf{k}),\ \forall q \geq 0. \tag{21}$$

Remark 2.4. There are analogues of Hochster-type decompositions for pullbacks from linear models in combinatorial commutative algebra, at least when $\mathcal{K}$ is Cohen-Macaulay. Let $\phi : V(\mathcal{K}) \to [n]$ be an n-coloring of such a complex, $A \subset [n]$ be a subset of colors and $\mathcal{K}_{\phi^{-1}(A)}$ be the full subcomplex colored in A. It is easy to show that: (1) $\mathcal{K}_{\phi^{-1}(A)}$ is Cohen-Macaulay again; (2) its top Betti number is the flag h-number $h_A(\mathcal{K})$ (see [17, Sec 3]); (3) we obviously have $h_i(\mathcal{K}) = \sum_{|A|=i} h_A(\mathcal{K})$ which is Hochster's formula for the h-vector of $\mathcal{K}$.

3. Stable decompositions of $X(\mathcal{K}, \lambda_\alpha)$

It is shown in [2] that the stable homotopy type of a polyhedral product $(\underline{\mathbb{X}}, \underline{\mathbb{A}})^{\mathcal{K}}$ is a wedge of spaces, which implies corresponding homological decompositions of $(\underline{\mathbb{X}}, \underline{\mathbb{A}})^{\mathcal{K}}$. In this section, we prove a parallel result for $X(\mathcal{K}, \lambda_\alpha)$. Our argument proceeds along the same line as [2].

Let $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ be a partition of the vertex set $V(\mathcal{K})$ of $\mathcal{K}$. For any $i \in [k]$, choose the base-point of $S^1_{(i)} = e^0_{(i)} \cup e^1_{(i)}$ to be $e^0_{(i)}$. So

- $e^0_{(i)}$ is a base-point of $S^1_{(i)} * \tau$ for any simplex $\tau \in \mathcal{K}$, and
- $(e^0_{(i)}, \cdots, e^0_{(i_s)})$ is a base-point of $S^1_{(i_1)} \times \cdots \times S^1_{(i_s)}$ for any $i_1, \cdots, i_s \in [k]$.

Then for any subset $\mathrm{L} \subset [k]$ and any simplex $\sigma \in \mathcal{K}$, it is meaningful to define

$$\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma) := \bigwedge_{i\in \mathrm{I}_\alpha(\sigma)\cap \mathrm{L}} S^1_{(i)} * (\sigma \cap \Delta^{\alpha_i}) \wedge \bigwedge_{i\in \mathrm{L}\setminus(\mathrm{I}_\alpha(\sigma)\cap\mathrm{L})} S^1_{(i)}$$

where $\wedge$ and $\bigwedge$ denote the smash product with respect to the based spaces. So for $\mathrm{L} = \{i_1, \cdots, i_s\}$, the base-point of $\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma)$ is $(e^0_{(i)}, \cdots, e^0_{(i_s)})$. We adopt the convention that the smash product of a space with the empty space is empty. Then the following lemma is immediate from the definition of $\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma)$.

Lemma 3.1. *For any subset* $\mathrm{L} \subset [k]$ *and any simplex* $\sigma \in \mathcal{K}$*, we have*

(i) $\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma) = \mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma \cap \mathcal{K}_{\alpha,\mathrm{L}})$,

(ii) $\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma)$ *is contractible whenever* $\mathrm{I}_\alpha(\sigma) \cap \mathrm{L} \neq \emptyset$,

(iii) $\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\hat{\mathbf{0}}) = \bigwedge_{i\in\mathrm{L}} S^1_{(i)} \cong S^{|\mathrm{L}|}$.

For any simplex $\sigma \in \mathcal{K}$, define

$$\mathbf{D}_\alpha(\sigma) := \prod_{i\in\mathrm{I}_\alpha(\sigma)} S^1_{(i)} * (\sigma \cap \Delta^{\alpha_i}) \times \prod_{i\in[k]\setminus\mathrm{I}_\alpha(\sigma)} S^1_{(i)}. \tag{22}$$

Note that all the $\mathbf{D}_\alpha(\sigma)$ have the same base-point $(e^0_{(1)}, \cdots, e^0_{(k)})$, which is the base-point of $X(\mathcal{K}, \lambda_\alpha)$.

By [2, Theorem 2.8], there are natural homotopy equivalences

$$\mathbf{\Sigma}(\mathbf{D}_\alpha(\sigma)) \simeq \mathbf{\Sigma}\Big(\bigvee_{\mathrm{L}\subset[k]} \mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma)\Big)$$

where $\mathbf{\Sigma}$ denotes the reduced suspension and $\bigvee$ denotes the wedge sum with respect to the base-point of $\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma)$. Now let

$$\mathbf{E}_\alpha(\sigma) := \bigvee_{\mathrm{L}\subset[k]} \mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma).$$

Let $\mathrm{Cat}(\mathcal{K})$ denote the *face category* of $\mathcal{K}$ whose objects are simplices $\sigma \in \mathcal{K}$ and there is a morphism from σ to τ whenever $\sigma \subseteq \tau$. Then we can consider $\mathbf{D}_\alpha$ and $\mathbf{E}_\alpha$ as functors from $\mathrm{Cat}(\mathcal{K})$ to the category CW_* of connected, based CW-complexes and based continuous maps. It is clear that

$$X(\mathcal{K}, \lambda_\alpha) = \bigcup_{\sigma\in\mathcal{K}} \mathbf{D}_\alpha(\sigma) = \mathrm{colim}(\mathbf{D}_\alpha(\sigma)).$$

For any subset $\mathrm{L} = \{l_1, \cdots, l_s\} \subset [k]$, define

$$\widehat{X}(\mathcal{K}_{\alpha,\mathrm{L}}, \lambda_\alpha) := \bigcup_{\sigma\in\mathcal{K}} \mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma) = \bigcup_{\sigma\in\mathcal{K}_{\alpha,\mathrm{L}}} \mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma).$$

For a fixed $\mathrm{L} \subset [k]$, all the spaces $\{\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma), \sigma \in \mathcal{K}_{\alpha,\mathrm{L}}\}$ share a base-point, which then defines the base-point of $\widehat{X}(\mathcal{K}_{\alpha,\mathrm{L}}, \lambda_\alpha)$. So $\widehat{X}(\mathcal{K}_{\alpha,\mathrm{L}}, \lambda_\alpha)$ is the colimit of $\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma)$ over the face category $\mathrm{Cat}(\mathcal{K}_{\alpha,\mathrm{L}})$ of $\mathcal{K}_{\alpha,\mathrm{L}}$. In addition, we clearly have

$$\mathrm{colim}(\mathbf{E}_\alpha(\sigma)) = \bigvee_{\mathrm{L}\subset[k]} \widehat{X}(\mathcal{K}_{\alpha,\mathrm{L}}, \lambda_\alpha).$$

Since the suspension commutes with the colimits up to homotopy equivalence (see [2, Theorem 4.3]), we obtain the following homotopy equivalences.

$$\begin{aligned} \mathbf{\Sigma}(X(\mathcal{K}, \lambda_\alpha)) &\simeq \mathrm{colim}(\mathbf{\Sigma}(\mathbf{D}_\alpha(\sigma))) \simeq \mathrm{colim}(\mathbf{\Sigma}(\mathbf{E}_\alpha(\sigma))) \\ &\simeq \mathbf{\Sigma}\big(\bigvee_{\mathrm{L}\subset[k]} \widehat{X}(\mathcal{K}_{\alpha,\mathrm{L}}, \lambda_\alpha)\big). \end{aligned} \tag{23}$$

Definition 3.2 (Order Complex). Given a poset (partially ordered set) $(\mathcal{P}, <)$, the *order complex* $\Delta(\mathcal{P})$ is the simplicial complex with vertices given by the set of points of $\mathcal{P}$ and k-simplices given by the ordered $(k+1)$-tuples $(p_1, p_2, ..., p_{k+1})$ in $\mathcal{P}$ with $p_1 < p_2 < \ldots < p_{k+1}$.

Lemma 3.3. *For any* $\mathrm{L} \subset [k]$*, there is a homotopy equivalence:*

$$\widehat{X}(\mathcal{K}_{\alpha,\mathrm{L}}, \lambda_\alpha) \simeq \bigvee_{\sigma\in\mathcal{K}_{\alpha,\mathrm{L}}} |\Delta((\mathcal{K}_{\alpha,\mathrm{L}})_{<\sigma})| * \mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma).$$

Here $\Delta((\mathcal{K}_{\alpha,\mathrm{L}})_{<\sigma})$ *is the order complex of the poset* $\{\tau \in \mathcal{K}_{\alpha,\mathrm{L}} \,|\, \tau \supsetneq \sigma\}$ *whose order is the reverse inclusion, and* $|\Delta((\mathcal{K}_{\alpha,\mathrm{L}})_{<\sigma})|$ *is the geometric realization of* $\Delta((\mathcal{K}_{\alpha,\mathrm{L}})_{<\sigma})$*.*

Proof. Note that the natural inclusion $S^1 \hookrightarrow S^1 * \Delta^{[m]}$ is null-homotopic for any $m \geq 1$. Then the same argument as in the proof of [2, Theorem 2.12] shows that there is a homotopy equivalence $H_{\mathrm{L}}(\sigma) : \mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma) \to \mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma)$ for each simplex $\sigma \in \mathcal{K}_{\alpha,\mathrm{L}}$ so that the following diagram commutes for any $\tau \subseteq \sigma \in \mathcal{K}_{\alpha,\mathrm{L}}$,

$$\begin{tikzcd}
\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\tau) \arrow[r, "H_{\mathrm{L}}(\tau)"] \arrow[d, "\zeta_{\sigma,\tau}"'] & \mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\tau) \arrow[d, "c_{\sigma,\tau}"] \\
\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma) \arrow[r, "H_{\mathrm{L}}(\sigma)"] & \mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma)
\end{tikzcd} \tag{24}$$

where $\zeta_{\sigma,\tau}$ is the natural inclusion and $c_{\sigma,\tau}$ is the constant map to the base-point. Then by [2, Theorem 4.1] and [2, Theorem 4.2], there is a homotopy equivalence

$$\widehat{X}(\mathcal{K}_{\alpha,\mathrm{L}}, \lambda_\alpha) = \mathrm{colim}(\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma)) \quad \simeq \quad \bigvee_{\sigma\in\mathcal{K}_{\alpha,\mathrm{L}}} |\Delta((\mathcal{K}_{\alpha,\mathrm{L}})_{<\sigma})| * \mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma).$$

So the lemma is proved. □

Proof of Theorem 1.3. Putting the homotopy equivalences in the equation (23) and Lemma 3.3 together gives us a homotopy equivalence:

$$\mathbf{\Sigma}(X(\mathcal{K}, \lambda_\alpha)) \simeq \mathbf{\Sigma}\Big(\bigvee_{\mathrm{L}\subset[k]} \bigvee_{\sigma\in\mathcal{K}_{\alpha,\mathrm{L}}} |\Delta((\mathcal{K}_{\alpha,\mathrm{L}})_{<\sigma})| * \mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma)\Big).$$

Moreover, the space in the left hand side can be simplified by the following facts.

- $\mathbf{W}^{S^1}_{\alpha,\mathrm{L}}(\sigma)$ is contractible whenever $\sigma \neq \hat{\mathbf{0}} \in \mathcal{K}_{\alpha,\mathrm{L}}$ (see Lemma 3.1(ii)).
- $\Delta((\mathcal{K}_{\alpha,\mathrm{L}})_{<\hat{\mathbf{0}}})$ is isomorphic to the barycentric subdivision $\mathcal{K}'_{\alpha,\mathrm{L}}$ of $\mathcal{K}_{\alpha,\mathrm{L}}$ as simplicial complexes. So the geometric realization $|\Delta((\mathcal{K}_{\alpha,\mathrm{L}})_{<\hat{\mathbf{0}}})|$ is homeomorphic to that of $\mathcal{K}_{\alpha,\mathrm{L}}$.

Then by Lemma 3.1, we have the following homotopy equivalences:

$$\mathbf{\Sigma}(X(\mathcal{K}, \lambda_\alpha)) \simeq \mathbf{\Sigma}\Big(\bigvee_{\mathrm{L}\subset[k]} |\mathcal{K}_{\alpha,\mathrm{L}}| * S^{|\mathrm{L}|}\Big) \simeq \bigvee_{\mathrm{L}\subset[k]} \mathbf{\Sigma}^{|\mathrm{L}|+2}(|\mathcal{K}_{\alpha,\mathrm{L}}|).$$

So Theorem 1.3 is proved. □

4. A description of moment-angle complexes of simplicial posets

A poset (partially ordered set) $\mathcal{S}$ with the order relation $\leq$ is called *simplicial* if it has an initial element $\hat{0}$ and for each $\sigma \in \mathcal{S}$ the lower segment

$$[\hat{0}, \sigma] = \{\tau \in \mathcal{S} : \hat{0} \leq \tau \leq \sigma\}$$

is the face poset of a simplex.

For each $\sigma \in \mathcal{S}$ we assign a geometric simplex Δ^σ whose face poset is $[\hat{0}, \sigma]$, and glue these geometric simplices together according to the order relation in $\mathcal{S}$. We get a cell complex $\Delta^{\mathcal{S}}$ in which the closure of each cell is identified with a simplex preserving the face structure, and all attaching maps are inclusions. We call $\Delta^{\mathcal{S}}$ the *geometric realization* of $\mathcal{S}$. For convenience, we still use Δ^σ to denote the image of each geometric simplex Δ^σ in $\Delta^{\mathcal{S}}$. Then Δ^σ is a *maximal simplex* of $\Delta^{\mathcal{S}}$ if and only if σ is a maximal element of $\mathcal{S}$.

The notion of moment-angle complex $\mathcal{Z}_{\mathcal{S}}$ associated to a simplicial poset $\mathcal{S}$ is introduced in [12] where $\mathcal{Z}_{\mathcal{S}}$ is defined via the categorical language. Note that the barycentric subdivision also makes sense for $\Delta^{\mathcal{S}}$. Let $P_{\mathcal{S}}$ denote the cone of the barycentric subdivision of $\Delta^{\mathcal{S}}$. Let the vertex set of $\Delta^{\mathcal{S}}$ be $V(\Delta^{\mathcal{S}}) = \{u_1, \cdots, u_k\}$. Let $\lambda_{\mathcal{S}} : V(\Delta^{\mathcal{S}}) \to \mathbb{Z}^k$ be a map so that $\{\lambda_{\mathcal{S}}(u_i),\ 1 \leq i \leq k\}$ is a unimodular basis of $\mathbb{Z}^k$. Then we can construct $\mathcal{Z}_{\mathcal{S}}$ from $P_{\mathcal{S}}$ and $\lambda_{\mathcal{S}}$ via the same rule in (2). So we also denote $\mathcal{Z}_{\mathcal{S}}$ by $X(\mathcal{S}, \lambda_{\mathcal{S}})$.

Define a map $\lambda : [m] = \{v_1, \cdots, v_m\} \to \mathbb{Z}^m$ by $\lambda(v_i) = e_i$, $1 \leq i \leq m$, where $\{e_1, \cdots, e_m\}$ is a unimodular basis of $\mathbb{Z}^m$. We identify $\Delta^{[m]}$ with $\Delta^{[m]} \times \{0\}$ in $\Delta^{[m]} \times [0,1]$ (considered as a product of simplices). Let the vertex set of $\Delta^{[m]} \times [0,1]$ be $\{v_1, \cdots, v_m, v_1', \cdots, v_m'\}$ where $v_i' = v_i \times \{1\}$, $1 \leq i \leq m$. Define a map $\tilde{\lambda} : \{v_1, \cdots, v_m, v_1', \cdots, v_m'\} \to \mathbb{Z}^m$ by

$$\tilde{\lambda}(v_i) = \tilde{\lambda}(v_i') = e_i,\ 1 \leq i \leq m.$$

It is clear that $X(\Delta^{[m]}, \lambda)$ can be considered as a subspace $X(\Delta^{[m]} \times [0,1], \tilde{\lambda})$.

Lemma 4.1. *There is a canonical deformation retraction from $X(\Delta^{[m]} \times [0,1], \tilde{\lambda})$ to $X(\Delta^{[m]}, \lambda) = \mathcal{Z}_{\Delta^{[m]}}$.*

Proof. Any m-simplex in $\Delta^{[m]} \times [0,1]$ can be written as

$$\sigma_j^m = [v_1, \cdots, v_j, v_j', \cdots, v_m'],\ 1 \leq j \leq m.$$

Note that $\sigma_j^m \cap \sigma_{j+1}^m = [v_1, \cdots, v_j, v_{j+1}', \cdots, v_m']$. So $\sigma_1^m, \sigma_2^m, \cdots, \sigma_m^m$ is a shelling of $\Delta^{[m]} \times [0,1]$. By the cell decomposition (10), we have

$$X(\Delta^{[m]}, \lambda) = (S^1_{(1)} * v_1) \times \cdots \times (S^1_{(m)} * v_m) \subset \prod_{1 \leq j \leq m} S^1_{(j)} * [v_j, v_j'].$$

$$X(\Delta^{[m]} \times [0,1], \tilde{\lambda}) = \bigcup_{1 \leq j \leq m} B_j \subset \prod_{1 \leq j \leq m} S^1_{(j)} * [v_j, v_j'],\ \text{where}$$

$$B_j = (S^1_{(1)} * v_1) \times \cdots \times (S^1_{(j-1)} * v_{j-1}) \times (S^1_{(j)} * [v_j, v_j']) \times (S^1_{(j+1)} * v_{j+1}') \times \cdots \times (S^1_{(m)} * v_m').$$

There is a canonical deformation retraction from $\Delta^{[m]} \times [0,1]$ to $\Delta^{[m]}$ along the above shelling of $\Delta^{[m]} \times [0,1]$ as follows. We first compress the edge $[v_1, v_1']$ to v_1, which induces a deformation retraction

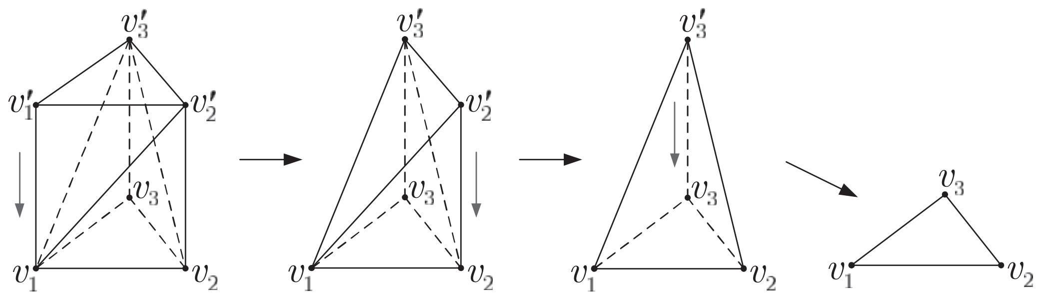

Fig. 2. Canonical deformation retraction from $\Delta^{[m]} \times [0,1]$ to $\Delta^{[m]}$

$$\sigma_1^m = [v_1, v_1', \cdots, v_m'] \longrightarrow [v_1, v_2' \cdots, v_m'] = \sigma_1^m \cap \sigma_2^m.$$

It compresses $\Delta^{[m]} \times [0,1] = \bigcup_{1\le j\le m} \sigma_j^m$ to $\bigcup_{2\le j\le m} \sigma_j^m$. Next, we compress the edge $[v_2, v_2']$ to v_2 which induces a deformation retraction from σ_2^m to $\sigma_2^m \cap \sigma_3^m$ and hence compresses $\bigcup_{2\le j\le m} \sigma_j^m$ to $\bigcup_{3\le j\le m} \sigma_j^m$, and so on. After m steps of retractions, $\Delta^{[m]} \times [0,1]$ is deformed to $\Delta^{[m]} \times \{0\} = \Delta^{[m]}$ (see Figure 2).

Using the above retractions of $\Delta^{[m]} \times [0,1]$, we obtain parallel deformation retractions from $\prod_{1\le j\le m} S^1_{(j)} * [v_j, v_j']$ to $X(\Delta^{[m]}, \lambda)$ in m steps. The j-th step is to compress $S^1_{(j)} * [v_j, v_j']$ to $S^1_{(j)} * v_j$ along the edge $[v_j, v_j']$, which compresses B_j to $B_j \cap B_{j+1}$. Then starting from the first step, we obtain a sequence of retractions

$$X(\Delta^{[m]} \times [0,1], \tilde{\lambda}) = \bigcup_{1\le j\le m} B_j \xrightarrow{\text{step 1}} \bigcup_{2\le j\le m} B_j \xrightarrow{\text{step 2}} \cdots \longrightarrow \bigcup_{m-1\le j\le m} B_j \xrightarrow{\text{step m-1}} B_m$$

$$\xrightarrow{\text{step m}} (S^1_{(1)} * v_1) \times \cdots \times (S^1_{(m)} * v_m) = X(\Delta^{[m]}, \lambda).$$

The above deformation process is canonical since it only depends on the ordering of the vertices of $\Delta^{[m]}$. □

The canonical deformation retraction from $X(\Delta^{[m]} \times [0,1], \tilde{\lambda})$ to $X(\Delta^{[m]}, \lambda)$ in the above lemma will serve as a model of homotopies in our argument below.

Theorem 4.2. *For any finite simplicial poset $\mathcal{S}$, there always exists a finite simplicial complex $\mathcal{K}$ and a partition α of $V(\mathcal{K})$ so that $\mathcal{Z}_{\mathcal{S}}$ is homotopy equivalent to $X(\mathcal{K}, \lambda_\alpha)$.*

Proof. We can construct $\mathcal{K}$ and α in the following way. Let $p : \Delta^{\mathcal{S}} \times [0,n] \to \Delta^{\mathcal{S}}$ be the projection where $\Delta^{\mathcal{S}}$ is identified with $\Delta^{\mathcal{S}} \times \{0\}$, and n is a large enough integer. For each maximal simplex $\Delta^\sigma \subset \Delta^{\mathcal{S}}$, we can choose a simplex $\widetilde{\Delta^\sigma} \subset \Delta^{\mathcal{S}} \times \{l_\sigma\}$ for some $0 \le l_\sigma \le n$ so that

- p maps $\widetilde{\Delta^\sigma}$ simplicially isomorphically onto Δ^σ.
- $\widetilde{\Delta^\sigma} \cap \widetilde{\Delta^\tau} = \varnothing$ for any maximal elements σ and τ in $\mathcal{S}$.

We call $\widetilde{\Delta^\sigma}$ a *horizontal lifting* of Δ^σ. We consider $\Delta^\sigma \times [0,n]$ as the Cartesian product of Δ^σ and $[0,n]$ as simplicial complexes (see [5, Construction 2.11]), where $[0,n]$ is considered as a 1-dimensional simplicial complex with vertices $\{0, \cdots, n\}$ and the set of 1-simplices $\{[i, i+1], 0 \le i \le n-1\}$. If σ and τ are both maximal, $\Delta^\sigma \cap \Delta^\tau$ is the geometric realization

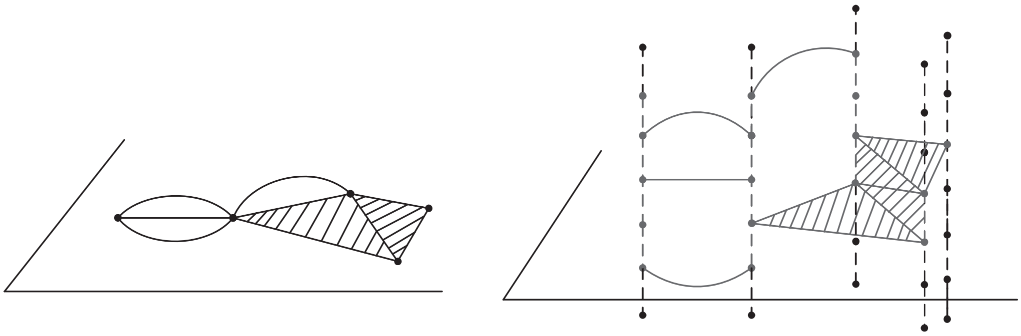

Fig. 3. A stretch of a simplicial poset

of $\sigma \wedge \tau$ (the greatest common lower bound of σ and τ). Now define

$$\mathcal{K} = \Big(\bigcup_{\substack{\sigma\in\mathcal{S}\\ \text{maximal}}} \widetilde{\Delta^\sigma} \Big) \bigcup \Big(\bigcup_{\substack{\sigma,\tau\in\mathcal{S}\\ \text{maximal}\\ l_\sigma<l_\tau}} (\Delta^\sigma \cap \Delta^\tau) \times [l_\sigma, l_\tau] \Big),$$

where $[l_\sigma, l_\tau]$ is considered as a simplicial subcomplex of $[0, n]$. Then by our construction, $\mathcal{K}$ is clearly a finite simplicial complex, called a *stretch of* $\Delta^{\mathcal{S}}$ (see Figure 3 for example). Let $V(\Delta^{\mathcal{S}}) = \{u_1, \cdots, u_k\}$ be the vertex set of $\Delta^{\mathcal{S}}$. Define a partition $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ of $V(\mathcal{K})$ by

$$\alpha_i = \{v \in V(\mathcal{K}) \,|\, p(v) = u_i\},\ 1 \le i \le k.$$

Then we get a $\mathbb{Z}^k$-coloring λ_α on $\mathcal{K}$. In the following we show that the space $X(\mathcal{K}, \lambda_\alpha)$ is homotopy equivalent to $X(\mathcal{S}, \lambda_{\mathcal{S}}) = \mathcal{Z}_{\mathcal{S}}$.

For a pair of maximal elements $\sigma, \tau \in \mathcal{S}$, we have a decomposition

$$(\Delta^\sigma \cap \Delta^\tau) \times [l_\sigma, l_\tau] = \bigcup_{l_\sigma \le s < l_\tau} (\Delta^\sigma \cap \Delta^\tau) \times [s, s+1] = \bigcup_{l_\sigma \le s < l_\tau} \bigcup_{\omega \in \sigma\wedge\tau} \Delta^\omega \times [s, s+1].$$

Given the shelling of each $\Delta^\omega \times [s, s+1]$ as we do for $\Delta^{[m]} \times [0, 1]$ in the proof of Lemma 4.1, we obtain a canonical shelling of $(\Delta^\sigma \cap \Delta^\tau) \times [l_\sigma, l_\tau]$. If for all pairs of maximal elements $\sigma, \tau \in \mathcal{S}$ we do the deformation retractions from $(\Delta^\sigma \cap \Delta^\tau) \times [l_\sigma, l_\tau]$ to $(\Delta^\sigma \cap \Delta^\tau) \times \{l_\sigma\}$ in $\mathcal{K}$ along their canonical shellings, we obtain a space that can be identified with $\Delta^{\mathcal{S}}$ in the end. Note that all these retractions are caused by compressing $\{v\}\times[s, s+1]$ to $\{v\}\times\{s\}$ step by step for each vertex $v \in \Delta^\sigma \cap \Delta^\tau$. So for different pairs of maximal elements of $\mathcal{S}$, the retractions we constructed are compatible with each other and hence can be done simultaneously at each $\{v\} \times [s, s + 1]$. In addition, the $\mathbb{Z}^m$-coloring λ_α on $\mathcal{K}$ naturally induces a $\mathbb{Z}^m$-coloring on the space in each step of the deformation and eventually recovers $\lambda_{\mathcal{S}}$ on $\Delta^{\mathcal{S}}$ in the end. So by applying Lemma 4.1 to every $\Delta^\omega \times [s, s + 1] \subset (\Delta^\sigma \cap \Delta^\tau) \times [l_\sigma, l_\tau]$, we see that the above deformation retractions on $\mathcal{K}$ induce a homotopy equivalence from $X(\mathcal{K}, \lambda_\alpha)$ to $X(\mathcal{S}, \lambda_{\mathcal{S}}) = \mathcal{Z}_{\mathcal{S}}$. This proves the theorem. $\square$

For any subset $\mathrm{L} \subset V(\mathcal{S})$, let $\mathcal{S}_{\mathrm{L}}$ denote the *full subposet* of $\mathcal{S}$ with vertex set L. By our construction of the stretch $\mathcal{K}$ of $\mathcal{S}$ in the proof of Theorem 1.2, the geometric realization $\Delta^{\mathcal{S}_{\mathrm{L}}}$ of $\mathcal{S}_{\mathrm{L}}$ is a deformation retraction of the subcomplex $\mathcal{K}_{\alpha,\mathrm{L}}$ of $\mathcal{K}$. So by Theorem 4.2 and

Theorem 1.2, we derive that for any $q \geq 0$,

$$H^q(\mathcal{Z}_{\mathcal{S}};\mathbf{k}) \cong H^q(X(\mathcal{K},\lambda_\alpha);\mathbf{k}) \cong \bigoplus_{\mathrm{L}\subset V(\mathcal{S})} \widetilde{H}^{q-|\mathrm{L}|-1}(\mathcal{K}_{\alpha,\mathrm{L}};\mathbf{k}) \cong \bigoplus_{\mathrm{L}\subset V(\mathcal{S})} \widetilde{H}^{q-|\mathrm{L}|-1}(\Delta^{\mathcal{S}_{\mathrm{L}}};\mathbf{k}).$$

The above equality can also be derived from [12, Theorem 3.5] easily.

5. Some generalizations

We can generalize our results on $X(\mathcal{K},\lambda_\alpha)$ to a wider range of spaces as follows. For a partition $\alpha = \{\alpha_1,\cdots,\alpha_k\}$ of $V(\mathcal{K})$, we can replace the $S^1_{(i)}$ in (10) by a sequence of spheres $\underline{\mathbb{S}} = (S^{d_1},\cdots,S^{d_k})$ and define

$$X(\mathcal{K},\alpha,\underline{\mathbb{S}}) = \bigcup_{\sigma\in\mathcal{K}} \Big(\prod_{i\in \mathrm{I}_\alpha(\sigma)} S^{d_i} * (\sigma\cap\Delta^{\alpha_i}) \times \prod_{i\in[k]\backslash \mathrm{I}_\alpha(\sigma)} S^{d_i} \Big) \subset \prod_{i\in[k]} S^{d_i} * \Delta^{\alpha_i}.$$

We have the following two theorems which are parallel to Theorem 1.2 and Theorem 1.3, respectively.

Theorem 5.1. *For any coefficients* $\mathbf{k}$*, there is a* $\mathbf{k}$*-module isomorphism:*

$$H^q(X(\mathcal{K},\alpha,\underline{\mathbb{S}});\mathbf{k}) \cong \bigoplus_{\mathrm{L}\subset[k]} \widetilde{H}^{q-1-\sum_{i\in\mathrm{L}} d_i}(\mathcal{K}_{\alpha,\mathrm{L}};\mathbf{k}),\ \forall q\geq 0.$$

Theorem 5.2. *There is a homotopy equivalence*

$$\mathbf{\Sigma}(X(\mathcal{K},\alpha,\underline{\mathbb{S}})) \simeq \bigvee_{\mathrm{L}\subset[k]} \mathbf{\Sigma}^{(\sum_{i\in\mathrm{L}} d_i)+2}(\mathcal{K}_{\alpha,\mathrm{L}}).$$

The proofs of the above two theorems are completely parallel to the proofs of Theorem 1.2 and Theorem 1.3. So we leave them to the reader. There is only one technical point in the proof here. The definition of $\kappa(i,\mathrm{L})$ and $\kappa(\sigma,\mathrm{L})$ (see (15)) in the proof of Theorem 1.2. need to be modified to be adapted to Theorem 5.1. For $X(\mathcal{K},\alpha,\underline{\mathbb{S}})$ we should redefine $\kappa(i,\mathrm{L})$ as follows and adjust the definition of $\kappa(\sigma,\mathrm{L})$ accordingly.

$$\kappa(i,\mathrm{L}) = (-1)^{r_{\underline{\mathbb{S}}}(i,\mathrm{L})},\ \text{where } r_{\underline{\mathbb{S}}}(i,\mathrm{L}) = \sum_{j\in\mathrm{L},\, j<i} d_j,\ \forall i\in\mathrm{L}\subset[k].$$

Remark 5.3. When α^* is the trivial partition of $V(\mathcal{K})$, $X(\mathcal{K},\alpha^*,\underline{\mathbb{S}})$ is nothing but the polyhedral product $\mathcal{K}^{(\underline{\mathbb{D}},\underline{\mathbb{S}})}$ where $(\underline{\mathbb{D}},\underline{\mathbb{S}}) = \{(D^{d_1+1},S^{d_1}),\cdots,(D^{d_k+1},S^{d_k})\}$. In this special case, Theorem 5.1 coincides with [8, Theorem 4.2].

If we let $\underline{\mathbb{S}} = (S^0,\cdots,S^0)$, the space $X(\mathcal{K},\alpha^*,\underline{\mathbb{S}})$ is the real moment-angle complex of K. Then similar to Example 1, we obtain Hochster's formula (with arbitrary coefficient $\mathbf{k}$) for small covers which are pullbacks from linear models from Theorem 5.1. More specifically, if $\phi : V(\mathcal{K}) \to [n]$ is an n-coloring of an $(n-1)$-dimensional simplicial complex $\mathcal{K}$. Let $\{\bar{e}_1,\cdots,\bar{e}_n\}$ be a basis of $(\mathbb{Z}_2)^n$. Then ϕ uniquely determines a $(\mathbb{Z}_2)^n$-coloring $\bar{\lambda}^\phi : V(\mathcal{K}) \to (\mathbb{Z}_2)^n$ where

$$\bar{\lambda}^\phi(v) = \bar{e}_{\phi(v)}.$$

We can construct a space $X(\mathcal{K},\bar{\lambda}^\phi)$ from $\mathcal{K}$ and $\bar{\lambda}^\phi$ in a similar fashion as we do for $\mathbb{Z}^n$-colorings of $\mathcal{K}$ (see [9]). The space $X(\mathcal{K},\bar{\lambda}^\phi)$ is called a *pullback from the linear model* in [9, Example 1.15] as well. It is clear that $X(\mathcal{K}, \bar{\lambda}^{\phi})$ is a quotient of the real moment-angle complex of K by a subgroup of $(\mathbb{Z}_2)^{|V(\mathcal{K})|}$ that satifies Theorem 5.1. So by Theorem 5.1, the cohomology groups of $X(\mathcal{K}, \bar{\lambda}^{\phi})$

$$H^q(X(\mathcal{K}, \bar{\lambda}^{\phi}); \mathbf{k}) \cong \bigoplus_{\mathrm{L}\subset[n]} \widetilde{H}^{q-1}(\mathcal{K}_{\alpha_\phi,\mathrm{L}}; \mathbf{k}),\ \forall q \geq 0. \tag{25}$$

where $\alpha_\phi := \{\phi^{-1}(1), \cdots \phi^{-1}(n)\}$ is the partition of $V(\mathcal{K})$ defined by ϕ.

The homology groups of a general small cover with $\mathbb{Z}_q$-coefficients (q is odd) and with rational coefficients are studied in [7, 16]. But here in (25) we can let $\mathbf{k}$ be arbitrary cofficients including $\mathbb{Z}$. Moreover, we have the stable decomposition of $X(\mathcal{K}, \bar{\lambda}^{\phi})$ from Theorem 5.2.

$$\mathbf{\Sigma}(X(\mathcal{K}, \bar{\lambda}^{\phi})) \simeq \bigvee_{\mathrm{L}\subset[k]} \mathbf{\Sigma}^2(\mathcal{K}_{\alpha_\phi,\mathrm{L}}).$$

Acknowledgements. The author wants to thank the anonymous referee for some helpful comments.

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