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ON CASTELNUOVO THEORY AND NON-EXISTENCE OF SMOOTH ISOLATED CURVES IN QUINTIC THREEFOLDS

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Abstract

We find some necessary conditions for a smooth irreducible curve $C \subset \mathbb{P}^4$ to be isolated in a smooth quintic threefold. As an application, we prove that Knutsen's list of examples of smooth isolated curves in general quintic threefolds is complete up to degree 9.

1. Introduction

We work over the complex number field $\mathbb{C}$. We say a smooth projective curve C is isolated in an ambient smooth projective variety Y if $h^0(\mathcal{N}_{C/Y}) = 0$, where $\mathcal{N}_{C/Y}$ is the normal bundle of C in Y. A Calabi–Yau threefold Y has the nice property that the expected dimension of the deformation space of any l.c.i curve lying in Y is zero. So it is quite reasonable to expect that Calabi–Yau threefolds contain isolated curves. More specifically, we can ask the following:

PROBLEM 1.1. Let $d > 0$ and $g \geq 0$ be integers. Does a general complete intersection Calabi–Yau (CICY) threefold (of a particular complete intersection type) contain a smooth isolated curve of degree d and genus g?

Problem 1.1 is interesting. In fact, embeddings of complex projective curves into CICY threefolds, and Calabi–Yau threefolds in general, have been extensively studied by mathematicians and physicists in the past decades. Both the development of quantum cohomology and the discovery of surprising relations between algebraic geometry and the theory of mirror symmetry are closely related to counting curves (especially rational) in Calabi–Yau threefolds.

Problem 1.1 is hard in general. It turns out that even for existence of smooth isolated rational curves (i.e., $g = 0$) a complete answer to Problem 1.1 requires hard work ([1], [6], [12], [3]). Building on results of Clemens and Kley ([2], [7]), Knutsen proved existence of many examples of smooth isolated curves of low genera in general CICY threefolds ([8]). By Knutsen's technique, more such examples have also been established ([16]). However, we are still very far from a full answer to Problem 1.1. In

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fact, the highest genus g known so far for which there exists a smooth isolated curve of genus g in a general CICY threefold is 29 (see [16]). It is conjectured that genera of smooth isolated curves in *generic* (i.e., complement to a countable union of proper closed subvarieties in moduli spaces) CICY threefolds should be unbounded.

In this note we consider non-existence of smooth isolated curves in smooth quintic threefolds. We find some necessary conditions (Lemma 2.1, Proposition 2.6, and Proposition 2.8) for curves to be isolated in smooth quintic threefolds and then combine certain results (Theorem 3.1) in Castelnuovo theory to prove a non-existence result of smooth isolated curves in smooth quintic threefolds (Theorem 3.5, which can be viewed as the main result of this note). As an application, we conclude that Knutsen's list ([8, Theorem 1.2]) of examples of smooth isolated curves in general quintic threefolds is complete up to degree 9 (Corollary 3.6). It is also hoped that the non-existence result in this note may be helpful for people to search for more existence results in the future.

2. Necessary conditions for curves to be isolated in quintics

Throughout this note, a *curve* means a smooth irreducible one dimensional projective variety.

Lemma 2.1. *Let $C \subset \mathbb{P}^4$ be a curve and $Y \subset \mathbb{P}^4$ be a smooth quintic threefold. Suppose $C \subset Y$ and C is isolated in Y. Then $h^i(\mathcal{N}_{C/\mathbb{P}^4}) = h^i(\mathcal{O}_C(5))$, $i = 0, 1$.*

Proof. Since C is isolated in Y, it follows that $h^0(\mathcal{N}_{C/Y}) = h^1(\mathcal{N}_{C/Y}) = 0$. Let us consider

$$0 \to \mathcal{N}_{C/Y} \to \mathcal{N}_{C/\mathbb{P}^4} \to \mathcal{O}_C(5) \to 0.$$

Taking cohomology groups, it's easy to see $H^i(C,\mathcal{N}_{C/\mathbb{P}^4}) \cong H^i(C,\mathcal{O}_C(5))$, $i = 0,1$. □

Lemma 2.2. *Let $C \subset \mathbb{P}^n$ be a curve. Suppose C is degenerate, i.e., C is contained in a hyperplane. Then $(n+1)h^1(\mathcal{O}_C(1)) \geq h^1(\mathcal{N}_{C/\mathbb{P}^n}) \geq h^1(\mathcal{O}_C(1))$. In particular, $h^1(\mathcal{N}_{C/\mathbb{P}^n}) = 0$ if and only if $h^1(\mathcal{O}_C(1)) = 0$.*

Proof. Notice that we have the following two exact sequences:

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{(n+1)} \to \mathcal{T}_{\mathbb{P}^n}|_C \to 0,$$
$$0 \to \mathcal{T}_C \to \mathcal{T}_{\mathbb{P}^n}|_C \to \mathcal{N}_{C/\mathbb{P}^n} \to 0.$$

Then clearly, $(n+1)h^1(\mathcal{O}_C(1)) \geq h^1(\mathcal{N}_{C/\mathbb{P}^n})$.

On the other hand, we have the following exact sequence:

$$0 \to \mathcal{N}_{C/\mathbb{P}^{n-1}} \to \mathcal{N}_{C/\mathbb{P}^n} \to \mathcal{O}_C(1) \to 0.$$

Therefore, $h^1(\mathcal{N}_{C/\mathbb{P}^n}) \geq h^1(\mathcal{O}_C(1))$. □

Lemma 2.3. *Let $X \subset \mathbb{P}^n$ be a reduced and irreducible variety. Let d be the smallest integer such that $h^0(\mathcal{I}_X(d)) \neq 0$, where $\mathcal{I}_X$ is the ideal sheaf of X. Suppose $0 \neq F \in H^0(\mathbb{P}^n, \mathcal{I}_X(d))$. Then F is irreducible and X is not contained in the singular locus of $V(F)$, where $V(F) \subset \mathbb{P}^n$ is the hypersurface defined by F.*

Proof. If F is not irreducible, then X must be contained in a hypersurface of degree less than d, but that is impossible by the definition of d. Similarly, the singular locus of $V(F)$ is defined by polynomials of degree $d-1$ (more explicitly, partial derivatives of F), and X is not contained in the singular locus of $V(F)$. □

The following lemma is critical to the rest of this note because it gives a nice lower bound for $h^1(\mathcal{N}_{C/\mathbb{P}^n})$.

Lemma 2.4. *Let $C \subset \mathbb{P}^n$ be a curve. Let m be the smallest integer such that $h^0(\mathcal{I}_C(m)) \neq 0$. Then $h^1(\mathcal{N}_{C/\mathbb{P}^n}) \geq h^1(\mathcal{O}_C(m))$.*

Proof. Let $F \in H^0(\mathbb{P}^n, \mathcal{I}_C(m))$ and $Y := V(F)$. Considering the following exact sequence of ideal sheaves:

$$0 \to \mathcal{I}_{Y/\mathbb{P}^n} \to \mathcal{I}_{C/\mathbb{P}^n} \to \mathcal{I}_{C/Y} \to 0.$$

Restricting the above exact sequence to C (i.e. tensoring $\mathcal{I}_{C/\mathbb{P}^n}$), we obtain

$$0 \to \mathcal{I}_{Y/\mathbb{P}^n}/(\mathcal{I}_{Y/\mathbb{P}^n}\mathcal{I}_{C/\mathbb{P}^n}) \xrightarrow{\phi} \mathcal{I}_{C/\mathbb{P}^n}/\mathcal{I}^2_{C/\mathbb{P}^n} \to \mathcal{I}_{C/Y}/\mathcal{I}^2_{C/Y} \to 0.$$

Notice that ϕ is injective because of Lemma 2.3. Actually, ϕ is obviously injective at the points where Y is smooth, so ϕ is injective generically by Lemma 2.3. Then ϕ is injective everywhere because $\mathcal{I}_{Y/\mathbb{P}^n}/(\mathcal{I}_{Y/\mathbb{P}^n}\mathcal{I}_{C/\mathbb{P}^n})$ is locally free.

Applying $\mathcal{H}om_{\mathcal{O}_C}(-, \mathcal{O}_C)$ to the above exact sequence, we get

$$0 \to \mathcal{N}_{C/Y} \to \mathcal{N}_{C/\mathbb{P}^n} \to \mathcal{N}_{Y/\mathbb{P}^n}|_C \to \mathcal{E}xt^1_{\mathcal{O}_C}(\mathcal{I}_{C/Y}/\mathcal{I}^2_{C/Y}, \mathcal{O}_C) \to 0.$$

Notice that $\mathcal{E}xt^1_{\mathcal{O}_C}(\mathcal{I}_{C/Y}/\mathcal{I}^2_{C/Y}, \mathcal{O}_C)$ is a torsion sheaf and $H^1(C, \mathcal{E}xt^1_{\mathcal{O}_C}(\mathcal{I}_{C/Y}/\mathcal{I}^2_{C/Y}, \mathcal{O}_C)) = 0$. Then it is easy to see $h^1(\mathcal{N}_{C/\mathbb{P}^n}) \geq h^1(\mathcal{N}_{Y/\mathbb{P}^n}|_C) = h^1(\mathcal{O}_C(m))$. □

Corollary 2.5. *Let $C \subset \mathbb{P}^n$ be a curve. Suppose C is contained in a hypersurface of degree d, then $h^1(\mathcal{N}_{C/\mathbb{P}^n}) \geq h^1(\mathcal{O}_C(d))$.*

The following theorem explains why $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = h^1(\mathcal{O}_C(5))$ is a strong constraint for a curve $C \subset \mathbb{P}^4$ and, essentially, it is one of the main ingredients of the proof of the non-existence results, namely, Theorem 3.5.

Proposition 2.6. *Let $C \subset \mathbb{P}^4$ be a curve. Suppose C is contained in a hypersurface of degree $d \leq 4$. Then $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = h^1(\mathcal{O}_C(5))$ if and only if $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = h^1(\mathcal{O}_C(5)) = h^1(\mathcal{O}_C(d)) = 0$.*

Proof. The "if" part is trivial, so we just need to show the "only if" part. Suppose we have $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = h^1(\mathcal{O}_C(5))$. Our goal is to show $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = h^1(\mathcal{O}_C(5)) = h^1(\mathcal{O}_C(d)) = 0$.

Notice that $h^1(\mathcal{O}_C(d)) \geq h^1(\mathcal{O}_C(5))$ by Serre duality and the fact that $d \leq 5$. By Corollary 2.5, $h^1(\mathcal{N}_{C/\mathbb{P}^4}) \geq h^1(\mathcal{O}_C(d))$. Therefore, $h^1(\mathcal{O}_C(d)) \geq h^1(\mathcal{O}_C(5)) = h^1(\mathcal{N}_{C/\mathbb{P}^4}) \geq h^1(\mathcal{O}_C(d))$. Thus, $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = h^1(\mathcal{O}_C(5)) = h^1(\mathcal{O}_C(d))$.

To finish the proof, we only need to show $h^1(\mathcal{O}_C(5)) = 0$. If $h^1(\mathcal{O}_C(5)) \neq 0$, then, by Serre duality, $h^0(\mathcal{K}_C(-5)) \neq 0$, where $\mathcal{K}_C$ is the canonical bundle of C. Then the complete linear system $|\mathcal{K}_C(-5)| \neq \emptyset$. Since $5 - d \geq 1$, it follows that $\mathcal{O}_C(5-d)$ is a very ample line bundle on C and $h^0(\mathcal{O}_C(5-d)) \geq 2$, in particular, $|\mathcal{O}_C(5-d)| \neq \emptyset$. Then by [5, Chapter IV, Lemma 5.5],

$$\dim|\mathcal{K}_C(-5)| + \dim|\mathcal{O}_C(5-d)| \leq \dim|\mathcal{K}_C(-d)|.$$

Thus,

$$h^0(\mathcal{K}_C(-5)) + h^0(\mathcal{O}_C(5-d)) \leq h^0(\mathcal{K}_C(-d)) + 1.$$

Since we have seen that $2 \leq h^0(\mathcal{O}_C(5-d))$, it follows that

$$h^0(\mathcal{K}_C(-5)) + 2 \leq h^0(\mathcal{K}_C(-5)) + h^0(\mathcal{O}_C(5-d)) \leq h^0(\mathcal{K}_C(-d)) + 1.$$

Then $h^0(\mathcal{K}_C(-5)) + 1 \leq h^0(\mathcal{K}_C(-d))$, equivalently, by Serre duality, $h^1(\mathcal{O}_C(5)) + 1 \leq h^1(\mathcal{O}_C(d))$, a contradiction to the fact $h^1(\mathcal{O}_C(5)) = h^1(\mathcal{O}_C(d))$. Therefore, we must have $h^1(\mathcal{O}_C(5)) = 0$. □

REMARK 2.7. Proposition 2.6 tells us that if a curve $C \subset \mathbb{P}^4$ is isolated in a smooth quintic threefold and C is contained in some hypersurface of degree ≤ 4, then C is even unobstructed as a curve in $\mathbb{P}^4$ (more precisely, $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = 0$) and $[C] \in \mathrm{Hilb}(\mathbb{P}^4)$ is a smooth point (cf. [9, Chapter I, Section 1.2]).

Let $C \subset \mathbb{P}^n$ be a curve of degree d and of genus g. Let $5 > k > 0$ be an integer. By Riemann–Roch, $h^1(\mathcal{O}_C(k)) = h^0(\mathcal{O}_C(k)) - kd - 1 + g$, this means, roughly speaking, if g is "very big" with respect to d (for example, $g > kd + 1$), then $h^1(\mathcal{O}_C(k))$ will be positive. Furthermore, if we hope C to satisfy $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = h^1(\mathcal{O}_C(5))$, then by Proposition 2.6 C can not be contained in a hypersurface of degree $\leq k$. More precisely, we have the following:

Proposition 2.8. *Let $C \subset \mathbb{P}^4$ be a curve such that C is not contained in any plane (i.e. two dimensional linear subspace of $\mathbb{P}^4$) and has degree d and genus g. Suppose $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = h^1(\mathcal{O}_C(5))$. Then:*

(i) *If* $g > d - 3$ *and* $d \geq 3$, *then* C *is non-degenerate, i.e.* $H^0(\mathbb{P}^4, \mathcal{I}_C(1)) = 0$;
(ii) *If* $g > 2d - 11$ *and* $d \geq 8$, *then* C *is not contained in any quadric hypersurfaces*;
(iii) *If* $g > 3d - 18$ *and* $d \geq 8$, *then* C *is not contained in any cubic hypersurfaces.*

Proof. (i): Assume $g > d - 3$ and $d \geq 3$. Suppose C is degenerate, then $h^0(\mathcal{I}_C(1)) = 1$ because C is not in any plane. By Riemann–Roch, $h^1(\mathcal{O}_C(1)) = h^0(\mathcal{O}_C(1)) - d - 1 + g \geq 4 - d - 1 + g = g - d + 3 > 0$. On the other hand, by Proposition 2.6 $h^1(\mathcal{O}_C(1)) = 0$, a contradiction. Therefore, C is non-degenerate.

(ii): Assume $g > 2d - 11$ and $d \geq 8$. Suppose C is contained in a quadric hypersurface. First of all, $d \geq 8$ implies $2d - 11 \geq d - 3$. Then by (i) C is non-degenerate. Then by [13, Corollary 1.5], $h^0(\mathcal{I}_C(2)) \leq 3$, it follows that $h^0(\mathcal{O}_C(2)) \geq 12$. By Riemann–Roch again, $h^1(\mathcal{O}_C(2)) = h^0(\mathcal{O}_C(2)) - 2d - 1 + g \geq 12 - 2d - 1 + g = g - 2d + 11 > 0$, a contradiction to Proposition 2.6.

(iii): Assume $g > 3d - 18$ and $d \geq 8$. Suppose C is contained in a cubic hypersurface. By (ii) C can not be in a quadric hypersurface, it follows that $h^0(\mathcal{I}_C(1)) = h^0(\mathcal{I}_C(2)) = 0$. Therefore $h^0(\mathcal{O}_C(1)) \geq 5$ and $h^0(\mathcal{O}_C(2)) \geq 15$. Then by [5, Chapter IV, Lemma 5.5] $h^0(\mathcal{O}_C(3)) \geq 19$. So $h^1(\mathcal{O}_C(3)) = h^0(\mathcal{O}_C(3)) - 3d - 1 + g \geq 19 - 3d - 1 + g = g - 3d + 18 > 0$, again a contradiction to Proposition 2.6. □

3. Castelnuovo theory and non-existence of isolated curves in quintics

Let $C \subset \mathbb{P}^n$ be a curve. Suppose that C has degree d and genus g. Roughly speaking, Castelnuovo theory tells us that if the g is “large” with respect to d, then C has to be contained in surfaces/hypersurfaces of “small” degree. More precisely, in the case of $n = 4$, we have the following:

Theorem 3.1 ([4, Theorem 3.7, Theorem 3.15, and Theorem 3.22])**.** *Let* $C \subset \mathbb{P}^4$ *be a curve of degree* d *and genus* g. *Then*:
(i) *If* $g > (d^2 - 5d + 6)/6$ *and* $d \geq 3$, *then* C *is degenerate.*
(ii) *If* C *is non-degenerate*, $g > (d^2 - 4d + 8)/8$ *and* $d \geq 9$, *then* C *is contained in a non-degenerate irreducible surface of degree* 3.
(iii) *If* C *is non-degenerate*, $g > (d^2 - 3d + 10)/10$, *and* $d \geq 144$, *then* C *is contained in a non-degenerate irreducible surface of degree* 4 *or less.*

If we want to use Proposition 2.8 to prove some non-existence results, we need to show that if the genus g is “large” with respect to degree d then the curve $C \subset \mathbb{P}^4$ has to be contained in a “low” degree hypersurface. But Theorem 3.1 (ii) and (iii) only tell us that curves with “large” genera are contained in “low” degree surfaces. Therefore, we need to show that “low” degree surfaces have to be contained in “low” degree hypersurfaces. Fortunately, we have the following:

Lemma 3.2 ([15, Lemma 3])**.** *Let* $W \subset \mathbb{P}^n$ *be an irreducible non-degenerate variety of dimension* m *and degree* d. *Let* $A \in W$ *be a point*; *and if* W *is a cone suppose* *that A is not a vertex of W. Let W_1 be the cone obtained by joining A to every point of W. Then W_1 does not lie in any hyperplane of $\mathbb{P}^n$, and it has dimension exactly $m+1$ and degree at most $d-1$; moreover, if A is a singular point of W then W_1 has degree at most $d-2$.*

Now the following is just an easy consequence of Lemma 3.2.

Proposition 3.3. *Let $X \subset \mathbb{P}^4$ be a non-degenerate irreducible surface of degree d. Then X is contained in a hypersurface of degree $d-1$; moreover, if X has a singular point which is not a vertex of X, then X is contained in a hypersurface of degree $d-2$.*

Proof. Let $A \in X$ be a point, and if X is a cone suppose A is not a vertex of X. Let X_1 be the cone obtained by joining A to every point of X. By Lemma 3.2, X_1 is a threefold of degree at most $d-1$ ($d-2$ if A is a singular point of X). □

REMARK 3.4. Notice if that the surface X in Proposition 3.3 is smooth, then X is even $(d-1)$-regular and hence the homogeneous ideal of X is even generated by polynomials of degree $d-1$ or less (cf. [10]).

Finally, we are ready to prove the following non-existence result:

Theorem 3.5. *Let $d \geq 3$ and $g \geq 0$ be integers. Let $C \subset \mathbb{P}^4$ be a curve of degree d and genus g. Then C cannot be isolated in any smooth quintic threefold if the pair (d, g) is in the following list*:
(1) $g > d-3$, $(d, g) \neq (3, 1)$ *and* $3 \leq d \leq 8$;
(2) $g > 2d-11$ *and* $9 \leq d \leq 12$;
(3) $g > (d^2-4d+8)/8$ *and* $12 < d < 144$;
(4) $g > (d^2-3d+10)/10$ *and* $d \geq 144$.

Proof. (i) Assume $g > d-3$, $(d, g) \neq (3, 1)$ and $3 \leq d \leq 8$. Notice that when $3 \leq d \leq 8$, then $d-3 \geq (d^2-5d+6)/6$. It follows that $g > (d^2-5d+6)/6$. Then by Theorem 3.1 (i) C is contained in a hyperplane. Therefore, by Proposition 2.8 (i) C has to be contained in a plane. But it is easy to check that if C is contained in a plane, $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = h^1(\mathcal{O}_C(5))$ only if $(d,g) = (3,1)$. But by assumption $(d,g) \neq (3,1)$, it follows that C cannot be isolated in any smooth quintic threefold by Lemma 2.1.

(ii) Assume $g > 2d-11$ and $9 \leq d \leq 12$. Notice that in this case, $g > (d^2-4d+8)/8$, then by Theorem 3.1 (ii) and Proposition 3.3, $h^0(\mathcal{I}_C(2)) \neq 0$. Thus, by Proposition 2.8 (ii) $h^1(\mathcal{N}_{C/\mathbb{P}^4}) \neq h^1(\mathcal{O}_C(5))$, it follows that C cannot be isolated in any smooth quintic threefold by Lemma 2.1.

(iii) Assume $g > (d^2-4d+8)/8$ and $12 < d < 144$. Notice that in this case $(d^2-4d+8)/8 \geq 2d-11$, and the rest of the argument is similar to the case (ii).

(iv) Similar to cases (ii) and (iii). □

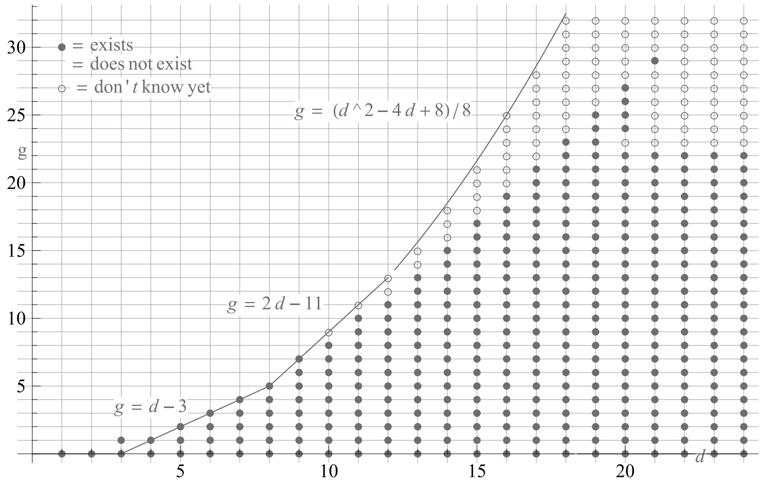

Fig. 1.

The non-existence/existence of smooth isolated curves in general quintic threefolds is described in Fig. 1.

As an application of Theorem 3.5, we get the following:

Corollary 3.6. *If there exists a smooth isolated curve of degree* $d \leq 9$ *and genus* g *in a general quintic threefold, then the pair of integers* (d, g) *must be in Knutsen's list* ([8, Theorem 1.2]). *In other words, Knutsen's list* ([8, Theorem 1.2]) *is complete for* $Y = (5) \subset \mathbb{P}^4$ *and* $d \leq 9$.

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