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ILL-POSEDNESS ISSUE FOR THE DRIFT DIFFUSION SYSTEM IN THE HOMOGENEOUS BESOV SPACES

TSUKASA IWABUCHI and TAKAYOSHI OGAWA

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Abstract

We consider the ill-posedness issue for the drift-diffusion system of bipolar type by showing that the continuous dependence on initial data does not hold generally in the scaling invariant Besov spaces. The scaling invariant Besov spaces are $\dot{B}^{-2+n/p}_{p,\sigma}(\mathbb{R}^n)$ with $1 \leq p, \sigma \leq \infty$ and we show the optimality of the case $p = 2n$ to obtain the well-posedness and the ill-posedness for the drift-diffusion system of bipolar type.

1. Introduction

We consider the ill-posedness issue for the initial value problems of a drift-diffusion equation of bipolar type:

$$(1.1)\qquad \begin{cases} \partial_t u - \Delta u + \kappa \nabla \cdot (u \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ \partial_t v - \Delta v - \kappa \nabla \cdot (v \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ -\Delta \psi = v - u, & t > 0, \ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), \ v(0, x) = v_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where u and v are the particle density of negative and positive electric charge, κ is the coupling constant and we assume $\kappa = \pm 1$. u_0 and v_0 are given initial data. The system (1.1) was originally considered for an initial boundary value problem with Dirichlet or Neumann boundary condition as simplest model of a semi-conductor device simulation and we refer to [1, 6, 8, 11, 19, 24] for the related results. As the model of the semiconductor device simulation, the mono-polar model is also considered;

$$(1.2)\qquad \begin{cases} \partial_t u - \Delta u + \kappa \nabla \cdot (u \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ -\Delta \psi = u, & t > 0, \ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

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The well-posedness issue was considered in both models (1.1) and (1.2) (see for instance, [9], [14], [15], [17], [18], [22], [30]). We note that the mono-polar model is considered as the limiting model of the Keller–Segel system in the chemotaxis and there are large literatures for this direction [3], [5], [9], [12], [14], [15], [20], [21], [22], [28].

The both of the problems (1.1) and (1.2) share the common scaling invariant property. Namely under the scaling transform

$$\begin{cases} u_\lambda(t, x) = \lambda^2 u(\lambda^2 t, \lambda x), \\ v_\lambda(t, x) = \lambda^2 v(\lambda^2 t, \lambda x), \\ \psi_\lambda(t, x) = \psi(\lambda^2 t, \lambda x) \end{cases} \tag{1.3}$$

with $\lambda > 0$, the equations in (1.1) and (1.2) remain invariant. Then the common invariant space by the scaling (1.3) in the Bochner space $L^\theta(\mathbb{R}_+; L^p(\mathbb{R}^n))$ is given by a restriction on (θ, p) with

$$2 = \frac{2}{\theta} + \frac{n}{p}.$$

In particular for $\theta = \infty$, the solution is consistent and we reach the invariant function spaces for (1.1) as $(u, v, \psi) \in L^{n/2}(\mathbb{R}^n) \times L^{n/2}(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$. Such a critical space can be generalized by term of the homogeneous Besov spaces with negative regularity indices such as $\dot{B}^{-2+n/p}_{p,\sigma}(\mathbb{R}^n)$.

According to the analytical and scaling structure of the nonlinear coupling term, the most of basic feature for the solutions to both the bipolar system (1.1) and the mono-polar system (1.2) are similar and common except the limiting class of the well-posedness. Namely, there appears a difference between (1.1) and (1.2) for the invariant limiting function spaces with low regularity. To specify the critical space for the well-posedness precisely, we necessarily introduce the scaling invariant Besov spaces $\dot{B}^{-2+n/p}_{p,\infty}(\mathbb{R}^n)$ with $1 \le p \le \infty$. It is shown by Iwabuchi [9] that the initial value problem (1.2) of the mono-polar type is well-posed for small initial data in the Besov spaces $\dot{B}^{-2+n/p}_{p,\infty}(\mathbb{R}^n)$ with $n/2 < p < \infty$ and ill-posed in $\dot{B}^{-2}_{\infty,\infty}(\mathbb{R}^n)$. This shows that the case $p = \infty$ is threshold for the wellposedness issue of the mono-polar type (1.2). On the other hand for the equation (1.1), Zhang–Liu–Ciu [30] showed that the problem is well-posed in $\dot{B}^{-2+n/p}_{p,\infty}(\mathbb{R}^n)$ with $n/2 < p < 2n$, however no ill-posed result can be found in the literatures.

In this paper we show that the critical space for the well-posedness and the ill-posedness to the equation (1.1) is identified as $p = 2n$ through the study of the ill-posedness in the Besov spaces $\dot{B}^{-2+n/p}_{p,\infty}(\mathbb{R}^n)$ $(2n \le p \le \infty)$.

We define the homogeneous Sobolev spaces and the Besov spaces and state our theorems.

We denote the function spaces of rapidly decreasing functions by $\mathcal{S}(\mathbb{R}^n)$, tempered distributions by $\mathcal{S}'(\mathbb{R}^n)$, and polynomials by $\mathcal{P}(\mathbb{R}^n)$.

DEFINITION (the homogeneous Sobolev spaces). For any $s \in \mathbb{R}$, $1 \le p \le \infty$, the homogeneous Sobolev space $\dot{H}^s_p(\mathbb{R}^n)$ is defined by

$$\dot{H}^s_p = \dot{H}^s_p(\mathbb{R}^n) := \{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) \mid \|f\|_{\dot{H}^s_p} := \|\mathcal{F}^{-1}[|\xi|^s \hat{f}(\xi)]\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

DEFINITION (the homogeneous Besov spaces). Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be satisfying the following:

$$\operatorname{supp} \hat{\phi} \subset \{\xi \in \mathbb{R}^n \mid 2^{-1} \le |\xi| \le 2\}, \quad \sum_{j \in \mathbb{Z}} \hat{\phi}\left(\frac{\xi}{2^j}\right) = 1 \quad \text{for} \quad \xi \in \mathbb{R}^n \setminus \{0\},$$

where $\hat{\phi}$ is the Fourier transform of ϕ, and let $\{\phi_j\}_{j \in \mathbb{Z}}$ be defined by

$$\phi_j(x) := 2^{nj} \phi(2^j x) \quad \text{for} \quad j \in \mathbb{Z},\ x \in \mathbb{R}^n.$$

Then, for any $s \in \mathbb{R}$, $1 \le p, \sigma \le \infty$, the homogeneous Besov space $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ is defined by

$$\dot{B}^s_{p,\sigma}(\mathbb{R}^n) := \{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) \mid \|f\|_{\dot{B}^s_{p,\sigma}} := \|\{2^{sj} \|\phi_j * f\|_{L^p(\mathbb{R}^n)}\}_{j \in \mathbb{Z}}\|_{l^\sigma(\mathbb{Z})} < \infty\}.$$

REMARK. One can regard the above homogeneous spaces as a subspace of $\mathcal{S}'(\mathbb{R}^n)$ for some s, p, q. Indeed, if s and p satisfy $s < n/p$, then the homogeneous Sobolev space $\dot{H}^s_p(\mathbb{R}^n)$ is equivalent to

$$\left\{ u \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; \|u\|_{\dot{H}^s_p} < \infty,\ u = \sum_{j \in \mathbb{Z}} \phi_j * u \text{ in } \mathcal{S}'(\mathbb{R}^n) \right\}.$$

If $s < n/p$ with $1 \le q \le \infty$, or $s = n/p$ with $q = 1$, the Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is also equivalent to

$$\left\{ u \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; \|u\|_{\dot{B}^s_{p,q}} < \infty,\ u = \sum_{j \in \mathbb{Z}} \phi_j * u \text{ in } \mathcal{S}'(\mathbb{R}^n) \right\}.$$

These equivalence are due to the argument by Kozono–Yamazaki [16].

Theorem 1.1. *Let $n \ge 2$, $\kappa = \pm 1$ and let p, σ satisfy*

$$2n < p \le \infty \quad \textit{and} \quad 1 \le \sigma \le \infty, \quad \textit{or} \quad p = 2n \quad \textit{and} \quad 2 < \sigma \le \infty. \tag{1.4}$$

Then, there exist a sequence of times $\{T_N\}_N$ *with* $T_N \to 0$ $(N \to \infty)$ *and a sequence of smooth and rapidly decreasing initial data* $\{u_{0,N}\}_N$, $\{v_{0,N}\}_N$ $(N = 1,2,\dots)$ *such that the corresponding sequence of smooth solutions* $\{u_N\}_N$, $\{v_N\}_N$ *to* (1.1) *with* $u_N(0) = u_{0,N}$ *and* $v_N(0) = v_{0,N}$ *satisfies*

$$\lim_{N\to\infty} \|u_{0,N}\|_{\dot{B}^{-2+n/p}_{p,\sigma}} = 0, \quad \lim_{N\to\infty} \|v_{0,N}\|_{\dot{B}^{-2+n/p}_{p,\sigma}} = 0,$$
$$\lim_{N\to\infty} \|u_N(T_N)\|_{\dot{B}^{-2+n/p}_{p,\sigma}} = \infty, \quad \lim_{N\to\infty} \|v_N(T_N)\|_{\dot{B}^{-2+n/p}_{p,\sigma}} = \infty.$$

REMARK. We should make the function space clear where the solution in the above theorem belongs to. There is no result of well-posedness to the system (1.1) in Besov spaces $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ under the condition (1.4). In the proof of theorem, we justify the solutions in the space $C([0, T], M_{n,1}(\mathbb{R}^n))$, where $M_{n,1}(\mathbb{R}^n)$ is the modulation space specified in Section 2. We note that these solutions are in the space $C([0,T], \dot{B}^{-1}_{n,1}(\mathbb{R}^n))$, and it is known that the local well-posedness is obtained in the space $\dot{B}^{-1}_{n,1}(\mathbb{R}^n)$ by the result [30]. Therefore initial data and corresponding solutions can be constructed with enough smoothness to guarantee justification of solutions.

REMARK. We note that when $n = 2$, $p = 4$ and $\sigma = 2$, the Besov space $\dot{B}^{-3/2}_{4,2}(\mathbb{R}^2)$ is the critical space for the well-posedness and ill-posedness. Indeed, one can show the global well-posedness for small initial data in $\dot{B}^{-3/2}_{4,2}(\mathbb{R}^2)$. In general, for $p = 2n$ with $1 \le \sigma \le 2$ except for $(n, \sigma) = (2, 2)$, one can show the ill-posedness in the analogous way in Theorem 1.1.

The main reason why the limitation of the well-posedness class is different in two problems (1.1) and (1.2) is because it depends on how much order the nonlinearity can exhibit derivatives and hence it depends on the symmetry of the nonlinear coupling. For the equation (1.2) of mono-polar type, the nonlinear term $u\nabla(-\Delta)^{-1}u$ satisfies

$$\begin{aligned}\partial_{x_j} u \partial_{x_j}(-\Delta)^{-1} u &= \frac{1}{2}\partial_{x_j}(-\Delta)\{((-\Delta)^{-1}u)(\partial_{x_j}(-\Delta)^{-1}u)\} \\ &\quad + \partial_{x_j}\nabla\cdot\{((-\Delta)^{-1}u)(\nabla\partial_{x_j}(-\Delta)^{-1}u)\} \\ &\quad + \frac{1}{2}\partial^2_{x_j}\{((-\Delta)^{-1}u)u\},\end{aligned} \tag{1.5}$$

which was observed in [9] and hence we can treat the nonlinear term $\nabla\cdot(u\nabla(-\Delta)^{-1}u)$ as a doubly divergence form such as $|\nabla|^2\{(|\nabla|^{-2}u)u\}$. This enables us to treat the equation in the weaker Besov space up to $\dot{B}^{-2+n/p}_{p,\infty}$ with $2 \le p < \infty$ and $\dot{B}^{-2}_{\infty,2}$. On the other hand for the bipolar type (1.1), the nonlinear term $\nabla\cdot(u\nabla(-\Delta)^{-1}v)$ has lack of symmetry in the nonlinear structure which prevents to have such a good expression as (1.5).

To be more precise, let χ be a characteristic function which support is $[-1,1]$ and $e_1 := (1, 0, \dots, 0)$, and we take initial data as

$$u_0 = N^{2-n/p}\mathcal{F}^{-1}[\chi(\,\cdot\, - Ne_1)], \quad v_0 = N^{2-n/p}\mathcal{F}^{-1}[\chi(\,\cdot\, + Ne_1)].$$

We note that the Fourier transforms of u_0 and v_0 are supported locally at particular frequency N and $-N$, respectively, and $\|u_0\|_{\dot{H}_p^{-2+n/p}}$, $\|v_0\|_{\dot{H}_p^{-2+n/p}}$ are independent of N, where $\dot{H}_p^s(\mathbb{R}^n)$ is the homogeneous Sobolev space. By the Duhamel formula, we write the solution by the integral equations:

$$\begin{aligned} u(t) &= e^{t\Delta}u_0 - \kappa \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(u\nabla(-\Delta)^{-1}(v-u))\,d\tau, \\ v(t) &= e^{t\Delta}v_0 + \kappa \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(v\nabla(-\Delta)^{-1}(v-u))\,d\tau. \end{aligned} \tag{1.6}$$

Since both of u and v are treated similarly, we consider the nonlinear term of u only. Then, we approximate the nonlinear part in the right hand side by a linear solution $u \cong e^{\tau\Delta}u_0$, $v \cong e^{\tau\Delta}v_0$ and it follows that

(1.7)

$$\begin{aligned} &\left\| \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(e^{\tau\Delta}u_0\nabla(-\Delta)^{-1}e^{t\Delta}(v_0-u_0))\,d\tau \right\|_{\dot{H}_p^{-2+n/p}} \\ &= \Bigg\| \mathcal{F}^{-1}\Bigg[|\xi|^{-2+n/p}\xi \\ &\qquad \cdot \int_{\mathbb{R}^n} \hat{u}_0(\xi-\eta)(\hat{v}_0(\eta)-\hat{u}_0(\eta))\frac{\eta}{|\eta|^2}\int_0^t e^{-(t-\tau)|\xi|^2}e^{-\tau|\xi-\eta|^2}e^{-\tau|\eta|^2}\,d\tau\,d\eta \Bigg]\Bigg\|_{L^p} \\ &= \left\| \mathcal{F}^{-1}\left[|\xi|^{-2+n/p}\xi\cdot\int_{\mathbb{R}^n}\hat{u}_0(\xi-\eta)(\hat{v}_0(\eta)-\hat{u}_0(\eta))\frac{\eta}{|\eta|^2}\frac{e^{-t|\xi|^2}(e^{2t(\xi-\eta)\cdot\eta}-1)}{2(\xi-\eta)\cdot\eta}\,d\eta \right]\right\|_{L^p} \\ &\cong \left\| \mathcal{F}^{-1}\left[|\xi|^{-2+n/p}\xi\cdot\int_{\mathbb{R}^n}\hat{u}_0(\xi-\eta)\widehat{v_0}(\eta)\frac{N}{N^2}\frac{1}{2N^2}\,d\eta \right]\right\|_{L^p} \\ &\cong \|\mathcal{F}^{-1}[|\xi|^{-2+n/p}\xi\chi(\xi)]\|_{L^p}N^{2(2-n/p)}N^{-1}N^{-2} \\ &\cong N^{1-2n/p}. \end{aligned}$$

The last term diverges as $N \to \infty$ if $p > 2n$. On the other hand for the part with the convolution of $\hat{u}_0$ and $\hat{u}_0$, we have from the structure $\nabla\cdot(u\nabla(-\Delta)^{-1}u) \cong |\nabla|^2\{(|\nabla|^{-2}u)u\}$

by (1.5) and the support of $\hat{u}_0 * \hat{u}_0$ being in the neighborhood of the frequency $2N$

$$
\begin{aligned}
&\left\| \mathcal{F}^{-1}\left[|\xi|^{-2+n/p}\xi \cdot \int_{\mathbb{R}^n} \hat{u}_0(\xi-\eta)\hat{u}_0(\eta)\frac{\eta}{|\eta|^2}\frac{e^{-t|\xi|^2}(e^{2t(\xi-\eta)\cdot\eta}-1)}{2(\xi-\eta)\cdot\eta}\,d\eta \right] \right\|_{L^p} \\
&\cong \|\mathcal{F}^{-1}[|\xi|^{n/p}\chi(\xi-2N)]\|_{L^p} N^{2(2-n/p)}N^{-2}N^{-2} \\
&\cong N^{-n/p}
\end{aligned}
\tag{1.8}
$$

and the last term is bounded for all $p \geq 1$ and $N \in \mathbb{N}$. Therefore, one can expect the divergence of the nonlinear term in (1.6) when $p > 2n$ as $N \to \infty$ by (1.7) and (1.8), while the norms of the initial data are bounded by some constant independent of N.

For the precise proof, we introduce an asymptotic expansion of solutions in terms of the order of the iterative succession by the initial data, and we justify it in the modulation space $M_{n,1}(\mathbb{R}^n)$ which is shown in Section 3. The usage of the modulation space is useful whenever we consider the nonlinear term since $M_{n,1}(\mathbb{R}^n)$ is a Banach algebra. Then we show the term observed in (1.7) tends to ∞ and the other terms in the asymptotic expansion are small.

In Section 4, we give a sequence of initial data and solutions satisfying the statement of our theorem by the use of the asymptotic expansion introduced in Section 3. We treat the system (1.1) with $\kappa = 1$ only in the following sections since the case $\kappa = -1$ can be treated analogously. Finally in Section 5, we compare the results on the local well-posedness for the incompressible Navier–Stokes system in three dimensions.

2. Preliminary

We introduce the modulation spaces $M_{p,\sigma}(\mathbb{R}^n)$ and show some facts for the bilinear term of functions in the modulation spaces which will be used in the proof of our theorem. In what follows, we denote various constants simply by $C > 0$.

DEFINITION (the modulation spaces). Let χ_k is the Fourier window function that satisfies

$$
\operatorname{supp}\hat{\chi}_k \subset \{\xi \in \mathbb{R}^n \mid k_j - 1 \leq \xi_j \leq k_j + 1 \text{ for } j = 1, 2, \dots, n\}, \quad \sum_{k\in\mathbb{Z}^n}\hat{\chi}_k(\xi) \equiv 1.
$$

Then, for any $1 \leq p, \sigma \leq \infty$, the modulation space $M_{p,\sigma} = M_{p,\sigma}(\mathbb{R}^n)$ is defined by

$$
M_{p,\sigma}(\mathbb{R}^n) := \{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \|f\|_{M_{p,\sigma}} := \|\{\|\chi_k * f\|_{L^p(\mathbb{R}^n)}\}_{k\in\mathbb{Z}^n}\|_{l^\sigma(\mathbb{Z}^n)} < \infty\}.
$$

Lemma 2.1 ([7], [25], [27]). (i) *$M_{p_1,\sigma_1}(\mathbb{R}^n) \subset M_{p_2,\sigma_2}(\mathbb{R}^n)$ if $p_1 \leq p_2$ and $\sigma_1 \leq \sigma_2$.*
(ii) *Let $1 \leq p, p' \leq \infty$ satisfy $1/p + 1/p' = 1$. Then, it holds that*

$$
M_{p,\min\{p,p'\}}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset M_{p,\max\{p,p'\}}(\mathbb{R}^n).
$$

(iii) *Let* $1 \leq p, \sigma \leq \infty$. *Then, there exists* $C > 0$ *such that*

$$\| |\nabla| e^{t\Delta} f \|_{M_{p,\sigma}} \leq C t^{-1/2} \| f \|_{M_{p,\sigma}}. \tag{2.1}$$

(iv) *Let* $1 \leq p \leq \infty$. *Then, there exists* $C > 0$ *such that*

$$\| fg \|_{M_{p,1}} \leq C \| f \|_{M_{p,1}} \| g \|_{M_{p,1}}. \tag{2.2}$$

See for the proof, [7], [25], [27]. The following is the lemma for the estimate in Besov spaces.

Lemma 2.2. (i) [13] *Let* $s_0 < s_1$ *and* $1 \leq p \leq \infty$. *Then, there exists* $C > 0$ *such that*

$$\| e^{t\Delta} f \|_{\dot{B}^{s_1}_{p,1}} \leq C t^{-(s_1 - s_0)/2} \| f \|_{\dot{B}^{s_0}_{p,\infty}}, \tag{2.3}$$

for all $f \in \dot{B}^{s_1}_{p,\infty}(\mathbb{R}^n)$.
(ii) [9] *Let* p, p_1, p_2 *satisfy* $1 \leq p, p_1, p_2 \leq \infty$, $1/p = 1/p_1 + 1/p_2$. *Then, there exists* $C > 0$ *such that*

$$\| f \nabla(-\Delta)^{-1} g + g \nabla(-\Delta)^{-1} f \|_{\dot{B}^{-1}_{p,\infty}} \leq C \| f \|_{\dot{B}^{-1}_{p_1,1}} \| g \|_{\dot{B}^{-1}_{p_2,1}}, \tag{2.4}$$

for all $f \in \dot{B}^{-1}_{p_1,1}(\mathbb{R}^n)$ *and* $g \in \dot{B}^{-1}_{p_2,1}(\mathbb{R}^n)$.

3. Asymptotic expansion

We introduce the asymptotic expansion of the solution to (1.1) by some small parameter $\varepsilon > 0$ as

$$u = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \cdots,$$
$$v = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \varepsilon^3 V_3 + \cdots,$$

with the initial data

$$u_0 = \varphi = \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \cdots,$$
$$v_0 = \rho = \rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \cdots.$$

For simplicity, let F be defined by

$$F(u, v) := -\nabla \cdot (u \nabla (-\Delta)^{-1} \nabla v),$$

and let us consider $\kappa = 1$ of the equation (1.1). If u, v satisfy the equation (1.1), we have the followings on the term of order ε^k with $k = 0, 1, 2, \ldots$

$$\varepsilon^0: \begin{cases} (\partial_t - \Delta)U_0 = F(U_0, V_0 - U_0), \\ (\partial_t - \Delta)V_0 = F(V_0, U_0 - V_0), \\ U_1(0) = \varphi_0, V_1(0) = \rho_0, \end{cases}$$

$$\varepsilon^1: \begin{cases} (\partial_t - \Delta)U_1 = F(U_0, V_1 - U_1) + F(U_1, V_0 - U_0), \\ (\partial_t - \Delta)V_1 = F(V_0, U_1 - V_1) + F(V_1, U_0 - V_0), \\ U_1(0) = \varphi_1, V_1(0) = \rho_1, \end{cases}$$

$$\varepsilon^2: \begin{cases} (\partial_t - \Delta)U_2 = F(U_0, V_2 - U_2) + F(U_1, V_1 - U_1) + F(U_2, V_0 - U_0), \\ (\partial_t - \Delta)V_2 = F(V_0, U_2 - V_2) + F(V_1, U_1 - V_1) + F(V_2, U_0 - V_0), \\ U_2(0) = \varphi_2, V_2(0) = \rho_2, \end{cases}$$

$$\cdots$$

respectively. For our proof of the theorem, let $\varphi_k = 0$ and $\rho_k = 0$ for $k = 0, 2, 3, 4, 5, \ldots$ without $k = 1$ and we consider the initial data $u_0 = \varepsilon\varphi_1$, $v_0 = \varepsilon\rho_1$. The term of order ε^k can be reduced as follows:

$$\varepsilon^0: \begin{cases} (\partial_t - \Delta)U_0 = 0, \\ (\partial_t - \Delta)V_0 = 0, \\ U_1(0) = 0, \quad V_1(0) = 0, \end{cases}$$

$$\varepsilon^1: \begin{cases} (\partial_t - \Delta)U_1 = 0, \\ (\partial_t - \Delta)V_1 = 0, \\ U_1(0) = \varphi_1, \quad V_1(0) = \rho_1, \end{cases}$$

$$\varepsilon^2: \begin{cases} (\partial_t - \Delta)U_2 = F(U_1, V_1 - U_1), \\ (\partial_t - \Delta)V_2 = F(V_1, U_1 - V_1), \\ U_2(0) = 0, \quad V_2(0) = 0, \end{cases}$$

$$\cdots$$

$$\varepsilon^k: \begin{cases} (\partial_t - \Delta)U_k = \displaystyle\sum_{k_1+k_2=k,\, k_1,k_2\ge 1} F(U_{k_1}, V_{k_2} - U_{k_2}), \\ (\partial_t - \Delta)V_k = \displaystyle\sum_{k_1+k_2=k,\, k_1,k_2\ge 1} F(V_{k_1}, U_{k_2} - V_{k_2}), \quad \text{for} \quad k \ge 2. \\ U_k(0) = 0, \quad V_k(0) = 0, \end{cases}$$

Then, we introduce $U_k = U_k[\varphi_1, \rho_1]$ and $V_k = V_k[\varphi_1, \rho_1]$ for $k = 1, 2, 3, \ldots$ inductively

(3.1)
$$\begin{cases} U_1[\varphi_1, \rho_1](t) := e^{t\Delta}\varphi_1, \\ U_k[\varphi_1, \rho_1](t) := -\displaystyle\sum_{k_1+k_2=k,\, k_1,k_2\geq 1} \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(U_{k_1}\cdot\nabla(-\Delta)^{-1}(V_{k_2}-U_{k_2}))\,d\tau \\ \text{for any} \quad k = 2, 3, \ldots, \end{cases}$$

$$\begin{cases} V_1[\varphi_1, \rho_1](t) := e^{t\Delta}\rho_1, \\ V_k[\varphi_1, \rho_1](t) := \displaystyle\sum_{k_1+k_2=k,\, k_1,k_2\geq 1} \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(V_{k_1}\cdot\nabla(-\Delta)^{-1}(V_{k_2}-U_{k_2}))\,d\tau \\ \text{for any} \quad k = 2, 3, \ldots. \end{cases} \tag{3.2}$$

Therefore, we obtain a formal expansion

$$u(t) = U[u_0, v_0](t) := \sum_{k=1} \varepsilon^k U_k[\varphi_1, \rho_1],$$
$$v(t) = V[u_0, v_0](t) := \sum_{k=1} \varepsilon^k V_k[\varphi_1, \rho_1],$$

as a solution to (1.1) with the initial data $u_0 = \varepsilon\varphi_1$, $v_0 = \varepsilon\rho_1$. Once we obtain the above expansion of the solution, the formal expansion can be justified by the use of the linear estimate of the propagator $e^{t\Delta}$ and the bilinear estimates in Section 2. Namely, we show the following result.

Proposition 3.1. *For any $u_0, v_0 \in M_{n,1}(\mathbb{R}^n)$ with $|\nabla|^{-1}u_0, |\nabla|^{-1}v_0 \in M_{n,1}(\mathbb{R}^n)$, there exist a small $T > 0$ and a unique local solution $u = u(t, x)$, $v = v(t, x)$ in $C([0, T), M_{n,1}(\mathbb{R}^n))$ to the Cauchy problem* (1.1) *with $|\nabla|^{-1}u, |\nabla|^{-1}v \in C([0, T), M_{n,1}(\mathbb{R}^n))$. Moreover they satisfy the following expansions in $C([0, T), M_{n,1}(\mathbb{R}^n))$: For $0 < \varepsilon \leq 1$,*

$$u(t) = \sum_{k=1}^{\infty} \varepsilon^k U_k[u_0, v_0](t), \quad v(t) = \sum_{k=1}^{\infty} \varepsilon^k V_k[u_0, v_0](t),$$

where $U_k[u_0, v_0]$ and $V_k[u_0, v_0]$ are defined by (3.1) *and* (3.2).

Proof. Let $U_k := U_k[u_0, v_0]$ and $V_k := V[u_0, v_0]$ for simplicity. We consider the case $\varepsilon = 1$ since the other case is the corollary of this case. Let $M, \tilde{M} > 0$ be constants satisfying

$$\|u_0\|_{M_{n,1}} + \|v_0\|_{M_{n,1}} \leq M, \quad \||\nabla|^{-1}u_0\|_{M_{n,1}} + \||\nabla|^{-1}v_0\|_{M_{n,1}} \leq \tilde{M}. \tag{3.3}$$

We claim that there exists $C_0 > 0$ such that for $k \geq 2$ and $\alpha \in \{0, -1\}$

$$
\begin{aligned}
&\| |\nabla|^\alpha U_k(t)\|_{M_{n,1}} + \| |\nabla|^\alpha V_k(t)\|_{M_{n,1}} \\
&\leq \frac{C_0^{k-1}}{(k+1)^2}\left(t^{(k-1)/2-\alpha/2} M\tilde{M}^{k-1} + t^{k-3/2-\alpha/2} M^{k-1}\tilde{M}\right).
\end{aligned} \tag{3.4}
$$

In the case $k = 2$, we have for $\alpha = 0$ and -1 from the smoothing effect of $e^{t\Delta}$ and (2.2)

$$
\begin{aligned}
\| |\nabla|^\alpha U_2\|_{M_{n,1}} &\leq C \left\| |\nabla|^{\alpha+1} \int_0^t e^{(t-\tau)\Delta} U_1 \nabla(-\Delta)^{-1}(V_1 - U_1)\, d\tau \right\|_{M_{n,1}} \\
&\leq C \int_0^t (t-\tau)^{-(1+\alpha)/2} \|U_1 \nabla(-\Delta)^{-1}(V_1 - U_1)\|_{M_{n,1}}\, d\tau \\
&\leq C \int_0^t (t-\tau)^{-(1+\alpha)/2} \|U_1\|_{M_{n,1}} (\| |\nabla|^{-1} V_1\|_{M_{n,1}} + \| |\nabla|^{-1} U_1\|_{M_{n,1}})\, d\tau \\
&\leq C \int_0^t (t-\tau)^{-(1+\alpha)/2}\, d\tau\, M\tilde{M}. \\
&\leq C t^{(1-\alpha)/2} M\tilde{M}.
\end{aligned} \tag{3.5}
$$

The estimate of V_2 is obtained in the same way as that of U_2 and we have (3.4) in the case $k = 2$. For the constant $C_1 > 1$ satisfying the last inequality, let C_0 of (3.4) satisfy $C_0 \geq 2^7 C_1$, and we show (3.4) in the case $k \geq 3$ by induction. Let $k \geq 3$ and we assume (3.4) for $2, 3, \dots, k-1$. Then, we have on the estimate of U_k from the boundedness of the Riesz transform in $M_{n,1}(\mathbb{R}^n)$, the smoothing effect of $e^{t\Delta}$ and (2.2)

$$
\begin{aligned}
&\| |\nabla|^\alpha U_k(t)\|_{M_{n,1}} \\
&\leq C_1 \sum_{k_1+k_2=k} \int_0^t (t-\tau)^{-(1+\alpha)/2} \|U_{k_1}\|_{M_{n,1}} (\| |\nabla|^{-1} V_{k_2}\|_{M_{n,1}} + \| |\nabla|^{-1} U_{k_2})\|_{M_{n,1}})\, d\tau.
\end{aligned} \tag{3.6}
$$

In the case $2 \leq k_j \leq k-1$ $(j = 1, 2)$, we have from the assumption of the induction

$$
\begin{aligned}
&C_1 \int_0^t (t-\tau)^{-(1+\alpha)/2} \|U_{k_1}\|_{M_{n,1}} (\| |\nabla|^{-1} V_{k_2}\|_{M_{n,1}} + \| |\nabla|^{-1} U_{k_2})\|_{M_{n,1}})\, d\tau \\
&\leq C_1 \int_0^t (t-\tau)^{-(1+\alpha)/2} \frac{C_0^{k_1-1}}{(k_1+1)^2} (\tau^{(k_1-1)/2} M\tilde{M}^{k_1-1} + \tau^{k_1-3/2} M^{k_1-1}\tilde{M}) \\
&\qquad \times \frac{C_0^{k_2-1}}{(k_2+1)^2} (\tau^{(k_2-1)/2+1/2} M\tilde{M}^{k_2-1} + \tau^{k_2-3/2+1/2} M^{k_2-1}\tilde{M})\, d\tau \\
&\leq \frac{C_1 C_0^{k-2}}{(k_1+1)^2(k_2+1)^2} \cdot 2t^{1/2-\alpha/2} (t^{k/2-1/2} M^2 \tilde{M}^{k-2} + t^{k/2+k_2/2-3/2} M^{k_2}\tilde{M}^{k_1} \\
&\qquad\qquad + t^{k/2+k_1/2-3/2} M^{k_1}\tilde{M}^{k_2} + t^{k-5/2} M^{k-2}\tilde{M}^2) \\
&\leq \frac{2C_1 C_0^{k-2}}{(k_1+1)^2(k_2+1)^2} (t^{k/2-\alpha/2} M^2 \tilde{M}^{k-2} + t^{k/2+k_2/2-1-\alpha/2} M^{k_2}\tilde{M}^{k_1} \\
&\qquad\qquad + t^{k/2+k_1/2-1-\alpha/2} M^{k_1}\tilde{M}^{k_2} + t^{k-2-\alpha/2} M^{k-2}\tilde{M}^2).
\end{aligned}
$$

We apply Young's inequalities to see that all 4 terms with t, M and $\tilde{M}$ in the last inequality is bounded by

$$t^{(k-1)/2-\alpha/2}M\tilde{M}^{k-1} + t^{k-3/2-\alpha/2}M^{k-1}\tilde{M}$$

which is in the right hand side of (3.4). Therefore, we obtain

$$\begin{aligned}
&C_1\int_0^t (t-\tau)^{-(1+\alpha)/2}\|U_{k_1}\|_{M_{n,1}}(\||\nabla|^{-1}V_{k_2}\|_{M_{n,1}} + \||\nabla|^{-1}U_{k_2})\|_{M_{n,1}})\,d\tau \\
&\le \frac{8C_1C_0^{k-2}}{(k_1+1)^2(k_2+1)^2}(t^{(k-1)/2-\alpha/2}M\tilde{M}^{k-1} + t^{k-3/2-\alpha/2}M^{k-1}\tilde{M}).
\end{aligned} \tag{3.7}$$

In the case $(k_1, k_2) = (1, k-1)$, we can not use (3.4) for $k_1 = 1$ since the power of $t^{k_1-3/2}$ is negative. Then, we have from (3.3) for $k_1 = 1$, (3.4) for $k_2 = k-1$ and Young's inequality

$$\begin{aligned}
&C_1\int_0^t (t-\tau)^{-(1+\alpha)/2}\|U_1\|_{M_{n,1}}(\||\nabla|^{-1}V_{k-1}\|_{M_{n,1}} + \||\nabla|^{-1}U_{k-1})\|_{M_{n,1}})\,d\tau \\
&\le C_1\int_0^t (t-\tau)^{-(1+\alpha)/2}M\cdot\frac{C_0^{k-2}}{(k-1+1)^2}(\tau^{(k-2)/2+1/2}M\tilde{M}^{k-2} \\
&\qquad\qquad + \tau^{k-1-3/2+1/2}M^{k-2}\tilde{M})\,d\tau \\
&\le \frac{C_1C_0^{k-2}}{k^2}\cdot 2t^{1/2-\alpha/2}(t^{k/2-1/2}M^2\tilde{M}^{k-2} + t^{k-2}M^{k-1}\tilde{M}) \\
&\le \frac{2C_1C_0^{k-2}}{k^2}(t^{k/2-\alpha/2}M^2\tilde{M}^{k-2} + t^{k-3/2-\alpha/2}M^{k-1}\tilde{M}) \\
&\le \frac{4C_1C_0^{k-2}}{k^2}(t^{(k-1)/2-\alpha/2}M\tilde{M}^{k-1} + t^{k-3/2-\alpha/2}M^{k-1}\tilde{M}) \\
&= \frac{16C_1C_0^{k-2}}{(k_1+1)^2(k_2+1)^2}(t^{(k-1)/2-\alpha/2}M\tilde{M}^{k-1} + t^{k-3/2-\alpha/2}M^{k-1}\tilde{M}).
\end{aligned} \tag{3.8}$$

Similarly, for the case $(k_1, k_2) = (k-1, 1)$, we also have from (3.4) for $k_1 = k-1$, (3.3) for $k_2 = 1$ and Young's inequality

$$
\begin{aligned}
&C_1 \int_0^t (t-\tau)^{-(1+\alpha)/2} \|U_{k-1}\|_{M_{n,1}} (\||\nabla|^{-1} V_1\|_{M_{n,1}} + \||\nabla|^{-1} U_1)\|_{M_{n,1}})\, d\tau \\
&\le C_1 \int_0^t (t-\tau)^{-(1+\alpha)/2} \frac{C_0^{k-2}}{(k-1+1)^2} (\tau^{(k-2)/2} M\tilde{M}^{k-2} + \tau^{k-1-3/2} M^{k-2}\tilde{M}) \cdot \tilde{M}\, d\tau \\
&\le \frac{C_1 C_0^{k-2}}{k^2} \cdot 2t^{1/2-\alpha/2} (t^{k/2-1} M\tilde{M}^{k-1} + t^{k-5/2} M^{k-2}\tilde{M}^2) \\
&\le \frac{2C_1 C_0^{k-2}}{k^2} (t^{k/2-1/2-\alpha/2} M\tilde{M}^{k-1} + t^{k-2-\alpha/2} M^{k-2}\tilde{M}^2) \\
&\le \frac{4C_1 C_0^{k-2}}{k^2} (t^{(k-1)/2-\alpha/2} M\tilde{M}^{k-1} + t^{k-3/2-\alpha/2} M^{k-1}\tilde{M}) \\
&= \frac{16 C_1 C_0^{k-2}}{(k_1+1)^2 (k_2+1)^2} (t^{(k-1)/2-\alpha/2} M\tilde{M}^{k-1} + t^{k-3/2-\alpha/2} M^{k-1}\tilde{M}).
\end{aligned}
\tag{3.9}
$$

We take the sum of the above 3 inequalities over k_1, k_2 with $k_1 + k_2 = k$ and $k_1, k_2 \ge 1$. It follows from the symmetry of k_1 and k_2 that

$$
\begin{aligned}
\sum_{k_1+k_2=k,\, k_1,k_2\ge 1} \frac{1}{(k_1+1)^2(k_2+1)^2} &\le 2 \sum_{1\le k_1\le k/2} \frac{1}{(k_1+1)^2(k-k_1+1)^2} \\
&= 2 \sum_{1\le k_1\le k/2} \left\{ \frac{1}{k+2} \left(\frac{1}{k_1+1} + \frac{1}{k-k_1+1} \right) \right\}^2 \\
&\le \frac{2}{(k+2)^2} \sum_{1\le k_1\le k/2} \left(\frac{2}{k_1+1} \right)^2 \\
&\le \frac{8}{(k+2)^2} \int_1^{k/2} \frac{1}{x^2}\, dx \\
&\le \frac{8}{(k+2)^2}.
\end{aligned}
$$

By (3.7), (3.8), (3.9), the last inequality and $C_0 \ge 2^7 C_1$, we obtain

$$
\begin{aligned}
&\||\nabla|^{\alpha} U_k(t)\|_{M_{n,1}} \\
&\le \sum_{k_1+k_2=k} \frac{16 C_1 C_0^{k-2}}{(k_1+1)^2(k_2+1)^2} (t^{(k-1)/2-\alpha/2} M\tilde{M}^{k-1} + t^{k-3/2-\alpha/2} M^{k-1}\tilde{M}) \\
&\le \frac{8}{(k+2)^2} \cdot 16 C_1 C_0^{k-2} (t^{(k-1)/2-\alpha/2} M\tilde{M}^{k-1} + t^{k-3/2-\alpha/2} M^{k-1}\tilde{M}) \\
&\le \frac{C_0^{k-1}}{(k+1)^2} (t^{(k-1)/2-\alpha/2} M\tilde{M}^{k-1} + t^{k-3/2-\alpha/2} M^{k-1}\tilde{M}).
\end{aligned}
\tag{3.10}
$$

It is possible to show the same estimate as (3.10) for V_k in the same way to obtain (3.4).

Let $T > 0$ satisfy

$$C_0 T^{1/2}\tilde{M} + C_0 T M \le 1 \quad \text{and} \quad T \le 1, \tag{3.11}$$

and we have on the series $\sum_{k\ge 1} U_k$ and $\sum_{k\ge 1} V_k$

$$\begin{aligned}
&\sup_{t\in(0,T)} \sum_{k\ge 1} \| |\nabla|^{\alpha} U_k \|_{M_{n,1}} + \sup_{t\in(0,T)} \sum_{k\ge 1} \| |\nabla|^{\alpha} V_k \|_{M_{n,1}} \\
&\le M + \sum_{k\ge 2} \frac{C_0^{k-1}}{(k+1)^2} (T^{(k-1)/2-\alpha/2} M \tilde{M}^{k-1} + T^{k-3/2-\alpha/2} M^{k-1} \tilde{M}) \\
&\le M + \sum_{k\ge 2} \frac{M+\tilde{M}}{(k+1)^2} \\
&< \infty.
\end{aligned}$$

Then, we conclude that $U := \sum_{k\ge 1} U_k$ and $V := \sum_{k\ge 1} V_k$ are well defined and note that

$$\begin{aligned}
U &= \sum_{k=1}^{\infty} U_k \\
&= U_1 - \sum_{k=2}^{\infty} \sum_{k_1+k_2=k} \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (U_{k_1} \nabla(-\Delta)^{-1}(V_{k_2} - U_{k_2}))\, d\tau \\
&= U_1 - \sum_{k=2}^{\infty} \sum_{k_1=1}^{k-1} \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (U_{k_1} \nabla(-\Delta)^{-1}(V_{k-k_1} - U_{k-k_1}))\, d\tau \\
&= U_1 - \sum_{k_1=1}^{\infty} \sum_{k=k_1+1}^{\infty} \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (U_{k_1} \nabla(-\Delta)^{-1}(V_{k-k_1} - U_{k-k_1}))\, d\tau \\
&= U_1 - \int_0^t e^{(t-\tau)\Delta} \nabla \cdot \left\{ \left(\sum_{k_1=1}^{\infty} U_{k_1} \right) \nabla(-\Delta)^{-1} \sum_{k_2=1}^{\infty} (V_{k_2} - U_{k_2}) \right\} d\tau \\
&= U_1 - \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (U \nabla(-\Delta)^{-1}(V - U))\, d\tau,
\end{aligned}$$

and V also satisfies the integral equation:

$$V = V_1 + \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (V \nabla(-\Delta)^{-1}(V - U))\, d\tau.$$

This shows that (U, V) is subject to the problem (1.1) with the initial data (u_0, v_0) for small time interval $[0, T)$.

We finally show the uniqueness of the solution in the class $C([0,T); M_{n,1})$. Let (u,v) and $(\tilde{u},\tilde{v})$ be solutions in $C([0,T); M_{n,1}) \cap L^\infty([0,T); \dot{M}_{n,1}^{-1})$ with the same initial data (u_0, v_0), where $\dot{M}_{n,1}^{-1} = \{ f \in \mathcal{S}'; |\nabla|^{-1} f \in M_{n,1}(\mathbb{R}^n)\}$. We define R and $w(t)$ by

$$
\begin{aligned}
R &:= \sup_{\alpha\in\{0,-1\},\tau\in[0,T]} (\| |\nabla|^\alpha u(\tau)\|_{M_{n,1}} + \| |\nabla|^\alpha v(\tau)\|_{M_{n,1}} \\
&\qquad\qquad + \| |\nabla|^\alpha \tilde{u}(\tau)\|_{M_{n,1}} + \| |\nabla|^\alpha \tilde{v}(\tau)\|_{M_{n,1}}),\\
w(t) &:= \sup_{\alpha\in\{0,-1\},\tau\in[0,t]} (\| |\nabla|^\alpha (u(\tau)-\tilde{u}(\tau))\|_{M_{n,1}} + \| |\nabla|^\alpha (v(\tau)-\tilde{v}(\tau))\|_{M_{n,1}}).
\end{aligned}
$$

By the equality

$$
\begin{aligned}
u(t)-\tilde{u}(t) = \int_0^t e^{(t-\tau)\Delta}\nabla\cdot ((u-\tilde{u})\nabla(-\Delta)^{-1}(u-v) \\
+ \tilde{u}\nabla(-\Delta)^{-1}(u-v-(\tilde{u}-\tilde{v})))\,d\tau
\end{aligned}
$$

and the analogous estimate to (3.5), we have

$$
\begin{aligned}
&\| |\nabla|^\alpha (u(t)-\tilde{u}(t))\|_{M_{n,1}} \\
&\le C\int_0^t (t-\tau)^{-(1+\alpha)/2}\{\|u-\tilde{u}\|_{M_{n,1}} \| |\nabla|^{-1}(u-v)\|_{M_{n,1}} \\
&\qquad\qquad + \|\tilde{u}\|_{M_{n,1}}(\| |\nabla|^{-1}(u-\tilde{u})\|_{M_{n,1}} + \| |\nabla|^{-1}(v-\tilde{v})\|_{M_{n,1}})\}\,d\tau \\
&\le Ct^{(1-\alpha)/2} R w(t),\\
&\| |\nabla|^\alpha (v(t)-\tilde{v}(t))\|_{M_{n,1}} \le Ct^{(1-\alpha)/2} R w(t).
\end{aligned}
$$

It follows from the above two estimates that

$$
w(t) \le C(t^{1/2}+t)Rw(t) \quad \text{for all} \quad t\in[0,T].
$$

Combining $w(0)=0$, we obtain $w(t)=0$ if $t\in[0,T]$ satisfies $C(t^{1/2}+t)R \le 1/2$. Repeating this procedure, we obtain the uniqueness of the solution. □

4. Proof of Theorem 1.1

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be radial and satisfy

$$
\operatorname{supp}\varphi \subset \{\xi\in\mathbb{R}^n \mid |\xi| \le n\}
$$

and

$$
\varphi(\xi) = 1 \quad \text{if} \quad |\xi|\le 1,
$$

and let $\{\varphi_j^\pm\}$ be defined by

$$
\varphi_j^\pm(\xi) := \varphi(\xi \mp 2^j(1,0,\dots,0)) \quad \text{for} \quad \xi\in\mathbb{R}^n.
$$

Let initial data $\{u_{0,N}\}_{N=1}^{\infty}$, $\{v_{0,N}\}_{N=1}^{\infty}$ be defined by

$$u_{0,N} := N^{-1/2} \log N \sum_{N \le j \le (1+\delta)N} 2^{(3/2)j} \mathcal{F}^{-1}[\varphi_j^+],$$
$$v_{0,N} := N^{-1/2} \log N \sum_{N \le j \le (1+\delta)N} 2^{(3/2)j} \mathcal{F}^{-1}[\varphi_j^-],$$

where $\delta > 0$ satisfy $0 \le \delta \le 1/7$. On the estimate of $U_1[u_{0,N}, v_{0,N}]$ and $V_1[u_{0,N}, v_{0,N}]$, we have for $p > 2n$ and $1 \le \sigma \le \infty$

$$\begin{aligned} &\|U_1[u_{0,N}, v_{0,N}]\|_{\dot{B}^{-2+n/p}_{p,\sigma}} + \|V_1[u_{0,N}, v_{0,N}]\|_{\dot{B}^{-2+n/p}_{p,\sigma}} \\ &\le CN^{-1/2} \log N \left\{ \sum_{N \le j \le (1+\delta)N} (2^{(-2+n/p)j} 2^{(3/2)j})^{\sigma} \right\}^{1/\sigma} \\ &\le CN^{-1/2} \log N \\ &\to 0 \quad \text{as} \quad N \to \infty, \end{aligned} \tag{4.1}$$

and we also have for $p = 2n$ and $2 < \sigma \le \infty$

$$\begin{aligned} &\|U_1[u_{0,N}, v_{0,N}]\|_{\dot{B}^{-2+n/p}_{p,\sigma}} + \|V_1[u_{0,N}, v_{0,N}]\|_{\dot{B}^{-2+n/p}_{p,\sigma}} \\ &\le CN^{-1/2} \log N \left\{ \sum_{N \le j \le (1+\delta)N} (2^{-(3/2)j} 2^{(3/2)j})^{\sigma} \right\}^{1/\sigma} \\ &\le C\delta^{1/\sigma} N^{-1/2+1/\sigma} \log N \\ &\to 0 \quad \text{as} \quad N \to \infty. \end{aligned} \tag{4.2}$$

On the estimate of $\{U_k[u_{0,N}, v_{0,N}]\}_{k\ge2}$ and $\{V_k[u_{0,N}, v_{0,N}]\}_{k\ge2}$, we use the following propositions.

Proposition 4.1. *Let* $\alpha \in \{-1, 0\}$. *Then, there exists* $C > 0$ *such that*

$$\begin{aligned} &\||\nabla|^{\alpha} U_1[u_{0,N}, v_{0,N}](t)\|_{M_{n,1}} + \||\nabla|^{\alpha} V_1[u_{0,N}, v_{0,N}](t)\|_{M_{n,1}} \\ &\le C 2^{(3/2+\alpha)(1+\delta)N} N^{-1/2} \log N, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\||\nabla|^{\alpha} U_k[u_{0,N}, v_{0,N}](t)\|_{M_{n,1}} + \||\nabla|^{\alpha} V_k[u_{0,N}, v_{0,N}](t)\|_{M_{n,1}} \\ &\le C^k (t^{(k-1)/2-\alpha/2} 2^{(k/2+1)(1+\delta)N} \\ &\qquad + t^{k-3/2-\alpha/2} 2^{((3/2)k-1)(1+\delta)N}) N^{-k/2} (\log N)^k \quad \textit{for} \quad k \ge 2. \end{aligned} \tag{4.4}$$

Proof. To show (4.3), we apply the boundedness of $e^{t\Delta}$ in $M_{n,1}(\mathbb{R}^n)$ to obtain

$$\begin{aligned}
&\||\nabla|^{\alpha}U_1[u_{0,N}, v_{0,N}]\|_{M_{n,1}} + \||\nabla|^{\alpha}U_1[u_{0,N}, v_{0,N}]\|_{M_{n,1}} \\
&\le CN^{-1/2}\log N \sum_{N\le j\le(1+\delta)N} 2^{(3/2+\alpha)j}(\|\mathcal{F}^{-1}[\varphi_j^{+}]\|_{L^n} + \|\mathcal{F}^{-1}[\varphi_j^{-}]\|_{L^n}) \\
&\le C2^{(3/2+\alpha)(1+\delta)N}N^{-1/2}\log N.
\end{aligned}$$

We prove (4.4) with (3.4). Let $C_1 > 0$ be a constant which satisfies the above last inequality, and we take $M, \tilde{M}$ of (3.4) as

$$\begin{aligned}
M &:= 2C_1 2^{(3/2)(1+\delta)N}N^{-1/2}\log N, \\
\tilde{M} &:= 2C_1 2^{(1/2)(1+\delta)N}N^{-1/2}\log N.
\end{aligned}$$

Then, we have (3.3) for $u_{0,N}$ and $v_{0,N}$ instead of u_0 and v_0, and apply (3.4) to obtain (4.4). □

Proposition 4.2. *Let* $\phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ *satisfy*

$$\begin{aligned}
&\hat{\phi} \ge 0, \\
&\operatorname{supp}\hat{\phi} \subset \{\xi \in \mathbb{R}^n \mid 3/4 \le \xi_1 \le 1,\ 0 \le \xi_j \le 1/n \ \text{for}\ j = 2, 3, \dots, n\},
\end{aligned} \tag{4.5}$$

and let p, σ *satisfy* (1.4). *Then, there exist* $c, C > 0$ *such that for* $t = 2^{-2N}$

$$\|\phi * U_2[u_{0,N}, v_{0,N}](t)\|_{\dot{B}^{-2+n/p}_{p,\infty}} \ge c(\log N)^2 - CN^{-1}(\log N)^2, \tag{4.6}$$

$$\|\phi * V_2[u_{0,N}, v_{0,N}](t)\|_{\dot{B}^{-2+n/p}_{p,\infty}} \ge c(\log N)^2 - CN^{-1}(\log N)^2. \tag{4.7}$$

Proof. For simplicity, let $U_k := U_k[u_{0,N}, v_{0,N}]$ and $V_k := V_k[u_{0,N}, v_{0,N}]$. We prove (4.6) only since (4.7) is shown analogously. We have from the triangle inequality

$$\begin{aligned}
\|\phi * U_2\|_{\dot{B}^{-2+n/p}_{p,\sigma}} &\ge \left\|\phi * \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(U_1\nabla(-\Delta)^{-1}V_1)\,d\tau\right\|_{\dot{B}^{-2+n/p}_{p,\sigma}} \\
&\quad - \left\|\phi * \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(U_1\nabla(-\Delta)^{-1}U_1)\,d\tau\right\|_{\dot{B}^{-2+n/p}_{p,\sigma}} \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{4.8}$$

On the estimate of I, we have for $t = 2^{-2N}$ and ξ with $3/4 \le \xi_1 \le 1$ and $0 \le \xi_m \le 1/n$

$(m = 2, 3, \dots, n)$

(4.9)

$$\begin{aligned}
&\left|\int_0^t e^{(t-\tau)\Delta}\nabla\cdot(U_1\nabla(-\Delta)^{-1}V_1)\,d\tau\right| \\
&= \left|\int_0^t e^{-(t-\tau)|\xi|^2}\xi\cdot\int_{\mathbb{R}^n} e^{-\tau|\xi-\eta|^2}\hat{u}_{0,N}(\xi-\eta)\frac{\eta}{|\eta|^2}e^{-\tau|\eta|^2}\hat{v}_{0,N}(\eta)\,d\eta\,d\tau\right| \\
&= \left|e^{-t|\xi|^2}\xi\cdot\int_{\mathbb{R}^n}\frac{1-e^{2t(\xi-\eta)\cdot\eta}}{2(\xi-\eta)\cdot\eta}\frac{\eta}{|\eta|^2}\hat{u}_{0,N}(\xi-\eta)\hat{v}_{0,N}(\eta)\,d\eta\,d\tau\right| \\
&\ge cN^{-1}(\log N)^2\left|\sum_{N\le j\le(1+\delta)N}\xi\cdot\int_{\mathbb{R}^n}\frac{1-e^{2t(\xi-\eta)\cdot\eta}}{2(\xi-\eta)\cdot\eta}\frac{\eta}{|\eta|^2}2^{(3/2)j}\varphi_j^+(\xi-\eta)2^{(3/2)j}\varphi_j^-(\eta)\,d\eta\right| \\
&\ge cN^{-1}(\log N)^2\sum_{N\le j\le(1+\delta)N}\int_{\mathbb{R}^n}2^{-3j}2^{3j}\varphi_j^+(\xi-\eta)\varphi_j^-(\eta)\,d\eta \\
&\ge c\delta(\log N)^2.
\end{aligned}$$

Then, we have from (4.9)

$$I \ge c\delta(\log N)^2. \tag{4.10}$$

On the estimate of II, we have from the embedding $\dot{B}^{-1}_{n,1}(\mathbb{R}^n) \hookrightarrow \dot{B}^{-2+n/p}_{p,\sigma}(\mathbb{R}^n)$, (2.3) and (2.4)

$$\begin{aligned}
\mathrm{II} &\le C\int_0^t \|e^{(t-\tau)\Delta}\nabla\cdot U_1\nabla(-\Delta)U_1\|_{\dot{B}^{-1}_{n,1}}\,d\tau \\
&\le C\int_0^t \|e^{(t-\tau)\Delta}U_1\nabla(-\Delta)U_1\|_{\dot{B}^{0}_{n,1}}\,d\tau \\
&\le C\int_0^t (t-\tau)^{-1/2}\|U_1\nabla(-\Delta)U_1\|_{\dot{B}^{-1}_{n,\infty}}\,d\tau \\
&\le C\int_0^t (t-\tau)^{-1/2}\|U_1\|^2_{\dot{B}^{-1}_{2n,1}}\,d\tau \\
&\le C\int_0^t (t-\tau)^{-1/2}\tau^{-1/2}\|u_{0,N}\|^2_{\dot{B}^{-3/2}_{2n,\infty}}\,d\tau \\
&\le CN^{-1}(\log N)^2.
\end{aligned} \tag{4.11}$$

Therefore, we obtain (4.6) by (4.8), (4.10) and (4.11). □

We consider the sequence $\{(u_N, v_N)\}_{N=1}^{\infty}$ of the solutions which are expanded by

$$u_N(t) = \sum_{k=1}^{\infty} U_k[u_{0,N}, v_{0,N}](t), \quad v_N(t) = \sum_{k=1}^{\infty} V_k[u_{0,N}, v_{0,N}](t),$$

with the initial data $u_N(0) = u_{0,N}$ and $v_N(0) = v_{0,N}$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a function satisfying (4.5). We consider the solution at time $t = 2^{-2N}$ and have from triangle inequality, (4.6) and $\phi * U_1 = 0$,

$$
\begin{aligned}
&\|u_N(t)\|_{\dot{B}^{-2+n/p}_{p,\sigma}} \\
&= \|u_N(2^{-2N})\|_{\dot{B}^{-2+n/p}_{p,\sigma}} \\
&\geq c\|\phi * u_N(2^{-2N})\|_{\dot{B}^{-2+n/p}_{p,\sigma}} \\
&\geq c\|\phi * U_2[u_{0,N}, v_{0,N}](2^{-2N})\|_{\dot{B}^{-2+n/p}_{p,\sigma}} - c\|\phi * U_1[u_{0,N}, v_{0,N}](2^{-2N})\|_{\dot{B}^{-2+n/p}_{p,\sigma}} \\
&\quad - c\sum_{k\geq 3}\|\phi * U_k[u_{0,N}, v_{0,N}](2^{-2N})\|_{\dot{B}^{-2+n/p}_{p,\sigma}} \\
&\geq c(\log N)^2 - CN^{-1}(\log N)^2 - C\sum_{k\geq 3}\|\phi * U_k[u_{0,N}, v_{0,N}](2^{-2N})\|_{\dot{B}^{-2+n/p}_{p,\sigma}}.
\end{aligned} \tag{4.12}
$$

Then, we have from $\operatorname{supp}\hat{\phi}$ being compact and away from the origin, (4.4) and $t = 2^{-2N}$

$$
\begin{aligned}
&\sum_{k\geq 3} C\|\phi * U_k[u_{0,N}, v_{0,N}](t)\|_{\dot{B}^{-2+n/p}_{p,\sigma}} \\
&\leq C\sum_{k\geq 3}\||\nabla|^{-1}U_k[u_{0,N}, v_{0,N}](t)\|_{M_{n,1}} \\
&\leq \sum_{k\geq 3} C^k\big(t^{(k-1)/2+1/2}2^{(k/2+1)(1+\delta)N} + t^{k-3/2+1/2}2^{((3/2)k-1)(1+\delta)N}\big)N^{-k/2}(\log N)^k \\
&\leq \sum_{k\geq 3} C^k\big(2^{-kN}2^{(k/2+1)(1+\delta)N} + 2^{-2(k-1)N}2^{((3/2)k-1)(1+\delta)N}\big)N^{-k/2}(\log N)^k.
\end{aligned} \tag{4.13}
$$

Since $\delta \leq 1/7$, we have

$$2^{-kN}2^{(k/2+1)(1+\delta)N} + 2^{-2(k-1)N}2^{((3/2)k-1)(1+\delta)N} \leq 2 \quad \text{if} \quad k \geq 3.$$

Then, it follows from (4.13) that

$$\sum_{k\geq 3} C\|\phi * U_k[u_{0,N}, v_{0,N}](2^{-2N})\|_{\dot{B}^{-2+n/p}_{p,\sigma}} \to 0 \quad \text{as} \quad N \to \infty.$$

Therefore, we have from (4.12) and the last estimate on the solution at time $t = 2^{-2N}$

$$\|u(2^{-2N})\|_{\dot{B}^{-2+n/p}_{p,\sigma}} \to \infty \quad \text{as} \quad N \to \infty.$$

The divergence of v at time $t = 2^{-2N}$ is obtained in the same way and we complete the proof of Theorem 1.1.

5. Concluding remark on Navier–Stokes system

We should mention finally that similar structures to (1.5) exist for the incompressible Navier–Stokes equations and the vorticity equations. For the incompressible Navier–Stokes equations

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0, & t > 0,\ x \in \mathbb{R}^3, \\ \operatorname{div} u = 0, & t > 0,\ x \in \mathbb{R}^3, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^3, \end{cases}$$

the scaling invariant Besov spaces are $\dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)$ with $1 \le p, q \le \infty$. One can regard the case $p = 3$ $(= n)$ for the Navier–Stokes equations as the case $p = 3/2$ $(= n/2)$ for the drift diffusion system (1.1) in the Besov spaces $\dot{B}^{-2+n/p}_{p,q}(\mathbb{R}^n)$ since the scaling invariant Lebesgue space for the incompressible Navier–Stokes equations and the drift diffusion equations is $L^3(\mathbb{R}^3)$ and $L^{3/2}\mathbb{R}^3)$, respectively.

The well-posedness for the Navier–Stokes equations in $\dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$ $(3 < p < \infty)$ was considered in Kozono–Yamazaki [16], Cannone–Planchon [4], and the ill-posedness in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ was shown by Bourgain–Pavlović [2] and Yoneda [29]. Therefore, the case $p = \infty$ is optimal for the well-posedness and the ill-posedness on the study of the Navier–Stokes equations, and the important structure of nonlinear term is $\operatorname{div} u = 0$ and $(u \cdot \nabla) u = \nabla \cdot (u \otimes u)$, which corresponds to the structure (1.5) of the equation (1.2). For the vorticity equations:

$$\begin{cases} \partial_t \omega - \Delta \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u = 0, & t > 0,\ x \in \mathbb{R}^3, \\ \operatorname{div} \omega = 0, & t > 0,\ x \in \mathbb{R}^3, \\ \omega(0, x) = \operatorname{rot} u_0(x), & x \in \mathbb{R}^3, \end{cases}$$

scaling invariant Besov spaces for the vorticity ω are $\dot{B}^{-2+3/p}_{p,\sigma}(\mathbb{R}^3)$ $(1 \le p, \sigma \le \infty)$ since $\omega = \operatorname{rot} u$. Critical case should be $p = \infty$ since the case $p = \infty$ for the Navier–Stokes equation is optimal for the well-posedness and the ill-posedness. In those system, the nonlinear structure is again has a special symmetry and one can find that the nonlinearity can be expressed by $(u \cdot \nabla)\omega - (\omega \cdot \nabla) u \cong |\nabla|^2 \{(|\nabla|^{-2}\omega)\omega\}$ similar to (1.5). Indeed, the first component of the nonlinear term can be seen that

$$\begin{aligned} &\sum_{j=1}^{3} (u_j \partial_{x_j} \omega_1 - \omega_j \partial_{x_j} u_1) \\ &= \sum_{j=1}^{3} \partial_{x_j} (u_j \omega_1 - \omega_j u_1) \\ &= \sum_{j=1}^{3} \partial_j [\{(-\Delta)^{-1} (\partial_{x_k} \omega_l - \partial_{x_l} \omega_k)\} \omega_1 - \omega_j (-\Delta)^{-1} (\partial_{x_2} \omega_3 - \partial_{x_3} \omega_2)], \end{aligned}$$

where we used $\operatorname{div} u = \operatorname{div} \omega = 0$, the Bio–Savart law $u = (-\Delta)^{-1} \operatorname{rot} \omega$, and $(j, k, l) \in \{1, 2, 3\}^3$ satisfy the property of cyclic change, namely $j + 2 \equiv k + 1 \equiv l \pmod 3$. For simplicity, let $I_j := \{(-\Delta)^{-1}(\partial_{x_k} \omega_{x_l} - \partial_{x_l} \omega_k)\}\omega_1 - \omega_j(-\Delta)^{-1}(\partial_{x_2}\omega_3 - \partial_{x_3}\omega_2)$. If $j = 1$, we have

$$I_1 = \{(-\Delta)^{-1}(\partial_{x_2}\omega_3 - \partial_{x_3}\omega_2)\}\omega_1 - \omega_1(-\Delta)^{-1}(\partial_{x_2}\omega_3 - \partial_{x_3}\omega_2) = 0.$$

If $j = 2$, we have from $\operatorname{div} \omega = 0$

$$\begin{aligned} I_2 &= \{(-\Delta)^{-1}(\partial_{x_3}\omega_1 - \partial_{x_1}\omega_3)\}\omega_1 - \omega_2(-\Delta)^{-1}(\partial_{x_2}\omega_3 - \partial_{x_3}\omega_2) \\ &= \omega_1(-\Delta)^{-1}\partial_{x_3}\omega_1 + \omega_2(-\Delta)^{-1}\partial_{x_3}\omega_2 + \omega_3(-\Delta)^{-1}\partial_{x_3}\omega_3 - (\omega \cdot (-\Delta)^{-1}\nabla)\omega_3 \\ &= \omega_1(-\Delta)^{-1}\partial_{x_3}\omega_1 + \omega_2(-\Delta)^{-1}\partial_{x_3}\omega_2 + \omega_3(-\Delta)^{-1}\partial_{x_3}\omega_3 - \nabla \cdot (\omega(-\Delta)^{-1}\omega_3). \end{aligned}$$

The first, second and third terms of the last right hand side can be regarded as $|\nabla|(\omega|\nabla|^{-2}\omega)$ analogous way of (1.5) and we can regard $\partial_{x_2} I_2$ as $|\nabla|^2(\omega|\nabla|^{-2}\omega)$. The case $j = 3$ is also treated in the similar way to the case $j = 2$. The other components are also treated analogously, and therefore we obtain the structure $(u \cdot \nabla)\omega - (\omega \cdot \nabla)u \cong |\nabla|^2\{(|\nabla|^{-2}\omega)\omega\}$ similarly to (1.5).

References

[1] P. Biler and J. Dolbeault: *Long time behavior of solutions of Nernst–Planck and Debye–Hückel drift-diffusion systems*, Ann. Henri Poincaré **1** (2000), 461–472.

[2] J. Bourgain and N. Pavlović: *Ill-posedness of the Navier–Stokes equations in a critical space in* 3*D*, J. Funct. Anal. **255** (2008), 2233–2247.

[3] P. Biler, M. Cannone, I.A. Guerra and G. Karch: *Global regular and singular solutions for a model of gravitating particles*, Math. Ann. **330** (2004), 693–708.

[4] M. Cannone and F. Planchon: *Self-similar solutions for Navier–Stokes equations in* $\mathbf{R}^3$, Comm. Partial Differential Equations **21** (1996), 179–193.

[5] L. Corrias, B. Perthame and H. Zaag: *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions*, Milan J. Math. **72** (2004), 1–28.

[6] W. Fang and K. Ito: *Global solutions of the time-dependent drift-diffusion semiconductor equations*, J. Differential Equations **123** (1995), 523–566.

[7] H.G. Feichtinger: *Modulation spaces on locally compact Abelian groups*, Technical Report, University of Vienna, 1983, in; Proc. Internat. Conf. on Wavelets and Applications, New Delhi Allied Publishers, 1–56, 2003.

[8] H. Gajewski and K. Gröger: *On the basic equations for carrier transport in semiconductors*, J. Math. Anal. Appl. **113** (1986), 12–35.

[9] T. Iwabuchi: *Global well-posedness for Keller–Segel system in Besov type spaces*, J. Math. Anal. Appl. **379** (2011), 930–948.

[10] T. Iwabuchi and T. Ogawa: *Ill-posedness for the nonlinear Schrödinger equation with quadratic non-linearity in low dimensions*, Trans. Amer. Math. Soc. **367** (2015), 2613–2630.

[11] A. Jüngel: *Qualitative behavior of solutions of a degenerate nonlinear drift-diffusion model for semiconductors*, Math. Models Methods Appl. Sci. **5** (1995), 497–518.

[12] E.F. Keller and L.A. Segel: *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol., **26** (1970), 399–415.

[13] H. Kozono, T. Ogawa and Y. Taniuchi: *Navier–Stokes equations in the Besov space near* L^∞ *and BMO*, Kyushu J. Math. **57** (2003), 303–324.

[14] H. Kozono and Y. Sugiyama: *Local existence and finite time blow-up of solutions in the* 2*-D Keller–Segel system*, J. Evol. Equ. **8** (2008), 353–378.

[15] H. Kozono and Y. Sugiyama: *Strong solutions to the Keller–Segel system with the weak* $L^{n/2}$ *initial data and its application to the blow-up rate*, Math. Nachr. **283** (2010), 732–751.

[16] H. Kozono and M. Yamazaki: *Semilinear heat equations and the Navier–Stokes equation with distributions in new function spaces as initial data*, Comm. Partial Differential Equations **19** (1994), 959–1014.

[17] M. Kurokiba and T. Ogawa: *Finite time blow-up of the solution for a nonlinear parabolic equation of drift-diffusion type*, Differential Integral Equations **16** (2003), 427–452.

[18] M. Kurokiba and T. Ogawa: *Well-posedness for the drift-diffusion system in* L^p *arising from the semiconductor device simulation*, J. Math. Anal. Appl. **342** (2008), 1052–1067.

[19] M.S. Mock: *An initial value problem from semiconductor device theory*, SIAM J. Math. Anal. **5** (1974), 597–612.

[20] T. Nagai: *Behavior of solutions to a parabolic-elliptic system modelling chemotaxis*, J. Korean Math. Soc. **37** (2000), 721–733.

[21] T. Nagai, T. Senba and K. Yoshida: *Application of the Trudinger–Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac. **40** (1997), 411–433.

[22] T. Ogawa and S. Shimizu: *The drift-diffusion system in two-dimensional critical Hardy space*, J. Funct. Anal. **255** (2008), 1107–1138.

[23] T. Ogawa and S. Shimizu: *End-point maximal regularity and its application to two-dimensional Keller–Segel system*, Math. Z. **264** (2010), 601–628.

[24] T.I. Seidman and G.M. Troianiello: *Time-dependent solutions of a nonlinear system arising in semiconductor theory*, Nonlinear Anal. **9** (1985), 1137–1157.

[25] J. Toft: *Continuity properties for modulation spaces, with applications to pseudo-differential calculus*, I, J. Funct. Anal. **207** (2004), 399–429.

[26] H. Triebel: Theory of Function Spaces, Birkhäuser, Basel, 1983.

[27] W. Baoxiang, Z. Lifeng and G. Boling: *Isometric decomposition operators, function spaces* $E^\lambda_{p,q}$ *and applications to nonlinear evolution equations*, J. Funct. Anal. **233** (2006), 1–39.

[28] A. Yagi: *Norm behavior of solutions to a parabolic system of chemotaxis*, Math. Japon. **45** (1997), 241–265.

[29] T. Yoneda: *Ill-posedness of the* 3*D-Navier–Stokes equations in a generalized Besov space near* BMO^{-1}, J. Funct. Anal. **258** (2010), 3376–3387.

[30] J. Zhao, Q. Liu and S. Cui: *Existence of solutions for the Debye–Hückel system with low regularity initial data*, Acta Appl. Math. **125** (2013), 1–10.

Tsukasa Iwabuchi
Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka, 558-8585
Japan
e-mail: iwabuchi@sci.osaka-cu.ac.jp

Takayoshi Ogawa
Mathematical Institute
Tohoku University
Sendai 980-8578
Japan
e-mail: ogawa@math.tohoku.ac.jp