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EXTRINSIC CIRCULAR TRAJECTORIES ON GEODESIC SPHERES IN A COMPLEX PROJECTIVE SPACE

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Abstract

We say a trajectory for a Sasakian magnetic field on a geodesic sphere in a complex projective space to be extrinsic circular if it can be seen as a circle in the ambient space. We study how the moduli space of extrinsic circular trajectories behaves in the moduli space of all circles in the ambient complex projective space. As an application we characterize the geodesic sphere of special radius which lies on the boundary position of the family of Berger spheres among all geodesic spheres and that has a characteristic properties from the viewpoint of lengths of circles.

1. Introduction

A smooth curve σ parameterized by its arclength on a Riemannian manifold N is said to be a *circle* if there is a nonnegative constant k and a field Y of unit tangent vectors along σ satisfying the system of differential equations $\nabla_{\dot\sigma}\dot\sigma = kY$, $\nabla_{\dot\sigma}Y = -k\dot\sigma$. This constant k is said to be the *geodesic curvature* of σ, and $\{\dot\sigma, Y\}$ to be its Frenet frame. Since circles of null geodesic curvature are geodesics, from the viewpoint of Frenet formula, there is no doubt that circles are simplest curve next to geodesics. On a real space form, which is one of a standard sphere, a Euclidean space and a real hyperbolic space, lengths of closed circles depend smoothly on their geodesic curvatures. But the situation is different on a complex projective space $\mathbb{C}P^n$. When we consider lengths of closed circles on $\mathbb{C}P^n$, circles whose Frenet frame forms a complex line at each point have different properties from others ([2] and see §2). In order to explain this, in [4] the second author took trajectories on geodesic spheres in $\mathbb{C}P^n$, which are curves closely related with almost contact metric structures on these geodesic spheres. He considered them as curves in $\mathbb{C}P^n$, and showed that every circles on $\mathbb{C}P^n$ is obtained from some trajectory on some geodesic sphere.

In this paper we refine the study on trajectories which can be seen as circles in $\mathbb{C}P^n$ given in [4]. We investigate how the moduli space, the set of all congruence classes, of these trajectories on a given geodesic sphere is contained in the moduli space of circles in $\mathbb{C}P^n$. We find that circles of geodesic curvature $\sqrt{2c}/4$ in $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c have a singular property compared with other circles from the viewpoint of ex-

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trinsic shapes of trajectories. These circles are also obtained as images of geodesics through a parallel isometric embedding $(S^1 \times S^{n-1})/\sim \to \mathbb{C}P^n$ given by Naitoh [14], and are characteristic circles from the viewpoint of length spectrum of circles (see §2 and [6]). As an application of our study we characterize a geodesic sphere $G(r_0)$ of special radius r_0 satisfying $\cot(\sqrt{c}\, r_0/2) = 1/\sqrt{2}$ among real hypersurfaces in $\mathbb{C}P^n(c)$. It contains all these characteristic circles as trajectories. We note that geodesic spheres of radius satisfying $\cot(\sqrt{c}\, r/2) < 1/\sqrt{2}$ are so-called Berger spheres (see [17]). Sectional curvatures of the geodesic sphere $G(r_0)$ lie in the interval $[K/9, K]$ with $K = 9c/8$, and the length of its shortest closed geodesic is $2\pi/\sqrt{K}$. This geodesic can be seen as a circle in $\mathbb{C}P^n(c)$. Thus our result shows that not only the shortest closed geodesics but also trajectories on this geodesic sphere which can be seen as circles in $\mathbb{C}P^n$ have characteristic properties.

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2. Circles on a complex projective space

In order to explain the background of our study, we shall start by recalling the moduli space of circles on a complex projective space $\mathbb{C}P^n$. For a circle σ on $\mathbb{C}P^n$ which satisfies $\nabla_{\dot\sigma}\dot\sigma = k_\sigma Y$, $\nabla_{\dot\sigma} Y = -k_\sigma \dot\sigma$, we set $\tau_\sigma = \langle \dot\sigma, JY\rangle$ with the complex structure J on $\mathbb{C}P^n$, and call it the *complex torsion* of σ. Since J is parallel, we see that τ_σ is constant along σ. We say two smooth curves σ_1, σ_2 on a Riemannian manifold N which are parameterized by their arclengths to be *congruent* to each other if there is an isometry φ of N and a constant t_c satisfying $\varphi \circ \sigma_1(t) = \sigma_2(t + t_c)$ for all t. It is known that circles on $\mathbb{C}P^n$ are classified into congruence classes by their geodesic curvatures and complex torsions (see [13]). Hence, the moduli space $\mathcal{M}(\mathbb{C}P^n)$ of circles of positive geodesic curvature on $\mathbb{C}P^n$, which is the set of all congruence classes of such circles, is set theoretically congruent to the band $(0, \infty)\times[0, 1]$ in $\mathbb{R}^2$.

A smooth curve σ parameterized by its arclength is said to be closed if there is a positive constant t_0 satisfying $\sigma(t + t_0) = \sigma(t)$ for all t. The minimum positive t_0 with this property is said to be the *length* of σ. For an open curve, a curve which is not closed, we consider that its length is infinity. Since two smooth curves which are congruent to each other have the same length, we can define a function $\mathcal{L} : \mathcal{M}(\mathbb{C}P^n(c)) \to (0, \infty]$ which shows lengths of circles. On $\mathbb{C}P^n(c)$, circles of geodesic curvature $\sqrt{2c}/4$ have characteristic properties. For $k > 0$ and τ with $0 \le \tau \le 1$, we denote by $[\sigma(k, \tau)]$ the congruence class of circles of geodesic curvature k and of complex torsion τ on $\mathbb{C}P^n(c)$. We put $\mathcal{M}_k = \{[\sigma(k, \tau)] \;\big|\; 0 \le \tau < 1\}$ $(\subset \mathcal{M}(\mathbb{C}P^n(c)))$ for a positive k. Then we have an injective map $\Phi_k : \mathcal{M}_k \to \mathcal{M}_{\sqrt{2c}/4}$ given as $[\gamma(k, \tau)] \mapsto [\gamma(\sqrt{2c}/4, 3\sqrt{3}ck\tau(4k^2 + c)^{-3/2})]$. This satisfies $\mathcal{L} = C_k \cdot \mathcal{L} \circ \Phi_k$ on $\mathcal{M}_k$ with $C_k = \sqrt{3c/\{2(4k^2 + c)\}}$ (see [2]). Thus we have a lamination on $\mathcal{M}(\mathbb{C}P^n(c))$ each of whose leaf is maximal with respect to the smooth property of $\mathcal{L}$ (see Fig. 1). Every leaf crosses to the set $\mathcal{M}_{\sqrt{2c}/4}$ of congruence classes of circles of geodesic curvature $\sqrt{2c}/4$.

3. Trajectories for Sasakian magnetic fields

Let M be a real hypersurface in a Kähler manifold $\widetilde{M}$. On this hypersurface we have an almost contact metric structure induced by the complex structure J on $\widetilde{M}$. If we denote by $\mathcal{N}$ a (local) unit normal vector field on M in $\widetilde{M}$, this structure is a quartet $(\eta, \xi, \phi, \langle\ ,\ \rangle)$

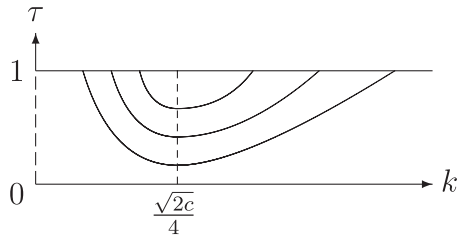

Fig. 1. Lamination on the moduli space of circles on $\mathbb{C}P^n(c)$

of a vector field ξ defined by $\xi = -J\mathcal{N}$, a 1-form η given by $\eta(v) = \langle v, \xi\rangle$, a $(1,1)$-tensor ϕ defined by $\phi(v) = Jv - \eta(v)\mathcal{N}$ and the metric $\langle\ ,\ \rangle$ induced by the metric on $\widetilde{M}$. These ξ and ϕ are called the characteristic vector field and the characteristic tensor of M, respectively. We define a 2-form $\mathbb{F}_\phi$ associated with this structure by $\mathbb{F}_\phi(v,w) = \langle v, \phi(w)\rangle$. One can easily find that it is a closed form (see [7]). Generally, a closed 2-form on a Riemannian manifold is said to be a magnetic field because it can be regarded as a generalization of a static magnetic field on a Euclidean 3-space $\mathbb{R}^3$ (see [15], for example). We therefore say that a constant multiple $\mathbb{F}_\kappa = \kappa\mathbb{F}_\phi$ ($\kappa \in \mathbb{R}$) of the form $\mathbb{F}_\phi$, which is also closed, a *Sasakian magnetic field* on M.

A smooth curve γ on M which is parameterized by its arclength is said to be a *trajectory* for $\mathbb{F}_\kappa$ if it satisfies the differential equation $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi(\dot\gamma)$. When $\kappa = 0$, which is the case that there is no influence of magnetic fields, trajectories are geodesics. Therefore, we may say that trajectories are extended objects of geodesics. To study properties of trajectories, we put $\rho_\gamma = \langle\dot\gamma, \xi_\gamma\rangle$, and call this function along a trajectory γ its *structure torsion*. We here recall Gauss and Weingarten formulae. If we denote by $\widetilde{\nabla}$ the Riemannian connection on $\widetilde{M}$, they are given as

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle A_M X, Y\rangle\mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_X\mathcal{N} = -A_M X$$

for vector fields X, Y tangent to M. Here A_M denotes the shape operator of M in $\widetilde{M}$ with respect to $\mathcal{N}$. By these formulae we have $\nabla_X\xi = \phi A_M X$, hence we get

$$\rho_\gamma' = \langle\kappa\phi\dot\gamma, \xi_\gamma\rangle + \langle\dot\gamma, \phi A_M\dot\gamma\rangle = \frac{1}{2}\langle\dot\gamma, (\phi A_M - A_M\phi)\dot\gamma\rangle,$$

because A_M is symmetric and ϕ is skew-symmetric. Therefore the structure torsion for each trajectory is a constant function if A_M and ϕ are simultaneously diagonalizable.

We denote by $\iota : M \to \widetilde{M}$ an isometric immersion. For a curve γ on M we have a curve $\iota\circ\gamma$ on $\widetilde{M}$. We call this the *extrinsic shape* of γ, and denote it also by γ for the sake of simplicity. In this paper we study extrinsic shapes of trajectories on a geodesic sphere $G(r)$ of radius r ($0 < r < \pi/\sqrt{c}$) in $\mathbb{C}P^n(c)$. It is well known that the characteristic vector ξ_p at each point $p \in M = G(r)$ is a principal curvature vector associated with $\delta_M = \sqrt{c}\cot(\sqrt{c}\,r)$, and that each tangent vector orthogonal to ξ_p is a principal curvature vector associated with $\lambda_M = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$. In particular, its shape operator and its characteristic tensor are simultaneously diagonalizable, hence each trajectory has constant structure torsion. By applying Gauss and Weingarten formulae, for a trajectory γ for $\mathbb{F}_\kappa$ on $G(r)$ we have

$$\begin{aligned}\widetilde{\nabla}_{\dot\gamma}\dot\gamma &= \kappa\phi\dot\gamma+\{\lambda_M(1-\rho_\gamma^2)+\delta_M\rho_\gamma^2\}\mathcal{N},\\ \widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma &= \kappa J\widetilde{\nabla}_{\dot\gamma}\dot\gamma+\{\lambda_M(1-\rho_\gamma^2)+\delta_M\rho_\gamma^2-\kappa\rho_\gamma\}\widetilde{\nabla}_{\dot\gamma}\mathcal{N}\\ &= -\{\kappa^2(1-\rho_\gamma^2)+\{\lambda_M+(\delta_M-\lambda_M)\rho_\gamma^2\}^2\}\dot\gamma\\ &\quad+\{\lambda_M-\kappa\rho_\gamma+(\delta_M-\lambda_M)\rho_\gamma^2\}\{\kappa+(\delta_M-\lambda_M)\rho_\gamma\}(\rho_\gamma\dot\gamma-\xi_\gamma).\end{aligned}$$

We shall call a curve on a real hypersurface *extrinsic circular* if its extrinsic shape is a circle of positive geodesic curvature. In view of the above computation we get the following.

Lemma 1. *A trajectory γ for $\mathbb{F}_\kappa$ on a geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$ is extrinsic circular if and only if it satisfies one of the following conditions*:

1) $\rho_\gamma = \pm 1$,
2) $\lambda_M - \kappa\rho_\gamma + (\delta_M - \lambda_M)\rho_\gamma^2 = 0$,
3) $\kappa + (\delta_M - \lambda_M)\rho_\gamma = 0$.

Corresponding to each case, the geodesic curvature k_γ and the complex torsion τ_γ of the extrinsic shape are as follows:

1) $k_\gamma = \delta_M,\ \tau_\gamma = \mp 1$,
2) $k_\gamma = |\kappa|,\ \tau_\gamma = -\mathrm{sgn}(\kappa)$,
3) $k_\gamma = \sqrt{\kappa^2 - 2\lambda_M\kappa\rho_\gamma + \lambda_M^2},\ \tau_\gamma = (2\kappa\rho_\gamma^2 - \kappa - \lambda_M\rho_\gamma)/k_\gamma$.

A geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$ of radius $r > (2/\sqrt{c})\sin^{-1}(\sqrt{6}/3)$ is known as a Berger sphere. Its sectional curvatures lie in the interval $[\epsilon K, K]$ with some $\epsilon \in (0, 1/9)$, and it has closed geodesics of length less than $2\pi/\sqrt{K}$. It is known that their structure torsions are ± 1. Hence extrinsic shapes of these geodesics are circles by Lemma 1. We should note that every trajectory γ of structure torsion ± 1 for an arbitrary Sasakian magnetic field $\mathbb{F}_\kappa$ is a geodesic because $\phi(\dot\gamma) = 0$.

4. Extrinsic circular trajectories in the moduli space of circles

In the first and the second cases in Lemma 1, the absolute values of complex torsions of extrinsic shapes are 1. We hence study the third case. For a geodesic sphere $M = G(r)$ in $\mathbb{C}P^n$, we denote by $\mathcal{E}(M)$ the moduli space of extrinsic circular trajectories of third type in Lemma 1 for some Sasakian magnetic field on M (for congruency of trajectories for Sasakian magnetic fields, see [3]). Since isometries of M is equivariant, that is for each isometry φ of M there is an isometry $\tilde\varphi$ of $\mathbb{C}P^n$ satisfying $\iota\circ\varphi = \tilde\varphi\circ\iota$ with an isometric immersion $\iota : M \to \mathbb{C}P^n$, we see that extrinsic shapes of two curves on M are congruent to each other in $\mathbb{C}P^n$ if they are congruent to each other in M. Therefore we have a map of $\mathcal{E}(M)$ into the moduli space $\mathcal{M}(\mathbb{C}P^n(c))$ of circles of positive geodesic curvature. From now on we use $\mathcal{E}(M)$ together with its image through this map.

First we study how $\mathcal{E}(M)$ is included in $\mathcal{M}(\mathbb{C}P^n(c))$. As we have $\delta_M - \lambda_M = -c/(4\lambda_M)$, in the case that $4\kappa\lambda_M = c\rho_\gamma$, we see by Lemma 1 that the geodesic curvature and the complex torsion of the extrinsic shape of circular trajectory γ are expressed as

$$k_\gamma = \sqrt{\lambda_M^2 - \frac{c\rho_\gamma^2}{2} + \frac{c^2\rho_\gamma^2}{16\lambda_M^2}}, \qquad \tau_\gamma = \frac{\rho_\gamma(2c\rho_\gamma^2 - c - 4\lambda_M^2)}{4\lambda_M k_\gamma}. \tag{4.1}$$

When $\lambda_M = \sqrt{2c}/4$, we have by (4.1) that $k_\gamma = \lambda_M$ and $\tau_\gamma = 4\rho_\gamma^3 - 3\rho_\gamma$. If we vary the parameter of structure torsion ρ_γ in the interval $[-1, 1]$ we find τ_γ takes all the values in the interval $[-1, 1]$. In this case, we find that the map of $\mathcal{E}(M)$ into $\mathcal{M}(\mathbb{C}P^n(c))$ is three to one on the image. When $\lambda_M \neq \sqrt{2c}/4$, the first equality of (4.1) shows that the map of $\mathcal{E}(M)$ into $\mathcal{M}(\mathbb{C}P^n(c))$ is one to one onto the image, hence we may consider that $\mathcal{E}(M)$ is the subset of $\mathcal{M}(\mathbb{C}P^n(c))$. In this case, by substituting the first equality of (4.1) to the second, we have

$$\tau_\gamma^2 = \frac{(k_\gamma^2 - \lambda_M^2)(32\lambda_M^2 k_\gamma^2 + 4c\lambda_M^2 - c^2)^2}{c(c - 8\lambda_M^2)^3 k_\gamma^2}. \tag{4.2}$$

Here, as $|\rho_\gamma| \leq 1$, the first equality of (4.1) shows that

$$\begin{cases} \lambda_M \leq k_\gamma \leq c/(4\lambda_M) - \lambda_M \ (= -\delta_M), & \text{when } \lambda_M < \sqrt{2c}/4, \\ c/(4\lambda_M) - \lambda_M \leq k_\gamma \leq \lambda_M, & \text{when } \sqrt{2c}/4 < \lambda_M < \sqrt{c}/2, \\ \lambda_M - c/(4\lambda_M) \leq k_\gamma \leq \lambda_M, & \text{when } \lambda_M \geq \sqrt{c}/2. \end{cases}$$

We consider the right hand side of (4.2) as a function $g(K) = g(K; \lambda_M)$ on $K = k_\gamma^2$. We then have

$$\frac{dg}{dK} = \frac{\lambda_M^2(8K - c)(8K + c - 4\lambda_M^2)(32\lambda_M^2 K + 4c\lambda_M^2 - c^2)}{c(c - 8\lambda_M^2)^3 K^2}.$$

Hence we find the following.

1) When $\lambda_M < \sqrt{2c}/4$, the function g is monotone increasing with respect to K in the intervals $[\lambda_M^2, c/8] \cup [(c(c - 4\lambda_M^2)/(32\lambda_M^2), (-\delta_M)^2]$, and is monotone decreasing in the interval $[c/8, (c(c - 4\lambda_M^2)/(32\lambda_M^2)]$. We have

$$g(\lambda_M^2) = g\Big(\frac{c(c - 4\lambda_M^2)}{32\lambda_M^2}\Big) = 0, \quad g\Big(\frac{c}{8}\Big) = g((-\delta_M)^2) = 1.$$

2) When $\sqrt{2c}/4 < \lambda_M < \sqrt{c}/2$, the function g is monotone decreasing with respect to K in the intervals $[(-\delta_M)^2, c(c-4\lambda_M^2)/(32\lambda_M^2)] \cup [c/8, \lambda_M^2]$, and is monotone increasing in the interval $[c(c - 4\lambda_M^2)/(32\lambda_M^2), c/8]$.
3) When $\sqrt{c}/2 \leq \lambda_M < \sqrt{2c}/2$, the function g is monotone decreasing with respect to K in the intervals $[\delta_M^2, (4\lambda_M^2 - c))/8] \cup [c/8, \lambda_M^2]$, and is monotone increasing in the interval $[(4\lambda_M^2 - c))/8, c/8]$. We have

$$g\Big(\frac{4\lambda_M^2 - c}{8}\Big) = \frac{(4\lambda_M^2 + c)^3(4\lambda_M^2 - c)}{c(8\lambda_M^2 - c)^3},$$

which satisfies $0 \leq g((4\lambda_M^2 - c)/8) < 1$ and is monotone increasing with respect to λ_M.
4) When $\lambda_M \geq \sqrt{2c}/2$, the function g is monotone decreasing with respect to K in the interval $[\delta_M^2, \lambda_M^2]$.

Thus for $M = G(r)$, the moduli space $\mathcal{E}(M)$ in $\mathcal{M}(\mathbb{C}P^n(c))$ is like the following figures (Figs. 2 – 5) corresponding to $\lambda_M = (\sqrt{c}/2)\cot(\sqrt{c}\, r/2)$.

Next we study the behavior of $\mathcal{E}(G(r))$ when we vary the radius r. We consider the right hand side of (4.2) as a function $h(\Lambda) = h(\Lambda; k_\gamma)$ on $\Lambda = \lambda_M^2$. We then have

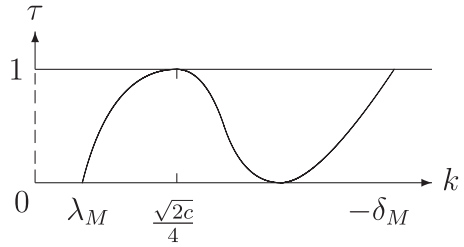

Fig. 2. $\lambda_M < \sqrt{2c}/4$

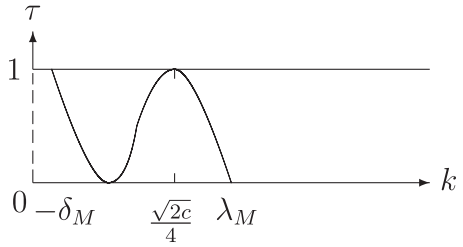

Fig. 3. $\sqrt{2c}/4 < \lambda_M < \sqrt{c}/2$

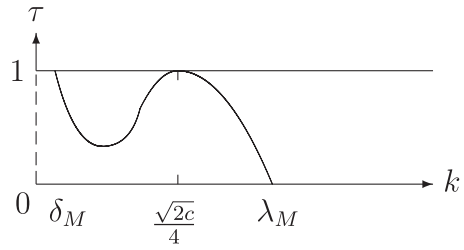

Fig. 4. $\sqrt{c}/2 \le \lambda_M < \sqrt{2c}/2$

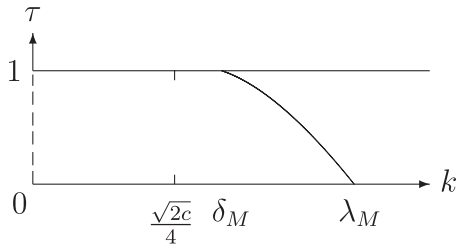

Fig. 5. $\lambda_M \ge \sqrt{2c}/2$

$$\frac{dh}{d\Lambda} = \frac{(8k_\gamma^2 - c)^2(32k_\gamma^2\Lambda + 4c\Lambda - c^2)(4\Lambda + c)}{c(8\Lambda - c)^4 k_\gamma^2}.$$

When $\Lambda > c/4$, as $32k_\gamma^2\Lambda + 4c\Lambda - c^2 > 0$, we have $dh/d\Lambda > 0$, and when $\Lambda < c/4$ we see $dh/d\Lambda$ changes its signature at $\Lambda = c^2/(32k_\gamma^2 + 4c)$. That is, when $\lambda_M > \sqrt{c}/2$, the function $h(\Lambda; k_\gamma)$ is increasing with respect to Λ for each k_γ, and when $\lambda_M < \sqrt{c}/2$ and $\lambda_M \neq \sqrt{2c}/4$, there is k_* satisfying that $h(\Lambda; k_\gamma)$ is decreasing with respect to Λ if $k_\gamma < k_*$ and is increasing with respect to Λ if $k_\gamma > k_*$. As we have $\lambda_M^2 = c^2/(32k_*^2 + 4c)$, we see $k_* > \sqrt{2c}/4$ when $\lambda_M < \sqrt{2c}/4$ and $k_* < \sqrt{2c}/4$ when $\sqrt{2c}/4 < \lambda_M < \sqrt{c}/2$. Since $\cot(\sqrt{c}\, r/2)$ is monotone decreasing with respect to r, if we take two geodesic spheres $M = G(r_1)$, $M' = G(r_2)$ $(r_1 > r_2)$ so that the difference $r_1 - r_2$ of their radii is sufficiently small, their moduli curves $\mathcal{E}(M), \mathcal{E}(M')$ of extrinsic circular trajectories are as Figs. 6 – 9.

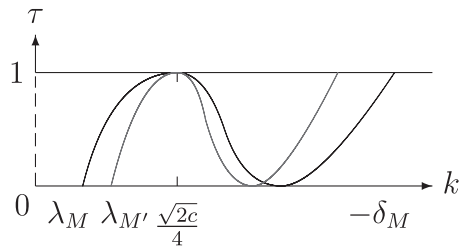

Fig. 6. $\lambda_M < \lambda_{M'} < \sqrt{2c}/4$

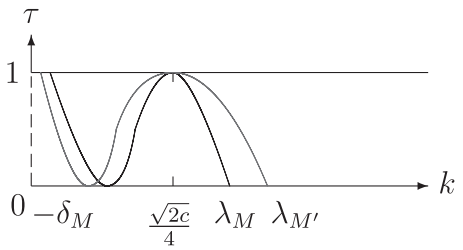

Fig. 7. $\sqrt{2c}/4 < \lambda_M < \lambda_{M'} < \sqrt{c}/2$

We divide the moduli space $\mathcal{M}(\mathbb{C}P^n(c))$ into three subsets $\mathcal{M}_-(\mathbb{C}P^n) \cup \mathcal{M}_{\sqrt{2c}/4} \cup \mathcal{M}_+(\mathbb{C}P^n)$, where $\mathcal{M}_-(\mathbb{C}P^n)$ is the moduli space of circles of geodesic curvature less than $\sqrt{2c}/4$, and $\mathcal{M}_+(\mathbb{C}P^n)$ is the moduli space of circles of geodesic curvature greater than $\sqrt{2c}/4$. For each k with $k < \sqrt{2c}/4$, if we vary λ in the interval $(0, k]$, we see $\tau^2(\lambda, k) = (k^2 - \lambda^2)(32k^2\lambda^2 + 4c\lambda^2 - c^2)^2/\{c(c - 8\lambda^2)^3k^2\}$ varies monotone decreasingly in the interval $[0, 1)$. Hence

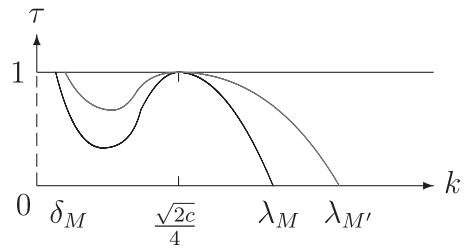

Fig. 8. $\sqrt{c}/2 \le \lambda_M < \lambda_{M'} < \sqrt{2c}/2$

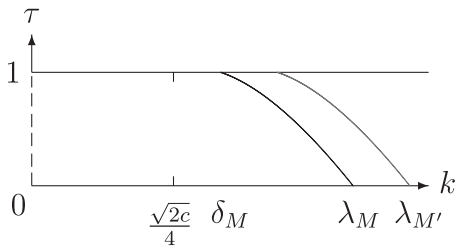

Fig. 9. $\lambda_{M'} > \lambda_M \ge \sqrt{2c}/2$

$\{\mathcal{E}(M) \cap \mathcal{M}_-(\mathbb{C}P^n) \mid \lambda_M < \sqrt{2c}/4\}$ is a foliation of $\mathcal{M}_-(\mathbb{C}P^n) \setminus \{[\gamma(k,1)] \mid k < \sqrt{2c}/4\}$. If we vary λ in the interval $[(-k + \sqrt{k^2 + c})/2, c/\{2\sqrt{8k^2 + c}\}]$, we see $\tau^2(\lambda, k)$ varies monotone decreasingly in the interval $[0, 1]$. Hence $\{\mathcal{E}(M) \cap \mathcal{M}_-(\mathbb{C}P^n) \mid \sqrt{2c}/4 < \lambda_M < \sqrt{2c}/2\}$ covers $\mathcal{M}_-(\mathbb{C}P^n)$. On the other hand, for each k with $k > \sqrt{2c}/4$, if we vary λ in the interval $[k, (k + \sqrt{k^2 + c})/2]$, we see $\tau^2(\lambda, k)$ varies monotone increasingly in the interval $[0, 1]$. Hence $\{\mathcal{E}(M) \cap \mathcal{M}_+(\mathbb{C}P^n) \mid \lambda_M > \sqrt{2c}/4\}$ is a foliation of $\mathcal{M}_+(\mathbb{C}P^n)$. If we vary λ in the interval $(0, k]$, we see $\tau^2(\lambda, k)$ takes all the values in the interval $[0, 1]$. Hence $\{\mathcal{E}(M) \cap \mathcal{M}_+(\mathbb{C}P^n) \mid \lambda_M < \sqrt{2c}/4\}$ covers $\mathcal{M}_+(\mathbb{C}P^n)$. Summarizing up we get the following.

Theorem 1. (1) *The family* $\{\mathcal{E}(G(r)) \cap \mathcal{M}_-(\mathbb{C}P^n) \mid r > (2/\sqrt{c}) \sin^{-1}\sqrt{2/3}\}$ *forms a foliation of* $\mathcal{M}_-(\mathbb{C}P^n) \setminus \{[\gamma(k,1)] \mid k < \sqrt{2c}/4\}$.

(2) *The family* $\{\mathcal{E}(G(r)) \cap \mathcal{M}_+(\mathbb{C}P^n) \mid r > (2/\sqrt{c}) \sin^{-1}\sqrt{2/3}\}$ *covers* $\mathcal{M}_+(\mathbb{C}P^n)$.

(3) $\mathcal{E}(G((2/\sqrt{c}) \sin^{-1}\sqrt{2/3})) = \mathcal{M}_{\sqrt{2c}/4}$.

(4) *The family* $\{\mathcal{E}(G(r)) \cap \mathcal{M}_+(\mathbb{C}P^n) \mid r < (2/\sqrt{c}) \sin^{-1}\sqrt{2/3}\}$ *forms a foliation of* $\mathcal{M}_+(\mathbb{C}P^n)$.

(5) *The family* $\left\{\mathcal{E}(G(r)) \cap \mathcal{M}_-(\mathbb{C}P^n) \mid \frac{2}{\sqrt{c}} \sin^{-1}\frac{1}{\sqrt{3}} < r < \frac{2}{\sqrt{c}} \sin^{-1}\sqrt{\frac{2}{3}}\right\}$ *covers* $\mathcal{M}_-(\mathbb{C}P^n)$.

5. A characterization of the geodesic sphere

In the previous section we see that in $\mathbb{C}P^n(c)$ geodesic spheres of radius $r = (2/\sqrt{c}) \sin^{-1}\sqrt{2/3}$ have a specific property on extrinsic circular trajectories. In this section we give its characterizations among real hypersurfaces in $\mathbb{C}P^n$ from this point of view.

We take a trajectory γ for $\mathbb{F}_\kappa$ on a general real hypersurface M of a Kähler manifold $\widetilde{M}$. By use of Gauss and Weingarten formulae, its extrinsic shape satisfies

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma + \langle A_M\dot\gamma, \dot\gamma\rangle\mathcal{N} = \kappa J\dot\gamma + (\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho)\mathcal{N}, \tag{5.1}$$

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma = -\kappa^2\dot\gamma - (\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho)(A_M\dot\gamma + \kappa\xi) + \frac{d}{dt}(\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho)\mathcal{N}. \tag{5.2}$$

Thus, if the extrinsic shape of γ is a circle of geodesic curvature k_γ, as we have $\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma = -k_\gamma^2\dot\gamma$, we obtain the following by (5.1) and by comparing (5.2) with this equality:

$$k_\gamma^2 = \kappa^2(1 - \rho_\gamma^2) + \langle A_M\dot\gamma, \dot\gamma\rangle^2, \tag{5.3}$$

$$(k_\gamma^2 - \kappa^2)\dot\gamma = (\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho_\gamma)(A_M\dot\gamma + \kappa\xi). \tag{5.4}$$

A real hypersurface M is said to be Hopf if its characteristic vector field ξ is principal at each point of M. It is known that the principal curvature associated with the characteristic

vector field of a Hopf real hypersurface in $\mathbb{C}P^n$ is locally constant ([10]). We here consider the following condition at $p \in M$:

(**ET**) The extrinsic shape of a trajectory γ_0 for some Sasakian magnetic field $\mathbb{F}_{\kappa_0}$ with $\dot{\gamma}_0(0) = \xi_p$ is a circle of geodesic curvature $k_{\gamma_0} \neq |\kappa_0|$.

Theorem 2. *A connected real hypersurface M in a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c is locally congruent to the geodesic sphere of radius $r = (2/\sqrt{c})\sin^{-1}\sqrt{2/3}$ if and only if the following conditions hold:*

i) *At each point of $p \in M$,*
   a) *the condition (ET) holds,*
   b) *there exist constants κ_p, ρ_p with $\kappa_p \neq 0, \sqrt{2c}/4$ and $|\rho_p| < 1$ and linearly independent unit tangent vectors $v_1, \ldots, v_{2n-2} \in U_pM$ with $\langle v_i, \xi_p\rangle = \rho_p$ which satisfy that the extrinsic shapes of trajectories γ_i $(i = 1, \ldots, 2n-2)$ for $\mathbb{F}_{\kappa_p}$ with $\dot{\gamma}_i(0) = v_i$ are circles of geodesic curvature $k_i \neq |\kappa_p|$;*

ii) *There is a trajectory whose extrinsic shape is a circle of geodesic curvature $\sqrt{2c}/4$ and of complex torsion $\tau \neq \pm 1$.*

Proof. We are enough to show the "if" part. By the first condition, we have from (5.3) and (5.4) that

$$k_{\gamma_0} = |\langle A_M\dot{\gamma}, \dot{\gamma}\rangle| \quad \text{and} \quad k_{\gamma_0}^2\xi_p = (\langle A_M\xi_p, \xi_p\rangle - \kappa_0)A_M\xi_p + \kappa_0\langle A_M\xi_p, \xi_p\rangle\xi_p.$$

As $k_{\gamma_0} \neq |\kappa_0|$, we find that ξ_p is principal. We denote by δ_p the principal curvature associated with ξ_p, which is locally constant.

By the second condition we decompose the both sides of (5.4) to components parallel to ξ_p and orthogonal to it, and get the following:

$$(k_i^2 - \kappa_p^2)\rho_p = (\langle A_M(v_i - \rho_p\xi_p), v_i - \rho_p\xi_p\rangle + \delta_p\rho_p^2 - \kappa_p\rho_p)(\rho_p\delta_p + \kappa_p), \tag{5.5}$$

$$(k_i^2 - \kappa_p^2)(v_i - \rho_p\xi_p) = (\langle A_M(v_i - \rho_p\xi_p), v_i - \rho_p\xi_p\rangle + \delta_p\rho_p^2 - \kappa_p\rho_p)A_M(v_i - \rho_p\xi_p). \tag{5.6}$$

Since $k_i \neq |\kappa_p|$, by (5.6) we find that $v_i - \rho_p\xi_p$ is principal. We denote by α_i the principal curvature of $v_i - \rho_p\xi_p$. Then (5.5) and (5.6) turn to

$$(k_i^2 - \kappa_p^2)\rho_p = \{\alpha_i(1 - \rho_p^2) + \rho_p^2\delta_p - \kappa_p\rho_p\}(\rho_p\delta_p + \kappa_p), \tag{5.7}$$

$$k_i^2 - \kappa_p^2 = \{\alpha_i(1 - \rho_p^2) + \rho_p^2\delta_p - \kappa_p\rho_p\}\alpha_i. \tag{5.8}$$

Hence we have $\rho_p\alpha_i = \rho_p\delta_p + \kappa_p$. If $\rho_p = 0$, we see $\kappa_p = 0$, which is a contradiction. Thus we have $\rho_p \neq 0$ and obtain $\alpha_i = \delta_p + (\kappa_p/\rho_p)$. We hence have $\alpha_1 = \cdots = \alpha_{2n-2}$ $(= \alpha_p)$. This shows that all tangent vectors at p which are orthogonal to ξ_p are principal. Due to the classification of homogeneous Hopf real hypersurfaces in $\mathbb{C}P^n$ by Takagi [16], we find that M is locally congruent to a geodesic sphere. Thus the third condition shows that the radius is $r = (2/\sqrt{c})\sin^{-1}\sqrt{2/3}$. □

In order to study more about geodesic spheres of special radius, we need to recall principal curvatures of homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$. Such real hypersurfaces are classified by Takagi [16]. A homogeneous real hypersurface in $\mathbb{C}P^n(c)$ is congruent to one of the following:

1) a geodesic sphere $G(r)$ of raius r $(0 < r < \pi/\sqrt{c})$,
2) a tube $T_\ell(r)$ of radius r $(0 < r < \pi/\sqrt{c})$ around totally geodesic $\mathbb{C}P^\ell$ $(1 \le \ell \le n-2)$,
3) a tube $Q(r)$ of radius r $(0 < r < \pi/(2\sqrt{c}))$ around complex hyperquadric $\mathbb{C}Q^{n-1}$,
4) tubes of radius r $(0 < r < \pi/(2\sqrt{c}))$ around $\mathbb{C}P^1 \times \mathbb{C}P^{(n-1)/2}$, $G_{2,5}(\mathbb{C})$ and $SO(10)/U(5)$.

The principal curvature δ_M corresponding to ξ is $\delta_M = \sqrt{c}\cot\sqrt{c}\,r$. When $M = T_\ell(r)$, it has two principal curvatures

$$\lambda_M^{(1)} = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2) \quad \text{and} \quad \lambda_M^{(2)} = -(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$$

for tangent vectors orthogonal to ξ. When $M = Q(r)$, it also has two principal curvatures

$$\lambda_M^{(1)} = \frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}\,r - \frac{\pi}{4}\Big) \quad \text{and} \quad \lambda_M^{(2)} = \frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}\,r + \frac{\pi}{4}\Big)$$

for tangent vectors orthogonal to ξ. For other homogeneous real hypersurfaces, they have four principal curvatures

$$\lambda_M^{(1)} = \frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}\,r - \frac{\pi}{4}\Big), \quad \lambda_M^{(2)} = \frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}\,r + \frac{\pi}{4}\Big),$$
$$\lambda_M^{(3)} = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}\,r}{2}, \qquad \lambda_M^{(4)} = -\frac{\sqrt{c}}{2}\tan\frac{\sqrt{c}\,r}{2}$$

for tangent vectors orthogonal to ξ. It is known that a Hopf real hypersurface all of whose principal curvatures are constant in $\mathbb{C}P^n$ is homogeneous.

Theorem 3. *A connected real hypersurface M in a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c is locally congruent to the geodesic sphere of radius $r = (2/\sqrt{c})\sin^{-1}\sqrt{2/3}$ if and only if it satisfies the following conditions with some constant κ with $|\kappa| \neq \sqrt{2c}/4$ at each point $p \in M$:*

i) *The condition (ET) holds*;
ii) *There exist linearly independent tangent vectors $v_1, \ldots, v_{2n-2} \in U_pM$ satisfying that the extrinsic shapes of trajectories γ_i $(i = 1, \ldots, 2n-2)$ for $\mathbb{F}_\kappa$ with $\dot\gamma_i(0) = v_i$ are circles of geodesic curvature $\sqrt{2c}/4$ and of complex torsion $\tau_i \neq \pm 1$.*

Proof. We are enough to show the "if" part. By the first condition we see M is a Hopf hypersurface. We denote by δ_M the principal curvature associated with ξ. We put $\rho_i = \langle v_i, \xi_p\rangle$. As $|\kappa| \neq \sqrt{2c}/4$, we find along the same lines as of the proof of Theorem 2 that each $v_i - \rho_i\xi_p$ is principal and that $\alpha_i\rho_i = \delta_M\rho_i + \kappa$ with the principal curvature α_i of $v_i - \rho_i\xi_p$. Moreover, (5.3) shows

$$c/8 = \kappa^2(1-\rho_i^2) + \{\alpha_i + (\delta_M - \alpha_i)\rho_i^2\}^2. \tag{5.9}$$

When $\rho_i = 0$, we have $\kappa = 0$. (5.9) shows that $\alpha_i = \pm\sqrt{2c}/4$. When $\rho_i \neq 0$, substituting $\alpha_i = \delta_M + (\kappa/\rho_i)$ into (5.9), we find that ρ_i satisfies the following equation

$$2\kappa\delta_M\rho_i^3 + \{(c/8) + \kappa^2 - \delta_M\}\rho_i^2 - 2\kappa\delta_M\rho_i - \kappa^2 = 0.$$

Thus ρ_i is one of the three solutions of this cubic equation. Therefore, by perturbation theory ([9]) we find that each α_i is locally constant. This means that M is a Hopf real hypersurface all of whose principal curvatures are constant.

We shall check that homogeneous real hypersurfaces except geodesic spheres of radius $r = (2/\sqrt{c})\sin^{-1}\sqrt{2/3}$ do not satisfy the second condition. We take $v_1, \ldots, v_{2n-2} \in U_pM$ as in the second condition. By the above argument, we see that $v_i - \rho_i\xi_p$ with $\rho_i = \langle v_i, \xi_p\rangle$ is principal. We denote by λ_i the principal curvature of $v_i - \rho_i\xi_p$. As $v_1 - \rho_1\xi_p, \ldots, v_{2n-2} - \rho_{2n-2}\xi_p$ span the tangent space T_p^0M orthogonal to ξ_p, these $\lambda_1, \ldots, \lambda_{2n-2}$ are all the principal curvatures for T_p^0M. Let γ_i denote the trajectory for $\mathbb{F}_\kappa$ with $\dot\gamma_i(0) = v_i$. Since the complex torsion τ_i of the extrinsic shape of γ_i is not ±1, by (5.1), (5.3) and by the same computation as for Lemma 1 we have

$$\kappa + (\delta_M - \lambda_i)\rho_i = 0, \tag{5.10}$$

$$c/8 = \kappa^2 - 2\lambda_i\kappa\rho_i + \lambda_i^2, \tag{5.11}$$

$$\tau_i = 4(2\kappa\rho_i^2 - \kappa - \lambda_i\rho_i)/\sqrt{2c}. \tag{5.12}$$

When $M = G(r)$ with $r \neq (2/\sqrt{c})\sin^{-1}\sqrt{2/3}$, we see in §4 that if the extrinsic shape of an extrinsic circular trajectory has geodesic curvature $\sqrt{2c}/4$ then its complex torsion is ±1, hence it does not satisfy the second condition.

When $M = T_\ell(r)$ we have $\delta_M = \lambda_M^{(1)} + \lambda_M^{(2)}$, $\lambda_M^{(2)} = -c/(4\lambda_M^{(1)})$, in particular one of $\lambda_M^{(1)}$, $\lambda_M^{(2)}$ is not $\sqrt{2c}/4$. For a circular trajectory γ_i corresponding to $\lambda_i \neq \sqrt{2c}/4$ we have $\tau_i^2 = 1$ by the same computation for (4.2). This is a contradiction. Hence $T_\ell(r)$ does not satisfies the second condition. Similarly, when M is one of tubes around $\mathbb{C}P^1 \times \mathbb{C}P^{(n-1)/2}$, $G_{2,5}(\mathbb{C})$ and $SO(10)/U(5)$, then $\lambda_M^{(3)}$, $\lambda_M^{(4)}$ satisfy $\delta_M = \lambda_M^{(3)} + \lambda_M^{(4)}$, $\lambda_M^{(4)} = -c/(4\lambda_M^{(3)})$. Hence we can conclude that it does not satisfy the second condition neither.

When $M = Q(r)$, we have

$$\lambda_M^{(1)} = (\sqrt{c}/2)\nu, \quad \lambda_M^{(2)} = -(\sqrt{c}/2)\nu^{-1}, \quad \delta_M = -2\sqrt{c}/(\nu - \nu^{-1})$$

with $\nu = \cot(\sqrt{c}\,r/2)$. When $\lambda_i = \lambda_M^{(1)}$, we have $\kappa = \sqrt{c}(\nu^2+3)\rho_i/\{2(\nu - \nu^{-1})\}$. Substituting this into (5.11) we find

$$\kappa^2 = c(2\nu^2 - 1)(\nu^2 + 3)/\{8(\nu^2 - 5)\}.$$

When $\lambda_i = \lambda_M^{(2)}$, we have $\kappa = \sqrt{c}(3 + \nu^{-2})\rho_i/\{2(\nu - \nu^{-1})\}$. Substituting this into (5.11) we find

$$\kappa^2 = c(\nu^2 - 2)(3\nu^2 + 1)/\{8\nu^2(5\nu^2 - 1)\}.$$

Comparing these we obtain

$$0 = \nu^2(5\nu^2 - 1)(2\nu^2 - 1)(\nu^2 + 3) - (\nu^2 - 5)(\nu^2 - 2)(3\nu^2 + 1) = 10(\nu^2 + 1)^3(\nu^2 - 1).$$

Since $0 < r < \pi/(2\sqrt{c})$, we have $\nu > 1$. Hence, the above is a contradiction. Thus $Q(r)$ does not satisfies the second condition. □

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