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MEASURE-EXPANSIVE HOMOCLINIC CLASSES

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Abstract

Let p be a hyperbolic periodic point of a diffeomorphism f on a compact C^∞ Riemannian manifold M. In this paper we introduce the notion of C^1 stably measure expansiveness of closed f-invariant sets, and prove that (i) the chain recurrent set $\mathcal{R}(f)$ of f is C^1 stably measure expansive if and only if f satisfies both Axiom A and no-cycle condition, and (ii) the homoclinic class $H_f(p)$ of f associated to p is C^1 stably measure expansive if and only if $H_f(p)$ is hyperbolic.

1. Introduction

In this paper we study the case when the homoclinic class $H_f(p)$ of a diffeomorphism f on a compact C^∞ Riemannian manifold M associated to a hyperbolic periodic point p is C^1 stably measure expansive.

Let $\mathrm{Diff}(M)$ be the space of C^1 diffeomorphisms on M endowed with the C^1-topology, and let d denote the distance on M induced from a Riemannian metric on the tangent bundle TM. For any closed f-invariant set $\Lambda \subset M$, we say that Λ is *expansive* for f (or $f|_\Lambda$ is *expansive*) if there is $\alpha > 0$ such that for any pair of distinct points $x, y \in \Lambda$ there is $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > \alpha$. It is clear that Λ is expansive for f if and only if there is $\alpha > 0$ such that $\Gamma_\alpha^f(x) = \{x\}$ for all $x \in \Lambda$, where

$$\Gamma_\alpha^f(x) = \{y \in \Lambda \colon d(f^i(x), f^i(y)) \leq \alpha \text{ for all } i \in \mathbb{Z}\}.$$

Moreover, we say that the Λ-*germ of f is expansive* if there is $\alpha > 0$ such that if $x \in \Lambda$, $y \in M$ and $d(f^n(x), f^n(y)) \leq \alpha$ for all $n \in \mathbb{Z}$ then $x = y$. Expansiveness is a dynamical property which is shared by a large class of dynamical systems exhibiting chaotic behavior.

It is well known that if p is a hyperbolic periodic point of f with period k then the sets

$$W^s(p) = \{x \in M \colon f^{kn}(x) \to p \text{ as } n \to \infty\}$$

and

$$W^u(p) = \{x \in M \colon f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

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are C^1-injectively immersed submanifolds of M. A point $x \in W^s(p) \cap W^u(p) \setminus \{p\}$ is called a *homoclinic point* of f associated to p, and it is said to be a *transversal homoclinic point* of f if the above intersection is transversal at x; i.e., $x \in W^s(p) \pitchfork W^u(p) \setminus \{p\}$. The closure of the homoclinic points of f associated to p is called the *homoclinic class* of f associated to p, and it is denoted by $H_f(p)$. The closure of the transversal homoclinic points of f associated to p is called the *transversal homoclinic class* of f associated to p, and it is denoted by $H_f^T(p)$. It is clear that both $H_f(p)$ and $H_f^T(p)$ are compact and invariant sets. Homoclinic classes are the natural candidates to replace hyperbolic basic sets in nonhyperbolic theory. Several recent papers explore their "hyperbolic-like" properties, many of which hold only for generic diffeomorphisms.

Let q be a hyperbolic periodic point of f. We say that p and q are *homoclinically related*, and write $p \sim q$ if

$$W^s(p) \pitchfork W^u(q) \neq \emptyset$$

and

$$W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

It is clear that if $p \sim q$ then $\text{index}(p) = \text{index}(q)$; i.e., $\dim W^s(p) = \dim W^s(q)$. By the Smale's transverse homoclinic point theorem, $H_f^T(p)$ coincides with the closure of the set of hyperbolic periodic points q of f such that $p \sim q$. Note that if p is a hyperbolic periodic point of f then there are a neighborhood U of p and a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$ there exists a unique hyperbolic periodic point p_g of g in U with the same period as p and $\text{index}(p_g) = \text{index}(p)$. Such that point p_g is called the *continuation* of $p = p_f$.

Recently, many people investigated the dynamics of diffeomorphisms with C^1 robust, C^1 stable and C^1 persistent expansiveness on the homoclinic classes, and characterized the sets under such C^1 open conditions, respectively (for more details, see [5, 10, 11, 14, 15]). Let us be more precise. We say that a homoclinic class $H_f(p)$ is C^1 *persistently expansive* if there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $g|_{H_g(p_g)}$ is expansive. Sambarino and Vieitez [14] proved that if the homoclinic class $H_f(p)$ is C^1 persistently expansive and the $H_f(p)$-germ of f is expansive then $H_f(p)$ is hyperbolic. However the following problem is still open: *Are the C^1 persistently expansive homoclinic classes hyperbolic*?

A closed f-invariant set $\Lambda \subset M$ is said to be C^1 *stably expansive* if there exist a compact neighborhood U of Λ and a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, Λ_g is expansive for g, where $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ and $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$. Recently Lee and Lee [5] proved that the homoclinic class $H_f(p)$ of f associated to p is C^1 stably expansive if and only if $H_f(p)$ is hyperbolic.

Very recently, Morales et al. [9] introduced a notion of measure expansiveness which generalize the usual concept of expansiveness. Let $\mathcal{M}(M)$ be the set of all Borel probability measures on M endowed with the weak* topology, and let $\mathcal{M}^*(M)$ be the set of nonatomic measures $\mu \in \mathcal{M}(M)$. For any $\mu \in \mathcal{M}^*(M)$, we say that Λ is μ-*expansive* for f if there is $\delta > 0$ such that $\mu(\Gamma_\delta^f(x)) = 0$ for all $x \in \Lambda$. An f-invariant set Λ is said to be *measure expansive* for f (or $f|_\Lambda$ is *measure expansive*) if Λ is μ-expansive for all $\mu \in \mathcal{M}^*(M)$; that is, there is a constant $\delta > 0$ such that for any $\mu \in \mathcal{M}^*(M)$ and $x \in \Lambda$, $\mu(\Gamma_\delta^f(x)) = 0$. Here δ is called a *measure expansive constant* of $f|_\Lambda$. Clearly, the expansiveness implies the measure expansiveness, but the converse does not hold in general. Note that f is measure expansive if and only if f^n is measure expansive for all $n \in \mathbb{Z} \setminus \{0\}$.

In this paper we introduce a notion of C^1 stable measure expansiveness which is general than that of C^1 stable expansiveness in [5], and study the dynamics of diffeomorphisms with C^1 stably measure expansiveness on homoclinic classes and chain recurrent sets.

DEFINITION 1.1. We say that a closed f-invariant set $\Lambda \subset M$ is C^1 *stably measure expansive* (or $f|_\Lambda$ is C^1 *stably measure expansive*) if there exist a compact neighborhood U of Λ and a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, Λ_g is measure expansive for g, where $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$ and $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$.

Recall that a closed f-invariant set Λ is said to be *hyperbolic* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$, $0 < \lambda < 1$ such that

$$\|Df^n|_{E^s(x)}\| \le C\lambda^n$$

and

$$\|Df^{-n}|_{E^u(x)}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. We say that f is *Anosov* if M is hyperbolic for f. f is said to be *quasi-Anosov* if for any $v \in TM \setminus \{0\}$, the set $\{\|Df^n(v)\| \colon n \in \mathbb{Z}\}$ is unbounded. Note that every Anosov diffeomorphism is quasi-Anosov, but the converse is not true in general. Mañé [6] proved that f is quasi-Anosov if and only if f belongs to the C^1 interior of the set of expansive diffeomorphisms in Diff(M). Moreover Sakai et al. [13] showed that if f belongs to the C^1 interior of the set of measure expansive diffeomorphisms, then f is quasi-Anosov. Thus we can restate the above facts as follows.

Theorem A. *M is C^1 stably measure expansive for f if and only if f is quasi-Anosov.*

For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^b$ in M $(-\infty \le a < b \le \infty)$ is called a δ-*pseudo-orbit* (or δ-*chain*) of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$. For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo-orbit $\{x_i\}_{i=a_\delta}^{b_\delta}$ $(a_\delta < b_\delta)$ of f such that $x_{a_\delta} = x$ and $x_{b_\delta} = y$. The set $\{x \in M \colon x \rightsquigarrow x\}$ is called the *chain* *recurrent set* of f and is denoted by $\mathcal{R}(f)$. It is easy to see that the set is closed and $f(\mathcal{R}(f)) = \mathcal{R}(f)$.

If we denote the set of periodic points of f, the set of nonwandering points of f and the set of chain recurrent points of f by $P(f)$, $\Omega(f)$ and $\mathcal{R}(f)$, respectively, then we have $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$. It is well known that the map $f \mapsto \mathcal{R}(f)$ is upper semi-continuous. More precisely, for any neighborhood U of $\mathcal{R}(f)$, there is $\delta > 0$ such that if $\rho_0(f, g) < \delta$, $(g \in \mathrm{Diff}(M))$, then $\mathcal{R}(g) \subset U$. Here ρ_0 is the usual C^0 metric on $\mathrm{Diff}(M)$. From this fact, we can obtain the first result of this paper based on the techniques in [2].

Theorem B. *The chain recurrent set $\mathcal{R}(f)$ of f is C^1 stably measure expansive if and only if f satisfies both* Axiom A *and no-cycle condition.*

Suppose f satisfy Axiom A. Then we know that f satisfies no-cycle condition if and only if $\Omega(f) = \mathcal{R}(f)$. Consequently the C^1 stable measure expansiveness on the chain recurrent set $\mathcal{R}(f)$ is characterized by the Ω stability of the system by Theorem B.

Let $D^2 \subset S^2$ be a two disk, and let f be the Smale's hyperbolic horseshoe map on D^2 with a (hyperbolic) saddle fixed point p. Then the homoclinic class $H_f(p)$ coincides with the hyperbolic horseshoe containing p. Since f is Ω-stable, we can see that the homoclinic class $H_f(p)$ is C^1 stably measure expansive by Theorem B. Moreover we can easily check that the horseshoe with a homoclinic tangency is expansive, but it is not C^1 stably measure expansive (see Example 2.2 in [12])

The main purpose of this paper is to characterize homoclinic classes $H_f(p)$ containing a hyperbolic periodic point p by making use of the measure expansiveness under C^1 open condition. This is a generalization of the main result in [5].

Theorem C. *Let p be a hyperbolic periodic point of f. Then the homoclinic class $H_f(p)$ of f associated to p is C^1 stably measure expansive if and only if $H_f(p)$ is hyperbolic.*

2. Proof of Theorem B

For any subset A of M and $\epsilon > 0$, let $B_\varepsilon(A) = \{x \in M : d(x, A) \leq \varepsilon\}$. Denote by $\mathcal{H}(M)$ the set of homeomorphisms of M. For the proof of the following lemma, see [4].

Lemma 2.1. *Let $f \in \mathcal{H}(M)$, and let $\mathcal{R}(f)$ be the chain recurrent set of f. For any $\varepsilon > 0$, there is $\delta > 0$ such that if $\rho_0(f, g) < \delta$ $(g \in \mathcal{H}(M))$ then $\mathcal{R}(g) \subset B_\varepsilon(\mathcal{R}(f))$.*

The following Franks' lemma will play essential roles in our proofs.

Lemma 2.2 ([1]). *Let $\mathcal{U}(f)$ be a C^1 neighborhood of f. Then there exist $\epsilon > 0$ and a C^1 neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \dots, x_N\}$, a neighborhood U of $\{x_1, x_2, \dots, x_N\}$ and linear maps $L_i\colon T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \le \epsilon$ for all $1 \le i \le N$, there exists $\hat{g} \in \mathcal{U}(f)$ such that $\hat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \dots, x_N\} \cup (M \setminus U)$ and $D_{x_i}\hat{g} = L_i$ for all $1 \le i \le N$.*

Denote by $\mathcal{F}(M)$ the set of $f \in \mathrm{Diff}(M)$ such that there is a C^1 neighborhood $\mathcal{U}(f)$ of f with property that every $p \in P(g)$ $(g \in \mathcal{U}(f))$ is hyperbolic. It is proved by Hayashi [2] that $f \in \mathcal{F}(M)$ if and only if f satisfies both Axiom A and no-cycle condition. Therefore, to complete the proof of Theorem B, it is enough to show that $\mathcal{R}(f)$ is C^1 stably measure expansive if and only if $f \in \mathcal{F}(M)$.

Proof of Theorem B. First we suppose that f satisfies both Axiom A and no-cycle condition. Then $\mathcal{R}(f) = \Omega(f) = \overline{P(f)}$ is hyperbolic, and so $\mathcal{R}(f)$ is locally maximal. By the stability of locally maximal hyperbolic sets, we can choose a compact neighborhood U of $\mathcal{R}(f)$ and a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is hyperbolic for g. Thus Λ_g is (measure) expansive for g. This means that $\mathcal{R}(f)$ is C^1 stably measure expansive.

Next we suppose that $\mathcal{R}(f)$ is C^1 stably measure expansive for f. Then there are a compact neighborhood U of $\mathcal{R}(f)$ and a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g}$ is measure expansive. Choose $\varepsilon > 0$ satisfying $B_\varepsilon(\mathcal{R}(f)) \subset U$. By Lemma 2.1, there is $\delta > 0$ such that if $\rho_1(f, g) < \delta$ for $g \in \mathcal{U}(f)$ then

$$\mathcal{R}(g) \subset B_\varepsilon(\mathcal{R}(f)) \subset U, \tag{2.1}$$

where ρ_1 is the usual C^1 metric on $\mathrm{Diff}(M)$. Put $\mathcal{U}_0(f) = \{g \in \mathcal{U}(f)\colon \rho_1(f, g) < \delta\}$. Then for each $g \in \mathcal{U}_0(f)$, $\mathcal{R}(g) \subset U$ and so $\mathcal{R}(g) \subset g^n(U)$ for all $n \in \mathbb{Z}$. This means that $\mathcal{R}(g) \subset \Lambda_g$ for $g \in \mathcal{U}_0(f)$. Since $g|_{\Lambda_g}$ is measure expansive, $g|_{\mathcal{R}(g)}$ is measure expansive. Let $\varepsilon > 0$, and let $\tilde{\mathcal{U}}(f) \subset \mathcal{U}_0(f)$ be a C^1 neighborhood of f which is given by Lemma 2.2 with respect to $\mathcal{U}_0(f)$. Let $p \in P(f)$, and let $\pi(p)$ be the period of p.

To derive a contradiction, we assume that $f \notin \mathcal{F}(M)$. Then there exist $g \in \mathcal{U}(f)$, a nonhyperbolic periodic point p of g and an eigenvalue λ of $D_pg^{\pi(p)}$ with $|\lambda| = 1$. Let $T_pM = E^c(p) \oplus E^s(p) \oplus E^u(p)$ be the $D_pg^{\pi(p)}$-invariant splitting of T_pM, where $E^\sigma(p)$, $\sigma = c, s, u$, are subspaces T_pM corresponding to eigenvalues λ of $D_pg^{\pi(p)}$ for $|\lambda| = 1$, $|\lambda| < 1$ and $|\lambda| > 1$, respectively. Choose $\varepsilon_0 > 0$ with $\mathcal{U}_{\varepsilon_0}(f) \subset \tilde{\mathcal{U}}(f)$, where $\mathcal{U}_{\varepsilon_0}(f) = \{g \in \mathrm{Diff}(M)\colon \rho_1(f, g) < \varepsilon_0\}$. Set $C = \sup_{x\in M}\{\|D_xg\|\}$. For $0 < \varepsilon_1 < \varepsilon_0$, we can obtain a linear automorphism $\mathcal{O}\colon T_pM \to T_pM$ such that

(i) $\|\mathcal{O} - id\| < \varepsilon_1/C$,

(ii) $\mathcal{O}$ keeps E^σ invariant, where $\sigma = c, s, u$, and

(iii) all eigenvalues of $\mathcal{O} \circ D_pg^{\pi(p)}|_{E^c(p)}$, say $\tilde{\lambda}_j$, $j = 1, 2, \dots, c$, are roots of unity.

Let $F = \{p, f(p), \dots, f^{\pi(p)-1}(p)\}$. Define

$$G_j = \begin{cases} D_{g^j(p)}g, & j = 0, 1, \dots, \pi(p) - 2, \\ \mathcal{O} \circ D_{g^{\pi(p)-1}(p)}g, & j = \pi(p) - 1. \end{cases}$$

Observe that $\|G_{\pi(p)-1} - D_{g^{\pi(p)-1}(p)}g\| \le \|\mathcal{O} - Id\| \, \|D_{f^{\pi(p)-1}(p)}g\| < \varepsilon_0$. Thus $\|G_j - D_{g^j(p)}g\| < \varepsilon_0$ for all $j = 0, 1, \dots, \pi(p) - 1$. By Lemma 2.2, we can find a diffeomorphism $g_1 \in \tilde{\mathcal{U}}(f)$ and $\delta_0 > 0$ such that

(a) $B_{4\delta_0}(g^j(p)) \subset U$, and $B_{4\delta_0}(g^j(p)) \cap B_{4\delta_0}(p) = \emptyset$, where $0 \le i \ne j \le \pi(p) - 1$,

(b) $g_1 = g$ on $F \cup \left(M - \bigcup_{j=0}^{\pi(p)-1} B_{4\delta_0}(g^j(p))\right)$, and

(c) $g_1 = \exp_{g^{j+1}(p)} \circ G_j \circ \exp^{-1}_{g^j(p)}$ on $B_{\delta_0}(g^j(p))$, $0 \le j \le \pi(p) - 1$.

Define

$$G = \mathcal{O} \circ D_p g^{\pi(p)} = \prod_{j=0}^{\pi(p)-1} G_j.$$

Then by (iii) we can find $m > 0$ such that $G^m|_{E^c(p)} = id|_{E^c(p)}$. Choose a small δ_1 satisfying $0 < 4\delta_1 < \delta_0$ such that

$$G^{mk}(T_pM(4\delta_1)) \subset T_pM(\delta_0),$$

where $T_pM(\delta) = \{v \in T_pM \colon \|v\| \le \delta\}$. Then by (c) we have

$$(g_1^{\pi(p)})^m = g_1^{m\pi(p)} = \exp_p \circ G^m \circ \exp_p^{-1}$$

on $\exp_p(T_pM(4\delta_1))$. We write

$$T_pM(\delta_1) = E^c(p, \delta_1) \oplus E^s(p, \delta_1) \oplus E^u(p, \delta_1),$$

where $E^\sigma(p, \delta_1) = E^\sigma(p) \cap T_pM(\delta_1)$, $\sigma = c, s, u$. Then $\exp_p(E^c(p, 4\delta_1))$ is $(g_1^{\pi(p)})^m$-invariant.

If the eigenvalue λ is real then take $0 < \delta_2 < 4\delta_1$ such that $\exp_p(E^c(p, \delta_2)) \subset U$. Put $\exp_p(E^c(p, \delta_2)) = \mathfrak{I}_p$. Then $\mathfrak{I}_p$ is a closed arc with the center at p which satisfies the following;

(1) $\mathfrak{I}_p \subset \Lambda_{g_1} = \bigcap_{n\in\mathbb{Z}} g_1^n(U)$,

(2) $\mathfrak{I}_p \subset \exp_p(E^c(p, 4\delta_1)) \cap B_{4\delta_1}(p)$,

(3) $g_1^i(\mathfrak{I}_p) \cap g_1^j(\mathfrak{I}_p) = \emptyset$, for $0 \le i \ne j \le m\pi(p) - 1$,

(4) $g_1^{m\pi(p)}(\mathfrak{I}_p) = \mathfrak{I}_p$, and

(5) $g_1^{m\pi(p)} \colon \mathfrak{I}_p \to \mathfrak{I}_p$ is the identity map.

Let $m_{\mathfrak{I}_p}$ be the normalized Lebesgue measure on $\mathfrak{I}_p$. We define $\mu \in \mathcal{M}(M)$ by

$$\mu(C) = \frac{1}{m\pi(p)} \sum_{i=0}^{m\pi(p)-1} m_{\mathfrak{I}_p}(g_1^{-i}(C \cap g_1^i(\mathfrak{I}_p))),$$

for any Borel set C of M. Then it is clear that $\mu \in \mathcal{M}^*(M)$. For simplicity, we set $g_1^{m\pi(p)} = g_1$. Let $0 < e < \delta_1$ be a measure expansive constant of $g_1|_{\mathcal{R}(g_1)}$. Let $\Gamma_e^{g_1}(x) = \{y \in M\colon d(g_1^i(x), g_1^i(y)) \le e$, for all $i \in \mathbb{Z}\}$. Since $g_1\colon \mathfrak{I}_p \to \mathfrak{I}_p$ is the identity map, for any $y \in \mathfrak{I}_p$, we know that $d(g_1^i(y), g_1^i(p)) = d(y, p)$ for all $i \in \mathbb{Z}$. Thus we get

$$\{y \in \mathfrak{I}_p\colon d(g_1^i(y), g_1^i(p)) \le e \text{ for all } i \in \mathbb{Z}\} \subset \Gamma_e^{g_1}(p).$$

Then we have

$$0 < \mu(\{y \in \mathfrak{I}_p\colon d(g_1^i(y), g_1^i(p)) \le e \text{ for all } i \in \mathbb{Z}\}) \le \mu(\Gamma_e^{g_1}(p)).$$

Since Λ_{g_1} is measure-expansive for g_1, $\mathfrak{I}_p$ is also measure expansive for g_1. This is a contradiction.

If the eigenvalue is complex then $\exp_p(E^c(p, \delta_1))$ is a disk $\mathfrak{D}_p$ centered at p. Then we have

$$\mathfrak{D}_p \subset \Lambda_{g_1} = \bigcap_{n\in\mathbb{Z}} g_1^n(U).$$

By the same argument as above, we obtain that

(1) $\mathfrak{D}_p \subset \exp_p(E^c(p, 4\delta_1)) \cap B_{4\delta_1}(p)$,

(2) $g_1^i(\mathfrak{D}_p) \cap g_1^j(\mathfrak{D}_p) = \emptyset$, for $0 \le i \ne j \le m\pi(p) - 1$,

(3) $g_1^{m\pi(p)}(\mathfrak{D}_p) = \mathfrak{D}_p$, and

(4) $g_1^{m\pi(p)}\colon \mathfrak{D}_p \to \mathfrak{D}_p$ is the identity map.

Let $m_{\mathfrak{D}_p}$ be the normalized Lebesgue measure on $\mathfrak{D}_p$. We define $\mu \in \mathcal{M}(M)$ by

$$\mu(C) = \frac{1}{m\pi(p)} \sum_{i=0}^{m\pi(p)-1} m_{\mathfrak{D}_p}(g_1^{-i}(C \cap g_1^i(\mathfrak{D}_p))),$$

for any Borel subset C of M. Then we see that $\mu \in \mathcal{M}^*(M)$. As in the first case, we can show that Λ_{g_1} is not measure expansive for g_1. The contradiction completes the proof of Theorem B. □

3. Proof of Theorem C

To prove Theorem C, we will adapt the techniques in [5, 8, 14] which uses the notion of uniform hyperbolicity for a family of periodic sequences of linear isomorphisms of $\mathbb{R}^n$, where n is the dimension of M. For this we need several lemmas.

Lemma 3.1. *Let Λ be a closed f-invariant subset of M which is C^1 stably measure expansive. Then there exist a neighborhood U of Λ and a C^1 neighborhood $\mathcal{U}_0(f)$ of f such that for any $g \in \mathcal{U}_0(f)$, every periodic point of g in $\Lambda_g = \bigcap_{n\in Z} g^n(U)$ is hyperbolic.*

Proof. Since Λ is C^1 stably measure expansive, there exist a compact neighborhood U of Λ and a C^1 neighborhood $\mathcal{U}(f)$ of f such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$, and Λ_g is measure expansive for $g \in \mathcal{U}(f)$. Take a C^1 neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f as in Lemma 2.2. By applying the similar techniques as in the proof of Theorem B, we can show that U and $\mathcal{U}_0(f)$ are desired neighborhoods of Λ and f, respectively.

Suppose there is a non-hyperbolic periodic point $q \in \Lambda_g$ for some $g \in \mathcal{U}_0(f)$. We can choose smaller $\mathcal{U}_0(f)$ and U if necessary so that $q \in \operatorname{int} U$. Then there is an eigenvalue λ of $D_q g^{\pi(q)}$ such that $|\lambda| = 1$. Let $T_q M = E^c(q) \oplus E^s(q) \oplus E^u(q)$ be the $D_q g^{\pi(q)}$-invariant splitting of $T_q M$, where $E^\sigma(q)$, $\sigma = c, s, u$, are subspaces $T_q M$ corresponding to eigenvalues λ of $D_q g^{\pi(q)}$ for $|\lambda| = 1$, $|\lambda| < 1$ and $|\lambda| > 1$, respectively.

If the eigenvalue λ is real, then by making use of Lemma 2.2, we can choose $\delta > 0$ and a diffeomorphism $h \in \mathcal{U}_0(f)$ C^1 close to g such that

(a) $h^{\pi(q)}(q) = g^{\pi(q)}(q) = q$,

(b) $h(x) = \exp_{g^{i+1}(q)} \circ D_{g^i(q)} g \circ \exp^{-1}_{g^i(q)}(x)$ if $x \in B_\delta(g^i(q))$ for $0 \le i \le \pi(q) - 2$, and

(c) $h(x) = \exp_q \circ D_{g^{\pi(q)-1}(q)} g \circ \exp^{-1}_{g^{\pi(q)-1}(q)}(x)$ if $x \in B_\delta(g^{\pi(q)-1}(q))$.

Then we have an invariant small arc $\mathfrak{I}_q \subset B_\delta(q) \cap \exp_q(E^c_q(\delta))$ with center at q which satisfies the following:

(1) $\mathfrak{I}_q \subset \Lambda_h = \bigcap_{n\in\mathbb{Z}} h^n(U)$,

(2) $h^i(\mathfrak{I}_q) \cap h^j(\mathfrak{I}_q) = \emptyset$ for $0 \le i \neq j \le m\pi(q) - 1$,

(3) $h^{m\pi(q)}(\mathfrak{I}_q) = \mathfrak{I}_q$, and

(4) $h^{m\pi(q)} \colon \mathfrak{I}_q \to \mathfrak{I}_q$ is the identity map.

Let $m_{\mathfrak{I}_q}$ be the normalized Lebesgue measure on $\mathfrak{I}_q$. We define $\mu \in \mathcal{M}(M)$ by

$$\mu(C) = \frac{1}{m\pi(q)} \sum_{i=0}^{m\pi(q)-1} m_{\mathfrak{I}_p}(h^{-i}(C \cap h^i(\mathfrak{I}_q))),$$

for any Borel set C of M. Then it is easy to show that $\mu \in \mathcal{M}^*(M)$. For simplicity, we set $h^{m\pi(q)} = h$. Let $0 < e < \delta$ be a measure expansive constant of $h|_{\Lambda_h}$. Since $h \colon \mathfrak{I}_q \to \mathfrak{I}_q$ is the identity map, for any $y \in \mathfrak{I}_q$, we see that $d(h^i(y), h^i(q)) = d(y, q)$ for all $i \in \mathbb{Z}$. Thus

$$\{y \in \mathfrak{I}_q : d(h^i(y), h^i(q)) \le e, \text{ for all } i \in \mathbb{Z}\} \subset \Gamma^h_e(q).$$

Then we have

$$0 < \mu(\{y \in \mathfrak{I}_q : d(h^i(y), h^i(q)) \le e, \text{ for all } i \in \mathbb{Z}\}) \le \mu(\Gamma^h_e(q)).$$

Since Λ_h is measure-expansive for h, $\mathfrak{I}_\mathrm{q}$ is also measure expansive for h, and this contradicts the assumption.

If the eigenvalue λ is complex, we can get a contradiction by the same techniques as in the proof of Theorem B and the above argument. □

We say that a compact f-invariant set $\Lambda \subset M$ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that for all $x \in \Lambda$ and $n \geq 0$, we have

$$\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n.$$

By Lemma 3.1, we can apply Proposition 2.1 in [8] to obtain the following proposition.

Proposition 3.2. *Let $H_f(p)$ be the homoclinic class of f associated to a hyperbolic periodic point p. If $H_f(p)$ is C^1 stably measure expansive, then there exist a C^1 neighborhood $\mathcal{U}(f)$ of f, constants $C > 0$, $0 < \lambda < 1$ and $m \in \mathbb{Z}^+$ such that*
(1) *for each $g \in \mathcal{U}(f)$, if q is a periodic point of g in Λ_g with period $\pi(q,g)$ ($\pi(q,g) \geq m$) and $q \sim p_g$ then $\prod_{i=0}^{k-1}\|Dg^m|_{E^s(g^{im}(q))}\| < C\lambda^k$, $\prod_{i=0}^{k-1}\|Dg^{-m}|_{E^u(g^{-im}(q))}\| < C\lambda^k$, where $k = [\pi(q,g)/m]$,*
(2) *$H_f(p)$ admits a dominated splitting $T_{H_f(p)}M = E \oplus F$ with $\dim E = \mathrm{index}(p)$.*

Let p be a hyperbolic periodic point of f with $0 < \mathrm{index}(p) < \dim M$. For each $0 < i < \dim(M)$, we put

$$P_i(f|_{H_f(p)}) = \{q \in H_f(p) \cap P_h(f) \colon \mathrm{index}(q) = i\},$$

where $P_h(f)$ is the set of hyperbolic periodic points of f. Let $\Lambda_i(f) = \overline{P_i(f|_{H_f(p)})}$ for $i = 1, 2, \ldots, \dim M - 1$. If $\mathrm{index}(p) = j$, then we know that $\Lambda_j(f) = H_f(p)$.

If an invariant set Λ admits a dominated splitting, then Mañé [7] has shown the existence of locally invariant manifolds everywhere on Λ which are tangent to the invariant subspaces of the splitting. In fact, by the uniqueness of the dominated splitting, if $q \in H_f(p)$ is a periodic point with $q \sim p$ then we have $E(q) = E^s(q)$ and $F(q) = E^u(q)$. Let $\dim E = s$ and by $\dim F = u$, and put $D_r^j = \{x \in \mathbb{R}^j \colon \|x\| \leq r\}$ ($r > 0$), for $j = s, u$. Let $\mathrm{Emb}_\Lambda(D_1^j, M)$ be the space of C^1 embeddings $\beta \colon D_1^j \to M$ such that $\beta(0) \in \Lambda$ endowed with the C^1 topology. Then we have

Proposition 3.3 ([3, 7])**.** *Let $H_f(p)$ be the homoclinic class of f associated to a hyperbolic periodic point p, and let $\Lambda = H_f(p)$. Suppose that Λ has a dominated splitting $E \oplus F$. Then there exist sections $\phi^s \colon \Lambda \to \mathrm{Emb}_\Lambda(D_1^s, M)$ and $\phi^u \colon \Lambda \to \mathrm{Emb}_\Lambda(D_1^u, M)$ such that by defining $W_\varepsilon^{cs}(x) = \phi^s(x)D_\varepsilon^s$ and $W_\varepsilon^{cu}(x) = \phi^u(x)D_\varepsilon^u$, for each $x \in \Lambda$, we have*
(1) *$T_x W_\varepsilon^{cs}(x) = E(x)$ and $T_x W_\varepsilon^{cu}(x) = F(x)$,*
(2) *for every $0 < \varepsilon_1 < 1$ there exists $0 < \varepsilon_2 < 1$ such that $f(W_{\varepsilon_2}^{cs}(x)) \subset W_{\varepsilon_1}^{cs}(f(x))$ and $f^{-1}(W_{\varepsilon_2}^{cu}(x)) \subset W_{\varepsilon_1}^{cu}(f^{-1}(x))$,*
(3) *for every $0 < \varepsilon_1 < 1$ there exists $0 < \delta < 1$ such that if $d(x,y) < \delta$ ($x, y \in \Lambda$) then $W_{\varepsilon_1}^{cs}(x) \cap W_{\varepsilon_1}^{cu}(y) \neq \emptyset$, and this intersection is transverse.*

In Proposition 3.3, the set $W^{cs}_{\varepsilon}(x)$ and $W^{cu}_{\varepsilon}(x)$ are called the *local center stable* and *local center unstable manifolds* of x, respectively. The following lemma can be proved similarly to that of Lemma 4 in [14].

Lemma 3.4. *Let $H_f(p)$ be the homoclinic class of f associated to a hyperbolic periodic point p, and suppose that $H_f(p)$ is C^1 stably measure expansive. Then for C, λ as in* Proposition 3.2 *and $\delta > 0$ satisfying $\lambda' = \lambda(1+\delta) < 1$ and $q \sim p$, there exists $0 < \epsilon_1 < \epsilon$ such that if for all $0 \le n \le \pi(q)$ it holds that for some $\epsilon_2 > 0$, $f^n(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q))$ then*

$$f^{\pi(q)}(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{C\lambda'^{\pi(q)}\epsilon_2}(q).$$

Similarly, if $f^{-n}(W^{cu}_{\epsilon_2}(q)) \subset W^{cu}_{\epsilon_1}(f^{-n}(q))$ then

$$f^{-\pi(q)}(W^{cu}_{\epsilon_2}(q)) \subset W^{cu}_{C\lambda'^{\pi(q)}\epsilon_2}(q).$$

Recall that a compact f-invariant set Λ has a *local product structure* if given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d(x, y) < \delta$ and $x, y \in \Lambda$ then

$$\emptyset \neq W^s_{\varepsilon}(x) \cap W^u_{\varepsilon}(y) \subset \Lambda.$$

Note that by using the transverse homoclinic point theorem, we have $H^T_f(p) = \bar{S}_p$, where $S_p = \{q \in P(f) \colon q \sim p\}$.

Lemma 3.5. *Let $H_f(p)$ be the homoclinic class of f associated to a hyperbolic periodic point p, and suppose that $H_f(p)$ is C^1 stably measure expansive. Then $H_f(p)$ has a local product structure. Moreover, for any periodic point $q \in H_f(p)$,* $\mathrm{index}(p) = \mathrm{index}(q)$.

Proof. Let U be a locally maximal neighborhood of $H_f(p)$, and let $e > 0$ be a measure expansive constant of $f|_{H_f(p)}$. Since $H_f(p)$ is a closed set, there is $\epsilon > 0$ such that $B_\epsilon(H_f(p)) \subset U$. Let $\epsilon_1 > 0$ be constant such that
(i) $\epsilon_1 < \min\{e, \epsilon\}$, and
(ii) $\sup\{\mathrm{diam}\, W^{cs}_{\epsilon_1}(q) \colon q \in H_f(p)\} < \epsilon$.

For any $q \in H_f(p)$ with $q \sim p$, we let

$$\varepsilon(q) = \sup\{\varepsilon > 0 \colon f^n(W^{cs}_{\varepsilon}(q)) \subset W^{cs}_{\varepsilon_1}(f^n(q)) \text{ for all } n \ge 0\}.$$

Let $\varepsilon' = \inf\{\varepsilon(q) \colon q \in S_p\}$. If we prove that ε' is positive then for all $q \in S_p$, we can see that $f^n(W^{cs}_{\varepsilon'}(q)) \subset W^{cs}_{\varepsilon_1}(f^n(q))$ for all $n \ge 0$.

Suppose $\varepsilon' = 0$. Then there exists a sequence $\{q_n\}$ in S_p such that $\varepsilon(q_n) \to 0$ as $n \to \infty$. Hence we obtain $0 < m_n < \pi(q_n)$ with $m_n \to \infty$ as $n \to \infty$ and $y_n \in$

$W^{cs}_{\varepsilon(q_n)}(q_n)$ such that

$$d(f^{m_n}(q_n), f^{m_n}(y_n)) = \varepsilon_1.$$

Let $I_n = [f^{m_n}(q_n), f^{m_n}(y_n)]$ be an arc joining $f^{m_n}(q_n)$ with $f^{m_n}(y_n)$ in $W^{cs}_{\varepsilon_1}(f^{m_n}(q_n))$, and let $J_n = f^{-m_n}(I_n)$. Clearly J_n is contained in $W^{cs}_{\varepsilon(q_n)}(q_n)$ and we get

$$f^i(J_n) \subset W^{cs}_{\varepsilon_1}(f^i(q_n)),$$

where $0 \leq i \leq \pi(q_n)$. By Lemma 3.4, we have

$$f^{\pi(q_n)}(W^{cs}_{\varepsilon(q_n)}(q_n)) \subset W^{cs}_{C\lambda'^{\pi(q_n)}\varepsilon(q_n)}(q_n).$$

Observe that sequences $\{m_n\}$ and $\{\pi(q_n) - m_n\}$ tend to ∞ as $n \to \infty$. Take the limit points x and y from $f^{m_n}(q_n)$ and $f^{m_n}(y_n)$, respectively. We can assume that I_n converges to a closed arc joining x to y in the Hausdorff metric, say, I. Note that

$$\operatorname{diam} f^{m_n+j}(J_n) = \operatorname{diam} f^j(I_n) \leq \varepsilon_1,$$

where $-m_n \leq j \leq \pi(q_n) - m_n$. This means that $\operatorname{diam} f^j(I) \leq \varepsilon_1$ for all $j \in \mathbb{Z}$. Since $H_f(p)$ is locally maximal in U and $f^i(I) \subset B_\epsilon(H_f(p)) \subset U$ for all $i \in \mathbb{Z}$, we have $I \subset \bigcap_{i\in\mathbb{Z}} f^i(U) = H_f(p)$. Let m_I be the normalized Lebesgue measure on I. We define $\mu \in \mathcal{M}(M)$ by $\mu(C) = m_I(C \cap I)$ for any Borel set C of M. Then we can see that $\mu \in \mathcal{M}^*(M)$. Since $\operatorname{diam} f^j(I) \leq \varepsilon_1$ for all $j \in \mathbb{Z}$, we get $0 < \mu(\Gamma^f_{\varepsilon_1}(x) \cap I) \leq \mu(\Gamma_{\varepsilon_1}(x))$, and so arrive at a contradiction.

Next, we show that for any $y \in W^{cs}_{\varepsilon_2}(q)$ and $z \in W^{cu}_{\varepsilon_2}(q)$, where ε_2 is given by Lemma 3.4, we have

$$\lim_{n\to\infty} d(f^n(q), f^n(y)) = 0$$

and

$$\lim_{n\to\infty} d(f^{-n}(q), f^{-n}(z)) = 0.$$

Assume that for some y in $W^{cs}_{\varepsilon_2}(q)$,

$$\limsup_{n\to\infty} d(f^n(q), f^n(y)) = \alpha > 0.$$

Then we can find a sequence $m_n > 0$ such that $m_n \to \infty$ as $n \to \infty$, and $d(f^{m_n}(q), f^{m_n}(y)) > \alpha/2$ for all $n > 0$. Choose a geodesic arc I_n joining $f^{m_n}(q)$ and $f^{m_n}(y)$ in $W^{cs}_{\epsilon_1}(f^{m_n}(q))$. Then $\operatorname{diam} I_n > \alpha/2$. Let $J_n = f^{-m_n}(I_n)$, and let $\lim_{n\to\infty} I_n = I$ under the Hausdorff metric. Then I is also a closed arc with two end points. As in the above arguments, we can see that $\operatorname{diam} f^k(I_n) \leq \varepsilon_1$ for k satisfying $-m_n \leq k$. Since $m_n \to \infty$ as $n \to \infty$, we get $\operatorname{diam} f^k(I_n) \leq \varepsilon_1$ for all $k \in \mathbb{Z}$. But this gives

$$\operatorname{diam} f^k(I) \leq \varepsilon_1 \quad \text{for all} \quad k \in \mathbb{Z}.$$

If $\varepsilon_1 > 0$ a measure expansive constant of $f|_{H_f(p)}$ then by the same arguments as the above, we can get a contradiction.

Finally, we show that $H_f(p)$ has a local product structure. Take $\varepsilon' > 0$ as above. Note that for any $x \in H_f(p)$, the local center stable and unstable manifolds of x are true stable and unstable manifolds of x, respectively. Indeed, if $\epsilon = \min\{\epsilon', \epsilon_1\}$, by the continuity of $W^{cs}_{\epsilon'}(x)$ with respect to x and the fact $H_f(p) = \overline{\{q \in P_h(f): q \sim p\}}$, we can see that $f^n(W^{cs}_{\epsilon}(x)) \subset W^{cs}_{\epsilon_1}(f^n(x))$ for $x \in H_f(p)$ and all $n \geq 0$. Moreover, if $y \in W^{cs}_{\epsilon}(x) \cap H_f(p)$ then $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$. Consequently we have that $W^{cs}_{\epsilon}(x) = W^s_{\epsilon}(x)$ for any $x \in H_f(p)$. This means that the local center stable manifolds are true stable manifolds. Simiarly we can show the same results for the center unstable manifolds. Even though this part was essentially proved in [14], we mention it here for safety.

By Proposition 3.3, we can take $\delta > 0$ such that

$$W^s_{\epsilon'}(x) \cap W^u_{\epsilon'}(y) \neq \emptyset,$$

whenever $d(x, y) < \delta$ and $x, y \in H_f(p)$. Since $W^s_{\epsilon'}(x) \subset \overline{W^s(p)}$ and $W^u_{\epsilon'}(y) \subset \overline{W^u(p)}$, by the λ-lemma, we can see that

$$W^s_{\epsilon'}(x) \cap W^u_{\epsilon'}(y) \subset H_f(p).$$

This establishes that $H_f(p)$ has a local product structure. Since $H_f(p) = \overline{\{q \in P_h(f): q \sim p\}}$, and $H_f(p)$ has a local product structure, by Proposition 3.3 (3), for any periodic point q in $H_f(p)$, we know that $W^s(p) \cap W^u_{\epsilon'}(q) \neq \emptyset$ and $W^u(p) \cap W^s_{\epsilon'}(q) \neq \emptyset$ which are transverse intersections. Thus, we have index$(q) =$ index(p). $\square$

To prove Theorem C, we use the famous Mañé's ergodic closing lemma in [8]. Let $B_\varepsilon(f, x) = \{y \in M: d(f^n(x), y) < \varepsilon$ for some $n \in \mathbb{Z}\}$, and let Σ_f be the set of points $x \in M$ such that for any C^1 neighborhood $\mathcal{U}(f)$ of f and for every $\varepsilon > 0$ there exist $g \in \mathcal{U}(f)$ and $y \in M$ such that $y \in P(g)$, $g = f$ on $M \setminus B_\varepsilon(f, x)$ and $d(f^j(x), g^j(y)) < \varepsilon$ for all $0 \leq j \leq m$, where m is the g-period of y. Then the Mañé's ergodic closing lemma states that Σ_f is a total Probability set, i.e., for any f-invariant probability measure μ, $\mu(\Sigma_f) = 1$.

End of proof of Theorem C. Suppose the homoclinic class $H_f(p)$ is C^1 stably measure expansive. Then there exist a compact neighborhood U of Λ and a C^1 neighborhood $\mathcal{U}(f)$ of f such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$, and Λ_g is measure-expansive for $g \in \mathcal{U}(f)$. As observed earlier, we have $H_f(p) = \Lambda_j(f)$, where $j =$ index(p). If necessary, we can shrink the neighborhood $\mathcal{U}(f)$ to satisfy the assumptions of Lemma 3.1 and Proposition 3.2.

To show that $\Lambda_j(f)$ is hyperbolic, choose an open set U_j such that $\Lambda_j(f) \subset U_j \subset U$. By Lemma 3.5, we have $\Lambda_i(f) = \overline{P_i(f|_{H_f(p)})} = \emptyset$ if $i \neq j$. For a sufficiently small neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f, if $g \in \mathcal{U}_0(f)$ satisfies $g = f$ on $M \setminus U_j$, then we see that $\mathrm{index}(q) = \mathrm{index}(p_g)$ for any $q \in \Lambda_g \cap P(g)$. Suppose not. Then there exist $h \in \mathcal{U}_0(f)$ and $q \in \Lambda_h \cap P(h)$ such that $h = f$ on $M \setminus U_j$, $\mathrm{index}(q) \neq \mathrm{index}(p_h)$ and $h^{\pi(p)}(p_h) = p_h$. Suppose $\mathrm{index}(q) = l$. Define $\eta\colon \mathcal{U}_0(f) \to \mathbb{Z}$ by

$$\eta(g) = \#\{y \in \Lambda_g \cap P(g)\colon g^{\pi(q)}(y) = y,\ \mathrm{index}(y) = l\},$$

where $\#A$ denote the cardinal number of the set A. Then we see that η is a constant function on $\mathcal{U}_0(f)$ by Lemma 3.1. But this is a contradiction due to the fact $\eta(h) > \eta(f)$.

By Proposition 3.2 (2), we know that $H_f(p)$ admits a dominated splitting $T_{H_f(p)}M = E \oplus F$ with $\dim E = \mathrm{index}(p)$. Let $\epsilon > 0$, and let $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ be as in Lemma 2.2.

To prove that $H_f(p)$ is hyperbolic, we are going to use the techniques of the proof of Theorem B in [8]. That is, we show that

$$\liminf_{n\to\infty} \|Df^n|_{E(x)}\| = 0$$

and

$$\liminf_{n\to\infty} \|Df^{-n}|_{F(x)}\| = 0,$$

for all $x \in H_f(p)$. Suppose $\liminf_{n\to\infty} \|Df^n|_{E(x)}\| \neq 0$ for some $x \in H_f(p)$. For the constant $m \in \mathbb{Z}^+$ taken in Proposition 3.2, let $\psi(x) = \log\|D_x f^m|_{E(x)}\|$. Then we have a sequence $\{j_n\}$ and a f^m-invariant probability measure μ on $H_f(p)$ satisfying

$$\int_{H_f(p)} \psi\, d\mu = \lim_{n\to\infty} \frac{1}{j_n} \sum_{i=0}^{j_n-1} \log\|D_{f^{mi}(x)} f^m|_{E(f^{mi}(x))}\| \geq 0.$$

By Mañé's ergodic closing lemma and Birkhoff's theorem, we can find $q \in \Sigma_f \cap H_f(p)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\|D_{f^{mi}(q)} f^m|_{E(f^{mi}(q))}\| \geq 0.$$

By Proposition 3.2 (1), we conclude that q is not a periodic point of f. Let $C > 0$ and λ be as in Proposition 3.2. Choose $\lambda < \gamma < 1$ and n_0 such that

$$\frac{1}{n} \sum_{i=0}^{n-1} \log\|D_{f^{mi}(q)} f^m|_{E(f^{mi}(q))}\| \geq \log\gamma$$

when $n > n_0$. By Mañé's ergodic closing lemma we can find $\tilde{f} \in \mathcal{U}_0(f)$ and $\tilde{q} \in \Lambda_{\tilde{f}} \cap P(\tilde{f})$ such that the $\tilde{f}$-orbit of $\tilde{q}$ ϵ-shadows a part of the f-orbit of g for arbitrarily small $\epsilon > 0$. By Lemmas 3.1 and 3.5, $\tilde{q}$ is hyperbolic and index$(\tilde{q})$ = index(p). By applying Lemma 2.2, we can obtain $g \in \mathcal{V}(\tilde{f}) \subset \mathcal{U}_0(f)$ such that

$$\prod_{i=0}^{k-1} \| D_{g^{im}(\tilde{q})} g^m|_{E(g^{mi}(\tilde{q}))}\| \geq \gamma^k.$$

By Proposition 3.2 (1), we have

$$\prod_{i=0}^{k-1} \| D_{g^{im}(\tilde{q})} g^m|_{E(g^{mi}(\tilde{q}))}\| < C\lambda^k.$$

Observe that we can choose the period $\pi(\tilde{q})$ of $\tilde{q}$ large enough so that $\gamma^k \geq C\lambda^k$, where $k = [\pi(\tilde{q})/m]$. This is a contradiction and hence $\liminf_{n\to\infty} \|Df^n|_{E(x)}\| = 0$ for each $x \in H(p, f)$. Similarly we can show that $\liminf_{n\to\infty} \|Df^{-n}|_{F(x)}\| = 0$ for each $x \in H_f(p)$.

The converse is clear by Theorem C in [5], and so completes the proof of Theorem C. □

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