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RANK-ONE PERTURBATION OF WEIGHTED SHIFTS ON A DIRECTED TREE: PARTIAL NORMALITY AND WEAK HYPONORMALITY

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Abstract
A special rank-one perturbation $S_{t,n}$ of a weighted shift on a directed tree is constructed. Partial normality and weak hyponormality (including quasinormality, p-hyponormality, p-paranormality, absolute-p-paranormality and $A(p)$-class) of $S_{t,n}$ are characterized.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For nonzero vectors u and v in $\mathcal{H}$ we shall write $u \otimes v$ for the rank-one operator in $B(\mathcal{H})$ defined by $(u \otimes v)(x) = \langle x, v \rangle u$, $x \in \mathcal{H}$. For $X, Y \in B(\mathcal{H})$, we denote by $[X, Y] = XY - YX$ the *commutator* of X and Y. An operator $T \in B(\mathcal{H})$ is *normal* if $[T^*, T] = 0$, *subnormal* if it is (unitarily equivalent to) the restriction of a normal operator to an invariant subspace, and *hyponormal* if $[T^*, T] \geq 0$. An operator $T \in B(\mathcal{H})$ is said to be *p-hyponormal* ($0 < p < \infty$) if $(T^*T)^p \geq (TT^*)^p$. In particular, if $p = \frac{1}{2}$, then T is said to be *semi-hyponormal* ([26]). And $T \in B(\mathcal{H})$ is ∞-hyponormal if T is p-hyponormal for all $p \in (0, \infty)$. According to the Löwner-Heinz inequality ([16],[26]), every q-hyponormal operator is p-hyponormal for $p \leq q$. Recall that an operator $T \in B(\mathcal{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry satisfying $\ker U = \ker |T| = \ker T$ and $\ker U^* = \ker T^*$. An operator T is *absolute-p-paranormal* if $\||T|^p T x\| \geq \|Tx\|^{p+1}$ for all unit vectors x in $\mathcal{H}$. Note that every absolute-q-paranormal operator is absolute-p-paranormal for $q \leq p$ ([16]). And for each $p > 0$, an operator T is *p-paranormal* if $\| |T|^p U |T|^p x\| \geq \| |T|^p x\|^2$ for all unit vectors x in $\mathcal{H}$. Every q-paranormal operator is p-paranormal for $q \leq p$. Note that absolute-1-paranormality and 1-paranormality coincide; we call this property *paranormality* for simplicity. An operator T is *A(p)-class* if $(T^* |T|^{2p} T)^{\frac{1}{p+1}} \geq |T|^2$. There are relations among the classes of operators mentioned above as follows:

- subnormal $\Rightarrow$ p-hyponormal $\Rightarrow$ p-paranormal $\Rightarrow$ absolute-p-paranormal (when $0 < p < 1$);
- subnormal $\Rightarrow$ p-hyponormal $\Rightarrow$ absolute-p-paranormal $\Rightarrow$ p-paranormal (when $p > 1$);
- $A(p)$-class $\Rightarrow$ absolute-p-paranormal (when $p > 0$).

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The operator classes between subnormal and normaloid have been studied for more than 40 years (see [2],[3],[8],[16],[26]). Also, some operator models have been studied to detect those classes. For example, some block matrix operators induced by composition operators on discrete measure spaces were considered to exemplify some classes above (cf. [6],[7],[22],[23]). In [19] the notion of weighted shifts S_λ on directed trees was introduced and has been developed well for recently for several years. But this operator S_λ is not enough to differentiate the above classes; for example, S_λ is p-paranormal if and only if S_λ is absolute-p-paranormal (cf. Section 4). But a rank-one perturbation $S_{t,n}$ of S_λ which will be defined below (Section 2.2) is a good operator model to detect gaps of weak hyponormalities. In fact, the weighted shifts on directed trees have been discussed as a special model of weighted adjacency operators on directed graphs which generalizes Fujii-Sasaoka-Watatani's operator models; see [9],[12],[13],[14],[15] for related results. Note that the rank-one perturbations of a bounded (unbounded) operator can be applied to several related areas in mathematical physics as well as operator theory ([5],[10],[11],[21],[24]). In this paper we characterize the quasinormality, p-hyponormality, p-paranormality, absolute-p-paranormality and $A(p)$-class of operators $S_{t,n}$ which exemplify some operator gaps between normal and nomaloid operators.

The paper consists of five sections. In Section 2, we assemble some useful observations and recall some terminology and notation concerning weighted shifts on directed trees. And also we construct the rank-one perturbation $S_{t,n}$ of the weighted shift S_λ on a certain directed tree $\mathcal{T}_{2,\kappa}$. In Section 3, we characterize p-hyponormality of $S_{t,n}$ and discuss some related remarks. In Section 4, we also characterize absolute-p-paranormality, p-paranormality and $A(p)$-class property of $S_{t,n}$. In Section 5, we consider some related examples.

Throughout this paper we write $\mathbb{C}[\mathbb{R}, \mathbb{R}_+, \mathbb{Z}_+, \mathbb{N}$, resp.] for the set of complex numbers [real numbers, positive real numbers, nonnegative integers, positive integers, resp.]. Some of the calculations in this paper were obtained through computer experiments using the software tool *Mathematica* [25].

2. Preliminaries and notations

2.1. Some basic observations. In what follows we will frequently have use for certain elementary observations which we record here and use with little or no further comment. First, if a, b and p are positive real numbers, then $ab^p - (p+1)s^p a + ps^{p+1} \geq 0$ for all $s \geq 0$ if and only if $b \geq a$. Second, it is the standard Nested Determinant test ([4, p.213]) that a real symmetric matrix M is non-negative if the determinants of its principal submatrices are positive and $\det(M) \geq 0$. For a two-by-two real symmetric matrix A, A is positive semi-definite if and only if both its diagonal entries are non-negative and $\det(A) \geq 0$.

Third, we will frequently have occasion to find powers q of a real symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, which we do as usual by transforming to a diagonal matrix of eigenvalues using the associated eigenvectors. The eigenvalues are

$$\frac{1}{2}\left((a+c) \mp \sqrt{(a+c)^2 - 4(ac-b^2)}\right).$$

We will frequently call these names such as ρ_1 and ρ_2, with associated eigenvectors e_1 and e_2, and abbreviate the square root term by some name such as γ. If we express the eigenvectors as $e_1 = ((a - \rho_2)/b, 1)^T$ and $e_2 = ((a - \rho_1)/b, 1)^T$, the resulting expression for A^q, which we call form 1, is

$$\begin{pmatrix} \frac{(\rho_2-a)\rho_1^q-(\rho_1-a)\rho_2^q}{\gamma} & \frac{b(\rho_2^q-\rho_1^q)}{\gamma} \\ \frac{b(\rho_2^q-\rho_1^q)}{\gamma} & \frac{(\rho_2-a)\rho_2^q-(\rho_1-a)\rho_1^q}{\gamma} \end{pmatrix}.$$

If instead we express the eigenvectors as $e_1 = ((\rho_1 - c)/b, 1)^T$ and $e_2 = ((\rho_2 - c)/b, 1)^T$, the resulting expression for A^q, which we call form 2, is

$$\begin{pmatrix} \frac{(\rho_2-c)\rho_2^q-(\rho_1-c)\rho_1^q}{\gamma} & \frac{b(\rho_2^q-\rho_1^q)}{\gamma} \\ \frac{b(\rho_2^q-\rho_1^q)}{\gamma} & \frac{(\rho_2-c)\rho_1^q-(\rho_1-c)\rho_2^q}{\gamma} \end{pmatrix}.$$

When we apply this process we will indicate the form, the eigenvalues, and the square root term for the reader's convenience.

2.2. Directed trees. In this section we recall some definitions and terminology in graph theory which will be used in this paper ([19],[20]). First of all, we look at some basic notions of graph theory. A pair $\mathcal{G} = (V, E)$ is a *directed graph* if V is a nonempty set and E is a subset of $V \times V \setminus \{(v, v) \mid v \in V\}$. We set

$$\widetilde{E} = \{\{u, v\} \subseteq V \mid (u, v) \in E \text{ or } (v, u) \in E\}.$$

An element of V is called a *vertex* of $\mathcal{G}$, a member of E is called an *edge* of $\mathcal{G}$, and a member of $\widetilde{E}$ is called an *undirected edge*. A directed graph $\mathcal{G}$ is said to be *connected* if for any two distinct vertices u and v of $\mathcal{G}$, there exists a finite sequence $v_1, \cdots, v_n$ of vertices of $\mathcal{G}(n \geq 2)$ such that $u = v_1$, $\{v_j, v_{j+1}\} \in \widetilde{E}$ for all $j = 1, \cdots, n-1$, and $v_n = v$. Such a sequence will be called an *undirected path* joining u and v. For $u \in V$, put

$$\mathrm{Chi}(u) = \{v \in V \mid (u, v) \in E\}.$$

An element of $\mathrm{Chi}(u)$ is called a *child* of u. If, for a given vertex $u \in V$, there exists a unique vertex $v \in V$ such that $(v, u) \in E$, then we say that u has a *parent* v and write $\mathrm{par}(u)$ for v. A vertex v of $\mathcal{G}$ is called a *root* of $\mathcal{G}$, or briefly $v \in \mathrm{Root}(\mathcal{G})$, if there is no vertex u of $\mathcal{G}$ such that (u, v) is an edge of $\mathcal{G}$. If $\mathrm{Root}(\mathcal{G})$ is a one-element set, then its unique element is denoted by $\mathrm{root}(\mathcal{G})$, or simply by root if this causes no ambiguity. We write $V^\circ = V \backslash \mathrm{Root}(\mathcal{G})$. A finite sequence $\{u_j\}_{j=1}^n$ $(n \geq 2)$ of distinct vertices is said to be a *circuit* of $\mathcal{G}$ if $(u_j, u_{j+1}) \in E$ for all $j = 1, \cdots, n-1$, and $(u_n, u_1) \in E$. A directed graph $\mathcal{T}$ is a *directed tree* if $\mathcal{T}$ is connected, has no circuits and each vertex in $v \in V^\circ$ has a parent. From now on, $\mathcal{T} = (V, E)$ is assumed to be a directed tree. Note that $\ell^2(V)$ is the Hilbert space of all square summable complex functions on V with the standard inner product

$$\langle f, g \rangle = \sum_{u \in V} f(u)\overline{g(u)}, \; f, g \in \ell^2(V).$$

For $u \in V$, we define $e_u \in \ell^2(V)$ by

$$e_u(v) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

Then the set $\{e_u\}_{u \in V}$ is an orthonormal basis of $\ell^2(V)$. For $\lambda = \{\lambda_v\}_{v \in V^\circ} \subset \mathbb{C}$, we define the operator S_λ on $\ell^2(V)$ with the domain $D(S_\lambda)$ such that

$$\begin{aligned} D(S_\lambda) &= \{f \in \ell^2(V) : \sum_{u\in V}\Bigg(\sum_{v\in \mathrm{Chi}(u)} |\lambda_v|^2\Bigg)|f(u)|^2 < \infty\}, \\ S_\lambda f &= \Lambda_{\mathcal{T}} f,\ f \in D(S_\lambda), \end{aligned}$$

where $\Lambda_{\mathcal{T}}$ is the mapping defined on functions $f : V \to \mathbb{C}$ by

$$(\Lambda_{\mathcal{T}} f)(v) = \begin{cases} \lambda_v \cdot f(\mathrm{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \text{root}. \end{cases}$$

In this case the operator S_λ is called a *weighted shift* on the directed tree $\mathcal{T}$ with weights $\{\lambda_v\}_{v\in V^\circ}$. In particular, if $S_\lambda \in B(\ell^2(V))$, then

$$S_\lambda e_u = \sum_{v\in \mathrm{Chi}(u)} \lambda_v e_v$$

(cf. [19, Prop. 3.1.3]) and

$$S_\lambda^* e_u = \begin{cases} \overline{\lambda_u} e_{\mathrm{par}(u)} & \text{if } u \in V^\circ, \\ 0 & \text{if } u \text{ is root;} \end{cases}$$

these formulas are used frequently in this paper (cf. [19, Prop. 3.4.1]). Recall that S_λ is bounded if and only if $\sup_{u\in V} \sum_{v\in \mathrm{Chi}(u)} |\lambda_v|^2 < \infty$. In this paper we only consider the operators S_λ in $B(\ell^2(V))$. We deal with weighted shifts associated to the following models and this model is closely related to the subnormality of weighted shifts on directed trees (cf. [19]).

Definition 2.1 ([19]). Given $\eta, \kappa \in \mathbb{Z}_+ \cup \{\infty\}$ with $\eta \geq 2$, we define the directed tree $T_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ by

$$\begin{aligned} V_{\eta,\kappa} &= \{-k : k \in J_\kappa\} \cup \{0\} \cup \{(i, j) : i \in J_\eta, j \in \mathbb{N}\}, \\ E_{\eta,\kappa} &= E_\kappa \cup \{(0, (i, 1)) : i \in J_\eta\} \cup \{((i, j), (i, j + 1)) : i \in J_\eta, j \in \mathbb{N}\}, \end{aligned}$$

where $E_\kappa = \{(-k, -k + 1) : k \in J_\kappa\}$ and $J_\iota = \{k \in \mathbb{N} : k \leq \iota\}$ for $\iota \in \mathbb{Z}_+ \cup \{\infty\}$. The the directed tree $\mathcal{T}_{\eta,\kappa}$ is called an *(η, κ)-type directed tree*.

If $\kappa < \infty$, then the directed tree $\mathcal{T}_{\eta,\kappa}$ has a root and $\mathrm{root}(\mathcal{T}_{\eta,\kappa}) = -\kappa$. In turn, if $\kappa = \infty$, then the directed tree $\mathcal{T}_{\eta,\infty}$ is rootless. In the case of $\kappa < \infty$, the (η, κ)-type directed tree can be illustrated as in Figure 2.1 below.

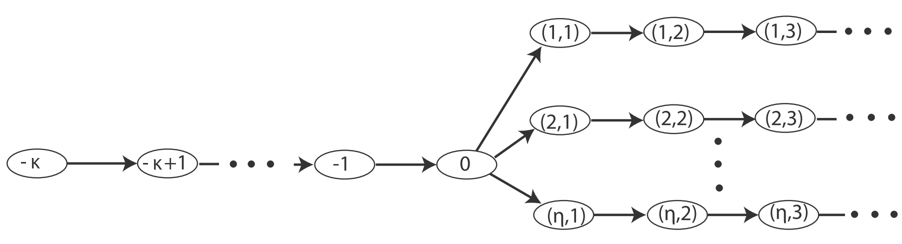

Fig. 2.1

2.3. Basic construction. Let S_λ be a weighted shift on the directed tree $\mathcal{T}_{2,\kappa}$ with weights $\{\lambda_v\}_{v\in V_{2,\kappa}^\circ}$ consisting of positive real numbers. Let $\{e_u\}_{u\in V_{2,\kappa}}$ be the usual orthonormal basis of $\ell^2(V_{2,\kappa})$. For a fixed $n \in \mathbb{N}$ and parameter $t \in \mathbb{R}$, we consider a rank-one perturbation of S_λ on the directed tree $\mathcal{T}_{2,\kappa}$

$$S_{t,n} := S_\lambda + te_{(2,n)} \otimes e_{(1,n)}. \tag{2.1}$$

Unless $t = 0$, $S_{t,n}$ is not a weighted shift on a directed tree. A special case of $S_{t,n}$ is $S_{\lambda_{(2,n)},n}$ which is a weighted adjacency operator in the sense of [9] on the directed graph below, although we do not take this point of view. But if $t \neq 0$ and $t \neq \lambda_{(2,n)}$ the rank-one perturbation $S_{t,n}$ is more general than either type. We consider an ordered orthonormal basis of $\ell^2(V_{2,\kappa})$

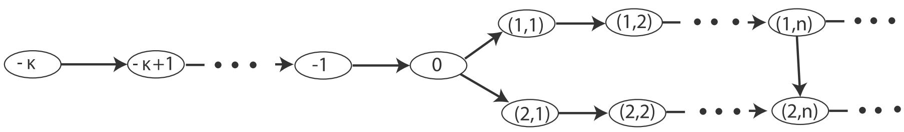

Fig.2.2

by taking the following ordering of the standard basis:

$$e_{-\kappa}, e_{-\kappa+1}, \cdots, e_0, e_{(1,1)}, e_{(2,1)}, e_{(1,2)}, e_{(2,2)}, e_{(1,3)}, e_{(2,3)}, \cdots, \tag{2.2}$$

and consider throughout this paper the matrices corresponding to operators $S_{t,n}$ relative to the ordered orthonormal basis in (2.2).

First, we begin with the following computational lemma.

Lemma 2.1. *Let $S_{t,n}$ be as in* (2.1). *Suppose that $p \in (0, \infty)$. If $t \neq 0$, then the following assertions hold:*

(i) $(S^*_{t,1}S_{t,1})^p = Diag\{\lambda^{2p}_{-\kappa+1}, \cdots, \lambda^{2p}_{-1}, \lambda^{2p}_0, A^p_1, \lambda^{2p}_{(2,2)}, \lambda^{2p}_{(1,3)}, \lambda^{2p}_{(2,3)}, \lambda^{2p}_{(1,4)}, \cdots\}$, *where A^p_1 is unitarily equivalent to a 2×2 matrix $(a_{ij}(1, p))_{1\le i,j\le 2}$ with*

$$a_{11}(1, p) = \{(\beta_1 - \lambda^2_{(1,1)} - \lambda^2_{(2,1)})\alpha^p_1 + (\lambda^2_{(1,1)} + \lambda^2_{(2,1)} - \alpha_1)\beta^p_1\}/\gamma_1, \tag{2.3a}$$

$$a_{12}(1, p) = a_{21}(1, p) = t\lambda_{(2,1)}(\beta^p_1 - \alpha^p_1)/\gamma_1, \tag{2.3b}$$

$$a_{22}(1, p) = \{(\lambda^2_{(1,1)} + \lambda^2_{(2,1)} - \alpha_1)\alpha^p_1 + (\beta_1 - \lambda^2_{(1,1)} - \lambda^2_{(2,1)})\beta^p_1\}/\gamma_1, \tag{2.3c}$$

$$\alpha_1 = (t^2 + \lambda^2_{(1,1)} + \lambda^2_{(2,1)} + \lambda^2_{(1,2)} - \gamma_1)/2, \tag{2.3d}$$

$$\beta_1 = (t^2 + \lambda^2_{(1,1)} + \lambda^2_{(2,1)} + \lambda^2_{(1,2)} + \gamma_1)/2, \tag{2.3e}$$

$$\gamma_1 = [(t^2 + \lambda^2_{(1,1)} + \lambda^2_{(1,2)} + \lambda^2_{(2,1)})^2 - 4\{\lambda^2_{(1,2)}\lambda^2_{(2,1)} + \lambda^2_{(1,1)}(t^2 + \lambda^2_{(1,2)})\}]^{1/2}, \tag{2.3f}$$

(ii) *for $n \geq 2$,*

$$(S^*_{t,n}S_{t,n})^p = Diag\{\lambda^{2p}_{-\kappa+1}, \cdots, \lambda^{2p}_0, (\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^p, \lambda^{2p}_{(1,2)}, \lambda^{2p}_{(2,2)}, \lambda^{2p}_{(1,3)}, \cdots$$
$$\cdots, \lambda^{2p}_{(1,n)}, A^p_n, \lambda^{2p}_{(2,n+1)}, \lambda^{2p}_{(1,n+2)}, \cdots\},$$

where A^p_n is unitarily equivalent to a 2×2 matrix $(a_{ij}(n, p))_{1\le i,j\le 2}$ with

$$a_{11}(n, p) = \{(\beta_n - \lambda^2_{(2,n)})\alpha^p_n + (\lambda^2_{(2,n)} - \alpha_n)\beta^p_n\}/\gamma_n, \tag{2.4a}$$

$$a_{12}(n, p) = a_{21}(n, p) = t\lambda_{(2,n)}(\beta^p_n - \alpha^p_n)/\gamma_n, \tag{2.4b}$$

(2.4c) $$a_{22}(n,p) = \{(\lambda^2_{(2,n)} - \alpha_n)\alpha_n^p + (\beta_n - \lambda^2_{(2,n)})\beta_n^p\}/\gamma_n,$$

(2.4d) $$\alpha_n = (t^2 + \lambda^2_{(1,n+1)} + \lambda^2_{(2,n)} - \gamma_n)/2,$$

(2.4e) $$\beta_n = (t^2 + \lambda^2_{(1,n+1)} + \lambda^2_{(2,n)} + \gamma_n)/2,$$

(2.4f) $$\gamma_n = [(t^2 + \lambda^2_{(1,n+1)} + \lambda^2_{(2,n)})^2 - 4\lambda^2_{(1,n+1)}\lambda^2_{(2,n)}]^{1/2}.$$

Proof. By simple computations, we have that

(2.5) $$S^*_{t,1}S_{t,1} = \mathrm{Diag}\{\lambda^2_{-\kappa+1}, \cdots, \lambda^2_{-1}, \lambda^2_0, A_1, \lambda^2_{(2,2)}, \lambda^2_{(1,3)}, \lambda^2_{(2,3)}, \lambda^2_{(1,4)}, \cdots\}$$

and for $n \geq 2$,

(2.6) $$\begin{aligned} S^*_{t,n}S_{t,n} = \mathrm{Diag}\{&\lambda^2_{-\kappa+1}, \cdots, \lambda^2_0, \lambda^2_{(1,1)} + \lambda^2_{(2,1)}, \lambda^2_{(1,2)}, \lambda^2_{(2,2)}, \lambda^2_{(1,3)}, \cdots \\ &\cdots, \lambda^2_{(1,n)}, A_n, \lambda^2_{(2,n+1)}\lambda^2_{(1,n+2)}, \cdots\}, \end{aligned}$$

with

(2.7) $$A_1 = \begin{pmatrix} \lambda^2_{(1,1)} + \lambda^2_{(2,1)} & t\lambda_{(2,1)} \\ t\lambda_{(2,1)} & t^2 + \lambda^2_{(1,2)} \end{pmatrix}$$

and

(2.8) $$A_n = \begin{pmatrix} \lambda^2_{(2,n)} & t\lambda_{(2,n)} \\ t\lambda_{(2,n)} & t^2 + \lambda^2_{(1,n+1)} \end{pmatrix}.$$

Since A_n is diagonalizable, we obtain that for $n \in \mathbb{N}$,

$$D_n := \mathrm{Diag}\{\alpha_n, \beta_n\} = P_n^{-1}A_nP_n,$$

where α_n, β_n and γ_n are as in (2.3d-f) and (2.4d-f),

$$P_1 = \begin{pmatrix} \frac{\lambda^2_{(1,1)}+\lambda^2_{(2,1)}-\beta_1}{t\lambda_{(2,1)}} & \frac{\lambda^2_{(1,1)}+\lambda^2_{(2,1)}-\alpha_1}{t\lambda_{(2,1)}} \\ 1 & 1 \end{pmatrix}$$

and

$$P_n = \begin{pmatrix} \frac{\lambda^2_{(2,n)}-\beta_n}{t\lambda_{(2,n)}} & \frac{\lambda^2_{(2,n)}-\alpha_n}{t\lambda_{(2,n)}} \\ 1 & 1 \end{pmatrix} \ (n \geq 2).$$

Clearly, α_n and β_n are eigenvalues of A_n, and P_n is a nonsingular matrix consisting of the associated eigenvectors of A_n, for each $n \in \mathbb{N}$. By calculating the matrix product $P_nD_n^pP_n^{-1}$, we obtain the entries of A_n^p as in (2.3) and (2.4), $n \in \mathbb{N}$. Observe that this is the construction of "form 1", with eigenvalues α_n, β_n and with "square root term" γ_n. □

Note that, in Lemma 2.1, if $t = 0$ then

$$(S^*_{0,n}S_{0,n})^p = \mathrm{Diag}\{\lambda^{2p}_{-\kappa+1}, \cdots, \lambda^{2p}_0, (\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^p, \lambda^{2p}_{(1,2)}, \lambda^{2p}_{(2,2)}, \lambda^{2p}_{(1,3)}, \cdots\},$$

which will be used in the later sections.

3. Partial normalities

We discuss the p-hyponormality, quasinormality and normality of $S_{t,n}$ in this section. We begin this section with the p-hyponormality of $S_{0,n}$ as follows.

Proposition 3.1 ([19]). *If $t = 0$, then $S_{0,n}(= S_\lambda)$ is p-hyponormal if and only if the following inequalities hold:*
(i) $\lambda_{m+1} \geq \lambda_m,\ -\kappa + 1 \leq m \leq -1$,
(ii) $\lambda^2_{(1,1)} + \lambda^2_{(2,1)} \geq \lambda^2_0$,
(iii) $\lambda^{2p}_{(1,2)}\lambda^{2p}_{(2,2)} \geq (\lambda^{2p}_{(1,2)}\lambda^2_{(2,1)} + \lambda^2_{(1,1)}\lambda^{2p}_{(2,2)})(\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^{p-1}$,
(iv) $\lambda_{(i,j+2)} \geq \lambda_{(i,j+1)}$, *for* $i = 1, 2,\ j \in \mathbb{N}$.

We now discuss the general case below.

Theorem 3.2. *Let $S_{t,n}$ be as in* (2.1) *and let the a_{ij}'s be as in* Lemma 2.1. *Suppose that $p \in (0, \infty)$ and $t \in \mathbb{R}\backslash\{0\}$. Then the following assertions hold.*
(i) *$S_{t,1}$ is p-hyponormal if and only if the following conditions are satisfied:*
(i-a) *it holds that*

$$\lambda_{m+1} \geq \lambda_m,\ -\kappa + 1 \leq m \leq -1, \tag{3.1}$$

$$\lambda_{(1,k+3)} \geq \lambda_{(1,k+2)},\ \lambda_{(2,k+2)} \geq \lambda_{(2,k+1)},\ k \in \mathbb{N}, \tag{3.2}$$

(i-b) *the following matrix is positive:*

$$\begin{pmatrix} a_{11}(1,p) - \lambda_0^{2p} & a_{12}(1,p) & 0 & 0 \\ a_{12}(1,p) & a_{22}(1,p) - b_{11}(1,p) & -b_{12}(1,p) & -b_{13}(1,p) \\ 0 & -b_{12}(1,p) & \lambda^{2p}_{(2,2)} - b_{22}(1,p) & -b_{23}(1,p) \\ 0 & -b_{13}(1,p) & -b_{23}(1,p) & \lambda^{2p}_{(1,3)} - b_{33}(1,p) \end{pmatrix},$$

where b_{ij}'s are as in Appendix A1.
(ii) *$S_{t,2}$ is p-hyponormal if and only if the following conditions are satisfied:*
(ii-a) *the inequalities in* (3.1) *hold,*
(ii-b) *it holds that*

$$\lambda^2_{(1,1)} + \lambda^2_{(2,1)} \geq \lambda^2_0, \tag{3.3}$$

$$\lambda_{(1,k+4)} \geq \lambda_{(1,k+3)},\ \lambda_{(2,k+3)} \geq \lambda_{(2,k+2)},\ k \in \mathbb{N}, \tag{3.4}$$

(ii-c) *the following matrix is positive:*

$$\begin{pmatrix} \lambda^{2p}_{(1,2)} - \lambda^2_{(1,1)}(\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^{-1+p} & -\lambda_{(1,1)}\lambda_{(2,1)}(\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^{-1+p} & 0 \\ -\lambda_{(1,1)}\lambda_{(2,1)}(\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^{-1+p} & a_{11}(2,p) - \lambda^2_{(2,1)}(\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^{-1+p} & a_{12}(2,p) \\ 0 & a_{12}(2,p) & a_{22}(2,p) - \lambda^{2p}_{(1,2)} \end{pmatrix},$$

(ii-d) *it holds that*

$$\lambda^{2p}_{(2,3)} \geq b_{11}(2,p),\ \lambda^{2p}_{(1,4)} \geq b_{22}(2,p),$$
$$(\lambda^{2p}_{(2,3)} - b_{11}(2,p))(\lambda^{2p}_{(1,4)} - b_{22}(2,p)) \geq b_{12}(2,p)^2,$$

where b_{ij}'s are as in Appendix A1.
(iii) *For $n \geq 3$, $S_{t,n}$ is p-hyponormal if and only if the following conditions are satisfied:*

(iii-a) *the inequalities in* (3.1) *and* (3.3) *hold,*
(iii-b) *it holds that*

$$\lambda_{(1,k+1)} \geq \lambda_{(1,k)},\ 2 \leq k \leq n-1;\ k \geq n+2, \tag{3.5}$$

$$\lambda_{(2,l+1)} \geq \lambda_{(2,l)},\ 2 \leq l \leq n-2;\ l \geq n+1, \tag{3.6}$$

(iii-c) $\lambda^{2p}_{(1,2)}\lambda^{2p}_{(2,2)} \geq (\lambda^{2p}_{(1,2)}\lambda^{2}_{(2,1)} + \lambda^{2}_{(1,1)}\lambda^{2p}_{(2,2)})(\lambda^{2}_{(1,1)} + \lambda^{2}_{(2,1)})^{-1+p}$,
(iii-d) *it holds that*

$$a_{11}(n,p) \geq \lambda^{2p}_{(2,n-1)},\ a_{22}(n,p) \geq \lambda^{2p}_{(1,n)},$$
$$(a_{11}(n,p) - \lambda^{2p}_{(2,n-1)})(a_{22}(n,p) - \lambda^{2p}_{(1,n)}) \geq a_{12}(n,p)^2,$$

(iii-e) *it holds that*

$$\lambda^{2p}_{(2,n+1)} \geq b_{11}(n,p),\ \lambda^{2p}_{(1,n+2)} \geq b_{22}(n,p),$$
$$(\lambda^{2p}_{(2,n+1)} - b_{11}(n,p))(\lambda^{2p}_{(1,n+2)} - b_{22}(n,p)) \geq b_{12}(n,p)^2,$$

where b_{ij}'s are as in Appendix A1.

Proof. By simple computations, we have that

$$S_{t,1}S^*_{t,1} = \mathrm{Diag}\{0, \lambda^2_{-\kappa+1}, \cdots, \lambda^2_0, B_1, \lambda^2_{(2,2)}, \lambda^2_{(1,3)}, \lambda^2_{(2,3)}, \lambda^2_{(1,4)}, \cdots\}$$

and for $n \geq 2$,

$$S_{t,n}S^*_{t,n} = \mathrm{Diag}\{0, \lambda^2_{-\kappa+1}, \cdots, \lambda^2_0, B_0, \lambda^2_{(1,2)}, \lambda^2_{(2,2)}, \lambda^2_{(1,3)}, \cdots$$
$$\cdots, \lambda^2_{(1,n)}, B_n, \lambda^2_{(2,n+1)}\lambda^2_{(1,n+2)}, \cdots\}$$

with

$$B_1 = \begin{pmatrix} \lambda^2_{(1,1)} & \lambda_{(1,1)}\lambda_{(2,1)} & 0 \\ \lambda_{(1,1)}\lambda_{(2,1)} & t^2 + \lambda^2_{(2,1)} & t\lambda_{(1,2)} \\ 0 & t\lambda_{(1,2)} & \lambda^2_{(1,2)} \end{pmatrix},$$

$$B_0 = \begin{pmatrix} \lambda^2_{(1,1)} & \lambda_{(1,1)}\lambda_{(2,1)} \\ \lambda_{(1,1)}\lambda_{(2,1)} & \lambda^2_{(2,1)} \end{pmatrix} \text{ and } B_n = \begin{pmatrix} t^2 + \lambda^2_{(2,n)} & t\lambda_{(1,n+1)} \\ t\lambda_{(1,n+1)} & \lambda^2_{(1,n+1)} \end{pmatrix}.$$

Then we can obtain the entries of B_1^p, B_0^p and B_n^p by using $\mathrm{Diag}\{0, \alpha_1, \beta_1\} = Q_1^{-1}B_1Q_1$, $\mathrm{Diag}\{0, \lambda^2_{(1,1)} + \lambda^2_{(2,1)}\} = Q_0^{-1}B_0Q_0$ and $\mathrm{Diag}\{\alpha_n, \beta_n\} = Q_n^{-1}B_nQ_n$, where

$$Q_1 = \begin{pmatrix} \frac{\lambda_{(1,2)}\lambda_{(2,1)}}{t\lambda_{(1,1)}} & \frac{\lambda_{(1,1)}(\lambda^2_{(1,1)}+\lambda^2_{(2,1)}-\beta_1)}{t\lambda_{(2,1)}\lambda_{(1,2)}} & \frac{\lambda_{(1,1)}(\lambda^2_{(1,1)}+\lambda^2_{(2,1)}-\alpha_1)}{t\lambda_{(2,1)}\lambda_{(1,2)}} \\ \frac{-\lambda_{(1,2)}}{t} & \frac{\alpha_1-\lambda^2_{(1,2)}}{t\lambda_{(1,2)}} & \frac{\beta_1-\lambda^2_{(1,2)}}{t\lambda_{(1,2)}} \\ 1 & 1 & 1 \end{pmatrix},$$

$$Q_0 = \begin{pmatrix} -\frac{\lambda_{(2,1)}}{\lambda_{(1,1)}} & \frac{\lambda_{(1,1)}}{\lambda_{(2,1)}} \\ 1 & 1 \end{pmatrix} \text{ and } Q_n = \begin{pmatrix} \frac{\alpha_n-\lambda^2_{(1,n+1)}}{t\lambda_{(1,n+1)}} & \frac{\beta_n-\lambda^2_{(1,n+1)}}{t\lambda_{(1,n+1)}} \\ 1 & 1 \end{pmatrix}$$

with the α_n and β_n as in Lemma 2.1, $n \in \mathbb{N}$. Set $B_1^p := (b_{ij}(1,p))_{1\leq i,j\leq 3}$ and $B_n^p := (b_{ij}(n,p))_{1\leq i,j\leq 2}$, where the b_{ij}'s are as in Appendix A1. Note that B_1^p and B_n^p are symmetric, so $b_{ij} = b_{ji}$. And also we have

$$B_0^p = \begin{pmatrix} \lambda_{(1,1)}^2(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{-1+p} & \lambda_{(1,1)}\lambda_{(2,1)}(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{-1+p} \\ \lambda_{(1,1)}\lambda_{(2,1)}(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{-1+p} & \lambda_{(2,1)}^2(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{-1+p} \end{pmatrix}.$$

Since $(S_{t,1}S_{t,1}^*)^p = \text{Diag}\{0, \lambda_{-\kappa+1}^{2p}, \cdots, \lambda_0^{2p}, B_1^p, \lambda_{(2,2)}^{2p}, \lambda_{(1,3)}^{2p}, \lambda_{(2,3)}^{2p}, \lambda_{(1,4)}^{2p}, \cdots\}$, using Lemma 2.1 (i), (i) follows.

Second, using Lemma 2.1 (ii) with $n = 2$, the statements of (ii-a), (ii-b) and (ii-c) are easily checked from $(S_{t,2}^*S_{t,2})^p - (S_{t,2}S_{t,2}^*)^p \geq 0$. We can see that the statement (ii-d) is a condition equivalent to $\text{Diag}\{\lambda_{(2,3)}^{2p}, \lambda_{(1,4)}^{2p}\} - B_2^p \geq 0$.

Finally, we consider $(S_{t,n}^*S_{t,n})^p - (S_{t,n}S_{t,n}^*)^p \geq 0$ for $n \geq 3$. Using Lemma 2.1 (ii), we can see (iii-a) and (iii-b) easily. And we know that the positivity of $\text{Diag}\{\lambda_{(1,2)}^{2p}, \lambda_{(2,2)}^{2p}\} - B_0^p$ is equivalent to the following conditions:

$$\lambda_{(1,2)}^{2p} \geq \lambda_{(1,1)}^2(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{-1+p},$$
$$\lambda_{(2,2)}^{2p} \geq \lambda_{(2,1)}^2(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{-1+p},$$
$$\lambda_{(1,2)}^{2p}\lambda_{(2,2)}^{2p} \geq (\lambda_{(1,2)}^{2p}\lambda_{(2,1)}^2 + \lambda_{(1,1)}^2\lambda_{(2,2)}^{2p})(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{-1+p}.$$

Since we only consider the weights $\{\lambda_v\}_{v\in V_{2,\kappa}^\circ}$ of positive real numbers, in the presence of the third condition the first two inequalities above are automatic. Also, the conditions (iii-d) and (iii-e) are equivalent to the positivities of $A_n^p - \text{Diag}\{\lambda_{(2,n-1)}^{2p}, \lambda_{(1,n)}^{2p}\}$ and $\text{Diag}\{\lambda_{(2,n+1)}^{2p}, \lambda_{(1,n+2)}^{2p}\} - B_n^p$, respectively. Hence the proof is complete. □

Remark 3.3. It is obvious that $\left\|S_{t,n} - S_{0,n}\right\| \to 0$ as $t \to 0$. Also it is worth mentioning that if we let t approach 0 in the conditions equivalent to p-hyponormality of $S_{t,n}$ in Theorem 3.2, then such conditions obtained by some direct computations coincide exactly with the conditions equivalent to p-hyponormality of $S_{0,n}$ in Proposition 3.1.

Proposition 3.3. *Let $S_{0,n} = S_\lambda$ be as usual. Then $S_{0,n}$ is ∞-hyponormal if and only if the following conditions hold:*

(i) $\lambda_{m+1} \geq \lambda_m$, $-\kappa + 1 \leq m \leq -1$,

(ii) $\lambda_{(i,j+2)} \geq \lambda_{(i,j+1)}$, *for* $i = 1, 2$, $j \in \mathbb{N}$,

(iii) $\min\{\lambda_{(1,2)}^2, \lambda_{(2,2)}^2\} \geq \lambda_{(1,1)}^2 + \lambda_{(2,1)}^2 \geq \lambda_0^2$.

Proof. Since (i), (ii) and (iv) in Proposition 3.1 are independent of p, we will show that Proposition 3.1(iii) is equivalent to the condition $\min\{\lambda_{(1,2)}^2, \lambda_{(2,2)}^2\} \geq \lambda_{(1,1)}^2 + \lambda_{(2,1)}^2$. Suppose Proposition 3.1(iii) holds for all $p > 0$, i.e.,

$$\left(\frac{\lambda_{(2,1)}^2}{\lambda_{(2,2)}^{2p}} + \frac{\lambda_{(1,1)}^2}{\lambda_{(1,2)}^{2p}}\right)\left(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2\right)^{p-1} \leq 1, \ p > 0. \tag{3.7}$$

Without loss of generality, we assume that $\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2 = 1$. To see the first inequality of (iii), suppose $\min\{\lambda_{(1,2)}^2, \lambda_{(2,2)}^2\} < 1$. Say $\lambda_{(1,2)} < 1$. Then

$$\frac{\lambda_{(2,1)}^2}{\lambda_{(2,2)}^{2p}} + \frac{\lambda_{(1,1)}^2}{\lambda_{(1,2)}^{2p}} \to \infty \text{ as } p \to \infty,$$

which contradicts (3.7). Thus $\min\{\lambda^2_{(1,2)}, \lambda^2_{(2,2)}\} \geq \lambda^2_{(1,1)} + \lambda^2_{(2,1)}$. Conversely, we suppose the first inequality of Proposition 3.3(iii) holds, i.e., $\lambda^2_{(i,2)} \geq \lambda^2_{(1,1)} + \lambda^2_{(2,1)}$ $(i = 1, 2)$. Then, for any $p > 0$, we have

$$\begin{aligned}
&\left(\frac{\lambda^2_{(2,1)}}{\lambda^{2p}_{(2,2)}} + \frac{\lambda^2_{(1,1)}}{\lambda^{2p}_{(1,2)}}\right)\left(\lambda^2_{(1,1)} + \lambda^2_{(2,1)}\right)^{p-1} \\
&= \left(\lambda^2_{(2,1)}\left(\frac{\lambda^2_{(1,1)} + \lambda^2_{(2,1)}}{\lambda^2_{(2,2)}}\right)^p + \lambda^2_{(1,1)}\left(\frac{\lambda^2_{(1,1)} + \lambda^2_{(2,1)}}{\lambda^2_{(1,2)}}\right)^p\right)\left(\lambda^2_{(1,1)} + \lambda^2_{(2,1)}\right)^{-1} \\
&\leq \left(\lambda^2_{(2,1)} + \lambda^2_{(1,1)}\right)\left(\lambda^2_{(1,1)} + \lambda^2_{(2,1)}\right)^{-1} = 1.
\end{aligned}$$

So Proposition 3.1(iii) holds for all $p > 0$. □

Remark 3.4 (Normality). Note that $S_{t,n}$ can not be normal because weights are strictly positive. However, if we consider a weight sequence $\{\lambda_v\}_{v\in V^\circ_{2,\kappa}}$ in the real numbers, we can obtain that $S_{t,n}$ is normal if and only if the following conditions hold:

(i) if $\kappa < \infty$, then $t = 0 = \lambda_v,\ v \in V^\circ_{2,\kappa}$,

(ii) if $\kappa = \infty$, then one of the following conditions holds:

(ii-a) $t = \lambda_{(1,j)} = 0,\ \lambda_0 = \lambda_{-j} = \lambda_{(2,j)},\ j \in \mathbb{N}$,

(ii-b) $t = \lambda_{(2,j)} = 0,\ \lambda_0 = \lambda_{-j} = \lambda_{(1,j)},\ j \in \mathbb{N}$,

(ii-c) $t = \lambda_0 = \lambda_{-j} = \lambda_{(1,k)} = \lambda_{(2,j+n)},\ \lambda_{(1,j+n)} = \lambda_{(2,k)} = 0,\ 1 \leq k \leq n,\ j \in \mathbb{N}$.

Remark 3.5 (Quasinormality). Let $S_{t,n}$ be as usual. If $S_{t,n}$ is quasinormal, by a direct computation, $t = 0$, and so $S_{t,n}$ must be $S_{0,n}$. And $S_{0,n}$ is quasinormal if and only if

$$\lambda^2_{(1,1)} + \lambda^2_{(2,1)} = \lambda^2_v,\ v \in V^\circ_{2,\kappa}\backslash\{(1,1),(2,1)\}.$$

Of course, if we consider a weight sequence $\{\lambda_v\}_{v\in V^\circ_{2,\kappa}}$ in the real numbers, we can obtain some equivalent conditions for quasinormality of $S_{t,n}$. We leave the detailed conditions to the interested readers.

4. Weak hyponormalities

There are several kinds of partial normalities that are weaker than p-hyponormality, for example, p-paranormality, absolute-p-paranormality, $A(p)$-class (cf.[16],[18]). In particular, $S_{0,n} = S_\lambda$ is p-paranormal if and only if S_λ is absolute-p-paranormal (if and only if S_λ is $A(p)$-class). By some direct computations, S_λ is p-paranormal if and only if the following conditions hold:

(i) $\lambda_{m+1} \geq \lambda_m,\ -\kappa + 1 \leq m \leq -1$,

(ii) $\lambda^2_{(1,1)} + \lambda^2_{(2,1)} \geq \lambda^2_0$,

(iii) $\lambda^2_{(1,1)}\lambda^{2p}_{(1,2)} + \lambda^2_{(2,1)}\lambda^{2p}_{(2,2)} \geq (\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^{p+1}$,

(iv) $\lambda_{(i,j+2)} \geq \lambda_{(i,j+1)}$, for $i = 1, 2,\ j \in \mathbb{N}$.

It is not known in general for $p \in (0,\infty)\backslash\{1\}$ whether p-paranormality is different from absolute-p-paranormality. It is worth discussing p-paranormality and absolute-p-paranormality of $S_{t,n}$.

4.1. Absolute-p-paranormality. Recall from [16, p.174] that $T \in B(\mathcal{H})$ is absolute-p-paranormal if and only if $T^*(T^*T)^pT - (p+1)T^*Ts^p + ps^{p+1}I \geq 0$ for all $s \in \mathbb{R}_+$.

Theorem 4.1. *Let $S_{t,n}$ be as in* (2.1) *and let the a_{ij}'s be as in* Lemma 2.1. *Suppose $p \in (0,\infty)$ and $t \in \mathbb{R}\backslash\{0\}$. Then*

(i) *$S_{t,1}$ is absolute-p-paranormal if and only if the following conditions hold:*

(i-a) *the inequalities in* (3.1) *and* (3.2) *hold,*

(i-b) *for all $s \in \mathbb{R}_+$, $\Omega_1 := \Omega_1(p,t,s) \geq 0$, where*

$$\Omega_1 := \begin{pmatrix} \omega_{11}(1,p) & \lambda_0\lambda_{(1,1)}a_{12}(1,p) & 0 \\ \lambda_0\lambda_{(1,1)}a_{12}(1,p) & \omega_{22}(1,p) & t\lambda_{(2,1)}(\lambda_{(2,2)}^{2p} - (p+1)s^p) \\ 0 & t\lambda_{(2,1)}(\lambda_{(2,2)}^{2p} - (p+1)s^p) & \omega_{33}(1,p) \end{pmatrix} \tag{4.1}$$

with ω_{ii}'s as in Appendix A2.

(ii) *$S_{t,2}$ is absolute-p-paranormal if and only if the following conditions hold:*

(ii-a) *the inequalities in* (3.1), (3.3) *and* (3.4) *hold,*

(ii-b) *it holds that*

$$\lambda_{(1,1)}^2\lambda_{(1,2)}^{2p} + a_{11}(2,p)\lambda_{(2,1)}^2 \geq (\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{1+p},\ a_{22}(2,p) \geq \lambda_{(1,2)}^{2p},$$
$$\omega_{11}(2,p)\omega_{22}(2,p) \geq a_{12}(2,p)^2\lambda_{(1,2)}^2\lambda_{(2,1)}^2,\ s \in \mathbb{R}_+,$$

(ii-c) *it holds that*

$$\lambda_{(2,3)} \geq \lambda_{(2,2)},\ \lambda_{(1,3)}^2\lambda_{(1,4)}^{2p} + t^2\lambda_{(2,3)}^{2p} \geq (t^2 + \lambda_{(1,3)}^2)^{1+p},$$
$$\widetilde{\omega}_{11}(2,p)\widetilde{\omega}_{22}(2,p) \geq t^2\lambda_{(2,2)}^2\{\lambda_{(2,3)}^{2p} - (1+p)s^p\}^2,\ s \in \mathbb{R}_+,$$

where ω_{ii}'s and $\widetilde{\omega}_{ii}$'s are as in Appendix A2.

(iii) *For $n \geq 3$, $S_{t,n}$ is absolute-p-paranormal if and only if the following conditions hold:*

(iii-a) *the inequalities in* (3.1), (3.3), (3.5) *and* (3.6) *hold,*

(iii-b) *it holds that*

$$\lambda_{(1,1)}^2\lambda_{(1,2)}^{2p} + \lambda_{(2,1)}^2\lambda_{(2,2)}^{2p} \geq (\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{p+1},$$

(iii-c) *it holds that*

$$a_{11}(n,p) \geq \lambda_{(2,n-1)}^{2p},\ a_{22}(n,p) \geq \lambda_{(1,n)}^{2p},$$
$$\omega_{11}(n,p)\omega_{22}(n,p) \geq a_{12}(n,p)^2\lambda_{(1,n)}^2\lambda_{(2,n-1)}^2,\ s \in \mathbb{R}_+,$$

(iii-d) *it holds that*

$$\lambda_{(2,n+1)} \geq \lambda_{(2,n)},\ \lambda_{(1,n+1)}^2\lambda_{(1,n+2)}^{2p} + t^2\lambda_{(2,n+1)}^{2p} \geq (t^2 + \lambda_{(1,n+1)}^2)^{1+p},$$
$$\widetilde{\omega}_{11}(n,p)\widetilde{\omega}_{22}(n,p) \geq t^2\lambda_{(2,n)}^2\{\lambda_{(2,n+1)}^{2p} - (1+p)s^p\}^2,\ s \in \mathbb{R}_+,$$

where ω_{ii}'s and $\widetilde{\omega}_{ii}$'s are as in Appendix A2.

Proof. By Lemma 2.1(i), it is easy to compute that

$$S_{t,1}^*(S_{t,1}^*S_{t,1})^pS_{t,1} = \operatorname{Diag}\{\lambda_{-\kappa+1}^2\lambda_{-\kappa+2}^{2p}, \cdots, \lambda_{-1}^2\lambda_0^{2p}, W_1, \lambda_{(2,2)}^2\lambda_{(2,3)}^{2p}, \lambda_{(1,3)}^2\lambda_{(1,4)}^{2p}, \cdots\},$$

where

$$W_1 = \begin{pmatrix} a_{11}(1,p)\lambda_0^2 & \lambda_0\lambda_{(1,1)}a_{12}(1,p) & 0 \\ \lambda_0\lambda_{(1,1)}a_{12}(1,p) & a_{22}(1,p)\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2\lambda_{(2,2)}^{2p} & t\lambda_{(2,1)}\lambda_{(2,2)}^{2p} \\ 0 & t\lambda_{(2,1)}\lambda_{(2,2)}^{2p} & \lambda_{(1,2)}^2\lambda_{(1,3)}^{2p} + t^2\lambda_{(2,2)}^{2p} \end{pmatrix}. \tag{4.2}$$

Using (2.5), we can obtain that

$$\begin{aligned} &S_{t,1}^*(S_{t,1}^*S_{t,1})^pS_{t,1} - (p+1)S_{t,1}^*S_{t,1}s^p + ps^{p+1}I \\ &\qquad = \mathrm{Diag}\{\theta_{-\kappa+1},\cdots,\theta_{-1},\Omega_1,\theta_{(2,2)},\theta_{(1,3)},\cdots\}, \end{aligned}$$

where

$$\theta_{-m} := \lambda_{-m}^2\lambda_{-m+1}^{2p} - (p+1)\lambda_{-m}^2s^p + ps^{p+1}, \tag{4.3}$$

$$\theta_{(i,j)} := \lambda_{(i,j)}^2\lambda_{(i,j+1)}^{2p} - (p+1)\lambda_{(i,j)}^2s^p + ps^{p+1} \tag{4.4}$$

with $\kappa - 1 \geq m \geq 1$, $i = 1$; $j \geq 3$, $i = 2$; $j \geq 2$ and Ω_1 is as in (4.1). So, for $\kappa - 1 \geq m \geq 1$, $k \in \mathbb{N}$, θ_{-m}, $\theta_{(1,k+2)}$ and $\theta_{(2,k+1)}$ are nonnegative for all $s > 0$ if and only if

$$\lambda_{-m+1} \geq \lambda_{-m},\ \lambda_{(1,k+3)} \geq \lambda_{(1,k+2)} \text{ and } \lambda_{(2,k+2)} \geq \lambda_{(2,k+1)}.$$

Hence (i) is proved.

Next, by applying Lemma 2.1(ii) with $n = 2$, we can also compute that

$$\begin{aligned} S_{t,2}^*(S_{t,2}^*S_{t,2})^pS_{t,2} = {} & \mathrm{Diag}\{\lambda_{-\kappa+1}^2\lambda_{-\kappa+2}^{2p},\cdots,\lambda_{-1}^2\lambda_0^{2p},\lambda_0^2(\lambda_{(1,1)}^2+\lambda_{(2,1)}^2)^p, \\ & W_2,\widetilde{W}_2,\lambda_{(2,3)}^2\lambda_{(2,4)}^{2p},\lambda_{(1,4)}^2\lambda_{(1,5)}^{2p},\cdots\}, \end{aligned}$$

where

$$W_2 = \begin{pmatrix} a_{11}(2,p)\lambda_{(2,1)}^2 + \lambda_{(1,1)}^2\lambda_{(1,2)}^{2p} & \lambda_{(1,2)}\lambda_{(2,1)}a_{12}(2,p) \\ \lambda_{(1,2)}\lambda_{(2,1)}a_{12}(2,p) & a_{22}(2,p)\lambda_{(1,2)}^2 \end{pmatrix} \tag{4.5}$$

and

$$\widetilde{W}_2 = \begin{pmatrix} \lambda_{(2,2)}^2\lambda_{(2,3)}^{2p} & t\lambda_{(2,2)}\lambda_{(2,3)}^{2p} \\ t\lambda_{(2,2)}\lambda_{(2,3)}^{2p} & \lambda_{(1,3)}^2\lambda_{(1,4)}^{2p} + t^2\lambda_{(2,3)}^{2p} \end{pmatrix}. \tag{4.6}$$

Using (2.6) with $n = 2$,

$$\begin{aligned} &S_{t,2}^*(S_{t,2}^*S_{t,2})^pS_{t,2} - (p+1)S_{t,2}^*S_{t,2}s^p + ps^{p+1}I \\ &\qquad = \mathrm{Diag}\{\theta_{-\kappa+1},\cdots,\theta_{-1},\theta_0,\Omega_2,\widetilde{\Omega}_2,\theta_{(2,3)},\theta_{(1,4)},\cdots\}, \end{aligned}$$

where θ_{-m}, $\kappa - 1 \geq m \geq 1$, $\theta_{(i,j)}$, $i = 1$, $j \geq 4$; $i = 2$, $j \geq 3$ are as in (4.3) and (4.4),

$$\theta_0 := \lambda_0^2(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^p - (p+1)\lambda_0^2s^p + ps^{p+1}, \tag{4.7}$$

$$\Omega_2 = \begin{pmatrix} \omega_{11}(2,p) & \lambda_{(1,2)}\lambda_{(2,1)}a_{12}(2,p) \\ \lambda_{(1,2)}\lambda_{(2,1)}a_{12}(2,p) & \omega_{22}(2,p) \end{pmatrix}$$

and

$$\widetilde{\Omega}_2 = \begin{pmatrix} \widetilde{\omega}_{11}(2,p) & t\lambda_{(2,2)}(\lambda_{(2,3)}^{2p} - (1+p)s^p) \\ t\lambda_{(2,2)}(\lambda_{(2,3)}^{2p} - (1+p)s^p) & \widetilde{\omega}_{22}(2,p) \end{pmatrix}$$

with ω_{ii}'s and $\widetilde{\omega}_{ii}$'s as in Appendix A2. It follows that the positivities of Ω_2 and $\widetilde{\Omega}_2$ are equivalent to (ii-b) and (ii-c), respectively. And (ii-a) can be checked easily.

Finally, by using Lemma 2.1(ii), we get that for $n \geq 3$,

$$\begin{aligned} &S_{t,n}^*(S_{t,n}^*S_{t,n})^pS_{t,n} - (p+1)S_{t,n}^*S_{t,n}s^p + ps^{p+1}I \\ &\quad = \text{Diag}\{\theta_{-\kappa+1}, \cdots, \theta_0, \theta_{(1,1)}, \theta_{(1,2)}, \theta_{(2,2)}, \cdots, \theta_{(2,n-2)}, \\ &\qquad\qquad \theta_{(1,n-1)}, \Omega_n, \widetilde{\Omega}_n, \theta_{(2,n+1)}, \theta_{(1,n+2)}, \cdots\}, \end{aligned}$$

where θ_{-m}, $\kappa - 1 \geq m \geq 1$, $\theta_{(1,j)}$, $2 \leq j \leq n-1$; $j \geq n+2$, and $\theta_{(2,j)}$, $2 \leq j \leq n-2$; $j \geq n+1$ are as in (4.3), (4.7) and (4.4),

$$\theta_{(1,1)} := \lambda_{(1,1)}^2\lambda_{(1,2)}^{2p} + \lambda_{(2,1)}^2\lambda_{(2,2)}^{2p} - (p+1)(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)s^p + ps^{p+1},$$

$$\Omega_n = \begin{pmatrix} \omega_{11}(n,p) & \lambda_{(1,n)}\lambda_{(2,n-1)}a_{12}(n,p) \\ \lambda_{(1,n)}\lambda_{(2,n-1)}a_{12}(n,p) & \omega_{22}(n,p) \end{pmatrix}$$

and

$$\widetilde{\Omega}_n = \begin{pmatrix} \widetilde{\omega}_{11}(n,p) & t\lambda_{(2,n)}(\lambda_{(2,n+1)}^{2p} - (1+p)s^p) \\ t\lambda_{(2,n)}(\lambda_{(2,n+1)}^{2p} - (1+p)s^p) & \widetilde{\omega}_{22}(n,p) \end{pmatrix},$$

where ω_{ii}'s and $\widetilde{\omega}_{ii}$'s are as in Appendix A2. It follows that (iii-c) and (iii-d) are equivalent to the positivities of Ω_n and $\widetilde{\Omega}_n$, respectively. For all $s > 0$, $\theta_{(1,1)}$ is nonnegative if and only if (iii-b) holds. And (iii-a) can be obtained by nonnegativity of θ_v, $v \in V_{2,\kappa}^\circ \setminus \{(1,1),(2,1),(2,n-1),(1,n),(2,n),(1,n+1)\}$ for all $s > 0$. Hence the proof is complete. □

4.2. p-Paranormality. For $T \in B(\mathcal{H})$, let $T = U\,|T|$ be the (unique) polar decomposition of T. Then it follows from [27, Prop. 3] that T is p-paranormal if and only if

$$|T|^p\, U^*\, |T|^{2p}\, U\, |T|^p - 2s\,|T|^{2p} + s^2I \geq 0, \;\; s \in \mathbb{R}_+.$$

To characterize the p-paranormality of $S_{t,n}$, we begin with the following lemma.

Lemma 4.2. *Let $S_{t,n}$ be as in* (2.1)*, where $t \in \mathbb{R}\setminus\{0\}$, and let the a_{ij}'s be as in* Lemma 2.1. *Let $S_{t,n} = U_{t,n}\left|S_{t,n}\right|$ be the polar decomposition of $S_{t,n}$. Then*

(i) $U_{t,1} = S_{\widetilde{\lambda}} + u_{12}(1)e_{(1,1)} \otimes e_{(1,1)} + u_{22}(1)e_{(2,1)} \otimes e_{(1,1)} + u_{31}(1)e_{(1,2)} \otimes e_0$, *where*

$$u_{11}(1) = a_{22}\left(1,\frac{1}{2}\right)\lambda_{(1,1)}\Big/\delta, \qquad u_{12}(1) = -a_{12}\left(1,\frac{1}{2}\right)\lambda_{(1,1)}\Big/\delta,$$

$$u_{21}(1) = \left\{a_{22}\left(1,\frac{1}{2}\right)\lambda_{(2,1)} - ta_{12}\left(1,\frac{1}{2}\right)\right\}\Big/\delta, \quad u_{22}(1) = \left\{ta_{11}\left(1,\frac{1}{2}\right) - a_{12}\left(1,\frac{1}{2}\right)\lambda_{(2,1)}\right\}\Big/\delta,$$

$$u_{31}(1) = -a_{12}\left(1,\frac{1}{2}\right)\lambda_{(1,2)}\Big/\delta, \qquad u_{32}(1) = a_{11}\left(1,\frac{1}{2}\right)\lambda_{(1,2)}\Big/\delta$$

with $\delta = (\lambda_{(1,2)}^2\lambda_{(2,1)}^2 + \lambda_{(1,1)}^2(t^2 + \lambda_{(1,2)}^2))^{1/2}$ and $\widetilde{\lambda} := \{\widetilde{\lambda}_v\}_{v \in V_{2,\kappa}^\circ}$ such that $\widetilde{\lambda}_{(1,1)} = u_{11}(1)$, $\widetilde{\lambda}_{(2,1)} = u_{21}(1)$, $\widetilde{\lambda}_{(1,2)} = u_{32}(1)$ and $\widetilde{\lambda}_v = 1$ (otherwise),

(ii) *if $n \geq 2$,*

$$U_{t,n} = S_{\widetilde{\lambda}} + u_{21}(n)e_{(1,n+1)} \otimes e_{(2,n-1)} + u_{12}(n)e_{(2,n)} \otimes e_{(1,n)},$$

where

$$u_{11}(n) = \left\{a_{22}\left(n, \frac{1}{2}\right)\lambda_{(2,n)} - ta_{12}\left(n, \frac{1}{2}\right)\right\} \Big/ (\lambda_{(1,n+1)}\lambda_{(2,n)}),$$
$$u_{12}(n) = \left\{ta_{11}\left(n, \frac{1}{2}\right) - a_{12}\left(n, \frac{1}{2}\right)\lambda_{(2,n)}\right\} \Big/ (\lambda_{(1,n+1)}\lambda_{(2,n)}),$$
$$u_{21}(n) = -a_{12}\left(n, \frac{1}{2}\right)\Big/\lambda_{(2,n)},\ u_{22}(n) = a_{11}\left(n, \frac{1}{2}\right)\Big/\lambda_{(2,n)}$$

with $\widetilde{\lambda} := \{\widetilde{\lambda}_v\}_{v\in V^\circ_{2,\kappa}}$ *such that* $\widetilde{\lambda}_{(1,1)} = \lambda_{(1,1)}(\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^{-1/2}$, $\widetilde{\lambda}_{(2,1)} = \lambda_{(2,1)}(\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^{-1/2}$, $\widetilde{\lambda}_{(2,n)} = u_{11}(n)$, $\widetilde{\lambda}_{(1,n+1)} = u_{22}(n)$ *and* $\widetilde{\lambda}_v = 1$ *(otherwise).*

Proof. Since the weights $\{\lambda_v\}_{v\in V^\circ_{2,\kappa}}$ are positive and the determinants of $A_1^{1/2}$ and $A_n^{1/2}$ ($n \geq 2$) are $(\lambda^2_{(1,2)}\lambda^2_{(2,1)} + \lambda^2_{(1,1)}(t^2 + \lambda^2_{(1,2)}))^{1/2}$ and $\lambda_{(1,n+1)}\lambda_{(2,n)}$, respectively, we see that $|S_{t,n}|$ is invertible for all $n \in \mathbb{N}$. Other proofs are routine. □

We now characterize the p-paranormality of $S_{t,n}$.

Theorem 4.3. *Let* $S_{t,n}$ *be as in* (2.1) *and let the* a_{ij}*'s be as in* Lemma 2.1. *Suppose that* $p \in (0, \infty)$ *and* $t \in \mathbb{R}\backslash\{0\}$. *Then*

(i) $S_{t,1}$ *is p-paranormal if and only if the inequalities in* (3.1) *and* (3.2) *hold, and for all* $s \in \mathbb{R}_+$, $\Psi_1 := (\varphi_{ij}(1, p))_{1\leq i,j\leq 3} \geq 0$, *where the* φ_{ij}*'s are as in Appendix* A3,

(ii) $S_{t,2}$ *is p-paranormal if and only if the following assertions hold:*

(ii-a) *the inequalities in* (3.1), (3.3) *and* (3.4) *hold,*

(ii-b) *it holds that*

$$\lambda^2_{(1,1)}\lambda^{2p}_{(1,2)} + a_{11}(2, p)\lambda^2_{(2,1)} \geq (\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^{p+1},\ a_{22}(2, p) \geq \lambda^{2p}_{(1,2)},$$
$$\varphi_{11}(2, p)\varphi_{22}(2, p) \geq a_{12}(2, p)^2\lambda^{2p}_{(1,2)}\lambda^2_{(2,1)}(\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^{p-1},\ s \in \mathbb{R}_+,$$

(ii-c) *it holds that*

$$\lambda^{2p}_{(2,3)}\phi_1(2)^2 + \lambda^{2p}_{(1,4)}\phi_4(2)^2 \geq a_{11}(2, p)^2,$$
$$\lambda^{2p}_{(2,3)}\phi_2(2)^2 + \lambda^{2p}_{(1,4)}\phi_6(2)^2 \geq a_{22}(2, p)^2,$$
$$\widetilde{\varphi}_{11}(2, p)\widetilde{\varphi}_{22}(2, p) \geq \widetilde{\varphi}_{12}(2, p)^2,\ s \in \mathbb{R}_+,$$

where φ_{ij}*'s and* $\widetilde{\varphi}_{ij}$*'s are as in Appendix* A3.

(iii) $S_{t,n}$ *for* $n \geq 3$ *is p-paranormal if and only if the following assertions hold:*

(iii-a) *the inequalities in* (3.1), (3.3), (3.5) *and* (3.6) *hold,*

(iii-b) $\lambda^2_{(1,1)}\lambda^{2p}_{(1,2)} + \lambda^2_{(2,1)}\lambda^{2p}_{(2,2)} \geq (\lambda^2_{(1,1)} + \lambda^2_{(2,1)})^{p+1}$,

(iii-c) *it holds that*

$$a_{11}(n, p) \geq \lambda^{2p}_{(2,n-1)},\ a_{22}(n, p) \geq \lambda^{2p}_{(1,n)},$$
$$\varphi_{11}(n, p)\varphi_{22}(n, p) \geq a_{12}(n, p)^2\lambda^{2p}_{(1,n)}\lambda^{2p}_{(2,n-1)},\ s \in \mathbb{R}_+,$$

(iii-d) *it holds that*

$$\lambda^{2p}_{(2,n+1)}\phi_1(n)^2 + \lambda^{2p}_{(1,n+2)}\phi_4(n)^2 \geq a_{11}(n, p)^2,$$
$$\lambda^{2p}_{(2,n+1)}\phi_2(n)^2 + \lambda^{2p}_{(1,n+2)}\phi_6(n)^2 \geq a_{22}(n, p)^2,$$
$$\widetilde{\varphi}_{11}(n, p)\widetilde{\varphi}_{22}(n, p) \geq \widetilde{\varphi}_{12}(n, p)^2,\ s \in \mathbb{R}_+,$$

where φ_{ij}'s and $\widetilde{\varphi}_{ij}$'s are as in Appendix A3.

Proof. (i) Applying Lemma 2.1(i) and Lemma 4.2(i), it follows that

$$\left|S_{t,1}\right|^p U_{t,1}^* \left|S_{t,1}\right|^{2p} U_{t,1} \left|S_{t,1}\right|^p - 2s\left|S_{t,1}\right|^{2p} + s^2 I = \mathrm{Diag}\{\psi_{-\kappa+1}, \cdots, \psi_{-1}, \Psi_1, \psi_{(2,2)}, \psi_{(1,3)}, \cdots\},$$

where

$$\psi_{-m} := \lambda_{-m}^{2p}\lambda_{-m+1}^{2p} - 2\lambda_{-m}^{2p}s + s^2, \tag{4.8}$$

$$\psi_{(i,j)} := \lambda_{(i,j)}^{2p}\lambda_{(i,j+1)}^{2p} - 2\lambda_{(i,j)}^{2p}s + s^2 \tag{4.9}$$

with $\kappa - 1 \geq m \geq 1$, $i = 1; j \geq 3$, $i = 2; j \geq 2$, and $\Psi_1 := (\varphi_{ij}(1,p))_{1\leq i,j\leq 3}$, with $\varphi_{ij}(1,p)$'s as in Appendix A3. So $S_{t,1}$ is p-paranormal if and only if ψ_{-m}, $\psi_{(i,j)}$ and Ψ_1 are nonnegative for all $s \in \mathbb{R}_+$. It is obvious that ψ_{-m} and $\psi_{(i,j)}$ are nonnegative for all $s \in \mathbb{R}_+$ if and only if (3.1) and (3.2) hold, respectively.

(ii) By Lemma 2.1(ii) and Lemma 4.2(ii) with $n = 2$, we have

$$\left|S_{t,2}\right|^p U_{t,2}^* \left|S_{t,2}\right|^{2p} U_{t,2} \left|S_{t,2}\right|^p - 2s\left|S_{t,2}\right|^{2p} + s^2 I = \mathrm{Diag}\{\psi_{-\kappa+1}, \cdots, \psi_{-1}, \psi_0, \Psi_2, \widetilde{\Psi}_2, \psi_{(2,3)}, \psi_{(1,4)}, \cdots\},$$

where ψ_{-m}, $\kappa - 1 \geq m \geq 1$, $\psi_{(1,j)}$, $j \geq 4$, and $\psi_{(2,j)}$, $j \geq 3$ are as in (4.8) and (4.9), respectively,

$$\psi_0 := \lambda_0^{2p}(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^p - 2\lambda_0^{2p}s + s^2, \tag{4.10}$$

$$\Psi_2 = \begin{pmatrix} \varphi_{11}(2,p) & \lambda_{(1,2)}^p\lambda_{(2,1)}a_{12}(2,p)(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{\frac{p-1}{2}} \\ \lambda_{(1,2)}^p\lambda_{(2,1)}a_{12}(2,p)(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{\frac{p-1}{2}} & \varphi_{22}(2,p) \end{pmatrix}$$

and $\widetilde{\Psi}_2 := (\widetilde{\varphi}_{ij}(2,p))_{1\leq i,j\leq 2}$ with the φ_{ij}'s and $\widetilde{\varphi}_{ij}$'s as in Appendix A3. For all $s \in \mathbb{R}_+$, $\psi_v \geq 0$, $v \in V_{2,\kappa}^\circ \setminus \{(1,1),(2,1),(1,2),(2,2),(1,3)\}$ if and only if (ii-a) holds. It follows that the matrices Ψ_2 and $\widetilde{\Psi}_2$ are positive semi-definite for all $s \in \mathbb{R}_+$ if and only if (ii-b) and (ii-c) hold, respectively.

(iii) By Lemma 2.1(ii) and Lemma 4.2(ii) with $n \geq 3$, we have

$$\begin{aligned}\left|S_{t,n}\right|^p U_{t,n}^* \left|S_{t,n}\right|^{2p} U_{t,n} \left|S_{t,n}\right|^p - 2s\left|S_{t,n}\right|^{2p} + s^2 I &= \mathrm{Diag}\{\psi_{-\kappa+1}, \cdots, \psi_{-1}, \psi_0, \psi_{(1,1)}, \psi_{(1,2)}, \psi_{(2,2)}, \cdots, \psi_{(2,n-2)}, \\ &\qquad \psi_{(1,n-1)}, \Psi_n, \widetilde{\Psi}_n, \psi_{(2,n+1)}, \psi_{(1,n+2)}, \cdots\},\end{aligned}$$

where ψ_{-m}, $\kappa - 1 \geq m \geq 1$, ψ_0, $\psi_{(1,j)}$, $2 \leq j \leq n-1$; $j \geq n+2$, $\psi_{(2,j)}$, $2 \leq j \leq n-2$; $j \geq n+1$ are as in (4,8), (4.10) and (4.9),

$$\begin{aligned}\psi_{(1,1)} &= (\lambda_{(1,1)}^2\lambda_{(1,2)}^{2p} + \lambda_{(2,1)}^2\lambda_{(2,2)}^{2p})(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{p-1} - 2(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^p s + s^2, \\ \Psi_n &= \begin{pmatrix} \varphi_{11}(n,p) & \lambda_{(1,n)}^p\lambda_{(2,n-1)}^p a_{12}(n,p) \\ \lambda_{(1,n)}^p\lambda_{(2,n-1)}^p a_{12}(n,p) & \varphi_{22}(n,p) \end{pmatrix}, \\ \widetilde{\Psi}_n &= (\widetilde{\varphi}_{ij}(n,p))_{1\leq i,j\leq 2}\end{aligned}$$

with the φ_{ij}'s and $\widetilde{\varphi}_{ij}$'s as in Appendix A3. For all $s \in \mathbb{R}_+$, $\psi_v \geq 0$, $v \in V_{2,\kappa}^\circ \setminus \{(2,1),(2,n-$

$1),(1,n),(2,n),(1,n+1)\}$ if and only if (iii-a) and (iii-b) hold. It follows that Ψ_n and $\widetilde{\Psi}_n$ are positive semi-definite for all $s \in \mathbb{R}_+$ if and only if (iii-c) and (iii-d) hold, respectively. Hence the proof is complete. □

4.3. A(p)-class. Recall that an operator $T \in B(\mathcal{H})$ is a *class A operator* if $\left|T^2\right| \geq |T|^2$. The class A operators have been developed well for several decades. Note that the $A(1)$ class property is equivalent to the class A property. In [17], one shows that there exists an absolute-2-paranormal operator T which is not $A(2)$-class by using some block matrices. However the models for $A(p)$-class operators have not been developed completely. In this section we characterize the class $A(p)$-class property of our operator model $S_{t,n}$.

Theorem 4.4. *Let $S_{t,n}$ be as in* (2.1) *and let the a_{ij}'s be as in* Lemma 2.1. *Suppose that $p \in (0,\infty)$ and $t \in \mathbb{R}\backslash\{0\}$. Then the following assertions hold.*

(i) $S_{t,1}$ *is* an $A(p)$*-class operator if and only if the inequalities in* (3.1), (3.2) *and* $W_1^{\frac{1}{p+1}} \geq \mathrm{Diag}\{\lambda_0^2, A_1\}$ *hold, where W_1 is as in* (4.2) *and A_1 is as in* (2.7).

(ii) $S_{t,2}$ *is* an $A(p)$*-class operator if and only if the following conditions hold:*

(ii-a) *the inequalities in* (3.1), (3.3) *and* (3.4) *hold,*

(ii-b) *it holds that*

$$f_{11}(2,p) \geq \lambda_{(1,1)}^2 + \lambda_{(1,2)}^2,\ f_{22}(2,p) \geq \lambda_{(1,2)}^2,$$
$$(f_{11}(2,p) - \lambda_{(1,1)}^2 - \lambda_{(1,2)}^2)(f_{22}(2,p) - \lambda_{(1,2)}^2) \geq f_{12}(2,p)^2,$$

where f_{ij}'s are as in Appendix A4,

(ii-c) *it holds that*

$$g_{11}(2,p) \geq \lambda_{(2,2)}^2,\ g_{22}(2,p) \geq \lambda_{(1,3)}^2 + t^2,$$
$$(g_{11}(2,p) - \lambda_{(2,2)}^2)(g_{22}(2,p) - \lambda_{(1,3)}^2 - t^2) \geq (g_{12}(2,p) - t\lambda_{(2,2)})^2,$$

where g_{ij}'s are as in Appendix A4.

(iii) *For $n \geq 3$, $S_{t,n}$ is* an $A(p)$*-class operator if and only if the following conditions hold:*

(iii-a) *the inequalities in* (3.1), (3.3), (3.5) *and* (3.6) *hold,*

(iii-b) $\lambda_{(1,1)}^2\lambda_{(1,2)}^{2p} + \lambda_{(2,1)}^2\lambda_{(2,2)}^{2p} \geq (\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{p+1}$,

(iii-c) *it holds that*

$$f_{11}(n,p) \geq \lambda_{(2,n-1)}^2,\ f_{22}(n,p) \geq \lambda_{(1,n)}^2,$$
$$(f_{11}(n,p) - \lambda_{(2,n-1)}^2)(f_{22}(n,p) - \lambda_{(1,n)}^2) \geq f_{12}(n,p)^2,$$

where f_{ij}'s are as in Appendix A5,

(iii-d) *it holds that*

$$g_{11}(n,p) \geq \lambda_{(2,n)}^2,\ g_{22}(n,p) \geq \lambda_{(1,n+1)}^2 + t^2,$$
$$(g_{11}(n,p) - \lambda_{(2,n)}^2)(g_{22}(n,p) - \lambda_{(1,n+1)}^2 - t^2) \geq (g_{12}(n,p) - t\lambda_{(2,n)})^2,$$

where g_{ij}'s are as in Appendix A5.

Proof. See (4.2) in the proof of Theorem 4.1 for the matrix form of $S_{t,1}^*\left|S_{t,1}\right|^{2p}S_{t,1}$, and also (2.5) for the matrix form of $\left|S_{t,1}\right|^2$. The statement (i) then follows naturally. Since W_1 is diagonalized by its eigenvectors, we can also find the matrix form of $W_1^{\frac{1}{p+1}}$ by direct computation.

Applying the proof of Theorem 4.3 with $S_{t,2}^* \left|S_{t,2}\right|^{2p} S_{t,2}$, where W_2 and $\widetilde{W}_2$ are as in (4.5) and (4.6), we have that

$$(S_{t,2}^* \left|S_{t,2}\right|^{2p} S_{t,2})^{\frac{1}{p+1}} = \mathrm{Diag}\{ \left(\lambda_{-\kappa+1}^2 \lambda_{-\kappa+2}^{2p}\right)^{\frac{1}{p+1}}, \cdots, \left(\lambda_{-1}^2 \lambda_0^{2p}\right)^{\frac{1}{p+1}}, \left(\lambda_0^2 (\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^p\right)^{\frac{1}{p+1}},$$
$$W_2^{\frac{1}{p+1}}, \widetilde{W}_2^{\frac{1}{p+1}}, \left(\lambda_{(2,3)}^2 \lambda_{(2,4)}^{2p}\right)^{\frac{1}{p+1}}, \left(\lambda_{(1,4)}^2 \lambda_{(1,5)}^{2p}\right)^{\frac{1}{p+1}}, \cdots\},$$

where $W_2^{\frac{1}{p+1}} := (f_{ij}(2, p))$ and $\widetilde{W}_2^{\frac{1}{p+1}} := (g_{ij}(2, p))$ with the f_{ij}'s and g_{ij}'s as in Appendix A4. Hence (ii-b) and (ii-c) are equivalent to $W_2^{\frac{1}{p+1}} \geq \mathrm{Diag}\{\lambda_{(1,1)}^2 + \lambda_{(1,2)}^2, \lambda_{(1,2)}^2\}$ and $\widetilde{W}_2^{\frac{1}{p+1}} \geq A_2$, respectively, where A_2 is as in (2.8) with $n = 2$. And (ii-a) is obtained easily. For $n \geq 3$, we obtain that

$$\left(S_{t,n}^* \left|S_{t,n}\right|^{2p} S_{t,n}\right)^{\frac{1}{p+1}}$$
$$= \mathrm{Diag}\{\left(\lambda_{-\kappa+1}^2 \lambda_{-\kappa+2}^{2p}\right)^{\frac{1}{p+1}}, \cdots, \left(\lambda_{-1}^2 \lambda_0^{2p}\right)^{\frac{1}{p+1}}, \left(\lambda_0^2 (\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^p\right)^{\frac{1}{p+1}},$$
$$\left(\lambda_{(1,1)}^2 \lambda_{(1,2)}^{2p} + \lambda_{(2,1)}^2 \lambda_{(2,2)}^{2p}\right)^{\frac{1}{p+1}}, \left(\lambda_{(1,2)}^2 \lambda_{(1,3)}^{2p}\right)^{\frac{1}{p+1}}, \cdots, \left(\lambda_{(2,n-2)}^2 \lambda_{(2,n-1)}^{2p}\right)^{\frac{1}{p+1}},$$
$$\left(\lambda_{(1,n-1)}^2 \lambda_{(1,n)}^{2p}\right)^{\frac{1}{p+1}}, W_n^{\frac{1}{p+1}}, \widetilde{W}_n^{\frac{1}{p+1}}, \left(\lambda_{(2,n+1)}^2 \lambda_{(2,n+2)}^{2p}\right)^{\frac{1}{p+1}}, \cdots\},$$

where

$$W_n = \begin{pmatrix} a_{11}(n,p)\lambda_{(2,n-1)}^2 & \lambda_{(1,n)}\lambda_{(2,n-1)}a_{12}(n,p) \\ \lambda_{(1,n)}\lambda_{(2,n-1)}a_{12}(n,p) & a_{22}(n,p)\lambda_{(1,n)}^2 \end{pmatrix}$$

and

$$\widetilde{W}_n = \begin{pmatrix} \lambda_{(2,n)}^2 \lambda_{(2,n+1)}^{2p} & t\lambda_{(2,n)}\lambda_{(2,n+1)}^{2p} \\ t\lambda_{(2,n)}\lambda_{(2,n+1)}^{2p} & \lambda_{(1,n+1)}^2 \lambda_{(1,n+2)}^{2p} + t^2 \lambda_{(2,n+1)}^{2p} \end{pmatrix}.$$

By direct computations, we have that $W_n^{\frac{1}{p+1}} := (f_{ij}(n, p))$ and $\widetilde{W}_n^{\frac{1}{p+1}} := (g_{ij}(n, p))$ with the f_{ij} and g_{ij} as in Appendix A5. Thus $S_{t,n}$ is an $A(p)$-class operator if and only if (iii-a) and (iii-b) hold, $W_n^{\frac{1}{p+1}} \geq \mathrm{Diag}\{\lambda_{(2,n-1)}^2, \lambda_{(1,n)}^2\}$ and $\widetilde{W}_2^{\frac{1}{p+1}} \geq A_n$, where A_n is as in (2.8). And (iii-c) and (iii-d) are equivalent to $W_n^{\frac{1}{p+1}} \geq \mathrm{Diag}\{\lambda_{(2,n-1)}^2, \lambda_{(1,n)}^2\}$ and $\widetilde{W}_2^{\frac{1}{p+1}} \geq A_n$, respectively. Hence the proof is complete. □

5. Examples

We consider some examples related to theorems in the previous sections.

Let S_λ be a weighted shift on the directed tree $\mathcal{T}_{2,\kappa}$ with the λ_v below and consider $S_{t,1} := S_\lambda + te_{(2,1)} \otimes e_{(1,1)}$, with

$$\lambda_{(1,1)} = \lambda_{(2,1)} = 2,\ \lambda_m = 1,\ -\kappa + 1 \leq m \leq 0,$$
$$\lambda_{(1,2)} = 3,\ \lambda_{(1,k+2)} = \lambda_{(2,k+1)} = 4,\ k \in \mathbb{N}.$$

p-**Hyponormality.** According to Theorem 3.2(i), we obtain that $S_{t,1}$ is p-hyponormal $(0 < p < \infty)$ if and only if $\Delta(p,t) \geq 0$, where

$$\Delta(p,t) := \begin{pmatrix} 8 & 2t & 0 & 0 \\ 2t & t^2+9 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}^p - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 4 & t^2+4 & 3t \\ 0 & 0 & 3t & 9 \end{pmatrix}^p,$$

which is equivalent to the positivity of the 4×4 matrix in Theorem 3.2 (i-b).

If we give a positive number p, we can estimate the range of t in $\mathbb{R}$ for the p-hyponormality of $S_{t,1}$. For example, a direct computation proves that $S_{t,1}$ is 2-hyponormal if and only if $t \in [-\delta, \delta]$, where δ is the unique positive root of $\det \Delta(2,t) = 0$. For some $p > 0$, it is not easy to find the range in t for the p-hyponormality of $S_{t,1}$, but we can find a subrange for the p-hyponormality of $S_{t,1}$. For example, taking $p = \frac{1}{2}$ and $t = \frac{207}{100}$, we have $\Delta(\frac{1}{2}, \frac{207}{100}) \geq 0$, i.e., $S_{\frac{207}{100},1}$ is $\frac{1}{2}$-hyponormal.

Absolute-p-paranormality. We compute W_1 appearing in (4.2) using instead part of the result from the computation of $\Delta(p,t)$ above and direct computation from $S_{t,1}^*(S_{t,1}^* S_{t,1})^p S_{t,1}$. According to Theorem 4.1(i), we recall that $S_{t,1}$ is absolute-p-paranormal $(0 < p < \infty)$ if and only if $\Omega_1(p,t,s) \geq 0$ for all $s > 0$, where

$$\Omega_1(p,t,s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & t & 3 \end{pmatrix} \begin{pmatrix} 8 & 2t & 0 & 0 \\ 2t & t^2+9 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}^p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & t \\ 0 & 0 & 3 \end{pmatrix}$$

$$-(p+1)s^p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 2t \\ 0 & 2t & t^2+9 \end{pmatrix} + ps^{p+1} I$$

as in (4.1). By some computations, we have that $S_{\frac{53}{25},1}$[$S_{\frac{21}{10},1}$, or $S_{\frac{209}{100},1}$, resp.] is absolute-2-paranormal[absolute-1-paranormal, or absolute-$\frac{1}{2}$-paranormal, resp.].

p-**Paranormality.** In what follows, we use for convenience of computation an alternative form of the relevant matrix obtained using the polar decomposition of $S_{t,1}$. According to Theorem 4.3(i), we obtain that $S_{t,1}$ is p-paranormal $(0 < p < \infty)$ if and only if $\Psi_1(p,t,s) \geq 0$ for all $s > 0$, where

$$\Psi_1(p,t,s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 2t \\ 0 & 2t & t^2+9 \end{pmatrix}^{p/2} \widetilde{U}^* \begin{pmatrix} 8 & 2t & 0 & 0 \\ 2t & t^2+9 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}^p \widetilde{U} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 2t \\ 0 & 2t & t^2+9 \end{pmatrix}^{p/2}$$

$$-2s \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 2t \\ 0 & 2t & t^2+9 \end{pmatrix}^p + s^2 I$$

with

$$\widetilde{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_{11}(1) & u_{12}(1) \\ 0 & u_{21}(1) & u_{22}(1) \\ 0 & u_{31}(1) & u_{32}(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & t \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 2t \\ 0 & 2t & t^2+9 \end{pmatrix}^{-1/2},$$

where $u_{ij}(1)$ are as in Lemma 4.2(i). By some computations, we have that $S_{\frac{11}{5},1}$[$S_{\frac{21}{10},1}$, or $S_{\frac{52}{25},1}$, resp.] is 2-paranormal[1-paranormal, or $\frac{1}{2}$-paranormal, resp.].

$A(p)$-**class operator.** We compute W_1 appearing in (4.2) using instead part of the result from the computation of $\Delta(p,t)$ above. According to Theorem 4.4(i), we obtain that $S_{t,1}$ is an $A(p)$-class operator ($0 < p < \infty$) if and only if $(W_1(p,t))^{\frac{1}{p+1}} - \mathrm{Diag}\{1, A_1\} \geq 0$, where

$$\begin{aligned} W_1(p,t) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & t & 3 \end{pmatrix} \begin{pmatrix} 8 & 2t & 0 & 0 \\ 2t & t^2+9 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}^p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & t \\ 0 & 0 & 3 \end{pmatrix}, \\ A_1 &= \begin{pmatrix} 8 & 2t \\ 2t & t^2+9 \end{pmatrix}, \end{aligned}$$

as in (4.2) and (2.7). (In the examples which follow, what is required for the Löwner-Heinz inequality is the positivity of a certain matrix difference. However, using the Nested Determinant Test the positivity condition arising from the determinant of the full difference matrix is the most restrictive, as is shown by an easy computation, so we omit the other conditions.) To consider the case of an $A(1)$-class operator, if we take any $t \in [-\delta, \delta]$, where δ is the unique positive root of polynomial

$$\det(W_1(1,t) - (\mathrm{Diag}\{1, A_1\})^2) = 15876 - 2548t^2 - 212t^4 - 12t^6,$$

by the Löwner-Heinz inequality, $(W_1(1,t))^{\frac{1}{2}} \geq \mathrm{Diag}\{1, A_1\}$, i.e., $S_{t,1}$ is an $A(1)$-class operator. Similarly, for an $A(\frac{1}{2})$-class operator, if we take any t satisfying

$$\det\left(\left(W_1\left(\frac{1}{2}, t\right)\right)^2 - (\mathrm{Diag}\{1, A_1\})^3\right) \geq 0,$$

then by the Löwner-Heinz inequality, $W_1(\frac{1}{2}, t)^{\frac{2}{3}} \geq \mathrm{Diag}\{1, A_1\}$, i.e., $S_{t,1}$ is an $A(\frac{1}{2})$-class operator. Also, for an $A(2)$-class operator, if we take any t satisfying

$$\det(W_1(2,t) - (\mathrm{Diag}\{1, A_1\})^3) \geq 0,$$

then by the Löwner-Heinz inequality, $W_1(2,p)^{\frac{1}{3}} \geq \mathrm{Diag}\{1, A_1\}$, i.e., $S_{t,1}$ is an $A(2)$-class operator. For example, $S_{\frac{103}{50},1}$[$S_{\frac{207}{100},1}$, or $S_{\frac{9}{5},1}$, resp.] is an $A(2)$-class operator[$A(1)$-class operator, or $A(\frac{1}{2})$-class operator, resp.].

Finally we give some remarks related to the topics on partial normality and weak hyponormality.

Remark 5.1. If we consider other values $p \in (0, \infty)$ instead of $p = \frac{1}{2}, 1, 2$ in the above discussion about the operator $S_{t,1}$, we may compare the range of t for the p-hyponormality, p-paranormality and absolute-p-paranormality of $S_{t,1}$ to show such classes are distinct. We leave them to interested readers.

The notion of n-contractivity has played an important role to detect the gaps between subnormality and hyponormality. The following remark records some information about the connection between n-contractivity and absolute-p-paranormality.

Remark 5.2. Recall that $T \in B(\mathcal{H})$ is 2-*contractive* if $T^{*2}T^2 - 2T^*T + I \geq 0$ ([1]). Clearly, if T is absolute-1-paranormal then it is 2-contractive. Our model $S_{t,1}$ can show these properties are distinct. For example, $S_{\frac{11}{5},1}$ is 2-contractive but not absolute-1-paranormal because the matrix Ω_1 is not positive when $s = 15$.

The following example related to our operator model is interesting in its own right.

Remark 5.3. If we allow $t = 0$ in the model, we may create some examples of 2-isometries which we believe to be new. Recall that $T \in B(\mathcal{H})$ is a 2-*isometry* if $I - 2T^*T + T^{*2}T^2 = 0$, that every isometry is a 2-isometry (including the unilateral shift), and that the standard non-isometric 2-isometry is the Dirichlet shift W_D, with weights $\sqrt{2}, \sqrt{3/2}, \sqrt{4/3}, \sqrt{5/4}, \ldots$. Observe that the W_D is a strict expansion, in the sense that $\|W_D x\| > \|x\|$ for all $x \neq 0$ (any 2-isometry is at least a weak expansion). If we use (for example) $\kappa = 4$ with

$$\lambda_{-3} = \sqrt{2},\ \lambda_{-2} = \sqrt{3/2},\ \lambda_{-1} = \sqrt{4/3},\ \lambda_0 = \sqrt{5/4},$$
$$\lambda_{2,n} = 1\ (n \geq 1),\ \lambda_{1,1} = \sqrt{1/5},\ \lambda_{1,n} = \sqrt{n/(n-1)}\ (n \geq 2),$$

we produce a 2-isometry which is neither an isometry, strictly expansive, nor a trivial direct sum of the Dirichlet shift with an isometry.

Appendix - expressions of polynomials

We give the exact expressions of polynomials which appeared in the previous sections.

A1. Polynomials in Theorem 3.2:

$$b_{11}(1,p) = \lambda_{(1,1)}^2((\beta_1 - \lambda_{(1,1)}^2 - \lambda_{(2,1)}^2)\alpha_1^p\beta_1 + (\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2 - \alpha_1)\beta_1^p\alpha_1)/(\gamma_1\delta^2),$$
$$b_{12}(1,p) = \lambda_{(1,1)}\lambda_{(2,1)}((\lambda_{(1,2)}^2 - \alpha_1)\alpha_1^p\beta_1 + (\beta_1 - \lambda_{(1,2)}^2)\alpha_1\beta_1^p)/(\gamma_1\delta^2),$$
$$b_{13}(1,p) = t\lambda_{(1,1)}\lambda_{(1,2)}\lambda_{(2,1)}(\alpha_1\beta_1^p - \alpha_1^p\beta_1)/(\gamma_1\delta^2),$$
$$\begin{aligned} b_{22}(1,p) = {} & ((\lambda_{(1,2)}^2 - \alpha_1)(\lambda_{(1,2)}^2\lambda_{(2,1)}^2 + \lambda_{(1,1)}^2(\alpha_1 - \lambda_{(1,1)}^2 - \lambda_{(2,1)}^2))\alpha_1^p \\ & + (\beta_1 - \lambda_{(1,2)}^2)(\lambda_{(1,2)}^2\lambda_{(2,1)}^2 + \lambda_{(1,1)}^2(\beta_1 - \lambda_{(1,1)}^2 - \lambda_{(2,1)}^2))\beta_1^p)/(\gamma_1\delta^2), \end{aligned}$$
$$\begin{aligned} b_{23}(1,p) = {} & t\lambda_{(1,2)}((\lambda_{(1,1)}^2(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2 - \alpha_1) - \lambda_{(1,2)}^2\lambda_{(2,1)}^2)\alpha_1^p \\ & + (\lambda_{(1,1)}^2(\beta_1 - \lambda_{(1,1)}^2 - \lambda_{(2,1)}^2) + \lambda_{(1,2)}^2\lambda_{(2,1)}^2)\beta_1^p)/(\gamma_1\delta^2), \end{aligned}$$
$$\begin{aligned} b_{33}(1,p) = {} & \lambda_{(1,2)}^2(((\beta_1 - \lambda_{(1,2)}^2)\lambda_{(2,1)}^2 + \lambda_{(1,1)}^2(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2 - \alpha_1))\alpha_1^p \\ & + ((\lambda_{(1,2)}^2 - \alpha_1)\lambda_{(2,1)}^2 + \lambda_{(1,1)}^2(\beta_1 - \lambda_{(1,1)}^2 - \lambda_{(2,1)}^2))\beta_1^p)/(\gamma_1\delta^2), \end{aligned}$$
$$b_{11}(n,p) = ((\lambda_{(1,n+1)}^2 - \alpha_n)\alpha_n^p + (\beta_n - \lambda_{(1,n+1)}^2)\beta_n^p)/\gamma_n;\ b_{12}(n,p) = t\lambda_{(1,n+1)}(\beta_n^p - \alpha_n^p)/\gamma_n,$$
$$b_{22}(n,p) = ((\beta_n - \lambda_{(1,n+1)}^2)\alpha_n^p + (\lambda_{(1,n+1)}^2 - \alpha_n)\beta_n^p)/\gamma_n,$$

where $\delta = (\lambda_{(1,2)}^2\lambda_{(2,1)}^2 + \lambda_{(1,1)}^2(t^2 + \lambda_{(1,2)}^2))^{1/2}$.

A2. Polynomials in Theorem 4.1:

$$\omega_{11}(1,p) = a_{11}(1,p)\lambda_0^2 - (p+1)\lambda_0^2 s^p + ps^{p+1},$$

$$\omega_{22}(1,p) = a_{22}(1,p)\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2\lambda_{(2,2)}^{2p} - (p+1)(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)s^p + ps^{p+1},$$
$$\omega_{33}(1,p) = t^2\lambda_{(2,2)}^{2p} + \lambda_{(1,2)}^2\lambda_{(1,3)}^{2p} - (p+1)(t^2 + \lambda_{(1,2)}^2)s^p + ps^{p+1},$$
$$\omega_{11}(2,p) = \lambda_{(1,1)}^2\lambda_{(1,2)}^{2p} + a_{11}(2,p)\lambda_{(2,1)}^2 - (1+p)s^p(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2) + ps^{1+p},$$
$$\omega_{11}(n,p) = a_{11}(n,p)\lambda_{(2,n-1)}^2 - (1+p)s^p\lambda_{(2,n-1)}^2 + ps^{1+p},$$
$$\omega_{22}(n,p) = a_{22}(n,p)\lambda_{(1,n)}^2 - (1+p)s^p\lambda_{(1,n)}^2 + ps^{1+p},$$
$$\widetilde{\omega}_{11}(n,p) = \lambda_{(2,n)}^2\lambda_{(2,n+1)}^{2p} - (1+p)s^p\lambda_{(2,n)}^2 + ps^{1+p},$$
$$\widetilde{\omega}_{22}(n,p) = \lambda_{(1,n+1)}^2\lambda_{(1,n+2)}^{2p} + t^2\lambda_{(2,n+1)}^{2p} - (1+p)s^p(t^2 + \lambda_{(1,n+1)}^2) + ps^{1+p}.$$

A3. Polynomials in Theorem 4.3:

$$\varphi_{11}(1,p) = \lambda_0^{2p}\, a_{11}(1,p) - 2s\lambda_0^{2p} + s^2;\ \varphi_{12}(1,p) = \lambda_0^p a_{12}(1,p)\phi_1(1),$$
$$\varphi_{13}(1,p) = \lambda_0^p a_{12}(1,p)\phi_2(1),$$
$$\varphi_{22}(1,p) = a_{22}(1,p)\phi_1(1)^2 + \lambda_{(1,3)}^{2p}\phi_3(1)^2 + \lambda_{(2,2)}^{2p}\phi_4(1)^2 - 2sa_{11}(1,p) + s^2,$$
$$\varphi_{23}(1,p) = a_{22}(1,p)\phi_1(1)\phi_2(1) + \lambda_{(1,3)}^{2p}\phi_3(1)\phi_5(1) + \lambda_{(2,2)}^{2p}\phi_4(1)\phi_6(1) - 2sa_{12}(1,p),$$
$$\varphi_{33}(1,p) = a_{22}(1,p)\phi_2(1)^2 + \lambda_{(1,3)}^{2p}\phi_5(1)^2 + \lambda_{(2,2)}^{2p}\phi_6(1)^2 - 2sa_{22}(1,p) + s^2,$$
$$\varphi_{11}(2,p) = (\lambda_{(1,1)}^2\lambda_{(1,2)}^{2p} + a_{11}(2,p)\lambda_{(2,1)}^2)(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^{p-1} - 2s(\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2)^p + s^2,$$
$$\varphi_{11}(n,p) = a_{11}(n,p)\lambda_{(2,n-1)}^{2p} - 2s\lambda_{(2,n-1)}^{2p} + s^2;\ \varphi_{22}(n,p) = a_{22}(n,p)\lambda_{(1,n)}^{2p} - 2s\lambda_{(1,n)}^{2p} + s^2,$$
$$\widetilde{\varphi}_{11}(n,p) = \lambda_{(2,n+1)}^{2p}\phi_1(n)^2 + \lambda_{(1,n+2)}^{2p}\phi_4(n)^2 - 2sa_{11}(n,p) + s^2,$$
$$\widetilde{\varphi}_{12}(n,p) = \lambda_{(1,n+2)}^{2p}\phi_4(n)\phi_6(n) + \lambda_{(2,n+1)}^{2p}\phi_1(n)\phi_2(n) - 2sa_{12}(n,p),$$
$$\widetilde{\varphi}_{22}(n,p) = \lambda_{(2,n+1)}^{2p}\phi_2(n)^2 + \lambda_{(1,n+2)}^{2p}\phi_6(n)^2 - 2sa_{22}(n,p) + s^2,$$

where

$$\phi_1(n) = a_{11}\left(n,\frac{p}{2}\right)u_{11}(n) + a_{12}\left(n,\frac{p}{2}\right)u_{12}(n);\ \phi_2(n) = a_{12}\left(n,\frac{p}{2}\right)u_{11}(n) + a_{22}\left(n,\frac{p}{2}\right)u_{12}(n),$$
$$\phi_3(n) = a_{11}\left(n,\frac{p}{2}\right)u_{31}(n) + a_{12}\left(n,\frac{p}{2}\right)u_{32}(n),$$
$$\phi_4(n) = a_{11}\left(n,\frac{p}{2}\right)u_{21}(n) + a_{12}\left(n,\frac{p}{2}\right)u_{22}(n);\ \phi_5(n) = a_{12}\left(n,\frac{p}{2}\right)u_{31}(n) + a_{22}\left(n,\frac{p}{2}\right)u_{32}(n),$$
$$\phi_6(n) = a_{12}\left(n,\frac{p}{2}\right)u_{21}(n) + a_{22}\left(n,\frac{p}{2}\right)u_{22}(n).$$

A4. Polynomials in Theorem 4.4 (f_{ij} form 2, g_{ij} form 1):

$$f_{11}(2,p) = ((a_{22}(2,p)\lambda_{(1,2)}^2 - \rho_{2,1})\rho_{2,1}^{1/(p+1)} + (\rho_{2,2} - a_{22}(2,p)\lambda_{(1,2)}^2)\rho_{2,2}^{1/(p+1)})/\xi_2,$$
$$f_{12}(2,p) = -a_{12}(2,p)\lambda_{(1,2)}\lambda_{(2,1)}(\rho_{2,1}^{1/(p+1)} - \rho_{2,2}^{1/(p+1)})/\xi_2,$$
$$f_{22}(2,p) = ((\rho_{2,2} - a_{22}(2,p)\lambda_{(1,2)}^2)\rho_{2,1}^{1/(p+1)} + (a_{22}(2,p)\lambda_{(1,2)}^2 - \rho_{2,1})\rho_{2,2}^{1/(p+1)})/\xi_2,$$
$$g_{11}(2,p) = ((\widetilde{\rho}_{2,2} - \lambda_{(2,2)}^2\lambda_{(2,3)}^{2p})\widetilde{\rho}_{2,1}^{1/(p+1)} + (\lambda_{(2,2)}^2\lambda_{(2,3)}^{2p} - \widetilde{\rho}_{2,1})\widetilde{\rho}_{2,2}^{1/(p+1)})/\widetilde{\xi}_2,$$
$$g_{12}(2,p) = -t\lambda_{(2,2)}\lambda_{(2,3)}^{2p}(\widetilde{\rho}_{2,1}^{1/(p+1)} - \widetilde{\rho}_{2,2}^{1/(p+1)})/\widetilde{\xi}_2,$$
$$g_{22}(2,p) = ((\lambda_{(2,2)}^2\lambda_{(2,3)}^{2p} - \widetilde{\rho}_{2,1})\widetilde{\rho}_{2,1}^{1/(p+1)} + (\widetilde{\rho}_{2,2} - \lambda_{(2,2)}^2\lambda_{(2,3)}^{2p})\widetilde{\rho}_{2,2}^{1/(p+1)})/\widetilde{\xi}_2,$$

where

$$\begin{aligned}
\rho_{2,1} &= (a_{11}(2,p)\lambda^2_{(2,1)} + \lambda^2_{(1,1)}\lambda^{2p}_{(1,2)} + a_{22}(2,p)\lambda^2_{(1,2)} - \xi_2)/2 \text{ (eigenvalue)},\\
\rho_{2,2} &= (a_{11}(2,p)\lambda^2_{(2,1)} + \lambda^2_{(1,1)}\lambda^{2p}_{(1,2)} + a_{22}(2,p)\lambda^2_{(1,2)} + \xi_2)/2 \text{ (eigenvalue)},\\
\xi_2 &= ((a_{11}(2,p)\lambda^2_{(2,1)} + \lambda^2_{(1,1)}\lambda^{2p}_{(1,2)} + a_{22}(2,p)\lambda^2_{(1,2)})^2\\
&\quad - 4\lambda^2_{(1,2)}(a_{22}(2,p)\lambda^2_{(1,1)}\lambda^{2p}_{(1,2)} + \lambda^2_{(2,1)}\lambda^{2p}_{(1,3)}\lambda^{2p}_{(2,2)}))^{1/2}\\
&\qquad \text{(“square root” term – see Note following A5)},\\
\widetilde{\rho}_{2,1} &= (\lambda^2_{(1,3)}\lambda^{2p}_{(1,4)} + t^2\lambda^{2p}_{(2,3)} + \lambda^2_{(2,2)}\lambda^{2p}_{(2,3)} - \widetilde{\xi}_2)/2,\\
\widetilde{\rho}_{2,2} &= (\lambda^2_{(1,3)}\lambda^{2p}_{(1,4)} + t^2\lambda^{2p}_{(2,3)} + \lambda^2_{(2,2)}\lambda^{2p}_{(2,3)} + \widetilde{\xi}_2)/2 \text{ (eigenvalues)},\\
\widetilde{\xi}_2 &= ((\lambda^2_{(1,3)}\lambda^{2p}_{(1,4)} + (t^2 + \lambda^2_{(2,2)})\lambda^{2p}_{(2,3)})^2 - 4\lambda^2_{(1,3)}\lambda^{2p}_{(1,4)}\lambda^2_{(2,2)}\lambda^{2p}_{(2,3)})^{1/2}\text{(“square root” term)},
\end{aligned}$$

A5. Polynomials in Theorem 4.4 (f_{ij} form 1, g_{ij} form 2): for $n \geq 3$,

$$\begin{aligned}
f_{11}(n,p) &= ((\rho_{n,2} - a_{11}(n,p)\lambda^2_{(2,n-1)})\rho_{n,1}^{1/(p+1)} + (a_{11}(n,p)\lambda^2_{(2,n-1)} - \rho_{n,1})\rho_{n,2}^{1/(p+1)})/\xi_n,\\
f_{12}(n,p) &= -a_{12}(n,p)\lambda_{(1,n)}\lambda_{(2,n-1)}(\rho_{n,1}^{1/(p+1)} - \rho_{n,2}^{1/(p+1)})/\xi_n,\\
f_{22}(n,p) &= ((a_{11}(n,p)\lambda^2_{(2,n-1)} - \rho_{n,1})\rho_{n,1}^{1/(p+1)} + (\rho_{n,2} - a_{11}(n,p)\lambda^2_{(2,n-1)})\rho_{n,2}^{1/(p+1)})/\xi_n,\\
g_{11}(n,p) &= ((\widetilde{\rho}_{n,2} - \lambda^2_{(2,n)}\lambda^{2p}_{(2,n+1)})\widetilde{\rho}_{n,1}^{1/(p+1)} + (\lambda^2_{(2,n)}\lambda^{2p}_{(2,n+1)} - \widetilde{\rho}_{n,1})\widetilde{\rho}_{n,2}^{1/(p+1)})/\widetilde{\xi}_n,\\
g_{12}(n,p) &= -t\lambda_{(2,n)}\lambda^{2p}_{(2,n+1)}(\widetilde{\rho}_{n,1}^{1/(p+1)} - \widetilde{\rho}_{n,2}^{1/(p+1)})/\widetilde{\xi}_n,\\
g_{22}(n,p) &= ((\lambda^2_{(2,n)}\lambda^{2p}_{(2,n+1)} - \widetilde{\rho}_{n,1})\widetilde{\rho}_{n,1}^{1/(p+1)} + (\widetilde{\rho}_{n,2} - \lambda^2_{(2,n)}\lambda^{2p}_{(2,n+1)})\widetilde{\rho}_{n,2}^{1/(p+1)})/\widetilde{\xi}_n,
\end{aligned}$$

where

$$\begin{aligned}
\rho_{n,1} &= (a_{22}(n,p)\lambda^2_{(1,n)} + a_{11}(n,p)\lambda^2_{(2,n-1)} - \xi_n)/2,\\
\rho_{n,2} &= (a_{22}(n,p)\lambda^2_{(1,n)} + a_{11}(n,p)\lambda^2_{(2,n-1)} + \xi_n)/2 \text{ (eigenvalues)},\\
\xi_n &= ((a_{22}(n,p)\lambda^2_{(1,n)} + a_{11}(n,p)\lambda^2_{(2,n-1)})^2\\
&\quad - 4\lambda^2_{(1,n)}\lambda^{2p}_{(1,n+1)}\lambda^2_{(2,n-1)}\lambda^{2p}_{(2,n)})^{1/2} \text{ (“square root” term – see Note below)},\\
\widetilde{\rho}_{n,1} &= (\lambda^2_{(1,n+1)}\lambda^{2p}_{(1,n+2)} + t^2\lambda^{2p}_{(2,n+1)} + \lambda^2_{(2,n)}\lambda^{2p}_{(2,n+1)} - \widetilde{\xi}_n)/2 \text{ (eigenvalue)},\\
\widetilde{\rho}_{n,2} &= (\lambda^2_{(1,n+1)}\lambda^{2p}_{(1,n+2)} + t^2\lambda^{2p}_{(2,n+1)} + \lambda^2_{(2,n)}\lambda^{2p}_{(2,n+1)} + \widetilde{\xi}_n)/2 \text{ (eigenvalue)},\\
\widetilde{\xi}_n &= ((\lambda^2_{(1,n+1)}\lambda^{2p}_{(1,n+2)} + (t^2 + \lambda^2_{(2,n)})\lambda^{2p}_{(2,n+1)})^2\\
&\quad - 4\lambda^2_{(1,n+1)}\lambda^{2p}_{(1,n+2)}\lambda^2_{(2,n)}\lambda^{2p}_{(2,n+1)})^{1/2} \text{ (“square root” term)}.
\end{aligned}$$

Note. To simplify ξ_2 and ξ_n, we use that $a_{11}(n,p)a_{22}(n,p) - (a_{12}(n,p))^2 = \lambda^{2p}_{(1,n+1)}\lambda^{2p}_{(2,n)}$, $n \geq 2$.

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