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LONGTIME CONVERGENCE FOR EPITAXIAL GROWTH MODEL UNDER DIRICHLET CONDITIONS

Dedicated to the Memory of Professor Alfredo Lorenzi

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Abstract

This paper continues our study on the initial-boundary value problem for a semilinear parabolic equation of fourth order which has been presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [12] to describe the large-scale features of a growing crystal surface under molecular beam epitaxy. In the preceding paper [1], we already constructed a dynamical system generated by the problem and verified that the dynamical system has a finite-dimensional attractor (especially, every trajectory has nonempty ω-limit set) and admits a Lyapunov function (of the form (3.1)). This paper is then devoted to showing longtime convergence of trajectory. We shall prove that every trajectory converges to some stationary solution as $t \to \infty$.

As a matter of fact, we have obtained in [10] the similar result for the equation but under the Neumann like boundary conditions $\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\varDelta u = 0$ on the unknown function u. In this paper, we want as in [1] to handle the Dirichlet boundary conditions $u = \frac{\partial u}{\partial n} = 0$, maybe physically more natural conditions than before.

1. Introduction

We are concerned with the initial-boundary value problem for a fourth order nonlinear parabolic equation

$$(1.1) \qquad \begin{cases} 2\dfrac{\partial u}{\partial t} = -a\Delta^2 u - \mu\nabla\cdot\left(\dfrac{\nabla u}{1+|\nabla u|^2}\right) & \text{in} \quad \Omega\times(0,\infty), \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

in a two-dimensional bounded domain Ω. Such a problem was presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr[12] in order to describe the growing process of a crystal surface under molecular beam epitaxy. For the physical backgrounds, see [6, 14, 16, 20], Here, $u = u(x,t)$ denotes a displacement of surface height from the standard level at position $x \in \Omega$ and time $t \geq 0$.

In the papers [7, 8, 9, 10], we already studied the same equation but under the Neumann like boundary conditions $\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\varDelta u = 0$. In such a case, it is possible to reduce the fourth order differential operator $\varDelta^2$ into a product $(-\varDelta)^2$ of the negative Laplace operator

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$-\Delta$ equipped with the usual Neumann boundary conditions which is a positive definite self-adjoint operator of $L_2(\Omega)$. But these boundary conditions seem to be somewhat artificial. In this paper, we want to handle the same equation but under the Dirichlet boundary conditions $u = \frac{\partial u}{\partial n} = 0$. Because of loss of any convenient reductions of the fourth order operator to a second order one, we have to handle a very fourth order elliptic operator.

In the preceding paper [1], we have already constructed a dynamical system generated by (1.1) having a finite-dimensional attractor and showed that the dynamical system admits a Lyapunov function of the form (3.1) whose values are monotone decreasing along trajectories. This paper is then devoted to showing longtime convergence of trajectories to some stationary solution of (1.1) depending on initial functions. As in [10], we will employ the theory of Łojasiewicz-Simon inequality in infinite-dimensional spaces, We cannot, however, apply this theory to the present problem by any parallel arguments to [10]. Some modifications are needed. These modifications may be rather significant in the sense that, thanks to these, one can prove the same longtime convergence of solutions for the Keller-Segel equations, too. Remember that the Lyapunov function for the Keller-Segel equations also contains a logarithmic function, see [15].

In proving the longtime convergence for (1.1), the property that $\Delta^2 u \in L_2(\Omega)$ together with conditions $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ implies $\nabla u \in \mathcal{C}(\overline{\Omega})$ is indispensable for verifying analyticity of the function $u \mapsto \int_\Omega \log(1 + |\nabla u|^2)dx$. For this reason, we will assume that Ω is a rectangular domain

$$\Omega = \{(x_1, x_2);\ 0 < x_1 < \ell_1,\ 0 < x_2 < \ell_2\} \qquad (\ell_1 > 0 \text{ and } \ell_2 > 0) \tag{1.2}$$

or a $\mathcal{C}^4$ bounded domain. Then, if $\Delta^2 u \in L_2(\Omega)$ with $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, then $u \in H^4(\Omega)$ and hence $u \in \mathcal{C}^2(\overline{\Omega})$.

Throughout this paper, Ω is a rectangular or $\mathcal{C}^4$, bounded domain in $\mathbb{R}^2$. The outer normal vector of boundary at $x \in \partial\Omega$ is denoted by $n(x)$. For $1 \leq p \leq \infty$, $L_p(\Omega)$ is the space of real valued L_p functions in Ω. For $s \geq 0$, $H^s(\Omega)$ is the real Sobolev space in Ω with exponent s. For $m = 0, 1, 2, \ldots$, $\mathcal{C}^m(\overline{\Omega})$ is a space of real valued functions on $\overline{\Omega}$ of class $\mathcal{C}^m$.

Even when Ω is of the from (1.2), one can verify the similar results on the Sobolev spaces $H^s(\Omega)$ as for the $\mathcal{C}^4$ domains. In fact, when Ω is a bounded domain with Lipschitz boundary, the trace operator $u \mapsto u_{|\partial\Omega}$ is defined and is continuous from $H^1(\Omega)$ into $L_2(\partial\Omega)$ (see [11, Theorem 1.5.1.3] and notice that $H^1(\Omega) = W_2^1(\Omega)$). When Ω is a bounded domain with Lipschitz boundary, there exists a linear operator $\mathcal{E}$ extending functions u in Ω to functions $\mathcal{E}u$ in $\mathbb{R}^2$ that is continuous from $H_p^m(\Omega)$ into $H_p^m(\mathbb{R}^2)$ for every integer $m = 0, 1, 2, \ldots$ and every $1 \leq p \leq \infty$ (cf. [21, Theorem 1.33]). This then yields that the Sobolev embedding theorems in the whole space $\mathbb{R}^2$ hold true even in the Ω. Finally, usual integration by parts is available even in Ω of the from (1.2), because, for any fixed $0 < x_2 < \ell_2$, the function $u(\cdot, x_2)$ for $u \in H^1(\Omega)$ is defined on the interval $(0, \ell_1)$ and hence one can use the integration by parts for the variable x_1. It is the same for the variable x_2.

2. Dynamical System

2.1. Abstract Formulation. We rewrite (1.1) into the form

$$\begin{cases} \dfrac{du}{dt} + Au = f(u), & 0 < t < \infty, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

in the underlying space $X = L_2(\Omega)$. Here, A is a realization of $a\Delta^2$ in $L_2(\Omega)$ under the Dirichlet boundary conditions.

In fact, A is defined in the following way. Consider the symmetric sesquilinear form

$$a(u, v) = a \int_\Omega \Delta u \cdot \Delta v \, dx, \qquad u, \ v \in H_0^2(\Omega).$$

Here, $H_0^2(\Omega)$ is the closure of $C_0^\infty(\Omega)$ (space of infinitely differentiable functions in Ω with compact support) in $H^2(\Omega)$. If $u \in H_0^2(\Omega)$, then $\nabla u \in H_0^1(\Omega)$; consequently, u satisfies $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Since it is clear that $u = 0$ on $\partial\Omega$, $u \in H_0^2(\Omega)$ implies that u satisfies the Dirichlet boundary conditions in (1.1). Furthermore, the convexity of Ω when Ω is given by (1.2), or the C^4 regularity of $\partial\Omega$ in the alternative case yields that

$$\|u\|_{H^2} \le C\|\Delta u\|_{L_2}, \quad u \in H^2(\Omega) \cap H_0^1(\Omega).$$

This shows that the form $a(u, v)$ is coercive on $H_0^2(\Omega)$. Consequently, $a(u, v)$ determines a linear operator $\mathcal{A}$ from $H_0^2(\Omega)$ into $H^{-2}(\Omega)$ by the formula $a(u, v) = \langle \mathcal{A}u, v\rangle_{H^{-2}\times H_0^2}$ (see [5]), where $H^{-2}(\Omega)$ denotes the dual space of $H_0^2(\Omega)$ and these spaces compose a triplet $H_0^2(\Omega) \subset L_2(\Omega) \subset H^{-2}(\Omega)$.

The operator $\mathcal{A}$ thus defined is considered as a realization of $a\Delta^2$ in $H^{-2}(\Omega)$ under the Dirichlet boundary conditions which is a densely defined, closed operator in $H^{-2}(\Omega)$ with domain $\mathcal{D}(\mathcal{A}) = H_0^2(\Omega)$. Furthermore, its part in $L_2(\Omega)$ denoted by A ($= \mathcal{A}_{|L_2}$) is defined by

$$\begin{cases} \mathcal{D}(A) = \{u \in H_0^2(\Omega); \ \mathcal{A}u \in L_2(\Omega)\}, \\ Au = \mathcal{A}u. \end{cases} \tag{2.2}$$

Whence, A is a realization of $a\Delta^2$ in $L_2(\Omega)$ under the Dirichlet boundary conditions. It is easily seen that A is a positive definite self-adjoint operator of $L_2(\Omega)$.

Proposition 2.1. *The domain of A given by* (2.2) *can actually be characterized as $\mathcal{D}(A) = H^4(\Omega) \cap H_0^2(\Omega)$. Furthermore,*

$$\|u\|_{H^4} \le C\|Au\|_{L_2}, \qquad u \in \mathcal{D}(A). \tag{2.3}$$

Proof. If $u \in H^4(\Omega) \cap H_0^2(\Omega)$, then $a(u, v) = (a\Delta^2 u, v)$ for any $v \in H_0^2(\Omega)$. Therefore, $u \in \mathcal{D}(A)$. This shows that it is the case in general that $H^4(\Omega) \cap H_0^2(\Omega) \subset \mathcal{D}(A)$. So, what we have to prove is the converse inclusion $H^4(\Omega) \cap H_0^2(\Omega) \supset \mathcal{D}(A)$.

Let us first prove this in the case where Ω is rectangular. We use the Fourier expansion for the function of $L_2(\Omega)$. Any function $u \in L_2(\Omega)$ can be expanded as a series

$$u = \sum_{m,n=1}^{\infty} u_{mn} \sin\frac{m\pi}{\ell_1}x_1 \cdot \sin\frac{n\pi}{\ell_2}x_2$$

with Fourier coefficients u_{mn} satisfying $\sum_{m,n} |u_{mn}|^2 < \infty$. Then,

$$\Delta^2 u = \sum_{m,n=1}^{\infty} u_{mn} \left[\left(\frac{m\pi}{\ell_1}\right)^2 + \left(\frac{n\pi}{\ell_2}\right)^2\right]^2 \sin\frac{m\pi}{\ell_1}x_1 \cdot \sin\frac{n\pi}{\ell_2}x_2$$

in the distribution sense. So, if $\Delta^2 u \in L_2(\Omega)$, then there exists a double sequence f_{mn} satisfying $\sum_{m,n} |f_{mn}|^2 < \infty$ such that

$$u_{mn} = \left[\left(\frac{m\pi}{\ell_1}\right)^2 + \left(\frac{n\pi}{\ell_2}\right)^2\right]^{-2} f_{mn}, \qquad 1 \le m,\, n < \infty.$$

This yields that for $k = 0, 1, 2, 3, 4$, $D_1^k D_2^{4-k} u \in L_2(\Omega)$ as may be evident for $k = 0, 2, 4$. For $k = 1, 3$, say $k = 1$, we have

$$D_1 D_2^3 u = -\sum_{m,n=1}^{\infty} u_{mn} \frac{m\pi}{\ell_1}\left(\frac{n\pi}{\ell_2}\right)^3 \cos\frac{m\pi}{\ell_1}x_1 \cdot \cos\frac{n\pi}{\ell_2}x_2.$$

So, since $\cos\frac{m\pi}{\ell_1}x_1 \cdot \cos\frac{n\pi}{\ell_2}x_2$ are mutually orthogonal in Ω, it is seen that

$$\|D_1 D_2^3 u\|_{L_2}^2 = \frac{\ell_1 \ell_2}{4} \sum_{m,n=1}^{\infty} \left\{\frac{m\pi}{\ell_1}\left(\frac{n\pi}{\ell_2}\right)^3 \left[\left(\frac{m\pi}{\ell_1}\right)^2 + \left(\frac{n\pi}{\ell_2}\right)^2\right]^{-2}\right\}^2 |f_{mn}|^2 < \infty.$$

Furthermore,

$$\|D_1 D_2^3 u\|_{L_2}^2 \le C \sum_{m,n=1}^{\infty} |f_{mn}|^2 \le C\|\Delta^2 u\|_{L_2}^2.$$

Hence, $\Delta^2 u \in L_2(\Omega)$ implies $u \in H^4(\Omega)$.

Second, let us consider the case where Ω is a C^4 bounded domain. In this case, we have to appeal to a definitive existence result for the higher order elliptic operators. Among others, the arguments due to Tanabe [17, Section 3.8] are very comprehensible (cf. also [18, Section 5.2]). It is then asserted that for any $f \in L_2(\Omega)$, there exists a unique solution $u \in H^4(\Omega)$ for which it holds that $\Delta^2 u = f$ in Ω and $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ together with $\|u\|_{H^4} \le C\|f\|_{L_2}$, $C > 0$ being some constant. Furthermore, since $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ implies $u \in H_0^2(\Omega)$, we see that $u \in H^4(\Omega) \cap H_0^2(\Omega)\ (\subset \mathcal{D}(A))$ and $Au = f$. Then, since A is one-to-one from $\mathcal{D}(A)$ onto $L_2(\Omega)$, $\mathcal{D}(A)$ must coincide with $H^4(\Omega) \cap H_0^2(\Omega)$. □

Proposition 2.2. *For the square root $A^{\frac{1}{2}}$ of A, it holds true that $\mathcal{D}(A^{\frac{1}{2}}) = H_0^2(\Omega)$ together with the estimate*

$$\|u\|_{H^2} \le C\|A^{\frac{1}{2}} u\|_{L_2}, \qquad u \in \mathcal{D}(A^{\frac{1}{2}}). \tag{2.4}$$

Proof. Note that $a(u, v)$ is symmetric. It is then known (cf. [21, Theorem 2.34]) that the domain of the square root of the operator obtained from a symmetric form coincides with its form domain, i.e., $H_0^2(\Omega)$. □

By the interpolation of (2.3) and (2.4) (cf. [21, Chapter 16]), it is immediately verified that for $\frac{1}{2} \le \theta \le 1$,

$$\mathcal{D}(A^\theta) \subset H^{4\theta}(\Omega) \cap H_0^2(\Omega). \tag{2.5}$$

On the other hand, for $0 \le \theta < \frac{1}{2}$,

$$\mathcal{D}(A^{\theta}) \subset H^{4\theta}(\Omega).$$

It also holds true that for any $0 \le \theta \le 1$,

$$\|u\|_{H^{4\theta}} \le C\|A^{\theta}u\|_{L_2}, \qquad u \in \mathcal{D}(A^{\theta}). \tag{2.6}$$

The nonlinear operator $f(u)$ is defined by

$$\begin{aligned} f(u) &= -\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right) \\ &= -\mu\left[\frac{\Delta u}{1+|\nabla u|^2} - \frac{\nabla|\nabla u|^2\cdot\nabla u}{(1+|\nabla u|^2)^2}\right]. \end{aligned}$$

By direct calculations (as in the proof of [7, Proposition 2]) we observe that

$$\|f(u)-f(v)\|_{L_2} \le C[\|u-v\|_{H^2} + (\|u\|_{C^2}+\|v\|_{C^2})\|u-v\|_{H^1}].$$

In view of the inequality (2.6) (with $\theta = \frac{1}{4}$ and $\theta = \frac{7}{8}$) and the embedding $H^{\frac{7}{2}}(\Omega) \subset C^2(\overline{\Omega})$, it is verified that

$$\|f(u)-f(v)\|_{L_2} \le C[\|A^{\frac{1}{2}}(u-v)\|_{L_2} + (\|A^{\frac{7}{8}}u\|_{L_2}+\|A^{\frac{7}{8}}v\|_{L_2})\|A^{\frac{1}{4}}(u-v)\|_{L_2}]. \tag{2.7}$$

By the theory of abstract semilinear parabolic equations (see [21, Theorem 4.1]), we can state that, for any $u_0 \in \mathcal{D}(A^{\frac{1}{4}}) \subset H^1(\Omega)$, there exists a unique local solution to (2.1) in the function space:

$$u \in C([0,T_{u_0}];\mathcal{D}(A^{\frac{1}{4}})) \cap C^1((0,T_{u_0}];L_2(\Omega)) \cap C((0,T_{u_0}];\mathcal{D}(A)),$$

$T_{u_0} > 0$ being determined by the norm $\|A^{\frac{1}{4}}u_0\|_{L_2}$ alone.

2.2. Global solutions. In order to extend the local solution constructed above to a global solution, we show *a priori* estimate for the local solutions of (2.1). Consider a local solution u which is defined on interval $[0,T_u]$:

$$u \in C([0,T_u];\mathcal{D}(A^{\frac{1}{4}})) \cap C^1((0,T_u];L_2(\Omega)) \cap C((0,T_u];\mathcal{D}(A)). \tag{2.8}$$

We can then prove the following estimates.

Proposition 2.3. *There exist positive constants δ and C such that, for any local solution u in the space* (2.8)*, it holds true that*

$$\|A^{\frac{1}{4}}u(t)\|_{L_2} \le e^{-\delta t}\|A^{\frac{1}{4}}u_0\|_{L_2} + C, \qquad 0 \le t \le T_u. \tag{2.9}$$

Here, δ and C are independent of the interval $[0,T_u]$ on which u is defined.

Proof. Consider the inner product of the equation of (2.1) and $A^{\frac{1}{2}}u(t)$. Then, since $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, it follows that

$$\begin{aligned} \frac{d}{dt}\|A^{\frac{1}{4}}u(t)\|_{L_2}^2 + \|A^{\frac{3}{4}}u(t)\|_{L_2}^2 &= -\mu\int_{\Omega}\left[\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)\right]A^{\frac{1}{2}}u(t)dx \\ &= \mu\int_{\Omega}\left(\frac{\nabla u}{1+|\nabla u|^2}\right)\cdot\nabla A^{\frac{1}{2}}u(t)dx \end{aligned}$$

$$\leq \frac{\mu}{2}\|\nabla A^{\frac{1}{2}}u(t)\|_{L_2}.$$

Noting that $\|\nabla A^{\frac{1}{2}}u\|_{L_2} \leq C\|A^{\frac{3}{4}}u\|_{L_2}$ and $\|A^{\frac{1}{4}}u\|_{L_2} \leq C\|A^{\frac{3}{4}}u\|_{L_2}$, we conclude that

$$\frac{d}{dt}\|A^{\frac{1}{4}}u(t)\|_{L_2}^2 + \delta\|A^{\frac{1}{4}}u(t)\|_{L_2}^2 \leq C$$

with some constant $\delta > 0$. Solving this differential inequality, we obtain (2.9). □

By the standard arguments we can then construct for any $u_0 \in \mathcal{D}(A^{\frac{1}{4}})$, a unique global solution to (2.1) in the function space:

$$u \in \mathcal{C}([0,\infty);\mathcal{D}(A^{\frac{1}{4}})) \cap \mathcal{C}^1((0,\infty);L_2(\Omega)) \cap \mathcal{C}((0,\infty);H^4(\Omega)\cap H_0^2(\Omega)).$$

The global solution u as well satisfies the same estimates

$$\|A^{\frac{1}{4}}u(t)\|_{L_2} \leq e^{-\delta t}\|A^{\frac{1}{4}}u_0\|_{L_2} + C, \qquad 0 \leq t < \infty, \tag{2.10}$$

$$\|Au(t)\|_{L_2} \leq C(t^{-\frac{3}{4}} + 1)\|A^{\frac{1}{4}}u_0\|_{L_2}, \qquad 0 < t < \infty. \tag{2.11}$$

However, as shown in [1] (cf. also [7]), there is a local solution u to (2.1) for any initial value $u_0 \in L_2(\Omega)$. Indeed, we can apply [21, Theorem 4.1] again but to the Cauchy problem

$$\begin{cases} \dfrac{du}{dt} + \mathcal{A}u = \mathcal{F}(u), & 0 < t < \infty, \\ u(0) = u_0, \end{cases}$$

formulated in the space $\mathcal{X} = H^{-2}(\Omega)$. Here, $\mathcal{A}$ is the realization of $a\Delta^2$ in $H^{-2}(\Omega)$ with domain $\mathcal{D}(\mathcal{A}) = H_0^2(\Omega)$, and the nonlinear operator $\mathcal{F}(u) = -\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)$ is treated as a mapping from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ which is uniformly Lipschitz continuous. By the facts that $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = L_2(\Omega)$ and $\mathcal{D}(\mathcal{A}^{\frac{3}{4}}) \subset H_0^1(\Omega)$, the condition [21, (4.2)] is fulfilled with exponents $\beta = \frac{1}{2}$ and $\eta = \frac{3}{4}$. Consequently, for any $u_0 \in L_2(\Omega)$, there exists a unique local solution such that

$$u \in \mathcal{C}([0,T_{u_0}];L_2(\Omega)) \cap \mathcal{C}^1((0,T_{u_0}];H^{-2}(\Omega)) \cap \mathcal{C}((0,T_{u_0}];H_0^2(\Omega)).$$

Take now time $t_0 \in (0,T_{u_0})$; then, $\widetilde{u}_0 = u(t_0) \in \mathcal{D}(\mathcal{A}) = \mathcal{D}(A^{\frac{1}{2}}) \subset \mathcal{D}(A^{\frac{1}{4}})$; thereby, we can extend this local solution to a global one by considering (2.1) under the substituted initial condition $u(t_0) = \widetilde{u}_0$.

Ultimately, we conclude the following existence result. For any initial function $u_0 \in L_2(\Omega)$, (2.1) possesses a unique global solution in the function space:

$$u \in \mathcal{C}([0,\infty);L_2(\Omega)) \cap \mathcal{C}^1((0,\infty);L_2(\Omega)) \cap \mathcal{C}((0,\infty);H^4(\Omega)\cap H_0^2(\Omega)). \tag{2.12}$$

For $0 \leq t < \infty$, set $S(t)u_0 = u(t;u_0)$, where $u(t;u_0)$ is the global solution of (2.1) for initial value $u_0 \in L_2(\Omega)$. Then, $S(t)$ defines a family of nonlinear operators acting on $L_2(\Omega)$ with the semigroup property $S(t+s) = S(t)S(s)$ and $S(0) = I$. Moreover, the mapping $G\colon [0,\infty)\times L_2(\Omega) \to L_2(\Omega)$ defined by $G(t,u_0) = S(t)u_0$ is continuous, i.e., $S(t)$ is a continuous semigroup on $L_2(\Omega)$. In this way, (2.1) generates a dynamical system $(S(t), L_2(\Omega))$.

Let $u_0 \in L_2(\Omega)$. In view of (2.11), the trajectory $\{S(t)u_0;\ 1 \leq t < \infty\}$ is a bounded subset of $H^4(\Omega)$. Consequently, it is a relatively compact subset of $L_2(\Omega)$. In particular, its ω-limit set

$$\omega(u_0) = \{\overline{u};\ \exists t_n \uparrow \infty \ \text{ such that } \ S(t_n)u_0 \to \overline{u} \ \text{ in } \ L_2(\Omega)\}$$

is a nonempty set. In addition, if $S(t_n)u_0 \to \overline{u}$ in $L_2(\Omega)$, then it automatically observed that

$$S(t_n)u_0 \to \overline{u} \quad \text{in } H^s(\Omega) \tag{2.13}$$

for any $0 < s < 4$.

As verified in [1], $(S(t), L_2(\Omega))$ has furthermore a finite-dimensional attractor which attracts every trajectory at an exponential rate (cf, [2, 19, 21]).

3. Lyapunov Function

It is already proved by [1] that the following function

$$\Phi(u) = \frac{1}{2}\int_\Omega [a|\Delta u|^2 - \mu \log(1 + |\nabla u|^2)]dx, \qquad u \in H_0^2(\Omega), \tag{3.1}$$

becomes a Lyapunov function of our dynamical system $(S(t), L_2(\Omega))$.

In what follows, we will consider Φ to be a function from $H_0^2(\Omega)$ to $\mathbb{R}$ (although Φ may be defined on the whole space $H^2(\Omega)$). We furthermore handle it in the triplet

$$H_0^2(\Omega) \subset L_2(\Omega) \subset H^{-2}(\Omega) = H_0^2(\Omega)'. \tag{3.2}$$

This section is then devoted to verifying various properties of the derivatives $\Phi'(u) \in \mathcal{L}(H_0^2(\Omega), \mathbb{R}) = H^{-2}(\Omega)$ and $\Phi''(u) \in \mathcal{L}(H^2(\Omega), H^{-2}(\Omega))$.

3.1. Differentiability of $\Phi(u)$.

Let us begin with showing differentiability of $\Phi(u)$.

Proposition 3.1. *Φ: $H_0^2(\Omega) \to \mathbb{R}$ is differentiable with the derivative $\Phi'(u) = \mathcal{A}u - \mathcal{F}(u) \in H^{-2}(\Omega)$ for $u \in H_0^2(\Omega)$. Here, $\mathcal{F}(u) = -\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)$ is a nonlinear operator from $H_0^2(\Omega)$ into $H^{-1}(\Omega)\ (\subset H^{-2}(\Omega))$.*

Proof. For u, $h \in H_0^2(\Omega)$, we have

$$\|\Delta(u+h)\|_{L_2}^2 - \|\Delta u\|_{L_2}^2 = 2(\Delta u, \Delta h) + (\Delta h, \Delta h).$$

Therefore,

$$\|\Delta(u+h)\|_{L_2}^2 - \|\Delta u\|_{L_2}^2 - 2\langle \Delta^2 u, h\rangle_{H^{-2}\times H_0^2} = \|\Delta h\|_{L_2}^2. \tag{3.3}$$

In the meantime, for a.e. $x \in \Omega$, we have

$$\begin{aligned}
&\log\{1 + |\nabla[u(x) + h(x)]|^2\} - \log\{1 + |\nabla u(x)|^2\}\\
&\qquad = \int_0^1 \frac{d}{d\theta}\log\{1 + |\nabla[u(x) + \theta h(x)]|^2\}d\theta\\
&\qquad = \int_0^1 \frac{2\nabla u(x)\cdot\nabla h(x) + 2\theta|\nabla h(x)|^2}{1 + |\nabla[u(x) + \theta h(x)]|^2}d\theta.
\end{aligned}$$

Moreover, since

$$\frac{1}{1 + |\nabla[u(x) + \theta h(x)]|^2} = \frac{1}{1 + |\nabla u(x)|^2} - \frac{2\theta\nabla u(x)\cdot\nabla h(x) + \theta^2|\nabla h(x)|^2}{\{1 + |\nabla[u(x) + \theta h(x)]|^2\}(1 + |\nabla u(x)|^2)},$$

it follows that

$$\left|\log\{1+|\nabla[u(x)+h(x)]|^2\}-\log\{1+|\nabla u(x)|^2\}-\frac{2\nabla u(x)\cdot\nabla h(x)}{1+|\nabla u(x)|^2}\right| \\ \leq C\{|\nabla h(x)|^2+|\nabla h(x)|^4\}.$$

Therefore, integration in Ω yields that

$$\left|\int_\Omega\left[\log\{1+|\nabla(u+h)|^2\}-\log\{1+|\nabla u|^2\}-\frac{2\nabla u\cdot\nabla h}{1+|\nabla u|^2}\right]dx\right|\leq C\{\|\nabla h\|_{L_2}^2+\|\nabla h\|_{L_4}^4\}.$$

We here use Galiardo-Nireberg's inequality ([21, Theorem 1.37]) to obtain that

$$\|\nabla h\|_{L_4}\leq C\|\nabla h\|_{L_2}^{\frac12}\|\nabla h\|_{H^1}^{\frac12}\leq C\|h\|_{H^1}^{\frac12}\|h\|_{H^2}^{\frac12}\leq C\|h\|_{L_2}^{\frac14}\|h\|_{H^2}^{\frac34}.$$

Hence,

$$\left|\int_\Omega\left[\log\{1+|\nabla(u+h)|^2\}-\log\{1+|\nabla u|^2\}\right]dx+2\langle\nabla\cdot\left(\tfrac{\nabla u}{1+|\nabla u|^2}\right),h\rangle_{H^{-1}\times H_0^1}\right| \\ \leq C\|h\|_{L_2}(\|h\|_{H^2}+\|h\|_{H^2}^3). \tag{3.4}$$

Combining (3.3) and (3.4), we conclude that

$$|\Phi(u+h)-\Phi(u)-\langle\mathcal{A}u-\mathcal{F}(u),h\rangle_{H^{-2}\times H_0^2}|\leq C[\|\varDelta h\|_{L_2}^2+\|h\|_{L_2}(\|h\|_{H^2}+\|h\|_{H^2}^3)].$$

This shows that $\Phi(u)$ is differentiable and the derivative is given by $\Phi'(u)=\mathcal{A}u-\mathcal{F}(u)$ for any $u\in H_0^2(\Omega)$. □

On the domain $\mathcal{D}(A)\,(\subset H^4(\Omega))$, however, it is possible to observe that $\Phi(u)$ is differentiable in somewhat weak topology.

Proposition 3.2. *If $u\in\mathcal{D}(A)$, then $\Phi'(u)=Au-f(u)\in L_2(\Omega)$. In addition, when the variable h also runs only in $\mathcal{D}(A)$, it holds true that*

$$|\Phi(u+h)-\Phi(u)-(Au-f(u),h)|\leq C\|h\|_{L_2}(\|h\|_{H^4}+\|h\|_{H^2}+\|h\|_{H^2}^3). \tag{3.5}$$

Proof. Since $u\in\mathcal{D}(A)$ implies $\mathcal{A}u-\mathcal{F}(u)=Au-f(u)$, the first assertion is obvious. In addition, for $h\in\mathcal{D}(A)$, we observe that

$$\|\varDelta h\|_{L_2}^2=(\varDelta h,\varDelta h)=\langle\varDelta^2h,h\rangle_{H^{-2}\times H_0^2}=(\varDelta^2h,h)\leq\|h\|_{H^4}\|h\|_{L_2}.$$

Hence, (3.5) is also verified. □

Let $u_0\in L_2(\Omega)$. Let $\{u(t);\,0\leq t<\infty\}$ be the trajectory starting from u_0 and $\omega(u_0)$ be its ω-limit set. As an immediate consequence of (3.5), we observe that $\Phi(u(t))$ is differentiable for $t>0$ with the derivative

$$\frac{d}{dt}\Phi(u(t))=-\|Au(t)-f(u(t))\|_{L_2}^2. \tag{3.6}$$

Indeed, we apply (3.5) with $u=u(t)$ and $h=u(t+\varDelta t)-u(t)$. Then,

$$\left|\frac{\Phi(u(t+\varDelta t))-\Phi(u(t))}{\varDelta t}-\left(Au(t)-f(u(t)),\frac{u(t+\varDelta t)-u(t)}{\varDelta t}\right)\right| \\ \leq C\left\|\frac{u(t+\varDelta t)-u(t)}{\varDelta t}\right\|_{L_2}(\|h\|_{H^4}+\|h\|_{H^2}+\|h\|_{H^2}^3).$$

As $u(t+\Delta t) - u(t) \to 0$ in $H^4(\Omega)$ due to (2.12), we obtain (3.6). Therefore, along the trajectory $u(t)$, the values of Φ are monotone decreasing. Furthermore, if $\overline{u} \in \omega(u_0)$, then

$$\Phi(\overline{u}) = \lim_{n\to\infty} \Phi(u(t_n)) = \inf_{0<t<\infty} \Phi(u(t)). \tag{3.7}$$

In particular, Φ takes a constant value on the ω-limit set $\omega(u_0)$.

It is well known that $\omega(u_0)$ is an invariant set of $S(t)$. Indeed, if $\overline{u} \in \omega(u_0)$, then there exists $t_n \uparrow \infty$ such that $S(t_n)u_0 \to \overline{u}$ in $L_2(\Omega)$. Then, $S(t+t_n)u_0 = S(t)S(t_n)u_0 \to S(t)\overline{u}$; hence $S(t)\overline{u} \in \omega(u_0)$, i.e., $S(t)\omega(u_0) \subset \omega(u_0)$. Conversely, we have $S(t_n)u_0 = S(t)S(t_n - t)u_0$ for all t_n such that $t_n \geq t$. Since $S(t_n - t)u_0$ is a relatively compact subset of $L_2(\Omega)$, it is possible to assume that $S(t_n - t)u_0 \to \overline{v} \in \omega(u_0)$ in $L_2(\Omega)$, i.e., $\overline{u} = S(t)\overline{v}$. This means that $\omega(u_0) \subset S(t)\omega(u_0)$.

For any $\overline{u} \in \omega(u_0)$, consider the trajectory $S(t)\overline{u}$. As verified, $S(t)\overline{u} \in \omega(u_0)$; therefore, (3.7) implies that $\Phi(S(t)\overline{u}) \equiv \Phi(\overline{u})$; consequently, $\frac{d}{dt}\Phi(S(t)\overline{u}) \equiv 0$; in particular, $\frac{d}{dt}\Phi(S(0)\overline{u}) = 0$. Equality (3.6) then provides that $A\overline{u} - f(\overline{u}) = 0$. By virtue of Proposition 3.1, this is equivalent to $\Phi'(\overline{u}) = 0$. We have thus verified the following proposition.

Proposition 3.3. *For any $u_0 \in L_2(\Omega)$, its ω-limit set $\omega(u_0)$ consists of critical points of Φ. In particular, if $\overline{u} \in \omega(u_0)$ then $\Phi'(\overline{u}) = 0$.*

Let us next show that $\Phi(u)$ is twice differentiable.

Proposition 3.4. *$\Phi' : H_0^2(\Omega) \to H^{-2}(\Omega)$ is Fréchet differentiable with the derivative $\Phi''(u) = \mathcal{A} - \mathcal{F}'(u)$, where $\mathcal{F}'(u)$ is the Fréchet derivative of $\mathcal{F} : H_0^2(\Omega) \to H^{-2}(\Omega)$ which was introduced above. Precisely, for $u \in H_0^2(\Omega)$, $\mathcal{F}'(u) \in \mathcal{L}(H_0^2(\Omega), H^{-2}(\Omega))$ is given by*

$$\mathcal{F}'(u)h = -\mu\nabla\cdot\left(\frac{\nabla h}{1+|\nabla u|^2} - \frac{2(\nabla u\cdot\nabla h)\nabla u}{(1+|\nabla u|^2)^2}\right), \qquad h \in H_0^2(\Omega). \tag{3.8}$$

Proof. Noting that ∇ is a bounded linear operator from $L_2(\Omega)$ into $H^{-1}(\Omega)$, let us consider $\frac{\nabla u}{1+|\nabla u|^2}$. For $u,\ h \in H_0^2(\Omega)$, we have

$$\frac{\nabla(u+h)}{1+|\nabla(u+h)|^2} - \frac{\nabla u}{1+|\nabla u|^2} = \frac{(1+|\nabla u|^2)\nabla h - 2(\nabla u\cdot\nabla h)\nabla u - |\nabla h|^2\nabla u}{(1+|\nabla(u+h)|^2)(1+|\nabla u|^2)}.$$

Here, as seen before,

$$\frac{1}{1+|\nabla(u+h)|^2} = \frac{1}{1+|\nabla u|^2} - \frac{2\nabla u\cdot\nabla h + |\nabla h|^2}{(1+|\nabla(u+h)|^2)(1+|\nabla u|^2)}.$$

Therefore, it follows that

$$\left|\frac{\nabla(u+h)}{1+|\nabla(u+h)|^2} - \frac{\nabla u}{1+|\nabla u|^2} - \frac{(1+|\nabla u|^2)\nabla h - 2(\nabla u\cdot\nabla h)\nabla u}{(1+|\nabla u|^2)^2}\right| \leq C(|\nabla h|^2 + |\nabla h|^3),$$

and hence

$$\left\|\frac{\nabla(u+h)}{1+|\nabla(u+h)|^2} - \frac{\nabla u}{1+|\nabla u|^2} - \frac{(1+|\nabla u|^2)\nabla h - 2(\nabla u\cdot\nabla h)\nabla u}{(1+|\nabla u|^2)^2}\right\|_{L_2} \leq C(\|\nabla h\|_{L_4}^2 + \|\nabla h\|_{L_6}^3) \leq C(\|h\|_{H^2}^2 + \|h\|_{H^2}^3).$$

This shows the operator $u \mapsto \frac{\nabla u}{1+|\nabla u|^2}$ is Fréchet differentiable from $H_0^2(\Omega)$ into $L_2(\Omega)$. □

3.2. Gradient Estimates of $\Phi'(u)$. Let $u_0 \in L_2(\Omega)$ and let $\overline{u} \in \omega(u_0)$. As shown by Proposition 3.3, we know that $\Phi'(\overline{u}) = 0$. The goal of this subsection is to establish the Łojasiewicz-Simon inequality for $\Phi'(u)$ at $\overline{u}$ that plays a crucial role in proving convergence of $u(t)$ to $\overline{u}$. That is, there exists some exponent $0 < \theta \le \frac{1}{2}$ for which it holds true that

$$\|\Phi'(u)\|_{H^{-2}} \ge D|\Phi(u) - \Phi(\overline{u})|^{1-\theta}, \qquad u \in U(\overline{u}), \tag{3.9}$$

here $U(\overline{u})$ denotes a neighborhood of $\overline{u}$ in $H_0^2(\Omega)$ and $D > 0$ is some constant. For this purpose, we will follow the methods devised by Chill [4] in which the underlying space must be divided into a sum of the critical manifold and its supplement.

Put $L = \Phi''(\overline{u})$. As verified by Proposition 3.4, $L = \mathcal{A} - \mathcal{F}'(\overline{u})$ is a linear operator from $H_0^2(\Omega)$ into $H^{-2}(\Omega)$. As a general result of the calculus of variations (see [3, Théorème 5.1.1, p. 65]), or as is directly verified from (3.8), L is a symmetric operator, i.e.,

$$\langle Lu, v\rangle_{H^{-2}\times H_0^2} = \langle u, Lv\rangle_{H_0^2\times H^{-2}}, \qquad u, v \in H_0^2(\Omega). \tag{3.10}$$

In addition, L is observed to be a Fredholm operator. Indeed, writing $L = [I-\mathcal{F}'(\overline{u})\mathcal{A}^{-1}]\mathcal{A}$, we rather consider the operator $I - K$ acting on $H^{-2}(\Omega)$, where $K = \mathcal{F}'(\overline{u})\mathcal{A}^{-1}$. As $\mathcal{R}(K) \subset L_2(\Omega)$, K is a compact operator of $H^{-2}(\Omega)$. Therefore, by virtue of the Riesz-Schauder theory, $\mathcal{K}(I - K)$ is a finite-dimensional subspace of $H^{-2}(\Omega)$. In addition, $\mathcal{R}(I - K)$ is a closed subspace of $H^{-2}(\Omega)$ with finite-codimension such that $\dim \mathcal{K}(I-K) = \operatorname{codim} \mathcal{R}(I-K) = N$. Since $\mathcal{A}$ is an isomorphism from $H_0^2(\Omega)$ onto $H^{-2}(\Omega)$, it follows that $\mathcal{K}(L)$ is a finite-dimensional subspace of $H_0^2(\Omega)$ and $\mathcal{R}(L)$ is a closed subspace of $H^{-2}(\Omega)$ with $\dim \mathcal{K}(L) = \operatorname{codim} \mathcal{R}(L) = N$. That is, L satisfies the conditions of Fredholm operator.

Since $\mathcal{K}(L)$ is a finite-dimensional space, we can regard it as a closed subspace of any space of triplet $H_0^2(\Omega) \subset L_2(\Omega) \subset H^{-2}(\Omega)$. Furthermore, by the same reason, these topologies are mutually equivalent. In the arguments below, we may not clarify the topology of $\mathcal{K}(L)$ when it is easily presumed by the contexts.

We introduce the orthogonal projection $P: L_2(\Omega) \to \mathcal{K}(L)$ in $L_2(\Omega)$. We have a direct sum $L_2(\Omega) = H_0 + \mathcal{K}(L)$, where $H_0 = (I - P)L_2(\Omega)$ is the orthogonal supplement of $\mathcal{K}(L)$ in $L_2(\Omega)$. We notice that P is a bounded operator from $H_0^2(\Omega)$ into itself. So, P induces a projection from $H_0^2(\Omega)$ onto $\mathcal{K}(L)$ and a topological direct sum $H_0^2(\Omega) = H_2 + \mathcal{K}(L)$, where $H_2 = (I - P)H_0^2(\Omega)$ is a topological supplement of $\mathcal{K}(L)$ in $H_0^2(\Omega)$. On the other hand, it is easy to see that $\|Pf\|_{H^{-2}} \le C\|f\|_{H^{-2}}$ for all $f \in L_2(\Omega)$. This means that P can be extend by continuation over the space $H^{-2}(\Omega)$. Clearly, P is a bounded operator from $H^{-2}(\Omega)$ into itself and induces a projection from $H^{-2}(\Omega)$ onto $\mathcal{K}(L)$ which yields another topological direct sum $H^{-2}(\Omega) = H_{-2} + \mathcal{K}(L)$, $H_{-2} = (I - P)H^{-2}(\Omega)$ being a topological supplement of $\mathcal{K}(L)$ in $H^{-2}(\Omega)$. It is also clear that P is symmetric in the sense that

$$\langle Pu, \varphi\rangle_{H_0^2,H^{-2}} = \langle u, P\varphi\rangle_{H_0^2,H^{-2}}, \qquad u \in H_0^2(\Omega),\ \varphi \in H^{-2}(\Omega). \tag{3.11}$$

By definition, $LP = 0$ on $H_0^2(\Omega)$; then, (3.10) and (3.11) provide that $PL = LP = 0$ on $H_0^2(\Omega)$; in particular, $L = (I - P)L$ on $H_0^2(\Omega)$. This concludes that $\mathcal{R}(L) \subset H_{-2}$. But we remember that $\operatorname{codim} \mathcal{R}(L) = N = \operatorname{codim} H_{-2}$. Therefore, $\mathcal{R}(L)$ and H_{-2} must coincide and consequently

$$L \text{ must be an isomorphism from } H_2 \text{ onto } H_{-2}. \tag{3.12}$$

Following [4], we set the critical manifold by

$$S = \{u \in H_0^2(\Omega);\ (I-P)\Phi'(u) = 0\}.$$

Then, S is verified to be a C^1-manifold of dimension N in a neighborhood of $\overline{u}$. Indeed, apply the implicit function theorem to the operator $G : H_0^2(\Omega) \to H_{-2}$ given by $G(u_1, u_2) = (I-P)\Phi'(u_1 + u_2)$ for $u_1 \in H_2$, $u_2 \in \mathcal{K}(L)$. Then, since $D_1 G(u_1, u_2) = (I-P)\Phi''(u)_{|H_2}$, (3.12) yields that $D_1 G(\overline{u}) = L_{|H_2}$ is an isomorphism. So, in a neighborhood of $\overline{u}$, S can be represented as

$$S = \{(g(u_2), u_2);\ u_2 \in \mathcal{K}(L),\ g : \mathcal{K}(L) \to H_2\},$$

g being a C^1 mapping defined in a neighborhood of $\overline{u}_2 \in \mathcal{K}(L)$, where $\overline{u} = \overline{u}_1 + \overline{u}_2$.

According to [4, Theorem 2], we can state the following proposition.

Proposition 3.5. *Assume that the restriction of Φ on S satisfies* (3.9) *in a subset $U \cap S$, where U is some neighborhood of $\overline{u}$ in $H_0^2(\Omega)$, with exponent $\theta \in (0, \frac{1}{2}]$. Then, Φ itself satisfies* (3.9) *in a neighborhood of $\overline{u}$ in $H_0^2(\Omega)$ with the same exponent θ.*

The desired inequality (3.9) on S can generally be verified, as mentioned in [4, Corollary 3], from analyticity of the Lyapunov function $\Phi(u)$. This is, however, not true in the present case, for the correspondence $u \mapsto \int_\Omega \log(1+|\nabla u|^2)dx$ is not analytic in $H_0^2(\Omega)$ due to the fact that $H^1(\Omega) \not\subset C(\overline{\Omega})$. So, we have to utilize upper shifting of spaces.

Let $0 < \varepsilon < \frac{1}{2}$ be arbitrarily fixed. We introduce the domains $\mathcal{D}(\mathcal{A}^{1+\varepsilon})$ and $\mathcal{D}(\mathcal{A}^{\varepsilon})$. Naturally, $\mathcal{D}(\mathcal{A}^{1+\varepsilon}) \subset \mathcal{D}(\mathcal{A}) = H_0^2(\Omega)$ and $\mathcal{D}(\mathcal{A}^{\varepsilon}) \subset H^{-2}(\Omega)$. Since $\mathcal{A}^{1+\varepsilon} = \mathcal{A}\mathcal{A}^{\varepsilon}$, $\mathcal{A}$ is an isomorphism from $\mathcal{D}(\mathcal{A}^{1+\varepsilon})$ onto $\mathcal{D}(\mathcal{A}^{\varepsilon})$. Then, by the same reason as before, P is a bounded operator from $\mathcal{D}(\mathcal{A}^{1+\varepsilon})$ into itself and induces a topological direct sum $\mathcal{D}(\mathcal{A}^{1+\varepsilon}) = H_{2,\varepsilon} + \mathcal{K}(L)$, where $H_{2,\varepsilon} = (I-P)\mathcal{D}(\mathcal{A}^{1+\varepsilon})$. Similarly, P is a bounded operator from $\mathcal{D}(\mathcal{A}^{\varepsilon})$ into itself and induces a topological direct sum $\mathcal{D}(\mathcal{A}^{\varepsilon}) = H_{-2,\varepsilon} + \mathcal{K}(L)$, where $H_{-2,\varepsilon} = (I-P)\mathcal{D}(\mathcal{A}^{\varepsilon})$. Obviously, $H_{2,\varepsilon} \subset H_2$ and $H_{-2,\varepsilon} \subset H_{-2}$. We can verify that (3.12) still holds true in the shifted spaces.

Proposition 3.6. *L is an isomorphism from $H_{2,\varepsilon}$ onto $H_{-2,\varepsilon}$.*

Proof. As L is a bounded operator from $\mathcal{D}(\mathcal{A}^{1+\varepsilon})$ into $\mathcal{D}(\mathcal{A}^{\varepsilon})$, so is from $H_{2,\varepsilon}$ into $\mathcal{D}(\mathcal{A}^{\varepsilon})$. So, it suffices to prove that $L(H_{2,\varepsilon}) = H_{-2,\varepsilon}$. Let $\varphi \in L(H_{2,\varepsilon})$; then, $\varphi = Lu$ and $u = (I-P)v$ with some $v \in \mathcal{D}(\mathcal{A}^{1+\varepsilon})$; therefore, $\varphi = (I-P)Lv \in H_{-2,\varepsilon}$. Meanwhile, let $\varphi \in H_{-2,\varepsilon}$; then, $\varphi = L(I-P)u = Lu$ with some $u \in \mathcal{D}(\mathcal{A})$; furthermore, $\mathcal{A}u = \mathcal{F}'(\overline{u})u + \varphi \in \mathcal{D}(\mathcal{A}^{\varepsilon})$; therefore, $u \in \mathcal{D}(\mathcal{A}^{1+\varepsilon})$ and $\varphi = L(I-P)u \in L(H_{2,\varepsilon})$. □

We furthermore verify analyticity of $\Phi(u)$ for $u \in \mathcal{D}(\mathcal{A}^{1+\varepsilon})$.

Proposition 3.7. *$\Phi : \mathcal{D}(\mathcal{A}^{1+\varepsilon}) \to \mathbb{R}$ is analytic.*

Proof. Notice that $\mathcal{D}(\mathcal{A}^{1+\varepsilon}) = \mathcal{D}(A^{\frac{1}{2}+\varepsilon}) \subset H^{2+4\varepsilon}(\Omega)$ due to (2.5). Hence, $u \in \mathcal{D}(\mathcal{A}^{1+\varepsilon})$ implies $\nabla u \in C(\overline{\Omega})$. Then, for small variable $h \in \mathcal{D}(\mathcal{A}^{1+\varepsilon})$, it is possible to develop

$$\log(1+|\nabla(u+h)|^2) = \log(1+|\nabla u|^2) + \log\left(1 + \frac{2\nabla u \cdot \nabla h + |\nabla h|^2}{1+|\nabla u|^2}\right)$$

$$= \log(1 + |\nabla u|^2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{2\nabla u \cdot \nabla h + |\nabla h|^2}{1 + |\nabla u|^2} \right)^n .$$

This directly yields analyticity of $u \mapsto \int_\Omega \log(1 + |\nabla u|^2) dx$ on $\mathcal{D}(\mathcal{A}^{1+\varepsilon})$. □

It is now ready to show the inequality (3.9) on S. We first observe that S actually lies in $\mathcal{D}(\mathcal{A}^{1+\varepsilon})$. Indeed, if $u \in S$, then $\Phi'(u) = P\Phi'(u)$; therefore, $\mathcal{A}u = \mathcal{F}(u) + P\Phi'(u) \in L_2(\Omega)$; hence, by definition, $u \in \mathcal{D}(A) = \mathcal{D}(\mathcal{A}^{\frac{3}{2}})$. Thus, $S = \{u \in \mathcal{D}(\mathcal{A}^{1+\varepsilon});\ (I - P)\Phi'(u) = 0\}$. As before, S is determined by the operator $G : \mathcal{D}(\mathcal{A}^{1+\varepsilon}) \to H_{-2,\varepsilon}$ given by $G(u_1, u_2) = (I - P)\Phi'(u_1 + u_2)$ for $u_1 \in H_{2,\varepsilon}$ $u_2 \in \mathcal{K}(L)$. As we know that $D_1G(\overline{u}) = L_{|H_{2,\varepsilon}}$ is an isomorphism, S can be represented in a neighborhood of $\overline{u}$ as

$$S = \{(g(u_2), u_2);\ u_2 \in \mathcal{K}(L),\ g : \mathcal{K}(L) \to H_{2,\varepsilon}\}.$$

Now, as Φ is analytic, g is also analytic in a neighborhood of $\overline{u}_2$, where $\overline{u} = \overline{u}_1 + \overline{u}_2$, which means that S is an analytic manifold. Remembering that Φ is analytic on $\mathcal{D}(\mathcal{A}^{1+\varepsilon})$, we next apply Łojasiewicz' classical result [13] in finite-dimensional spaces to $\Phi_{|S}$. Then, for some exponent $\theta \in (0, \frac{1}{2}]$,

$$\|\Phi'(u)\|_{H^{-2}} \geq C|\Phi(u) - \Phi(\overline{u})|^{1-\theta}$$

for u in a neighborhood of $\overline{u}$ and on S.

As stated above, Proposition 3.5 thus provides the desired inequality (3.9) in a neighborhood of the whole space $H_0^2(\Omega)$ of $\overline{u}$.

4. Convergence Results

Let $u_0 \in L_2(\Omega)$ and $\overline{u} \in \omega(u_0)$. Due to (2.13), there exists a sequence $t_n \uparrow \infty$ such that $u(t_n) \to \overline{u}$ in $H_0^2(\Omega)$. We can then show that, once the trajectory approaches sufficiently close to $\overline{u}$, it must remain in a neighborhood forever.

Proposition 4.1. *Let $r > 0$ be the radius for which the gradient inequality* (3.9) *holds true in the ball $B^{H_0^2}(\overline{u}; r)$ and let t_N be such that $u(t_N) \in B^{H_0^2}(\overline{u}; r)$. Then, if $u(t) \in B^{H_0^2}(\overline{u}; r)$ for every $t \in [t_N, T]$, where $T\ (\geq t_N)$ is any time, then it holds that*

$$\|u(t) - u(t_N)\|_{H_0^2} \leq C[\Phi(u(t_N)) - \Phi(\overline{u})]^{\frac{\theta}{2}} \qquad \text{for every } t \in [t_N, T], \tag{4.1}$$

here $C > 0$ is a constant independent of T.

Proof. For $t_N \leq t \leq T$,

$$\begin{aligned} \frac{d}{dt}[\Phi(u(t)) - \Phi(\overline{u})]^\theta &= \theta[\Phi(u(t)) - \Phi(\overline{u})]^{\theta-1} \frac{d}{dt}\Phi(u(t)) \\ &= \theta[\Phi(u(t)) - \Phi(\overline{u})]^{\theta-1} \left(\Phi'(u(t)), \frac{du}{dt}(t) \right) \\ &= -\theta[\Phi(u(t)) - \Phi(\overline{u})]^{\theta-1} \|\Phi'(u(t))\|_{L_2} \left\| \frac{du}{dt}(t) \right\|_{L_2} . \end{aligned}$$

Here we used the equality $\frac{du}{dt}(t) = -Au(t) + f(u(t)) = -\Phi'(u(t))$. By virtue of (3.9),

$$-\frac{d}{dt}[\Phi(u(t) - \Phi(\overline{u})]^\theta \geq C[\Phi(u(t)) - \Phi(\overline{u})]^{\theta-1} \|\Phi'(u(t))\|_{H^{-2}} \left\| \frac{du}{dt}(t) \right\|_{L_2} \geq C \left\| \frac{du}{dt}(t) \right\|_{L_2} .$$

Integration in $[t_N, t]$ yields that

$$[\Phi(u(t_N)) - \Phi(\overline{u})]^\theta - [\Phi(u(t)) - \Phi(\overline{u})]^\theta \geq C \int_{t_N}^{t} \left\| \frac{du}{ds}(s) \right\|_{L_2} ds.$$

Therefore,

$$\begin{aligned} \|u(t) - u(t_N)\|_{L_2} &\leq \int_{t_N}^{t} \left\| \frac{du}{ds}(s) \right\|_{L_2} ds \\ &\leq C^{-1}\{[\Phi(u(t_N)) - \Phi(\overline{u})]^\theta - [\Phi(u(t)) - \Phi(\overline{u})]^\theta\}. \end{aligned} \tag{4.2}$$

Hence, $\|u(t) - u(t_N)\|_{L_2} \leq C^{-1}[\Phi(u(t_N)) - \Phi(\overline{u})]^\theta$.

We next apply the estimate

$$\|u\|_{H_0^2} \leq C\|Au\|_{L_2}^{\frac{1}{2}} \|u\|_{L_2}^{\frac{1}{2}}, \qquad u \in \mathcal{D}(A),$$

(which follows from (2.4)) to $u(t) - u(t_N)$. Then, in view of (2.11), we conclude (4.1). □

Choose a time t_N so that $\|u(t_N) - \overline{u}\|_{H_0^2} \leq \frac{r}{3}$ and $C[\Phi(u(t_N) - \Phi(\overline{u})]^{\frac{\theta}{2}} \leq \frac{r}{3}$, here C is the constant obtained in (4.1). Then, if $u(t) \in B^{H_0^2}(\overline{u}; r)$ for every $t \in [t_N, T]$, T $(\geq t_N)$ being any time, then

$$\begin{aligned} \|u(t) - \overline{u}\|_{H_0^2} &\leq \|u(t) - u(t_N)\|_{H_0^2} + \|u(t_N) - \overline{u}\|_{H_0^2} \\ &\leq C[\Phi(u(t_N)) - \Phi(\overline{u})]^{\frac{\theta}{2}} + \|u(t_N) - \overline{u}\|_{H_0^2} \leq \frac{2r}{3}, \end{aligned}$$

i.e., $u(t) \in \overline{B}^{H_0^2}(\overline{u}; \frac{2r}{3})$ for $t_N \leq t \leq T$. This means that the trajectory staring from u_0 is trapped in $B^{H_0^2}(\overline{u}; r)$ for all $t \geq t_N$.

We now arrive at the main result.

Theorem 4.1. *Let $u_0 \in L_2(\Omega)$ and $\overline{u} \in \omega(u_0)$. Let t_N be the time chosen above. Then,*

$$\|u(t) - \overline{u}\|_{L_2} \leq C[\Phi(u(t)) - \Phi(\overline{u})]^\theta \qquad \textit{for every } t \in [t_N, \infty). \tag{4.3}$$

Proof. We already know that, for all $t_N \leq t < \infty$, $u(t) \in B^{H_0^2}(\overline{u}; r)$. So, the same argument as in the proof of Proposition 4.1 is available to $u(t)$ for every $t \geq t_N$. Let $t_N \leq t \leq t_n$, where t_n is the sequence introduced above. Then, by the same way as for (**??**), we obtain that

$$\|u(t_n) - u(t)\|_{L_2} \leq C^{-1}\{[\Phi(u(t)) - \Phi(\overline{u})]^\theta - [\Phi(u(t_n)) - \Phi(\overline{u})]^\theta\}.$$

Fixing t, let t_n tend to infinity. Then, in view of (3.7), (4.3) is verified. □

5. Numerical Results

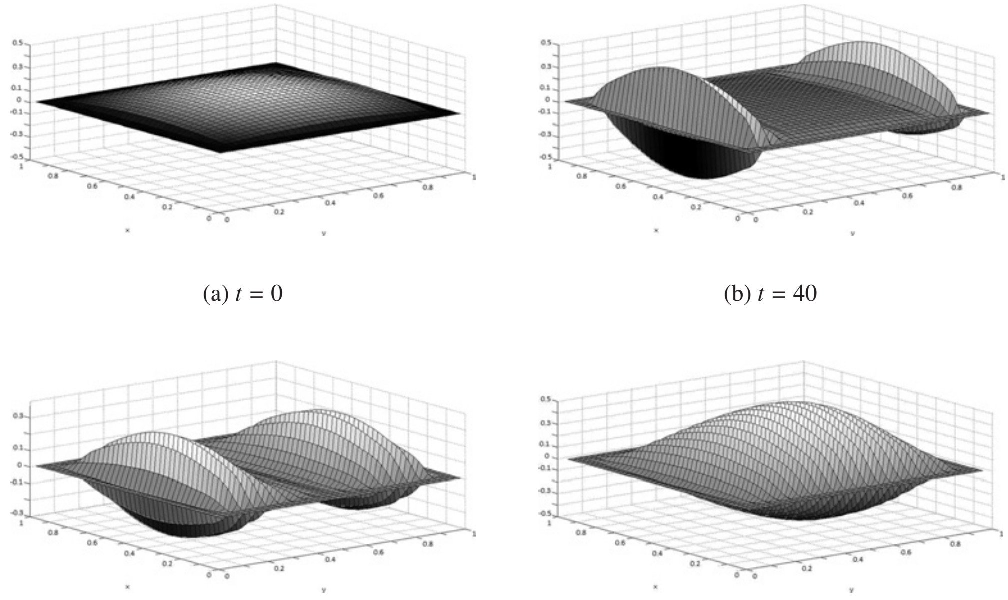
Let us conclude this paper with illustrating some numerical examples. We treat (1.1) in the square domain $\Omega = (0, 1) \times (0, 1)$. The coefficients a and μ are fixed as $a = 1$ and $\mu = 40$, respectively. We shall choose initial functions as

$$u_0(x_1, x_2) = 0.1[\sin(3.14kx_1) \times \sin(3.14x_2)], \qquad (x_1, x_2) \in \Omega, \tag{5.1}$$

where k is a positive integer varying from 1 to 4. These are a perturbation of the null stationary solution $u \equiv 0$ which is a unique homogeneous stationary solution to (1.1).

In Figures 1,3,5 and 7 below, the graphs show development of a surface that is determined by the solution $u = u(x_1, x_2, t)$ in the 3-dimensional space for (x_1, x_2, u) at each indicated time t. As observed, the surface consists of a few waves that a number of ridges ($u > 0$) and hollows ($u < 0$) of almost similar shape line up regularly one after the other.

First, let $k = 1$ in (5.1). The dynamics of the solution is illustrated by Figure 1. The small initial perturbation grows into a single wave. The graph of the Lyapunov function is given by Figure 2. At time about $t = 120$, the values of the Lyapunov function are stabilized. In view of Theorem 4.1, this suggests that a final profile of the trajectory may be given by that of time $t = 120$.



(a) $t = 0$ (b) $t = 40$

(c) $t = 80$ (d) $t = 120$

Fig. 1. Dynamics for $k = 1$

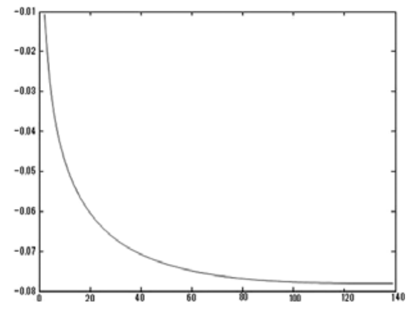
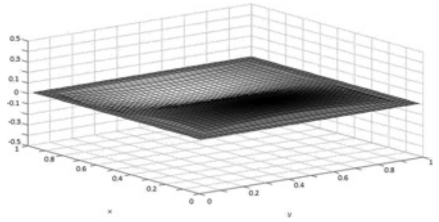
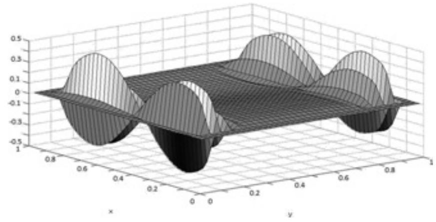


Fig. 2. Lyapunov function for $k = 1$

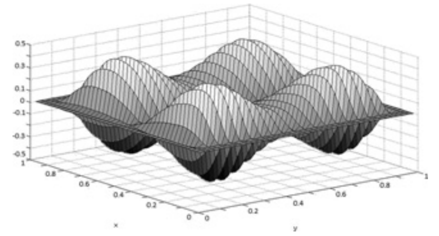
Secondly, let $k = 2$ in (5.1). As Figure 3 shows, the perturbation grows in this case into parallel waves. The profile of the solution is stabilized about time $t = 180$. The graph of the



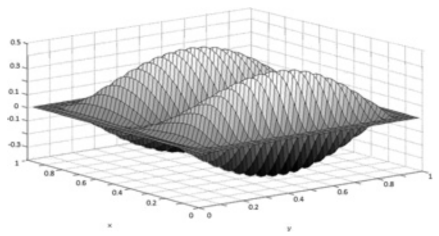
(a) $t = 0$



(b) $t = 60$



(c) $t = 120$



(d) $t = 180$

Fig. 3. Dynamics for $k = 2$

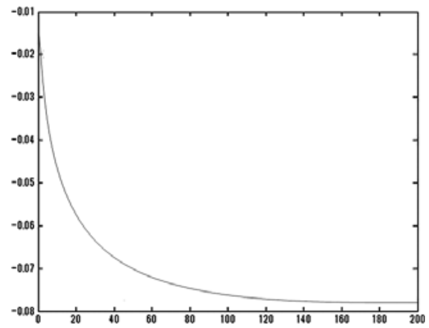
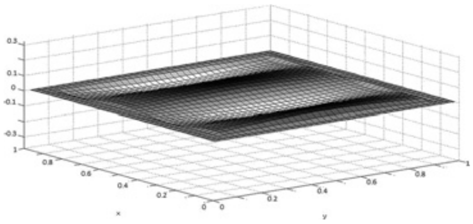


Fig. 4. Lyapunov function for $k = 2$

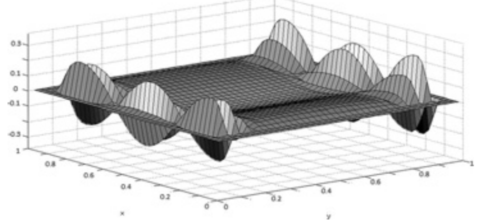
Lyapunov function is given by Figure 4.

Thirdly, consider the case where $k = 3$ in (5.1). As seen by Figure 5, the initial perturbation grows into triple waves. Figure 6 illustrates the graph of the Lyapunov function of trajectory.

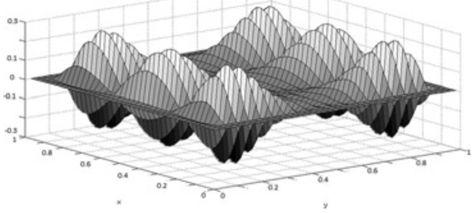
Finally, let $k = 4$ in (5.1). For a while, the small perturbation grows into four waves. Gradually, the states of four waves become unstable. Ultimately, one wave disappears and the trajectory converges to a stationary solution whose profile is the same as that of the case where $k = 3$, see Figures 7 and 8. Notice that in both cases the profiles of final states admit 18 ridges in each wave.



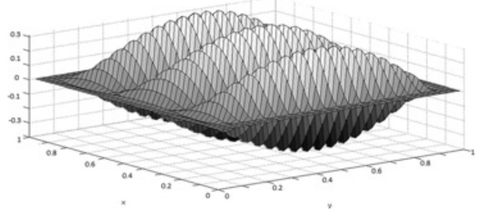
(a) $t = 0$



(b) $t = 80$



(c) $t = 160$



(d) $t = 240$

Fig. 5. Dynamics for $k = 3$

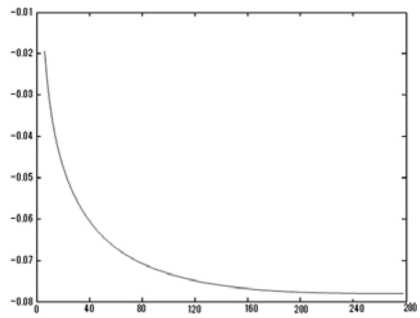
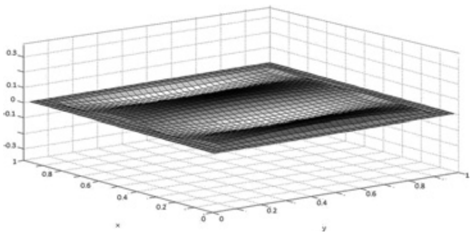
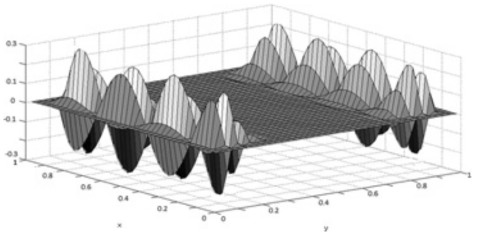


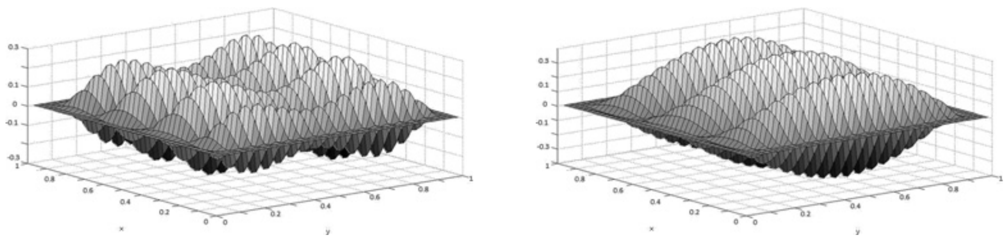
Fig. 6. Lyapunov function for $k = 3$



(a) $t = 0$



(b) $t = 100$



(c) $t = 200$ (d) $t = 300$

Fig. 7. Dynamics for $k = 4$

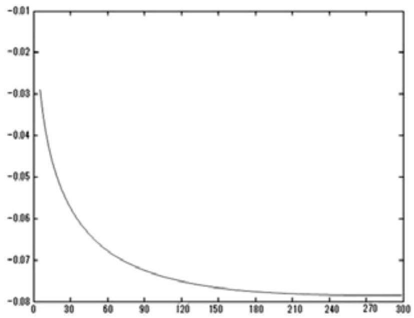


Fig. 8. Lyapunov function for $k = 4$

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