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A REMARK ON CONDITIONS THAT A DIFFUSION IN THE NATURAL SCALE IS A MARTINGALE

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Abstract

We consider a diffusion processes $\{X_t\}$ on an interval in the natural scale. Some results are known under which $\{X_t\}$ is a martingale, and we give simple and analytic proofs for them.

1. Introduction

Let $-\infty \le l_- < l_+ \le \infty$ and let m be a Borel measure with supp $m = (l_-, l_+)$. We denote by $\left\{\{X_t\}_{t\ge 0}, \{P_x\}_{x\in(l_-,l_+)}\right\}$ the minimal diffusion process on (l_-, l_+) with the speed measure m and the scale function $s(x) = x$. It is well known that a local martingale $\{X_t\}$ is a martingale if and only if $\{X_T : T$ is a stopping time with $T \le t\}$ is uniformly integrable for any $t \ge 0$. Here our aim is to have more explicit condition for the one-dimensional diffusions in the natural scale. If $|l_\pm| < \infty$, $\{X_t\}$ is bounded so that it is a martingale. If $l_- = -\infty$, $l_+ < \infty$, this can be reduced to the case of $l_- < \infty$, $l_+ = \infty$ by replacing X_t by $-X_t$. Hence it suffices to consider the following two cases.

$$\text{Case I} : -\infty < l_-, \, l_+ = +\infty, \qquad \text{Case II} : l_- = -\infty, \, l_+ = +\infty.$$

Let $P(l_-, l_+)$ be the set of Borel measures on (l_-, l_+), and for $\mu \in P(l_-, l_+)$ let $P_\mu(\cdot) := \int_{(l_-,l_+)} P_x(\cdot)\mu(dx)$. According to Lemma 4.1 ([1], Lemma 2), $\{X_t^\tau\}$ is a P_μ-martingale for some $\mu \in P(l_-, l_+)$ with $\int_{(l_-,l_+)} |x|\mu(dx) < \infty$ if and only if $\{X_t^\tau\}$ is P_x-martingale for any $x \in (l_-, l_+)$. We further set

$$\begin{aligned} \tau_a &:= \inf\left\{t \ge 0 \middle| X_t = a\right\}, \quad \tau_\pm := \lim_{a\to l_\pm} \tau_a, \quad \tau := \tau_+ \wedge \tau_- \\ X_t^\tau &:= X_{t\wedge\tau}. \end{aligned}$$

Kotani [1] showed the following theorem.

Theorem 1.1 ([1]). *$\{X_t^\tau\}$ is a P_x-martingale for any $x \in (l_-, l_+)$ if and only if*
Case I :

$$\int_{[r,l_+)} xm(dx) = \infty, \quad r \in (l_-, \infty)$$

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Case II :

$$\int_{[r,l_+)} xm(dx) = \infty \quad and \quad \int_{(l_-,r]} |x|m(dx) = \infty, \qquad r \in (-\infty, \infty).$$

By Feller's criterion, $P_x(\tau_\sharp = \infty) = 1$ if $|l_\sharp| = \infty$, $\sharp = \pm\infty$. Thus Theorem 1.1 implies that $\{X_t^\tau\}$ is a martingale if and only if the boundaries at infinity are natural. Hulley, Platen [2] derived another condition. Let

$$\mathcal{L} f := \frac{d^2}{dmdx} f$$

be the generator of $\{X_t\}$ and for $\lambda > 0$ let f_- (resp. f_+) be the positive increasing (resp. positive decreasing) solution to the equation $\mathcal{L} f = \lambda f$, which are unique up to constants unless the boundary is regular.

Theorem 1.2 ([2])**.** *$\{X_t^\tau\}$ is a P_x-martingale for any $x \in (l_-, l_+)$ if and only if*
Case I :

$$\lim_{z \to \infty} f_-'(z) = \infty$$

Case II :

$$\lim_{z \to \infty} f_-'(z) = \infty \quad and \quad \lim_{z \to -\infty} f_+'(z) = -\infty.$$

Gushchin, Urusov, and Zervos [3] derived a condition that $\{X_t^\tau\}$ is a submartingale or a supermartingale.

Theorem 1.3 ([3])**.** *$\{X_t^\tau\}$ is a P_x-submartingale if and only if $\int_r^\infty xm(dx) = \infty$, $r \in (l_-, l_+)$.*

By [2] Proposition 3.16, 3.17, this condition is equivalent to $\lim_{t\to\infty} f_-'(t) = \infty$. Together with Theorem 1.3 we thus have

Theorem 1.4. *$\{X_t^\tau\}$ is a P_x-submartingale if and only if* $\lim_{t\to\infty} f_-'(t) = \infty$.

Moreover in [3], they further derived a condition in Case I such that $\{X_t^\tau\}$ is a strict P_x supermartingale, that is, $\{X_t^\tau\}$ is a P_x-supermartingale but is not a P_x-martingale.

Theorem 1.5 ([3])**.** *Let $-\infty < l_-$, $l_+ = \infty$. Then $\{X_t^{\tau_-}\}$ is a strict P_x-supermartingale if and only if*

$$\lim_{t \to \infty} E_x[X_{t \wedge \tau_-}] = l_-$$

for any $x \in (l_-, l_+)$.

We believe that Theorem 1.5 is also true for $l_- = -\infty$. The goal of this paper is :
(1) To give a simple analytic proof of Theorem 1.4 without using the results in [2]. We note that the proofs of Proposition 3.16, 3.17 in [2] is more or less probabilistic using Tanaka's formula.
(2) To give a simple analytic proof of Theorem 1.5 ; the original proof of that in [3] is done by embedding $\{X_t\}$ into the geometric Brownian motion on the torus.

The rest of this paper is organized as follows. In Section 2 (resp. Section 3), we give a proof of Theorem 1.4 (resp. Theorem 1.5). In Appendix, we prepare some tools for these proofs.

2. A proof of Theorem 1.4

In Case I, the statement follows from Theorem 1.2, for $\{X_t^{\tau_-}\}$ is always a P_x-supermartingale being bounded from below. Henceforth we consider Case II.

Suppose $\{X_t\}$ is a P_x-submartingale and let $z < x$. Then $\{X_t^{\tau_z}\}$ is bounded from blow so that it is a P_x-martingale. For $\lambda > 0$, let f_-^z (resp. f_+^z) be the positive increasing (resp. positive decreasing) solution to the equation $\mathcal{L}f = \lambda f$ such that $f_-^z(z) = 0$. Then we have

$$f_-^z(x) = f_-(x) - \frac{f_-(z)}{f_+(z)} f_+(x), \quad f_+^z(x) = f_+(x).$$

Since f_+' is increasing, we have

$$f_-'(x) \quad = \quad f_-^{z\prime}(x) + \frac{f_-(z)}{f_+(z)} f_+'(x) \geq f_-^{z\prime}(x) + \frac{f_-(z)}{f_+(z)} f_+'(z), \quad x \in (z, \infty).$$

Applying Theorem 1.2 to $\{X_t^{\tau_z}\}$ yields $\lim_{t\to\infty} f_-^{z\prime}(t) = \infty$ and thus $\lim_{t\to\infty} f_-'(t) = \infty$.

Conversely, suppose $\lim_{t\to\infty} f_-'(t) = \infty$ and let $z < x$. Then

$$\begin{aligned} \lim_{z\to\infty} z \int_0^\infty e^{-\lambda t} P_x(\tau_z < t) dt &= \lim_{z\to\infty} \frac{z}{\lambda} E_x[e^{-\lambda \tau_z}] = \lim_{z\to\infty} \frac{z}{\lambda} \frac{f_-(x)}{f_-(z)} \\ &= \lim_{z\to\infty} \frac{f_-(x)}{\lambda} \frac{1}{f_-'(z)} = 0 \end{aligned}$$

where we used Lemma 4.3 and l'Hospital's rule. By Fatou's lemma,

$$\int_0^\infty e^{-\lambda t} \liminf_{z\to\infty} z P_x(\tau_z < t) dt = 0.$$

Hence $\liminf_{z\to\infty} z P_x(\tau_z < t) = 0$ so that we can find a sequence $\{z_n\} \subset (x, \infty)$ with $\lim_{n\to\infty} z_n = \infty$ such that

$$\lim_{n\to\infty} z_n P_x(\tau_{z_n} < t) = 0.$$

On the other hand $\{X_t^{\tau_{z_n}}\}$ is a P_x-submartingale being bounded from above and

$$x \leq E_x[X_{t\wedge\tau_{z_n}}] = z_n P_x(\tau_{z_n} < t) + E_x[X_t; \tau_{z_n} \geq t].$$

Since $\lim_{n\to\infty} P_x(\tau_{z_n} \geq t) = 1$, $x \leq E_x[X_t]$. Markov property implies $\{X_t\}$ is a P_x-submartingale. □

3. A proof of Theorem 1.5

Without losing generality, we may suppose $l_- < 0$. For $\lambda > 0$, let f_- (resp. f_+) be the positive increasing (resp. positive decreasing) solution to the equation $\mathcal{L}f = \lambda f$ such that $f_-(l_-) = 0$. Let G be Green's function of $\mathcal{L}$:

$$\begin{aligned} G(x, y, \lambda) &:= \begin{cases} \frac{1}{h} f_-(y) f_+(x) & (y < x) \\ \frac{1}{h} f_-(x) f_+(y) & (x \leq y) \end{cases} \\ h &:= f_+(x) f_-'(x) - f_-(x) f_+'(x). \end{aligned}$$

Then we have

$$\int_{l_-}^{\infty} G(x,y,\lambda)(y-l_-)m(dy) = E_x\Big[\int_0^{\infty} e^{-\lambda t}(X_{t\wedge\tau_-}-l_-)dt\Big]. \tag{3.1}$$

Let $\alpha_+ := \lim_{t\to\infty} f_+(t)$. Then $f_+' \in L^1(a,\infty)$ for $a \in (l_-,\infty)$ and

$$f_+(x) = \alpha_+ - \int_x^{\infty} f_+'(y)dy.$$

Therefore $\lim_{x\to\infty} f_+'(x) = 0$. The equation $\mathcal{L}f_+ = \lambda f_+$ yields

$$\begin{aligned} f_+'(x) &= -\lambda \int_x^{\infty} f_+(y)m(dy) \\ f_+(x) &= \alpha_+ + \lambda \int_x^{\infty} (y-x)f_+(y)m(dy) \end{aligned}$$

so that we have

$$\lambda \int_x^{\infty} yf_+(y)m(dy) = f_+(x) - \alpha_+ - xf_+'(x).$$

Similarly,

$$\begin{aligned} f_-'(y) &= f_-'(l_-) + \lambda \int_{l_-}^{y} f_-(z)m(dz) \\ f_-(x) &= f_-'(l_-)(x-l_-) + \lambda \int_{l_-}^{x} (x-y)f_-(y)m(dy) \\ \lambda \int_{l_-}^{x} yf_-(y)m(dy) &= f_-'(l_-)(x-l_-) - f_-(x) + \lambda x \int_{l_-}^{x} f_-(y)m(dy). \end{aligned}$$

Substituting them into (3.1) yields

$$\int_0^{\infty} e^{-\lambda t}E_x[X_{t\wedge\tau_-} - l_-]dt = \frac{x-l_-}{\lambda} - \frac{\alpha_+ f_-(x)}{\lambda h}. \tag{3.2}$$

We note that (3.2) and Lemma 4.1 also proves Theorem 1.1 in Case I.

Suppose $\{X_t^{\tau_-}\}$ is a strict P_x-supermartingale. The discussion above implies $\alpha_+ > 0$. We shall show below that

$$\lim_{\lambda\to 0}\Big(x - l_- - \frac{\alpha_+ f_-(x)}{h}\Big) = 0. \tag{3.3}$$

Let ϕ, ψ be the solution to $\mathcal{L}f = \lambda f$ with the initial condition

$$\begin{aligned} \phi(0) = 1, &\quad \phi'(0) = 0 \\ \psi(0) = 0, &\quad \psi'(0) = 1. \end{aligned}$$

Then $f_\pm$ satisfy

$$f_+(x) = \phi(x) - \left(\lim_{x\to\infty}\frac{\phi(x)}{\psi(x)}\right)\psi(x), \quad f_-(x) = \phi(x) - \left(\lim_{x\to l_-}\frac{\phi(x)}{\psi(x)}\right)\psi(x).$$

ψ, ψ can be decomposed by the method of successive approximation :

$$\phi(x) = 1 + \sum_{n=1}^{\infty} \lambda^n \phi_n(x), \quad \phi_0(x) = 1, \quad \phi_n(x) = \int_0^x (x-y)\phi_{n-1}(y)m(dy)$$
$$\psi(x) = x + \sum_{n=1}^{\infty} \lambda^n \psi_n(x), \quad \psi_0(x) = x, \quad \psi_n(x) = \int_0^x (x-y)\psi_{n-1}(x)m(dy)$$

which is convergent locally uniformly w.r.t. λ [4], which yields

$$\lim_{\lambda\to 0}\phi(x) = 1, \quad \lim_{\lambda\to 0}\phi'(x) = 0, \quad \lim_{\lambda\to 0}\psi(x) = x, \quad \lim_{\lambda\to 0}\psi'(x) = 1.$$

Moreover

$$\lim_{\lambda\to 0}\left(-\lim_{x\to l_-}\frac{\psi(x)}{\phi(x)}\right) = \lim_{\lambda\to 0}\left(\int_{l_-}^0 \frac{1}{(\phi(x))^2}dx\right) = \int_{l_-}^0 dx = -l_-$$

implies

$$\lim_{\lambda\to 0} f_-(x) = 1 - \frac{x}{l_-}, \quad \lim_{\lambda\to 0} f_-'(x) = -\frac{1}{l_-}.$$

On the other hand, by $\alpha_+ > 0$ and by Lemma 4.2, we have $\int_r^\infty xm(dx) < \infty$, $r \in (l_-, \infty)$ so that we can find g satisfying

$$g(x) = 1 + \lambda \int_x^\infty (y-x)g(y)m(dy)$$

by successive approximation. Using $\alpha_+ > 0$, $\lim_{t\to\infty} f_+'(t) = 0$, $\lim_{t\to\infty} g(t) = 1$ and $\lim_{t\to\infty} g'(t) = 0$, we have

$$f_+(x)g'(x) - f_+'(x)g(x) = 0$$

which implies $f_+(x) = Cg(x)$ for some positive constant C. Because $\lim_{\lambda\to 0} g(x) = 1$, $\lim_{\lambda\to 0} g'(x) = 0$, we have

$$\lim_{\lambda\to 0} f_+(x) = C, \quad \lim_{\lambda\to 0} f_+'(x) = 0.$$

Therefore

$$\lim_{\lambda\to 0}\left(x - l_- - \frac{\alpha_+ f_-(x)}{h}\right) = x - l_- - \frac{C\left(1 - \frac{x}{l_-}\right)}{C\cdot\left(\frac{-1}{l_-}\right) - 0\cdot\left(1 - \frac{x}{l_-}\right)} = 0$$

proving (3.3). Since $X_{t\wedge\tau_-}$ is a supermartingale, $f(t) := E_x[X_{t\wedge\tau_-} - l_-] \in C^1[0,\infty)$ is monotone decreasing which shows that $\lim_{t\to\infty} f(t)$ exists and $f' \in L^1(0,\infty)$. Thus by (3.2) and Lemma 4.4

$$\lim_{t\to\infty} E_x[X_{t\wedge\tau_-} - l_-] = 0.$$

Conversely, suppose that $\lim\limits_{t\to\infty} E_x[X_{t\wedge\tau_-} - l_-] = 0$. Then

$$\lim_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}E_x[X_{t\wedge\tau_-} - l_-]dt = 0$$

which implies $\alpha_+ > 0$ since otherwise it would contradict to (3.2), (3.3). Therefore $\{X_t^{\tau_-}\}$ is not a martingale. □

4. Appendix

Lemma 4.1 (Lemma 2 in [1]). *Suppose* $\{X_{t\wedge\tau_-}\}$ *is a* P_μ*-martingale for some* $\mu \in P(l_-, \infty)$. *Then for any* $t \geq 0$, $x \in (l_-, \infty)$,

$$E_x[X_{t\wedge\tau_-}] = x. \tag{4.1}$$

Conversely, if (4.1) *is valid, then* $\{X_{t\wedge\tau_-}\}$ *is a* P_μ*-martingale for any* $\mu \in P(l_-, \infty)$ *with* $\int_{l_-}^{\infty} |x|\mu(dx) < \infty$.

Lemma 4.2. *Let* $\lambda > 0$ *and let* f_+ *be the positive decreasing solution to* $\mathcal{L}f = \lambda f$ *with* $\alpha_+ := \lim_{x\to\infty} f_+(x)$. *Then the following three conditions are equivalent.*

$$\begin{aligned}
&(1) \qquad \alpha_+ = 0\\
&(2) \qquad \int_a^\infty y m(dy) = \infty\\
&(3) \qquad \lambda\int_x^\infty (y-x) f_+(y) m(dy) = f_+(x).
\end{aligned}$$

Lemma 4.3. *Let* $f_\pm$ *be the ones defined in the proof of* Theorem 1.5. *Then*

$$\begin{aligned}
E_x[e^{-\lambda\tau_a}] &= \frac{f_+(x)}{f_+(a)}, \quad a < x\\
E_x[e^{-\lambda\tau_b} : \tau_b < \tau_-] &= \frac{f_-(x)}{f_-(b)}, \quad -\infty \leq l_- < x < b.
\end{aligned}$$

Lemma 4.4. *Suppose* $f \in C^1[0, \infty)$ *and* $f' \in L^1(0, \infty)$. *Then*
(1) $\lim_{t\to\infty} f(t)$ *exists, and*
(2) $\lim_{t\to\infty} f(t) = \lambda \lim_{\lambda\downarrow 0} \int_0^\infty e^{-\lambda t} f(t)dt$.

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