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ON RAMIFIED TORSION POINTS ON A CURVE WITH STABLE REDUCTION OVER AN ABSOLUTELY UNRAMIFIED BASE

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Abstract

Let p be an odd prime number, W an *absolutely unramified* p-adically complete discrete valuation ring with algebraically closed residue field, and X a curve of genus at least two over the field of fractions K of W. In the present paper, we study, under the assumption that X has *stable reduction* over W, *torsion points* on X, i.e., torsion points of the Jacobian variety J of X which lie on the image of the Albanese embedding $X \hookrightarrow J$ with respect to a K-rational point of X. A consequence of the main result of the present paper is that if, moreover, J has good reduction over W, then every torsion point on X is *K-rational after multiplying p.* This result is closely related to a conjecture of *R. Coleman* concerning the ramification of torsion points. For instance, this result leads us to a solution of the conjecture in the case where a given curve is hyperelliptic and of genus at least p.

Contents

Introduction

Throughout the present paper, let p be an odd prime number and k an algebraically closed field of characteristic p. Write $W \overset{\text{def}}{=} W(k)$ for the ring of Witt vectors with coefficients in k and K for the field of fractions of W. Let $\overline{K}$ be an algebraic closure of K. Write $K^{\mathrm{tm}} \subseteq \overline{K}$ for the maximal tamely ramified extension of K in $\overline{K}$ and $\Gamma_K \overset{\text{def}}{=} \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group of K determined by the algebraic closure $\overline{K}$. Let $g \geq 2$ be an integer and X a curve over K (i.e., a scheme of dimension one which is projective, smooth, and geometrically connected over K) of genus g. Write J for the Jacobian variety of X. In the present paper, suppose that

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> the curve X over K has *stable reduction* over W, which thus implies that the abelian variety J over K has *semistable reduction* over W.

Write, moreover, $X_{\overline{K}} \overset{\mathrm{def}}{=} X \times_K \overline{K}$, $J_{\overline{K}} \overset{\mathrm{def}}{=} J \times_K \overline{K}$, and $X^{\mathrm{cl}}_{\overline{K}}$, $J^{\mathrm{cl}}_{\overline{K}}$ for the sets of closed points of $X_{\overline{K}}$, $J_{\overline{K}}$, respectively. Thus, we have natural bijections $X_{\overline{K}}(\overline{K}) \overset{\sim}{\to} X^{\mathrm{cl}}_{\overline{K}}$, $J_{\overline{K}}(\overline{K}) \overset{\sim}{\to} J^{\mathrm{cl}}_{\overline{K}}$, which thus determine natural actions of Γ_K on $X^{\mathrm{cl}}_{\overline{K}}$, $J^{\mathrm{cl}}_{\overline{K}}$, respectively.

Let $x_0 \in X(K)$ be a K-rational point of X. Then we have the *Albanese embedding* $X \hookrightarrow J$ with respect to $x_0 \in X(K)$, i.e., the closed immersion over K obtained by, roughly speaking, mapping "x" to the invertible sheaf corresponding to the divisor "$[x] - [x_0]$" — where we write "$[-]$" for the prime divisor determined by "$(-)$" — of degree zero. By this embedding, we have an injection

$$\varphi_{x_0} \colon X^{\mathrm{cl}}_{\overline{K}} \hookrightarrow J^{\mathrm{cl}}_{\overline{K}}.$$

In the present paper, we study a *torsion point* on $X_{\overline{K}}$, i.e., a closed point of $X_{\overline{K}}$ whose image, via φ_{x_0} for some $x_0 \in X(K)$, is a torsion point in $J^{\mathrm{cl}}_{\overline{K}}$. In particular, in the present paper, we study a *ramified torsion point* on $X_{\overline{K}}$, i.e., a non-K-rational torsion point on $X_{\overline{K}}$ (cf. Definition 3.5, (i)).

In Introduction, let us consider the following situation:

> (‡): Let $x_0 \in X(K)$ be a K-rational point of X. By means of the above injection $\varphi_{x_0} \colon X^{\mathrm{cl}}_{\overline{K}} \hookrightarrow J^{\mathrm{cl}}_{\overline{K}}$, we regard $X^{\mathrm{cl}}_{\overline{K}}$ as a subset of $J^{\mathrm{cl}}_{\overline{K}}$. Let $x \in X^{\mathrm{cl}}_{\overline{K}}$ ($\subseteq J^{\mathrm{cl}}_{\overline{K}}$) be a closed point of $X_{\overline{K}}$. Suppose that $x \in J^{\mathrm{cl}}_{\overline{K}}$ is *torsion*.

Let us first recall that, in [4], *R. Coleman* stated a conjecture concerning the *ramification of torsion points* on a curve which satisfies certain conditions (cf. [4, Conjecture B]). The following is the statement of a slightly stronger version of the conjecture. Note that the *original* conjecture of Coleman is the following conjecture in the case where the pair (X, x_0) can be descended to a subfield of K which is *finite over the field of rational numbers*.

Conjecture (Coleman)**.** *In the situation* (‡)*, suppose, moreover, that the following two conditions are satisfied*:

(1) *The inequality* $p \geq 5$ *holds.*

(2) *The curve X, hence also the abelian variety J, over K has* good reduction *over W.*

Then $x \in J^{\mathrm{cl}}_{\overline{K}}$ *is* K-rational. *In other words, there is* no ramified torsion *point on* $X_{\overline{K}}$.

Moreover, Coleman essentially proved the following result concerning the above conjecture (cf. [4, Corollary 20.2]):

Theorem (Coleman)**.** *In the situation of Conjecture, suppose, moreover, that one of the following three conditions is satisfied*:

(a) *The special fiber of the good model of J is an* ordinary *abelian variety over k.*

(b) *The special fiber of the good model of J is isomorphic to the direct product of* supersingular *elliptic curves over k.*

(c) *The inequality* $2g < p$ *holds.*

Then $x \in J^{\mathrm{cl}}_{\overline{K}}$ *is* K-rational.

Next, let us recall that *A. Tamagawa* studied, in [13], the ramification of torsion points in the case where the abelian variety-part of the special fiber of the semistable model of J is an *ordinary* abelian variety. Tamagawa proved, for instance, the following result (cf. [13, Theorem 0.1]):

Theorem (Tamagawa)**.** *In the situation* (‡)*, suppose, moreover, that the following three conditions are satisfied*:

(1) *The inequality* $p \geq 29$ *holds.*

(2) *The abelian variety-part of the special fiber of the semistable model of* J *is an* ordinary *abelian variety over* k*.*

(3) *The curve* X *over* K *is* not hyperelliptic.

Then $x \in J^{\mathrm{cl}}_{\overline{K}}$ *is* K-rational.

In the present paper, by combining the idea of Tamagawa that was applied in [13] with the study of the Galois representations associated to *finite flat commutative group schemes*, we prove the following result (cf. Theorem 3.4). This result concerns the ramification of torsion points *after multiplying* p without any assumption on the reduction of J.

Theorem A. *In the situation* (‡)*, the point* $p \cdot x \in J^{\mathrm{cl}}_{\overline{K}}$ *is* K^{tm}-rational.

In the case where J has *good reduction* over W, we obtain the following result (cf. Theorem 3.4, (ii)):

Theorem B. *In the situation* (‡)*, if, moreover, the abelian variety* J *over* K *has* good reduction *over* W*, then* $p \cdot x \in J^{\mathrm{cl}}_{\overline{K}}$ *is* K-rational.

In §3 of the present paper, by means of Theorems A and B, we study the geometry of curves which admit *ramified torsion* points. As one of consequences, we prove the following *nonexistence of ramified torsion points* (cf. Corollary 3.6):

Theorem C. *In the situation* (‡)*, suppose that the following two conditions are satisfied*:

(1) *The inequality* $g \geq p$ *holds.*

(2) *The abelian variety* J *over* K *has* good reduction *over* W*.*

Suppose, moreover, that one of the following three conditions is satisfied:

(a) *The curve* $X_{\overline{K}}$ *over* $\overline{K}$ *is* hyperelliptic (*i.e.,* of gonality 2).

(b) *The curve* $X_{\overline{K}}$ *over* $\overline{K}$ *is* of gonality $> p$*.*

(c) *Every* Weierstrass point *of* $X_{\overline{K}}$ *is* K-rational.

Then $x \in J^{\mathrm{cl}}_{\overline{K}}$ *is* K-rational. *In other words, there is* no ramified torsion *point on* $X_{\overline{K}}$*.*

Note that Theorem C yields some *conditional results* of the above conjecture of Coleman. Indeed, by, for instance, Theorem C in the case where the condition (a) is satisfied, we conclude that the conjecture of Coleman holds if X is *hyperelliptic* and *of genus* $\geq p$ (cf. also Remark 3.6.1).

The present paper is organized as follows: In §1, we consider the Galois representations associated to finite flat commutative group schemes. In particular, we discuss the relationship between the *level* of a finite flat commutative group scheme over W (cf. Definition 1.2, (i)) and the *ramification* of the Galois representation associated to the finite flat commutative group scheme (cf. Proposition 1.8, Lemma 1.9). In §2, we consider a *Galois module of type S* (cf. Definition 2.3, (i)), i.e., a Γ_K-module which is isomorphic to a finite Γ_K-submodule of the Γ_K-module obtained by considering torsion points of an abelian variety with semistable model over W. In particular, we prove the *triviality* of the Galois action on a subquotient of a Galois module of type S which satisfies a technical condition (cf. Lemma 2.7). In §3, we prove the main result of the present paper (cf. Theorem 3.4), which is closely related to the above conjecture due to *Coleman* (cf. Remark 3.4.1). Moreover, by means of the main result, we study the geometry of curves which admit *ramified torsion points* (cf. Corollaries 3.6, 3.8 and 3.9).

0. Notations and Conventions

Groups. Let G be a group and S a set on which G acts. Then we shall write $S^G \subseteq S$ for the subset of S of G-invariants, $G^S \subseteq G$ for the (necessarily normal and uniquely determined) maximal subgroup of G which acts on S trivially, and $G[S] \stackrel{\text{def}}{=} G/G^S$. Thus, the action of G on S *factors* through the natural surjection $G \twoheadrightarrow G^S$, and, moreover, the resulting action of G^S on S is *faithful*.

Modules. Let M be a module, $n \geq 0$ an integer, and l a prime number. Then we shall write $\mathrm{Aut}(M)$ for the group of automorphisms of the module M, $M[n] \subseteq M$ for the submodule of M obtained by forming the kernel of the endomorphism of M given by multiplication by n, and $M_l \stackrel{\text{def}}{=} \bigcup_{i\geq 1} M[l^i] \subseteq M$. If, moreover, M is *finite*, then we shall write $M_{\neq l} \subseteq M$ for the submodule of M generated by elements of the $M_{l'}$'s, where l' ranges over the prime numbers such that $l' \neq l$. Thus, if M is *finite*, then we have a natural decomposition $M = M_l \oplus M_{\neq l}$.

Let G be a group and M a G-module. Then we shall say that an element $x \in M$ of M is a *weak G-invariant* if, for every γ, $\delta \in G$, the following holds: If $(1-\gamma)^2(\delta \cdot x) = 0$, then $(1-\gamma)(\delta \cdot x) = 0$. (Thus, if $x \in M$ is a *G-invariant*, then $x \in M$ is a *weak G-invariant*.)

Varieties. Let k be a field. Then we shall say that a scheme over k is a *variety* over k if the scheme is separated and of finite type over k.

Let V be a variety over k and $\overline{k}$ an algebraic closure of k. Then we shall write $V_{\overline{k}} \stackrel{\text{def}}{=} V \times_k \overline{k}$ for the variety over $\overline{k}$ determined by V and $V^{\mathrm{cl}}_{\overline{k}}$ for the set of closed points of $V_{\overline{k}}$. Thus, if k is *perfect*, then we have a natural bijection $V_{\overline{k}}(\overline{k}) \xrightarrow{\sim} V^{\mathrm{cl}}_{\overline{k}}$, which thus determines a natural action of $\mathrm{Gal}(\overline{k}/k)$ on $V^{\mathrm{cl}}_{\overline{k}}$; moreover, the natural injection $V(k) \hookrightarrow V_{\overline{k}}(\overline{k})$ from the set $V(k)$ of k-rational points of V determines a bijection $V(k) \xrightarrow{\sim} (V^{\mathrm{cl}}_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}$.

Curves. Let k be a field. Then we shall say that a scheme over k is a *curve* over k if the scheme is of dimension one and, moreover, projective, smooth, and geometrically connected over k. Thus, a curve over k is a variety over k.

Let C be a curve over k and $g \geq 0$ an integer. We shall say that C is *of genus g* if $H^1(C, \mathcal{O}_C)$

is of dimension g over k. We shall say that C is *of gonality* g if the minimum among the degrees of finite morphisms from C to curves of genus zero over k is equal to g.

Suppose that the curve C is of genus $g \geq 2$, and that the field k is of characteristic zero. Let $\overline{k}$ be an algebraic closure of k, $c \in C^{\mathrm{cl}}_{\overline{k}}$, and $n \geq 0$ an integer. We shall say that the integer n is a *Weierstrass non-gap* at $c \in C^{\mathrm{cl}}_{\overline{k}}$ if there exists a section of $\mathcal{O}_{C_{\overline{k}}}$ on $C_{\overline{k}} \setminus \{c\}$ of order $-n$ at c (i.e., the integer n contains the Weierstrass monoid of $C_{\overline{k}}$ at $c \in C^{\mathrm{cl}}_{\overline{k}}$). We shall say that $c \in C^{\mathrm{cl}}_{\overline{k}}$ is a *Weierstrass point* of $C_{\overline{k}}$ if there exists an integer $1 \leq i \leq g$ such that i is a Weierstrass non-gap at $c \in C^{\mathrm{cl}}_{\overline{k}}$. Note that, as is well-known (cf., e.g., [1, Chapter I, Exercises E-8, (ii), and E-9]), if we write N for the number of Weierstrass points of $C_{\overline{k}}$, then the inequalities $2g + 2 \leq N \leq g^3 - g$ hold. We shall say that the pair (C, c) is *exceptional* (cf. [13], Definition in the discussion entitled "Weierstrass points on hyperelliptic curves") if 2 is a Weierstrass non-gap at $c \in C^{\mathrm{cl}}_{\overline{k}}$ (i.e., $C_{\overline{k}}$ is hyperelliptic, and the hyperelliptic involution of $C_{\overline{k}}$ is ramified at $c \in C^{\mathrm{cl}}_{\overline{k}}$).

1. Level and Ramification of Finite Flat Commutative Group Schemes

In the present §1, we consider the Galois representations associated to finite flat commutative group schemes. In particular, we discuss the relationship between the *level* of a finite flat commutative group scheme (cf. Definition 1.2, (i), below) and the *ramification* of the Galois representation associated to the finite flat commutative group scheme (cf. Proposition 1.8, Lemma 1.9, below).

In the present §1, let p be an odd prime number and k an algebraically closed field of characteristic p. Write $W \overset{\mathrm{def}}{=} W(k)$ for the ring of Witt vectors with coefficients in k and K for the field of fractions of W. Let $\overline{K}$ be an algebraic closure of K and $L \subseteq \overline{K}$ a(n) (possibly infinite) algebraic extension of K. Write $\Gamma_L \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/L)$ for the absolute Galois group of L determined by the algebraic closure $\overline{K}$, v_0 for the (uniquely determined) p-adic valuation on $\overline{K}$ such that $v_0(p) = 1$, and $\overline{W} \subseteq \overline{K}$, $V \subseteq L$ for the rings of integers of $\overline{K}$, L, respectively.

Definition 1.1. Suppose that $[L : K] < \infty$.

(i) Let M be a V-module which is *annihilated* by a power of p. Then we shall write

$$\mathrm{lv}_V(M) \overset{\mathrm{def}}{=} v_0(\mathrm{Ann}_V(M)).$$

(ii) We shall write

$$\mathrm{lv}^{\Omega}(L/K) \overset{\mathrm{def}}{=} \mathrm{lv}_V(\Omega^1_{V/W}).$$

Thus, it follows that

$$\mathrm{lv}^{\Omega}(L/K) = v_0(\delta_{L/K}),$$

where we write $\delta_{L/K}$ for the different of the finite extension L/K.

In the remainder of the present §1, let G be a *finite flat commutative group scheme* over W which is *annihilated* by a power of p. Thus, we have an exact sequence of finite flat commutative group schemes over W

$$0 \longrightarrow G^{\circ} \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 0,$$

where $G^{\circ} \subseteq G$ is *connected*, and $G^{\text{ét}}$ is *étale* over W. Write $K_G \subseteq \overline{K}$ for the (necessarily finite Galois) extension of K corresponding to the kernel of the natural action of Γ_K on $G(\overline{K})$, i.e., the finite Galois extension of K corresponding to the quotient $\Gamma_K \twoheadrightarrow \Gamma_K[G(\overline{K})]$ — and $W_G \subseteq K_G$ for the ring of integers of K_G.

Definition 1.2.

(i) We shall write

$$\mathrm{lv}(G) \overset{\text{def}}{=} \mathrm{lv}_W(G(\overline{K}) \otimes_{\mathbb{Z}_p} W) \quad (\in \mathbb{Z})$$

and refer to $\mathrm{lv}(G)$ as the *level* of G.

(ii) Let M be a W-module. Then we shall write

$$t^*_G(M) \overset{\text{def}}{=} (e^*_G \Omega^1_{G/W}) \otimes_W M$$

where we write e_G for the identity section of G/W — and

$$t_G(M) \overset{\text{def}}{=} \mathrm{Hom}_W(t^*_G(W), M).$$

We shall refer to $t^*_G(M)$ (respectively, $t_G(M)$) as the *M-valued cotangent* (respectively, *tangent*) *space* of G/W. Note that since G is *étale over K*, it follows that $t^*_G(M)$, hence also $t_G(M)$, is *annihilated* by a power of p.

(iii) We shall write

$$\mathrm{lv}^{\Omega}(G) \overset{\text{def}}{=} \mathrm{lv}_W(t^*_G(W)).$$

(iv) Suppose that L is *Galois* over K, and that $K_G \subseteq L$. (So $G(\overline{W}) = G(V)$). Then we shall define a homomorphism of $\overline{W}$-modules

$$\mathrm{ev}_L \colon\ G(\overline{W}) \otimes_{\mathbb{Z}_p} \overline{W} = G(V) \otimes_{\mathbb{Z}_p} \overline{W} \longrightarrow t_G(\Omega^1_{V/W} \otimes_V \overline{W})$$

as follows (cf. [7, §4.7]): Let $x \in G(V)$ be a V-valued point of G. Then, by considering the operation of restricting differential forms on G over W to x, we obtain a homomorphism $t^*_G(W) \to \Omega^1_{V/W}$, hence also a homomorphism of W-modules

$$e_x \colon\ t^*_G(W) \longrightarrow \Omega^1_{V/W} \otimes_V \overline{W}.$$

Thus, the assignment "$x \mapsto e_x$" determines a map

$$G(V) \longrightarrow t_G(\Omega^1_{V/W} \otimes_V \overline{W}).$$

Now since (one verifies easily that) this map is a homomorphism of $\mathbb{Z}_p$-modules, this map determines the homomorphism ev_L as above.

Remark 1.2.1. Thus, by definition, we obtain that $\underline{t}^*_G \leq \mathrm{lv}^{\Omega}(G)$ and $\underline{t}^*_G \not< \mathrm{lv}^{\Omega}(G)$ (cf. [9, Definition 1.3, (ii)]). If, moreover, G is *of p-rectangle-type* (cf. [9, Definition 2.1, (ii)]), then $\mathrm{lv}(G)$ of Definition 1.2, (i), *coincides* with $\mathrm{lv}(G)$ of [9, Definition 2.1, (ii)].

Theorem 1.3 (Fontaine)**.** *The following hold*:

(i) *The* $\overline{W}$*-module* $\mathrm{Coker}(\mathrm{ev}_{\overline{K}})$ *is* annihilated *by every element* $a \in \overline{W}$ *such that* $v_0(a) \geq \frac{1}{p-1}$.

(ii) *The inequality*

$$\mathrm{lv}^{\Omega}(K_G/K) \ < \ \mathrm{lv}(G) + \frac{1}{p-1}$$

holds.

Proof. Assertion (i) follows from [7, Corollaire to Théorème 3]. Assertion (ii) follows from [8, Corollaire to Théorème A]. □

Proposition 1.4. *The inequality*

$$\mathrm{lv}^{\Omega}(G) \ \leq \ \mathrm{lv}^{\Omega}(K_G/K) + \frac{1}{p-1}$$

holds.

Proof. Let us first observe that we have a commutative diagram of $\overline{W}$-modules

$$\begin{tikzcd}
G(K_G) \otimes_{\mathbb{Z}_p} \overline{W} \arrow[r, equal] \arrow[d, "\mathrm{ev}_{K_G}"'] & G(\overline{W}) \otimes_{\mathbb{Z}_p} \overline{W} \arrow[d, "\mathrm{ev}_{\overline{K}}"'] \\
t_G(\Omega^1_{W_G/W} \otimes_{W_G} \overline{W}) \arrow[r] & t_G(\Omega^1_{\overline{W}/W})
\end{tikzcd}$$

where the lower horizontal arrow is *injective* (cf., e.g., [7, Lemma 4]). Thus, it follows from Theorem 1.3, (i), that the $\overline{W}$-module

$$\mathrm{Coker}(t_G(\Omega^1_{W_G/W} \otimes_{W_G} \overline{W}) \hookrightarrow t_G(\Omega^1_{\overline{W}/W}))$$

is *annihilated* by every element $a \in \overline{W}$ such that $v_0(a) \geq \frac{1}{p-1}$. In particular, we conclude that the $\overline{W}$-module $t_G(\Omega^1_{\overline{W}/W})$ is *annihilated* by every element $b \in \overline{W}$ such that $v_0(b) \geq \mathrm{lv}^{\Omega}(K_G/K) + \frac{1}{p-1}$. Thus, it follows immediately from [7, Corollaire 1, (1)], that $\mathrm{lv}^{\Omega}(G) \leq \mathrm{lv}^{\Omega}(K_G/K) + \frac{1}{p-1}$, as desired. This completes the proof of Proposition 1.4. □

Theorem 1.5 (Raynaud)**.** *Every homomorphism over K between the generic fibers* (*i.e., the results of base-changing via $W \hookrightarrow K$*) *of finite flat commutative group schemes over W* uniquely extends *to a homomorphism between the original finite flat commutative group schemes over W. Moreover, the kernel of the resulting homomorphism between the original finite flat commutative group schemes over W is* flat *over W.*

Proof. This follows from [11, Corollaire 3.3.6, (1)]. □

Lemma 1.6. *Let $n \geq 0$ be an integer. Write $G[p^n] \subseteq G$ for the finite flat commutative group scheme over W obtained by forming the kernel of the endomorphism of G given by multiplication by p^n* (*cf.* Theorem 1.5)*. Then the exact sequence of finite flat commutative group schemes over W*

$$0 \longrightarrow G[p^n] \longrightarrow G \longrightarrow G/G[p^n] \longrightarrow 0$$

determines a commutative diagram of W-modules

$$\begin{tikzcd}
0 \arrow[r] & t^*_{G/G[p^n]}(W) \arrow[r] \arrow[d, "\wr"'] & t^*_G(W) \arrow[r] \arrow[d, equal] & t^*_{G[p^n]}(W) \arrow[r] \arrow[d, "\wr"'] & 0 \\
0 \arrow[r] & p^n \cdot t^*_G(W) \arrow[r] & t^*_G(W) \arrow[r] & t^*_G(W) \otimes_W W/p^n \arrow[r] & 0
\end{tikzcd}$$

where the horizontal sequences are exact, and the vertical arrows are isomorphisms.

Proof. Let us observe that one verifies immediately that the exact sequence of finite flat commutative group schemes over W

$$0 \longrightarrow G[p^n] \longrightarrow G \xrightarrow{p^n} G$$

determines an exact sequence

$$t^*_G(W) \xrightarrow{p^n} t^*_G(W) \longrightarrow t^*_{G[p^n]}(W) \longrightarrow 0,$$

which thus determines an isomorphism

$$t^*_G(W) \otimes_W W/p^n \xrightarrow{\sim} t^*_{G[p^n]}(W).$$

Thus, Lemma 1.6 follows from [9, Lemma 1.6]. This completes the proof of Lemma 1.6. □

Proposition 1.7. *The equality*

$$\mathrm{lv}^{\Omega}(G) = \mathrm{lv}(G^{\circ})$$

holds.

Proof. Write $n \overset{\mathrm{def}}{=} \mathrm{lv}(G^{\circ})$. Then it follows from a similar conclusion to the conclusion "$\underline{t}^*_G \leq \mathrm{lv}(G)$" of [9, Lemma 2.3] (cf. also Remark 1.2.1 of the present paper), that, to verify Proposition 1.7, it suffices to verify that $p^{n-1} \cdot t^*_G(W) \neq \{0\}$. To this end, assume that $p^{n-1} \cdot t^*_G(W) = \{0\}$. Then it follows from Lemma 1.6 that $t^*_{G/G[p^{n-1}]}(W) = \{0\}$, which thus implies that $G/G[p^{n-1}]$ is *étale* over W. Thus, the composite $G^{\circ} \hookrightarrow G \twoheadrightarrow G/G[p^{n-1}]$ is *trivial*, i.e., $G^{\circ} \subseteq G[p^{n-1}]$ — which *contradicts* our assumption that $\mathrm{lv}(G^{\circ}) = n$. This completes the proof of Proposition 1.7. □

Proposition 1.8. *The inequalities*

$$\mathrm{lv}(G^{\circ}) - \frac{1}{p-1} \leq \mathrm{lv}^{\Omega}(K_G/K) < \mathrm{lv}(G) + \frac{1}{p-1}$$

hold.

Proof. This follows from Theorem 1.3, (ii); Proposition 1.4, together with Proposition 1.7. □

The following result is the main result of the present §1:

Lemma 1.9. *Let H be a finite flat commutative group scheme over W which is* annihilated *by a power of p. Suppose that $K_H \subseteq K_G$ (cf. the notation introduced in the discussion preceding* Definition 1.2)*. Then the inequality*

$$\mathrm{lv}(H^\circ) \ \leq \ \mathrm{lv}(G)$$

holds.

Proof. It follows from Proposition 1.8, together with our assumption, that

$$\mathrm{lv}(H^\circ) - \frac{1}{p-1} \ \leq \ \mathrm{lv}^\Omega(K_H/K) \ \leq \ \mathrm{lv}^\Omega(K_G/K) \ < \ \mathrm{lv}(G) + \frac{1}{p-1}.$$

Thus, since (we have assumed that) $p \geq 3$, we obtain that $\mathrm{lv}(H^\circ) \leq \mathrm{lv}(G)$. This completes the proof of Lemma 1.9. □

Remark 1.9.1.

(i) One verifies immediately from Theorem 1.5 that the exact sequence $0 \to G^\circ(\overline{K}) \to G(\overline{K}) \to G^{\text{ét}}(\overline{K}) \to 0$ of (not Γ_K-modules but abstract) modules is *split*. One also verifies immediately that the action of $\Gamma_{K_{G^\circ}}$ ($\supseteq \Gamma_{K_G}$) on $G(\overline{K})$ determines and is determined by a homomorphism $\Gamma_{K_{G^\circ}} \to \mathrm{Hom}_{\mathbb{Z}_p}(G^{\text{ét}}(\overline{K}), G^\circ(\overline{K}))$. By means of these observations, one verifies easily that one may replace the respective "$\mathrm{lv}(G)$" of the right-hand sides of the displays of Proposition 1.8 and Lemma 1.9 by "$\mathrm{lv}(G^\circ)$". We leave the routine details to the interested reader.

(ii) Let us recall that we have worked in a situation *of absolute ramification index one*, i.e., a situation where the base discrete valuation field is *of absolute ramification index one*. Now let us observe that Theorem 1.5 may be applied in a situation *of absolute ramification index* $< p - 1$ (cf. [11, Corollaire 3.3.6, (1)]). In particular, even if we are in a situation *of absolute ramification index* $< p - 1$, one may obtain a similar result to Proposition 1.8, as well as a similar result to Lemma 1.9. We leave the routine details to the interested reader.

2. Galois Modules of Type S

In the present §2, we consider a *Galois module of type S* (cf. Definition 2.3, (i), below), i.e., a Γ_K-module which is isomorphic to a finite Γ_K-submodule of the Γ_K-module obtained by considering torsion points of an abelian variety with semistable model over W. In particular, we prove the *triviality* of the Galois action on a subquotient of a Galois module of type S which satisfies a technical condition (cf. Lemma 2.7 below).

In the present §2, we maintain the notation introduced at the beginning of §1. Write, moreover, $K^{\mathrm{tm}} \subseteq \overline{K}$ for the maximal tamely ramified extension of K in $\overline{K}$.

Lemma 2.1. *Let M be a finite module,* $\Gamma \subseteq \mathrm{Aut}(M)$ *a subgroup of* $\mathrm{Aut}(M)$*, and $x \in M$ an element of M. Write $x = x_p + x_{\neq p}$ for the representation of $x \in M$ with respect to the natural direct decomposition $M = M_p \oplus M_{\neq p}$ and $S_x \subseteq M$ for the subset of M consisting of the elements $y \in M$ which satisfy one of the following three conditions*:

(1) *There exist elements $\gamma_1, \gamma_2 \in \Gamma$ of Γ and an integer $i \geq 0$ such that $y = p^i(\gamma_1 - \gamma_2)x \in M$.*

(2) *There exist elements γ_1, $\gamma_2 \in \Gamma$ of Γ and an integer $i \geq 0$ such that $y = p^i(\gamma_1 - \gamma_2)x_p \in M$.*

(3) *There exist elements γ_1, γ_2, γ_3, $\gamma_4 \in \Gamma$ of Γ and an integer $i \geq 0$ such that $y = p^i(\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)x \in M$.*

Note that one verifies immediately that the subset $S_x \subseteq M$, hence also the subset $S_x[p] \overset{\text{def}}{=} S_x \cap M[p]$, of M is stable *under the action of Γ on M. Suppose that the following two conditions are satisfied*:

(a) *The element $x \in M$ is a* weak Γ-invariant.

(b) *For every $\gamma \in \Gamma$, the module $M_{\neq p}$ is* annihilated *by $(1 - \gamma)^2$.*

Then the following hold:

(i) *The element x_p is* contained *in $M^{\Gamma^{S_x[p]}}$.*

(ii) *Suppose, moreover, that the following condition is satisfied*:

(c) *The Γ-module M is* generated *by $x \in M$.*

Then $\Gamma^{S_x[p]} = \{1\}$.

Proof. To verify assertion (i), assume that $x_p \notin M^{\Gamma^{S_x[p]}}$. Write n for the smallest (necessarily *positive*) integer such that $p^n x_p \in M^{\Gamma^{S_x[p]}}$. Now since $p^{n-1}x_p \notin M^{\Gamma^{S_x[p]}}$, there exists a(n) (necessarily *nontrivial*) element $\gamma \in \Gamma^{S_x[p]}$ of $\Gamma^{S_x[p]}$ such that $p^{n-1}(1 - \gamma)x_p \neq 0$. For $1 \leq i \leq n$, write

$$y_i \overset{\text{def}}{=} p^{n-i}(1 - \gamma^{p^{i-1}})x_p \in M.$$

Now I claim that the following assertion holds:

> Claim 2.1.A: The element y_1 is *contained* in $S_x[p] \setminus \{0\}$.

Indeed, the fact that y_1 $(= p^{n-1}(1 - \gamma)x_p) \neq 0$ has already been verified. It follows from (2), together with the definition of y_i, that $y_i \in S_x$. Moreover, since $p^n x_p \in M^{\Gamma^{S_x[p]}}$, and $\gamma \in \Gamma^{S_x[p]}$, we obtain that $py_1 = p^n(1 - \gamma)x_p = 0$, i.e., that $y_1 \in M[p]$. This completes the proof of Claim 2.1.A.

Next, I claim that the following assertion holds:

> Claim 2.1.B: The element y_i *coincides* with the element y_1 for every $1 \leq i \leq n$.

We prove Claim 2.1.B by *induction on i*. Suppose that $y_i = y_1$ for $1 \leq i \leq n - 1$. Then it follows from Claim 2.1.A, together with the induction hypothesis, that $p^2 \cdot p^{n-i-1}(1 - \gamma^{p^{i-1}})x_p = py_i = py_1 = 0$, which thus implies that $p \cdot p^{n-i-1}(1 - \gamma^{p^{i-1}})x_p \in S_x[p]$ (cf. (2)). In particular, since $\gamma \in \Gamma^{S_x[p]}$, we obtain that

$$(*) \qquad p \cdot p^{n-i-1}(1 - \gamma^{p^{i-1}})^2 x_p = 0.$$

Thus, since $(1 - \gamma^{p^{i-1}})^2 x_{\neq p} = 0$ (cf. (b)), we obtain that $p \cdot p^{n-i-1}(1 - \gamma^{p^{i-1}})^2 x = 0$, which thus implies that $p^{n-i-1}(1 - \gamma^{p^{i-1}})^2 x_p = p^{n-i-1}(1 - \gamma^{p^{i-1}})^2 x \in S_x[p]$ (cf. (3)). Thus, since

$\gamma \in \Gamma^{S_x[p]}$, we obtain that

$$(**) \qquad p^{n-i-1}(1-\gamma^{p^{i-1}})^3 x_p \;=\; 0.$$

It follows from (∗), (∗∗), together with Lemma 2.2 below, that

$$(1-(\gamma^{p^{i-1}})^p)p^{n-i-1}x_p = p(1-\gamma^{p^{i-1}})p^{n-i-1}x_p,$$

i.e., that $y_{i+1} = y_i$, as desired. This completes the proof of Claim 2.1.B.

Next, let us observe that it follows from Claim 2.1.A and Claim 2.1.B that

$$(1-\gamma^{p^{n-1}})x_p \;=\; y_n \;=\; y_1 \;\in\; S_x[p] \setminus \{0\}.$$

Thus, since $\gamma^{p^{n-1}} \in \Gamma^{S_x[p]}$, and $(1-\gamma^{p^{n-1}})^2 x_{\neq p} = 0$ (cf. (b)), we obtain that

$$(1-\gamma^{p^{n-1}})^2 x \;=\; (1-\gamma^{p^{n-1}})^2 x_p \;=\; (1-\gamma^{p^{n-1}})y_n \;=\; 0,$$

which thus implies (cf. (a)) that $(1-\gamma^{p^{n-1}})x = 0$. In particular, we conclude that

$$y_n \;=\; (1-\gamma^{p^{n-1}})x_p \;=\; 0\,,$$

which *contradicts* Claim 2.1.A and Claim 2.1.B. This completes the proof of assertion (i).

Finally, we verify assertion (ii). Let $\gamma \in \Gamma^{S_x[p]}$ be an element of $\Gamma^{S_x[p]}$. Then since $x_p \in M^{\Gamma^{S_x[p]}}$ (cf. assertion (i)), and $\Gamma^{S_x[p]} \subseteq \Gamma$ is *normal*, we obtain that $\delta \cdot x_p \in M^{\Gamma^{S_x[p]}}$ for every $\delta \in \Gamma$. Thus, we conclude that $(1-\gamma)(\delta \cdot x) = (1-\gamma)(\delta \cdot x_{\neq p})$, which thus implies that $(1-\gamma)^2(\delta \cdot x) = (1-\gamma)^2(\delta \cdot x_{\neq p}) = 0$ (cf. (b)). In particular, it follows from (a) that $(1-\gamma)(\delta \cdot x) = 0$. Thus, it follows from (c) that the action of γ on M is *trivial*. This completes the proof of assertion (ii), hence also of Lemma 2.1. □

Lemma 2.2. *In the ring $\mathbb{Z}[T]$ of polynomials in T with coefficients in $\mathbb{Z}$, the congruence*

$$1-T^p \;\equiv\; p(1-T) \quad \mathrm{mod}\,(p(1-T)^2, (1-T)^3)$$

holds.

Proof. By "mod $(1-T)^3$", we obtain that

$$1-T^p \;=\; 1-(1-(1-T))^p \;\equiv\; 1-(1-p(1-T)+p(p-1)(1-T)^2/2)$$

$$=\; p(1-T)-p(p-1)(1-T)^2/2.$$

Thus, since (we have assumed that) $p \neq 2$, Lemma 2.2 holds. This completes the proof of Lemma 2.2. □

Definition 2.3. Let M be a finite module equipped with an action of Γ_K.

(i) We shall say that the Γ_K-module M is *of type G* (respectively, *of type S*) if there exist an abelian variety A over K which has good (respectively, semistable) reduction over W and a Γ_K-equivariant injection $M \hookrightarrow A(\overline{K})$.

(ii) We shall say that a Γ_K-submodule $N \subseteq M$ of M is a *G-part* of M if the following three conditions are satisfied:

(1) The Γ_K-module N is of type G (which thus implies that the action of Γ_K on $N_{\neq p}$

is trivial — cf. Remark 2.3.1, (i), (ii), below).

(2) The action of Γ_K on M/N is trivial.

(3) The action of Γ_K on every nontrivial Γ_K-stable subquotient of N_p is nontrivial (cf. Lemma 2.4, (ii), below).

(iii) We shall say that the action of Γ_K on M is *tame* if the (necessarily finite) quotient $\Gamma_K[M]$ of Γ_K is of order prime to p, i.e., the natural surjection $\Gamma_K \twoheadrightarrow \Gamma_K[M]$ factors through the quotient of Γ_K corresponding to the Galois extension K^{tm} ($\subseteq \overline{K}$) of K.

Remark 2.3.1.

(i) One verifies immediately from the various definitions involved that a Γ_K-module obtained by forming a subquotient of a finite Γ_K-module *of type G* (respectively, *of type S*) is *of type G* (respectively, *of type S*).

(ii) Let M be a finite Γ_K-module *of type G* such that $M = M_{\neq p}$. Then one verifies immediately that the action of Γ_K on M is *trivial* (cf., e.g., Lemma 2.4, (i), below).

(iii) It is well-known (cf., e.g., [10, Appendix A, Theorem A.6]) that, for a finite Γ_K-module M: The Γ_K-module M is *of type G* if and only if there exist a finite flat commutative group scheme G over W and a Γ_K-equivariant isomorphism $M \xrightarrow{\sim} G(\overline{K})$.

Lemma 2.4. *Let M be a finite* Γ_K*-module* of type G. *Thus, by* Remark 2.3.1, (iii)*, there exist a finite flat commutative group scheme G over W and a* Γ_K*-equivariant isomorphism* $M \xrightarrow{\sim} G(\overline{K})$. *Then the following hold*:

(i) *The action of* Γ_K *on M is* trivial *if and only if G is* étale *over W*.

(ii) *The action of* Γ_K *on every nontrivial* Γ_K*-stable subquotient of M is* nontrivial *if and only if G is* connected.

Proof. These assertions follow immediately from Theorem 1.5. □

Now let us recall the following *well-known* lemma:

Proposition 2.5. *Let M be a finite* Γ_K*-module* of type S. *Then the following hold*:

(i) *The* Γ_K*-module M has a* G-part.

(ii) *If* $M = M_{\neq p}$*, then the action of* Γ_K *on M is* tame.

Proof. Assertion (ii) follows immediately from assertion (i), together with conditions (1) and (2) of Definition 2.3, (ii). Thus, to complete the verification of Proposition 2.5, it suffices to verify assertion (i). On the other hand, assertion (i) follows from basic facts concerning Galois actions on torsion points of semi-abelian schemes (cf., e.g., [6, Chapter III], or [10, Appendix C], the discussion entitled "The Raynaud group") as follows.

To verify assertion (i), let us first review some consequences of the discussions of [6, Chapter III]. Let A be an abelian variety over K which has *semistable reduction* over W and n an integer such that $M \subseteq A(\overline{K})[n]$. Write A^D for the dual abelian variety of A. Then it follows from the discussions of [6, Chapter III], that there exist semi-abelian schemes $\widetilde{A}$, $\widetilde{A}_D$ over W; abelian schemes B, B_D over W; split tori T, T_D over W; free $\mathbb{Z}/n$-modules P,

P_D of finite rank equipped with the trivial actions of Γ_K which satisfy the following three conditions:

(a) The semi-abelian scheme $\widetilde{A}$ (respectively, $\widetilde{A}_D$) is an extension of B (respectively, B_D) by T (respectively, T_D). In particular, we have exact sequences of Γ_K-modules

$$0 \longrightarrow T(\overline{K})[n] \longrightarrow \widetilde{A}(\overline{K})[n] \longrightarrow B(\overline{K})[n] \longrightarrow 0,$$

$$0 \longrightarrow T_D(\overline{K})[n] \longrightarrow \widetilde{A}_D(\overline{K})[n] \longrightarrow B_D(\overline{K})[n] \longrightarrow 0.$$

(b) The Γ_K-modules of n-torsion points of A, $\widetilde{A}$, A^D, $\widetilde{A}_D$ fit into exact sequences of Γ_K-modules

$$0 \longrightarrow \widetilde{A}(\overline{K})[n] \longrightarrow A(\overline{K})[n] \longrightarrow P \longrightarrow 0,$$

$$0 \longrightarrow \widetilde{A}_D(\overline{K})[n] \longrightarrow A^D(\overline{K})[n] \longrightarrow P_D \longrightarrow 0.$$

(c) The natural pairing $A(\overline{K})[n] \times A^D(\overline{K})[n] \to \boldsymbol{\mu}_n(\overline{K})$ — where we write $\boldsymbol{\mu}_n(\overline{K}) \subseteq \overline{K}^\times$ for the group of n-th roots of unity in $\overline{K}$ — determines a Γ_K-equivariant isomorphism (cf. [6, Chapter III, Corollary 7.4])

$$A(\overline{K})[n]/T(\overline{K})[n] \overset{\sim}{\longrightarrow} \mathrm{Hom}_{\mathbb{Z}}(\widetilde{A}_D(\overline{K})[n], \boldsymbol{\mu}_n(\overline{K})).$$

Moreover, by (a), the quasi-finite flat commutative group schemes $G \overset{\text{def}}{=} \widetilde{A}[n]$, $\widetilde{A}_D[n]$ over W obtained by forming the respective kernels of the endomorphisms of $\widetilde{A}$, $\widetilde{A}_D$ given by multiplication by n is in fact *finite* over W. Thus, it follows from Remark 2.3.1, (iii), that the following holds:

(d) The finite Γ_K-modules $\widetilde{A}(\overline{K})[n]$, $\widetilde{A}_D(\overline{K})[n]$ are *of type G*. In particular, by (c), the finite Γ_K-module $A(\overline{K})[n]/T(\overline{K})[n]$ is *of type G*.

If $M = M_{\neq p}$, then it follows immediately — in light of Remark 2.3.1, (ii) — from (d), together with the various definitions involved, that the Γ_K-submodule of M determined by $T(\overline{K})[n] \subseteq A(\overline{K})[n]$ in the above discussion is a *G-part*. Thus, to complete the verification of assertion (i), we may assume without loss of generality that $M = M_p$, and that n is a *power of p*.

Since $G = \widetilde{A}[n]$ is a *finite flat commutative group scheme* over W (cf. the discussion preceding (d)), we have an exact sequence of finite flat commutative group schemes over W

$$0 \longrightarrow G^\circ \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 0,$$

where $G^\circ \subseteq G$ is *connected*, and $G^{\text{ét}}$ is *étale* over W. Now I claim that the following assertion holds:

Claim 2.5.A: The finite Γ_K-module $A(\overline{K})[n]/G^\circ(\overline{K})$ is *of type G*.

Indeed, by (d), to verify Claim 2.5.A, it suffices to verify that the inclusion $T(\overline{K})[n] \subseteq G^\circ(\overline{K})$ holds. On the other hand, this follows from the (easily verified) fact that the action of Γ_K on every nontrivial Γ_K-stable subquotient of $T(\overline{K})[n]$ is *nontrivial*. This completes the proof of

Claim 2.5.A.

Next, I claim that the following assertion holds:

> Claim 2.5.B: The Γ_K-submodule $N \subseteq M$ of M determined by $G^{\circ}(\overline{K}) \subseteq A(\overline{K})$ is a *G-part*.

Indeed, it follows from the various definitions involved, together with Lemma 2.4, (ii), that N satisfies conditions (1) and (3) of Definition 2.3, (ii). Next, to verify the assertion that N satisfies condition (2) of Definition 2.3, (ii), let us consider the following exact sequence of Γ_K-modules (which arises from the first exact sequence of (b))

$$0 \longrightarrow G^{\text{ét}}(\overline{K}) \longrightarrow A(\overline{K})[n]/G^{\circ}(\overline{K}) \longrightarrow P \longrightarrow 0.$$

Since the actions of Γ_K on $G^{\text{ét}}(\overline{K})$ and P are *trivial*, it follows immediately from Claim 2.5.A (cf. also Remark 2.3.1, (iii)) that the action of Γ_K on $A(\overline{K})[n]/G^{\circ}(\overline{K})$ is *trivial*. Thus, since M/N is a Γ_K-submodule of $A(\overline{K})[n]/G^{\circ}(\overline{K})$, it follows that N satisfies condition (2) of Definition 2.3, (ii). This completes the proof of Claim 2.5.B, hence also of assertion (i). □

Lemma 2.6. *Let M be a finite Γ_K-module* of type S. *Suppose that there exists a* weak Γ_K-invariant *$x \in M$ of M such that the Γ_K-module M is* generated *by $x \in M$. Then there exists a Γ_K-submodule $N \subseteq M$ of M which satisfies the following two conditions*:

(1) *The Γ_K-module N is* of type G *and* annihilated by *p*.

(2) *The natural surjection $\Gamma_K[M] \twoheadrightarrow \Gamma_K[N]$ is an* isomorphism.

Proof. Let $F \subseteq M$ be a *G-part* of M (cf. Proposition 2.5, (i)). Write $S_x[p] \subseteq M$ for the "$S_x[p]$" of Lemma 2.1 in the case where we take the "(M, Γ, x)" of Lemma 2.1 to be $(M, \Gamma_K[M], x)$. Then it follows from condition (2) of Definition 2.3, (ii), together with the definition of S_x, that $S_x[p] \subseteq F[p]$. Write $N \subseteq (F[p] \subseteq) M$ for the Γ_K-submodule of M generated by $S_x[p]$. Then it follows immediately from condition (1) of Definition 2.3, (ii), that N satisfies condition (1) of Lemma 2.6. Moreover, since (it follows from conditions (1) and (2) of Definition 2.3, (ii) that) (b) of Lemma 2.1 (in the case where we take the "(M, Γ, x)" of Lemma 2.1 to be $(M, \Gamma_K[M], x)$) holds, it follows from Lemma 2.1, (ii), that N satisfies condition (2) of Lemma 2.6. This completes the proof of Lemma 2.6. □

The following result is the main result of the present §2:

Lemma 2.7. *Let M be a finite Γ_K-module* of type S. *Suppose that there exists a* weak Γ_K-invariant *of M which generates the Γ_K-module M. Then the action of Γ_K on $p \cdot M_p$ is* trivial.

Proof. Let $F \subseteq M$ be a *G-part* of M (cf. Proposition 2.5, (i)) and $N \subseteq M$ a Γ_K-submodule of M which satisfies two conditions (1), (2) of Lemma 2.6. Then let us observe that both F_p and N are *of type G* and *annihilated* by a power of p. In particular, it follows from Remark 2.3.1, (iii), that there exist a finite flat commutative group scheme G_{F_p} (respectively, G_N) over W which is *annihilated* by a power of p and a Γ_K-equivariant isomorphism $F_p \xrightarrow{\sim} G_{F_p}(\overline{K})$ (respectively, $N \xrightarrow{\sim} G_N(\overline{K})$).

Now I claim that the following assertion holds:

> Claim 2.7.A: The inclusion $K_{G_{F_p}} \subseteq K_{G_N}$ (cf. the notation introduced in the discussion preceding Definition 1.2) holds. (In other words, the natural surjection $\Gamma_K \twoheadrightarrow \Gamma_K[F_p]$ *factors* through $\Gamma_K \twoheadrightarrow \Gamma_K[N]$.)

Indeed, since $F_p \subseteq M$, it is immediate that the natural surjection $\Gamma_K \twoheadrightarrow \Gamma_K[F_p]$ *factors* through $\Gamma_K \twoheadrightarrow \Gamma_K[M]$. Thus, it follows from condition (2) of Lemma 2.6 that Claim 2.7.A holds. This completes the proof of Claim 2.7.A.

Next, I claim that the following assertion holds:

> Claim 2.7.B: The module F_p is *annihilated* by p.

Indeed, let us first observe that it follows from Lemma 2.4, (ii), together with condition (3) of Definition 2.3, (ii), that G_{F_p} is *connected*, i.e., that $G_{F_p} = G^{\circ}_{F_p}$. Thus, it follows from Lemma 1.9, together with Claim 2.7.A, that

$$\mathrm{lv}_W(F_p \otimes_{\mathbb{Z}_p} W) = \mathrm{lv}(G_{F_p}) = \mathrm{lv}(G^{\circ}_{F_p}) \leq \mathrm{lv}(G_N) = \mathrm{lv}_W(N \otimes_{\mathbb{Z}_p} W).$$

Thus, since $\mathrm{lv}_W(N \otimes_{\mathbb{Z}_p} W) = 1$ (cf. condition (1) of Lemma 2.6), Claim 2.7.B holds. This completes the proof of Claim 2.7.B.

It follows from Claim 2.7.B that the natural surjection $M_p \twoheadrightarrow p \cdot M_p$ *factors* through $M_p \twoheadrightarrow M_p/F_p$. On the other hand, it follows from condition (2) of Definition 2.3, (ii), that the action of Γ_K on M_p/F_p is *trivial*. Thus, we conclude that the action of Γ_K on $p \cdot M_p$ is *trivial*, as desired. This completes the proof of Lemma 2.7. □

3. Ramified Torsion Points on Curves

In the present §3, we prove the main result of the present paper (cf. Theorem 3.4 below), which is closely related to a conjecture due to *R. Coleman* concerning the *ramification of torsion points* (cf. Remark 3.4.1 below). Moreover, by means of the main result, we study the geometry of curves which admit *ramified torsion points* (cf. Corollaries 3.6, 3.8 and 3.9 below).

In the present §3, we maintain the notation of §2. Let $g \geq 2$ be an integer and X a curve of genus g over K which has *stable reduction* over W. Write J for the Jacobian variety of X.

Let us first recall the following *well-known* result:

Proposition 3.1. *The abelian variety J over K has* semistable reduction *over W. Moreover, the abelian variety J over K has* good reduction *over W if and only if the dual graph of the special fiber of the stable model of X over W is a* tree.

Proof. This follows from, for instance, [3, §9.2, Example 8], and [3, §9.7, Corollary 2]. □

Definition 3.2. Let $x \in X^{\mathrm{cl}}_{\overline{K}}$ be a closed point of $X_{\overline{K}}$. Then we shall write $\varphi_x \colon X^{\mathrm{cl}}_{\overline{K}} \hookrightarrow J^{\mathrm{cl}}_{\overline{K}}$ for the injection between the sets of closed points determined by the Albanese embedding of X with respect to $x \in X^{\mathrm{cl}}_{\overline{K}}$.

Lemma 3.3. *Let $x_0 \in X(K)$ be a K-rational point of X and $x \in X^{\mathrm{cl}}_{\overline{K}}$ a closed point of $X_{\overline{K}}$. Suppose that $\varphi_{x_0}(x) \in J^{\mathrm{cl}}_{\overline{K}}$ is* torsion. *Write $M \subseteq J^{\mathrm{cl}}_{\overline{K}}$ for the* (*necessarily* finite) *Γ_K-submodule* *of* $J^{\mathrm{cl}}_{\overline{K}}$ *generated by* $\varphi_{x_0}(x) \in J^{\mathrm{cl}}_{\overline{K}}$. *Then the following hold*:

(i) *The* Γ_K*-module* M *is* of type S. *If, moreover, the abelian variety* J *over* K *has* good reduction *over* W, *then the* Γ_K*-module* M *is* of type G.

(ii) *If* (X, x) *is* not exceptional, *then the element* $\varphi_{x_0}(x) \in M$ *is a* weak Γ_K-invariant.

(iii) *The action of* Γ_K *on* $p \cdot M_p$ *is* trivial.

Proof. Assertion (i) follows from Proposition 3.1, together with the various definitions involved. Next, we verify assertion (ii). Let $\gamma, \delta \in \Gamma_K$ be such that $(1-\gamma)^2\delta \cdot \varphi_{x_0}(x) = 0$. Then since x_0 is *K-rational*, our assumption $(1-\gamma)^2\delta \cdot \varphi_{x_0}(x) = 0$ implies that $[\delta \cdot x] + [\gamma^2 \cdot \delta \cdot x] = 2[\gamma \cdot \delta \cdot x]$, where we write "$[-]$" for the prime divisor determined by "$(-)$". In particular, since (X, x), hence also $(X, \delta \cdot x)$, is *not exceptional*, we obtain that $\delta \cdot x = \gamma \cdot \delta \cdot x$, i.e., that $(1-\gamma)\delta \cdot \varphi_{x_0}(x) = 0$. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). If (X, x) is *not exceptional*, then assertion (iii) follows from Lemma 2.7, together with assertions (i), (ii). If (X, x) is *exceptional*, then it follows from [13, Proposition 3.1, (i)], that the action of Γ_K on $2 \cdot M$ is *trivial*. Thus, the action of Γ_K on $M_p \subseteq 2 \cdot M$, hence also on $p \cdot M_p$, is *trivial*. This completes the proof of assertion (iii), hence also of Lemma 3.3. □

The following result is the main result of the present paper:

Theorem 3.4. *In the notation introduced at the beginning of* §3, *let* $x_0 \in X(K)$ *be a* K*-rational point of* X *and* $x \in X^{\mathrm{cl}}_{\overline{K}}$ *a closed point of* $X_{\overline{K}}$. *Suppose that* $\varphi_{x_0}(x) \in J^{\mathrm{cl}}_{\overline{K}}$ *is* torsion. *Then the following hold*:

(i) *The residue field of* J *at* $p \cdot \varphi_{x_0}(x) \in J^{\mathrm{cl}}_{\overline{K}}$ *is* at most tamely ramified *over* K.

(ii) *Suppose, moreover, that one of the following two conditions is satisfied*:

(a) *There exists an integer* $n \geq 1$ *such that* $p^n \cdot \varphi_{x_0}(x) \in J^{\mathrm{cl}}_{\overline{K}}$ *is* K-rational.

(b) *The abelian variety* J *over* K *has* good reduction *over* W (*cf.* Proposition 3.1).

Then $p \cdot \varphi_{x_0}(x) \in J^{\mathrm{cl}}_{\overline{K}}$ *is* K-rational.

Proof. Assertion (i) follows immediately from Lemma 3.3, (i), (iii), together with Proposition 2.5, (ii). Assertion (ii) in the case where the condition (a) is satisfied follows immediately from Lemma 3.3, (iii). Assertion (ii) in the case where the condition (b) is satisfied follows immediately from Lemma 3.3, (i), (iii), together with Remark 2.3.1, (i), (ii). This completes the proof of Theorem 3.4. □

Remark 3.4.1.

(i) *R. Coleman* stated, in [4], a conjecture concerning the *ramification of torsion points* on a curve which satisfies certain conditions. Let us recall the statement of (a slightly stronger version of) the conjecture as follows (cf. [4, Conjecture B]):

> In the notation introduced at the beginning of §3, let $x_0 \in X(K)$ be a K-rational point of X and $x \in X^{\mathrm{cl}}_{\overline{K}}$ a closed point of $X_{\overline{K}}$. Suppose that $\varphi_{x_0}(x) \in J^{\mathrm{cl}}_{\overline{K}}$ is *torsion*. Suppose, moreover, that the following two conditions are

satisfied:

(1) The inequality $p \geq 5$ holds.

(2) The curve X, hence also the abelian variety J, over K has *good reduction* over W.

Then $\varphi_{x_0}(x) \in J^{\mathrm{cl}}_{\overline{K}}$ is *K-rational.*

As we discussed in Introduction of the present paper, Coleman himself proved the conjecture in the case where the given curve X satisfies a further assumption.

(ii) Observe that, in the situation of the conjecture of (i), we conclude from Theorem 3.4 that

at least $p \cdot \varphi_{x_0}(x) \in J^{\mathrm{cl}}_{\overline{K}}$ is *K-rational*

(cf. Theorem 3.4, (ii), in the case where the condition (b) is satisfied). Unfortunately, however, at the time of this writing, the author cannot derive a solution of the conjecture of (i) from Theorem 3.4.

Remark 3.4.2. One may give a *relatively simple alternative proof* of Theorem 3.4, (ii), in the case where the condition (b) is satisfied which does *not rely* on the main result of §2, i.e., Lemma 2.7. Such a proof is as follows:

Let us start with the notation of Lemma 3.3. Suppose that the condition (b) in the statement of Theorem 3.4, (ii), is satisfied (which thus implies that the finite Γ_K-modules $M_p \subseteq M$ and $A \overset{\mathrm{def}}{=} J(\overline{K})[p]$ are *of type G*). Let G_{M_p} (respectively, G_A) be a finite flat commutative group scheme over W which (is necessarily *annihilated* by a power of p and) admits a Γ_K-equivariant isomorphism $M_p \overset{\sim}{\to} G_{M_p}(\overline{K})$ (respectively, $A \overset{\sim}{\to} G_A(\overline{K})$) (cf. Remark 2.3.1, (iii)). Thus, we have an exact sequence of finite flat commutative group schemes over W

$$0 \longrightarrow G^{\circ}_{M_p} \longrightarrow G_{M_p} \longrightarrow G^{\text{ét}}_{M_p} \longrightarrow 0,$$

where $G^{\circ}_{M_p} \subseteq G_{M_p}$ is *connected*, and $G^{\text{ét}}_{M_p}$ is *étale* over W — and two finite Galois extensions $K_{G_{M_p}}$ and K_{G_A} of K contained in $\overline{K}$ (cf. the notation introduced in the discussion preceding Definition 1.2).

If (X, x) is *exceptional*, then it follows from [13], Proposition 3.1, (i) (cf. also the proof of Lemma 3.3, (iii), in the case where (X, x) is *exceptional*), that the action of Γ_K on $p \cdot M_p$ is *trivial*. In particular, we conclude from Remark 2.3.1, (i), (ii), that $p \cdot \phi_{x_0}(x)$ is *K-rational*, as desired.

Suppose that (X, x) is *not exceptional*. Write $\phi_{x_0}(x) = a_p + a_{\neq p}$ for the representation of $\phi_{x_0}(x) \in M$ with respect to the natural direct decomposition $M = M_p \oplus M_{\neq p}$. Now let us prove the *assertion* that

the element $a_p \in M_p$ is *fixed* by the action of $\Gamma_{K_{G_A}} \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K_{G_A}) \subseteq \Gamma_K$.

To this end, assume that $a_p \notin M^{\Gamma_{K_{G_A}}}$. Then it follows from [12, Lemma 2.4], that there exists $\gamma \in \Gamma_{K_{G_A}}$ such that $(1-\gamma)a_p \in A \setminus \{0\}$, which thus implies that $(1-\gamma)^2 a_p = 0$. Thus, since $(1-\gamma)a_{\neq p} = 0$ (cf. Remark 2.3.1, (i), (ii)), we obtain that $(1-\gamma)^2\phi_{x_0}(x) = 0$. In particular, it follows from Lemma 3.3, (ii), that $(1-\gamma)\phi_{x_0}(x) = 0$, which thus implies that $(1-\gamma)a_p = 0$. This *contradicts* the above fact that $(1-\gamma)a_p \in A \setminus \{0\}$. This completes the proof of the above *assertion*.

It follows immediately from the above *assertion* that $K_{G_{M_p}} \subseteq K_{G_A}$. Thus, it follows from Lemma 1.9 that $\mathrm{lv}(G^{\circ}_{M_p}) \leq \mathrm{lv}(G_A) = 1$. In particular, the natural surjection $M_p \twoheadrightarrow p \cdot M_p$ *factors* through $M_p \twoheadrightarrow M_p/G^{\circ}_{M_p}(\overline{K}) \xrightarrow{\sim} G^{\text{ét}}_{M_p}(\overline{K})$. In particular, since the action of Γ_K on $G^{\text{ét}}_{M_p}(\overline{K})$ is *trivial* (cf. Lemma 2.4, (i)), the action of Γ_K on $p \cdot M_p$ is *trivial*. Thus, we conclude from Remark 2.3.1, (i), (ii), that $p \cdot \phi_{x_0}(x)$ is *K-rational*, as desired.

Remark 3.4.3. Note that the proof of the main result of the present paper may be regarded as a *refinement* (in the *absolutely unramified* case) of an argument of [12] given by *D. Rössler*.

Definition 3.5.

(i) We shall say that a closed point $x \in X^{\mathrm{cl}}_{\overline{K}}$ of $X_{\overline{K}}$ is a *ramified torsion point* (respectively, *wildly ramified torsion point*) if the closed point $x \in X^{\mathrm{cl}}_{\overline{K}}$ is not K-rational (respectively, not K^{tm}-rational), and, moreover, there exists a K-rational point $x_0 \in X(K)$ of X such that $\varphi_{x_0}(x) \in J^{\mathrm{cl}}_{\overline{K}}$ is torsion.

(ii) We shall refer to an equivalence class with respect to the following equivalence relation "$\sim$" on $X^{\mathrm{cl}}_{\overline{K}}$ as a *torsion packet* on X: For $x, y \in X^{\mathrm{cl}}_{\overline{K}}$, write $x \sim y$ if $\varphi_x(y)$ $(= -\varphi_y(x))$ $\in J^{\mathrm{cl}}_{\overline{K}}$ is torsion.

(iii) We shall say that a torsion packet is a *ramified torsion packet* (respectively, *wildly ramified torsion packet*) if the torsion packet contains a ramified (respectively, wildly ramified) torsion point.

Remark 3.5.1. Thus, the conjecture of Coleman discussed in Remark 3.4.1, (i), is *equivalent* to the following assertion:

> In the notation introduced at the beginning of §3, suppose that the following two conditions are satisfied:
>
> (1) The inequality $p \geq 5$ holds.
>
> (2) The curve X, hence also the abelian variety J, over K has *good reduction* over W.
>
> Then there is *no ramified torsion point* (cf. Definition 3.5, (i)) on $X_{\overline{K}}$, or, equivalently, there is *no ramified torsion packet* (cf. Definition 3.5, (iii)) on X.

Corollary 3.6. *In the notation introduced at the beginning of* §3, *let $x \in X^{\mathrm{cl}}_{\overline{K}}$ be a* ramified torsion *point on $X_{\overline{K}}$. Suppose that one of the following two conditions is satisfied*:

(1) *The abelian variety J over K has* good reduction *over W.*

(2) *The closed point $x \in X^{\mathrm{cl}}_{\overline{K}}$ is a* wildly ramified torsion *point.*

Then the following hold:

(i) *Suppose that condition* (1) (*respectively,* (2)) *is satisfied. Let γ be an element of* Γ_K (*respectively, of the uniquely determined p-Sylow subgroup of* Γ_K). *Then the element $\varphi_{\gamma\cdot x}(x)$ is* annihilated *by p.*

(ii) *The prime number p is a* Weierstrass non-gap *at $x \in X^{\mathrm{cl}}_{\overline{K}}$. In particular, if $g \geq p$, then $x \in X^{\mathrm{cl}}_{\overline{K}}$ is a* Weierstrass point *of $X_{\overline{K}}$.*

(iii) *There is a finite morphism $X_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$* of degree *$p$ over $\overline{K}$ which is* totally ramified *at $x \in X^{\mathrm{cl}}_{\overline{K}}$. In particular, the curve $X_{\overline{K}}$ over $\overline{K}$ is* of gonality *$\leq p$.*

(iv) *If $g \geq p$, then the curve $X_{\overline{K}}$ over $\overline{K}$ is* not hyperelliptic.

Proof. First, we verify assertion (i). It follows immediately from the various definitions involved that there exists a K-rational point $x_0 \in X(K)$ of X such that $\varphi_{x_0}(x) \in J^{\mathrm{cl}}_{\overline{K}}$ is *torsion*. Thus, since (one verifies immediately that) $(1-\gamma)\varphi_{x_0}(x) = \varphi_{\gamma\cdot x}(x)$, assertion (i) follows from Theorem 3.4. This completes the proof of assertion (i).

Next, we verify assertion (ii). Suppose that condition (1) (respectively, (2)) is satisfied. Then it follows immediately from the various definitions involved that there exists an element γ of Γ_K (respectively, of the uniquely determined p-Sylow subgroup of Γ_K) such that $\varphi_{\gamma\cdot x}(x) \neq 0$, which thus implies (cf. assertion (i)) that $\varphi_{\gamma\cdot x}(x)$ is *of order p*. Thus, we conclude immediately from the various definitions involved that p is a *Weierstrass non-gap* at $x \in X^{\mathrm{cl}}_{\overline{K}}$. This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertion (ii).

Finally, we verify assertion (iv). It follows from assertion (iii) that there exists a finite morphism $X_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$ *of degree p* over $\overline{K}$. Thus, since (we have assumed that) $g \geq p \geq 3$, it follows from Lemma 3.7, (i), below that there is *no* finite morphism $X_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$ *of degree* 2 over $\overline{K}$. This completes the proof of assertion (iv), hence also of Corollary 3.6. □

Remark 3.6.1. Note that, in Corollary 3.6, (iv), one cannot remove the hypothesis "$g \geq p$". Indeed, if $p = 3$, then the *hyperelliptic* modular curve "$X_1(13)$" (*of genus* 2) over K has *good reduction* over W and admits a *ramified torsion* point (cf. [2, Appendix], the discussion following Conjecture 6.4).

Lemma 3.7. *Let $d \geq 1$ be an integer and ϕ, ψ: $X_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$ finite morphisms over $\overline{K}$. Suppose that ϕ is* of degree *p, that ψ is* of degree *d, and that $g > (p-1)(d-1)$. Then the following hold*:

(i) *The integer d is* contained *in $p\mathbb{Z}$. In particular, we obtain that $d \geq p$.*

(ii) *Suppose that $d = p$. Then the invertible sheaf $\phi^*\mathcal{O}_{\mathbb{P}^1_{\overline{K}}}(1)$ on $X_{\overline{K}}$ is* isomorphic *to the invertible sheaf $\psi^*\mathcal{O}_{\mathbb{P}^1_{\overline{K}}}(1)$ on $X_{\overline{K}}$.*

Proof. These assertions follow immediately from the *Castelnuovo-Severi inequality* (cf., e.g., [1, Chapter VIII, Exercise C-1]). □

Corollary 3.8. *In the situation of* Corollary 3.6, *suppose, moreover, that $g > (p-1)^2$. Then the following hold*:

(i) *The curve $X_{\overline{K}}$ over $\overline{K}$ is* of gonality *p.*

(ii) *Let ϕ: $X_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$ be a finite morphism* of degree *p over $\overline{K}$ (cf. (i)). Then ϕ is* totally ramified *at $x \in X^{\mathrm{cl}}_{\overline{K}}$.*

(iii) *If condition* (1) (*respectively,* (2)) *in the statement of* Corollary 3.6 *is satisfied, then the curve X has* exactly one ramified (*respectively,* wildly ramified) torsion packet.

Proof. Assertion (i) (respectively, (ii)) follows immediately from Corollary 3.6, (iii), together with Lemma 3.7, (i) (respectively, (ii)). Assertion (iii) follows immediately from assertions (i), (ii). This completes the proof of Corollary 3.8. □

Corollary 3.9. *In the situation of* Corollary 3.6*, let us suppose that condition* (1) (*respectively,* (2)) *in the statement of* Corollary 3.6 *is satisfied. Write d_x* (> 1) *for the extension degree over K of the residue field of X at $x \in X^{\mathrm{cl}}_{\overline{K}}$, $d_{x,p}$ for the "p-part" of d_x, i.e., $d_{x,p} \overset{\mathrm{def}}{=} \sharp(\mathbb{Z}_p/d_x)$, and $D_x \overset{\mathrm{def}}{=} d_x$ (respectively, $\overset{\mathrm{def}}{=} d_{x,p}$). Then the following hold*:

(i) *The inequality $D_x \leq g(p-1)^2$ holds.*

(ii) *Suppose, moreover, that $g > (p-1)^2$. If condition* (1) (*respectively,* (2)) *in the statement of* Corollary 3.6 *is satisfied, then the number of* ramified (*respectively,* wildly ramified) torsion *points on $X_{\overline{K}}$ is $\leq 2 + 2g/(p-1)$. In particular, the inequality $D_x \leq 2 + 2g/(p-1)$ holds.*

Proof. First, we verify assertion (i). Write $(x \in) \{x_1, x_2, \ldots, x_{D_x}\} \subseteq X^{\mathrm{cl}}_{\overline{K}}$ for the orbit of $x \in X^{\mathrm{cl}}_{\overline{K}}$ by the action of Γ_K (respectively, the uniquely determined p-Sylow subgroup of Γ_K). Then it follows from Corollary 3.6, (i), that, for every $i \in \{2, \ldots, D_x\}$, the element $\varphi_{x_1}(x_i) \in J^{\mathrm{cl}}_{\overline{K}}$ is *of order p*, which thus implies that $(1-p) \cdot \varphi_{x_1}(x_i) = \varphi_{x_1}(x_i)$. In particular, we conclude that

$$\{\varphi_{x_1}(x_1), \varphi_{x_1}(x_2), \ldots, \varphi_{x_1}(x_{D_x})\} \subseteq \varphi_{x_1}(X) \cap (1-p) \cdot \varphi_{x_1}(X).$$

Thus, it follows from [5, Lemma 4.1], that $D_x \leq g(1-p)^2$. This completes the proof of assertion (i).

Next, we verify assertion (ii). If condition (1) (respectively, (2)) in the statement of Corollary 3.6 is satisfied, then write N for the number of *ramified* (respectively, *wildly ramified*) *torsion* points on $X_{\overline{K}}$. Let $\phi \colon X_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$ be a finite morphism *of degree p* over $\overline{K}$ (cf. Corollary 3.8, (i)). Then, by applying Corollary 3.8, (ii), and the *Riemann-Hurwitz formula* to ϕ, we conclude that $2g - 2 \geq -2p + (p-1)N$, which thus implies that $N \leq 2 + 2g/(p-1)$. This completes the proof of Corollary 3.9. □

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References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris: Geometry of algebraic curves. Vol. I, Grundlehren der Mathematischen Wissenschaften **267**. Springer-Verlag, New York, 1985.

[2] M. Baker: *Torsion points on modular curves*, Ph.D. thesis, University of California, Berkeley, 1999.

[3] S. Bosch, W. Lütkebohmert and M. Raynaud: Néron models, Ergebnisse der Mathematik und ihrer Grenzgebiete (**3**), **21**. Springer-Verlag, Berlin, 1990.

[4] R.F. Coleman: *Ramified torsion points on curves*, Duke Math. J. **54** (1987), 615–640.

[5] R.F. Coleman, B. Kaskel and K.A. Ribet: Torsion points on $X_0(N)$, *Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996)*, 27–49, Proc. Sympos. Pure Math., **66**, Part **1**, Amer. Math. Soc., Providence, RI, 1999.

[6] G. Faltings and C.-L. Chai: Degeneration of abelian varieties. With an appendix by David Mumford, Ergebnisse der Mathematik und ihrer Grenzgebiete (**3**), **22**. Springer-Verlag, Berlin, 1990.

[7] J.-M. Fontaine: *Formes différentielles et modules de Tate des variétés abéliennes sur les corps locaux*, Invent. Math. **65** (1981/82), 379–409.

[8] J.-M. Fontaine: *Il n'y a pas de variété abélienne sur* $\mathbb{Z}$, Invent. Math. **81** (1985), 515–538.

[9] Y. Hoshi: *Tame-blind extension of morphisms of truncated Barsotti-Tate group schemes*, J. Math. Sci. Univ. Tokyo **16** (2009), 23–54.

[10] J.S. Milne: Arithmetic duality theorems, Second edition. BookSurge, LLC, Charleston, SC, 2006.

[11] M. Raynaud: *Schémas en groupes de type* $(p,\ldots,p)$, Bull. Soc. Math. France **102** (1974), 241–280.

[12] D. Rössler: *A note on the ramification of torsion points lying on curves of genus at least two*, J. Théor. Nombres Bordeaux **22** (2010), 475–481.

[13] A. Tamagawa: *Ramification of torsion points on curves with ordinary semistable Jacobian varieties*, Duke Math. J. **106** (2001), 281–319.

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