

# BEHAVIOR OF SOLUTIONS FOR RADIALLY SYMMETRIC SOLUTIONS FOR BURGERS EQUATION WITH A BOUNDARY CORRESPONDING TO THE RAREFACTION WAVE

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## Abstract

We investigate the large-time behavior of the radially symmetric solution for Burgers equation on the exterior of a small ball in multi-dimensional space, where the boundary data and the data at the far field are prescribed. In a previous paper [1], we showed that, for the case in which the boundary data is equal to 0 or negative, the asymptotic stability is the same as that for the viscous conservation law. In the present paper, it is proved that if the boundary data is positive, the asymptotic state is a superposition of the stationary wave and the rarefaction wave, which is a new wave phenomenon. The proof is given using a standard  $L^2$  energy method and the characteristic curve method.

## 1. Introduction

We consider Burgers equation for a multi-dimensional space,

$$(1.1) \quad \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \mu \Delta u, \quad (t > 0, x \in \mathbb{R}^n),$$

where  $\mu$  is a positive constant. In the present paper, we investigate a radially symmetric solution for (1.1) on the exterior domain  $|x| > r_0$  for some positive constant  $r_0$ , where the data on the boundary and at the far field are prescribed. For this purpose, we transform the unknown function  $u(t, x)$  in (1.1) to  $v(t, r)$  by means of  $u \equiv (x/r)v(t, r)$ , where  $r$  is defined by  $r := |x|$ . Then we have the initial boundary value problem for Burgers equation:

$$(1.2) \quad \begin{cases} v_t + vv_r = \mu \left( v_{rr} + (n-1) \left( \frac{v}{r} \right)_r \right), & r > r_0, t > 0, \\ v(t, r_0) = v_-, \quad \lim_{r \rightarrow +\infty} v(t, r) = v_+, & t > 0, \\ v(0, r) = v_0(r), & r > r_0, \end{cases}$$

where the initial data  $v_0$  is assumed to satisfy  $v_0(r_0) = v_-$  and  $\lim_{r \rightarrow +\infty} v_0(r) = v_+$  as the compatibility conditions. We are interested in the large-time behavior of the solution with condition  $v_- > 0$ .

For the viscous conservation law, for the case in which the flux is convex and the corresponding Riemann problem has the rarefaction wave, Liu–Matsumura–Nishihara [8] showed that, depending on the signs of the boundary condition  $v_{\pm}$ , the large-time behavior of the solution is classified into the three cases:

- (a)  $v_- < v_+ \leq 0$ ,
- (b)  $0 = v_- < v_+$ , and
- (c)  $v_- < 0 < v_+$ .

More precisely, they showed that in case (a), the solution tends toward the stationary solution. In case (b), the solution tends toward the rarefaction wave, and in case (c), the solution tends toward the linear superposition of the stationary solution and the rarefaction wave.

In the case of the viscous conservation law on the half-line, Nakamura [13] considered the case in which  $0 < v_- < v_+$  and demonstrated that the asymptotic state is a rarefaction wave, which connects  $v_-$  and  $v_+$  using the technical energy method. Nakamura also derived the decay rate to the rarefaction wave.

Liu and Yu [10], and Liu and Nishihara [9] considered the case where the boundary value satisfies  $v_- > v_+$ , and showed the asymptotic stability of viscous shock wave on the half space. Initial boundary value problem for the planer wave of conservation laws were investigated by Kawashima, Nishibata, and Nishikawa [4] on higher dimensional space. The problem of system for one-dimensional gas motion has been investigated by Matsumura and Nishihara [11, 12], Kawashima and Shizuta [6], and Kawashima [3]. On the other hand, the analysis of the large time behavior of radially symmetric solution for viscous conservation laws with Dirichlet boundary condition on multi-dimensional space had been open. Present research is one of the investigative research for radially symmetric problem of system for compressible viscous gas on multi-dimensional space.

For radially symmetric solutions for Burgers equation, we showed in [1] that the asymptotic states are divided into three cases depending on the signs of the boundary as (a), (b), and (c), and the asymptotic states are the same as those of the results reported by Liu–Matsumura–Nishihara [8].

In the present paper, we consider the case in which  $0 < v_- < v_+$  which is the same case of the research by Nakamura [13], and we showed that the asymptotic state is a superposition of the stationary wave  $\phi$  which connects from  $v_-$  to 0, and the rarefaction wave  $\psi^R$ , which connects from 0 to  $v_+$ . We emphasize that this asymptotic state differs from that of the viscous conservation law reported by Nakamura [13], and this asymptotic state is a new wave phenomena. Here,  $\phi$  is defined through the stationary

problem corresponding to (1.2), as follows:

$$(1.3) \quad \begin{cases} \left(\frac{1}{2}\phi^2\right)_r = \mu\left(\phi_{rr} + (n-1)\left(\frac{\phi}{r}\right)_r\right), & r > r_0, \\ \phi(r_0) = v_-, \quad \lim_{r \rightarrow +\infty} \phi(r) = 0. \end{cases}$$

On the other hand,  $\psi^R$  is defined as  $\psi^R((r - r_0)/t) = \psi^R(s)$  for  $t > 0$ , where  $\psi^R(s)$  is obtained as follows:

$$(1.4) \quad \psi^R(s) = \begin{cases} 0, & s \leq 0, \\ s, & 0 \leq s \leq v_+, \\ v_+, & v_+ \leq s. \end{cases}$$

We have the following theorem.

**Theorem 1.1.** *Suppose that  $0 < v_- < v_+$ ,  $n \geq 3$ , and  $v_-r_0 < \mu/2$ . Further, assume that  $v_0 - v_+ \in H^1$ . Let  $\phi(r)$  be the stationary wave satisfying problem (1.3), and let  $\psi^R((r - r_0)/t)$  be the rarefaction wave defined by (1.4). Then the initial-boundary value problem (1.2) has a unique solution  $v$  globally in time satisfying*

$$v - v_+ \in C^0([0, \infty); H^1), \quad (v - v_+)_r \in L^2(0, T; H^1), \quad T > 0,$$

and the asymptotic behavior

$$\limsup_{t \rightarrow \infty} \sup_{r > r_0} \left| v(r, t) - \phi(r) - \psi^R\left(\frac{r - r_0}{t}\right) \right| = 0.$$

Note that the assumption  $v_-r_0 < \mu/2$  in Theorem 1.1 is a natural condition, because we consider the case in which the boundary  $r_0$  is small.

The remainder of the present paper is organized as follows. After presenting the notation, we reformulate the problem in Section 2. In Section 3, we present an a priori estimate. Finally, in Section 4, we describe how to predict the asymptotic state.

NOTATION. We denote by  $L^2$  the usual Lebesgue space over  $r > r_0 > 0$  with the norm  $\|\cdot\|_{L^2}$  and by  $H^1$  the corresponding first-order Sobolev space with norm  $\|\cdot\|_{H^1}$ . We also denote by  $H_0^1 = H_0^1((0, 1))$  the space of functions  $f \in H^1$  with  $f(r_0) = 0$ .

For an interval  $I$  and a Banach space  $X$ ,  $C^k(I; X)$  denotes the space of  $k$ -times continuously differentiable functions on the interval  $I$  with values in  $X$ . Finally,  $C$  is used as a positive generic constant unless different constants need to be distinguished.

## 2. Reformulation of the problem

In this section, we present the preliminaries for the proof of Theorem 1.1. First, we consider the properties of the stationary solution  $\phi$ , which is given by the solution

to the boundary value problem for the ordinary differential equation (1.3). If we integrate the equation of (1.3) once, it is easy to see that (1.3) is equivalent to the problem

$$(2.1) \quad \begin{cases} \frac{1}{2}\phi^2 = \mu \left( \phi_r + (n-1)\frac{\phi}{r} \right), & r > r_0, \\ \phi(r_0) = v_-. \end{cases}$$

Then we have the following lemma.

**Lemma 2.1.** *Suppose  $v_-r_0 < 2\mu(n-2)$  and  $n \geq 3$ . Then the stationary problem (1.3) has a unique smooth solution  $\phi(r)$  satisfying  $0 \leq \phi(r) \leq v_-$ ,  $\phi_r(r) < 0$  and  $|\phi| \leq C/(r+1)$  for  $r > r_0$ .*

*Proof.* The first equality of (2.1) is rewritten as

$$(2.2) \quad \phi_r + \frac{n-1}{r}\phi = \frac{1}{2\mu}\phi^2,$$

and we introduce a new unknown function  $\xi$  as  $\xi = 1/\phi$ . As (2.2) is a Bernoulli-type differential equation, we can describe  $\xi$  as follows:

$$(2.3) \quad \begin{aligned} \xi &= e^{\int(n-1)/r dr} \left( -\int \frac{1}{2\mu} e^{-\int(n-1)/r dr} dr + K \right) \\ &= r^{n-1} \left( -\frac{1}{2\mu} \int \frac{1}{r^{n-1}} dr + K \right), \end{aligned}$$

where  $K$  is some constant. Now, we derive the solution of (2.1) for the case of  $n \geq 3$ . Note that when  $n = 2$ , (2.1) has no solution which satisfies the boundary condition. For the case  $n \geq 3$ , by direct calculation, we derive  $\phi$  as follows:

$$(2.4) \quad \phi(r) = \frac{1}{r/(2\mu(n-2)) + Kr^{n-1}}, \quad \text{where } K = -\frac{1}{2\mu(n-2)r_0^{n-2}} + \frac{1}{v_-r_0^{n-1}}.$$

As  $v_-r_0 < 2\mu(n-2)$  reduces to  $K > 0$ , we have the solution  $\phi$  which connects  $v_-$  and 0. By direct calculation, we also derive  $\phi_r > 0$ .  $\square$

Next, we generate a smooth approximation of the rarefaction wave  $\psi^R$  defined by (1.4). Because non-smoothness of  $\psi^R$  causes trouble in the process of handling the second derivative of the solution, we follow the arguments of Kawashima and Tanaka [7]. We define a smooth approximation  $\psi(t, r)$  of  $\psi^R((r-r_0)/t)$  by the solution of the viscous Burgers equation:

$$(2.5) \quad \begin{cases} \psi_t + \psi \psi_r = \psi_{rr}, & r \in \mathbb{R}, t > -1, \\ \psi(r, -1) = \begin{cases} -v_+, & r < r_0, \\ v_+, & r > r_0. \end{cases} \end{cases}$$

Note that the Hopf–Cole transformation gives an explicit formula for  $\psi(t, r)$ . We summarize the basic properties of  $\psi(t, r)$  in the next lemma. For its proof, we refer the reader to [7].

**Lemma 2.2.** *We have the following:*

- 1)  $\psi(t, r)$  is a smooth solution of (2.5) and verifies  $\psi(t, r_0) = 0$  for  $t \geq 0$ .
- 2)  $0 < \psi(t, r) < v_+$  and  $\psi_r(t, r) > 0$  for  $r > r_0$  and  $t \geq 0$ .
- 3) For  $1 \leq p \leq \infty$ , we have

$$\begin{aligned} \|\psi_r(t)\|_{L^p} &\leq C \min\{v_+(1+t)^{-\gamma}, v_+^{1/p}(1+t)^{-2\gamma}\}, \\ \|\psi_{rr}(t)\|_{L^p} &\leq C \min\{v_+(1+t)^{-\gamma-1/2}, (1+t)^{-\gamma-1}\}, \end{aligned}$$

where  $\gamma = (1/2)(1 - 1/p)$ , and  $C$  is a constant independent of  $v_+$ .

- 4)  $\psi(r, t)$  is an approximation of  $\psi^R(r, t)$  in the sense that

$$\|(\psi - \psi^R)(t)\|_{L^p} \leq C\sigma(t)(1+t)^{-\gamma},$$

for  $1 \leq p \leq \infty$ , where  $\gamma = (1/2)(1 - 1/p)$ ,  $\sigma(t) = \log(2+t)$  for  $p = 1$  and  $\sigma(t) = 1$  for  $1 < p \leq \infty$ , and  $C$  is a constant independent of  $v_+$ .

Next, we reformulate the problem. Let  $\phi$  and  $\psi$  be the stationary wave satisfying (2.1) and the smoothed rarefaction wave defined by (2.5), respectively. Now, we define  $\Phi(t, r)$  as the superposition of the stationary wave and the rarefaction wave as

$$\Phi(t, r) := \phi(r) + \psi(t, r),$$

which is an approximation of our solution. Using (2.1) and (2.5), we find that  $\Phi(t, r)$  satisfies

$$\begin{cases} \Phi_t + \left(\frac{1}{2}\Phi^2\right)_r = \mu\Phi_{rr} + \bar{R}, & r > r_0, t > 0, \\ \Phi(t, r_0) = v_-, & t > 0, \end{cases}$$

where  $\bar{R}$  is defined by

$$\bar{R} := \mu(n-1)\left(\frac{\phi}{r}\right)_r + (\phi\psi)_r.$$

Then we reformulate our problem (1.2) by introducing the perturbation  $w(t, r)$  by

$$v(t, r) = \Phi(t, r) + w(t, r).$$

Now, we rewrite our original problem (1.2) as

$$(2.6) \quad \begin{cases} w_t + \frac{1}{2}(w^2 + 2\Phi w)_r = \mu \left( w_{rr} + (n-1) \left( \frac{w + \psi}{r} \right)_r \right) - (\phi\psi)_r, \\ w(t, r_0) = 0, \\ w(0, r) = w_0(r), \end{cases} \quad \begin{array}{l} t > 0, \\ r > r_0. \end{array}$$

The theorem for the reformulated problem (2.6) we shall prove is

**Theorem 2.3.** *Suppose that  $0 < v_- < v_+$ ,  $n \geq 3$ , and  $v_{-r_0} < \mu/2$  hold. Assume that  $w_0 \in H^1$ . Then the initial boundary value problem (2.6) has a unique solution  $w$  globally in time*

$$w \in C([0, \infty); H^1), \quad w_r \in L^2(0, \infty; H^1),$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{r > r_0} |w(t, r)| = 0.$$

The main theorem, Theorem 1.1, is a direct consequence of Theorem 2.3. Theorem 2.3 itself is proved by combining the local existence theorem with the a priori estimate as in the previous papers.

To state the local existence theorem, we define the solution set for any interval  $I \subset \mathbb{R}$  and constant  $M > 0$  by

$$(2.7) \quad X_M(I) = \left\{ w \in C(I; H_0^1); w_r \in L^2(0, T; H^1), \sup_{t \in I} \|w(t)\|_{H^1} \leq M \right\}.$$

Then we state the local existence theorem.

**Proposition 2.4** (local existence). *For any positive constant  $M$ , there exists a positive constant  $t_0 = t_0(M)$  such that if  $\|w_0\|_{H^1} \leq M$ , the initial boundary value problem (2.6) has a unique solution  $w \in X_{2M}([0, t_0])$ .*

It is noted that the problem (2.6) is reduced to the integral equation

$$\begin{aligned} w(t, r) &= \int_{r_0}^{\infty} G(r, y, t) w_0(y) dy \\ &\quad + \int_0^t \int_{r_0}^{\infty} G(r, y; t-s) \left( -\frac{1}{2}(w^2 + 2\Phi w)_r + \tilde{R}(\phi, \psi) \right)(s) dy ds, \end{aligned}$$

where  $G(r, y; t)$  is the Green kernel of the Dirichlet zero boundary value problem for

the linear heat equation on the half line, which is concretely given by

$$G(r, y; t) = \frac{1}{\sqrt{4\pi\mu t}}(e^{-(r-y)^2/(4\mu t)} - e^{-(r+y)^2/(4\mu t)}),$$

and

$$(2.8) \quad \tilde{R} = \mu(n-1)\left(\frac{w+\psi}{r}\right)_r - (\phi\psi)_r.$$

Since we can prove the Proposition 2.4 by a standard iterative method, we omit the proof of the Proposition 2.4.

### 3. Proof of the a priori estimate

In this section, we present the a priori estimate of  $w(r, t)$ . The outline of the proof is similar to that of [1], but we also need to consider the boundary effects. First, we present a key lemma which plays an essential role in our energy method.

**Lemma 3.1.** *Under the condition  $n \geq 3$  and  $v_{-r_0} < \mu/2$ , we have the inequality*

$$(3.1) \quad \int_{r_0}^{\infty} |\phi_r| w^2 dr < \mu(n-1) \int_{r_0}^{\infty} \frac{1}{r^2} w^2 dr.$$

Proof. Differentiating (2.4) in terms of  $r$ , we can estimate  $|\phi_r|$  from above as

$$|\phi_r| < \frac{1}{2Kr^n} + \frac{n-1}{Kr^n} < \frac{1}{2(Kr_0^{n-2})r^2} + \frac{n-2}{(Kr_0^{n-2})r^2} < 2v_{-r_0}(n-1)\frac{1}{r^2}.$$

If  $v_{-r_0} < \mu/2$ , we have the desired inequality (3.1). □

Next, let us present the a priori estimate which is essential to the present study.

**Proposition 3.2** (a priori estimate). *Suppose that the same assumptions as in Theorem 2.3 hold. Then, if  $w \in X_{\infty}([0, T])$  is the solution of the problem (2.6) for some  $T > 0$ , it holds that*

$$(3.2) \quad \|w\|_{H^1}^2 + \int_0^t \|\sqrt{\psi_r}w(\tau)\|_{L^2}^2 + \|w_r(\tau)\|_{H^1}^2 + \left\| \frac{w(\tau)}{r} \right\|_{L^2}^2 d\tau \leq C(\|w_0\|_{H^1}^2 + 1),$$

for  $t \in [0, T]$ , where  $C$  is a positive constant independent of  $T$ .

Proof. Multiplying (2.6) by  $w$ , we obtain

$$(3.3) \quad \begin{aligned} & \left(\frac{1}{2}w^2\right)_t + F_r + \frac{1}{2}\Phi_r w^2 + \mu w_r^2 + \mu(n-1)\frac{w^2}{2r^2} \\ & = \mu(n-1)\left(\frac{\psi_r w}{r} - \frac{\psi w}{r^2}\right) - (\phi\psi)_r w, \end{aligned}$$

where

$$F := \frac{1}{3}w^3 + \frac{1}{2}\Phi w^2 - \mu w w_r - \mu(n-1)\left(\frac{w^2}{2r}\right).$$

Integrating (3.3) over  $[r_0, \infty]$  in terms of  $r$ , we have

$$\begin{aligned} (3.4) \quad & \left(\int_{r_0}^{\infty} \frac{1}{2}w^2 dr\right)_t + \frac{1}{2} \int_{r_0}^{\infty} \psi_r w^2 dr + \mu \int_{r_0}^{\infty} w_r^2 dr + \frac{\mu}{2}(n-1) \int_{r_0}^{\infty} \frac{w^2}{r^2} dr \\ & = -\frac{1}{2} \int_{r_0}^{\infty} \phi_r w^2 dr + \mu(n-1) \int_{r_0}^{\infty} \frac{\psi_r w}{r} - \frac{\psi w}{r^2} - (\phi\psi)_r w dr, \end{aligned}$$

where  $\phi_r < 0$ . Now, we estimate the right-hand side of (3.4). Note that the first term of the right-hand side of (3.4) is absorbed into the last term of the left-hand side of (3.4) by using Lemma 3.1. By Young’s inequality, the second term of the right-hand side of (3.4) is estimated as

$$\begin{aligned} (3.5) \quad & \int_{r_0}^{\infty} \frac{\psi_r w}{r} dr \leq \|w\|_{L^\infty} \int_{r_0}^{\infty} \frac{\psi_r}{r} dr \\ & \leq \epsilon \|w_r\|_{L^2}^2 + C_\epsilon \|w\|_{L^2}^{2/3} \left(\int_{r_0}^t \frac{\psi_r}{r} dr + \int_t^{\infty} \frac{\psi_r}{r} dr\right)^{4/3}. \end{aligned}$$

We introduce new symbols  $I_1$  and  $I_2$  as

$$I_1 := \int_{r_0}^t \frac{\psi_r}{r} dr, \quad I_2 := \int_t^{\infty} \frac{\psi_r}{r} dr.$$

Using the estimate of the rarefaction wave in Lemma 2.2-3), we can estimate  $I_1$  as

$$(3.6) \quad I_1 \leq \|\psi_r\|_{L^\infty} \int_{r_0}^t \frac{1}{r} dr \leq C(1+t)^{-1} \log(2+t).$$

On the other hand, by using the integration by parts,  $I_2$  is estimated as

$$(3.7) \quad I_2 = \int_t^{\infty} \frac{\psi_r}{r} dr \leq \left[\frac{\psi}{r}\right]_t^{\infty} + \int_t^{\infty} \frac{\psi}{r^2} dr \leq C(v_+)(1+t)^{-1}.$$

By virtue of these two estimates, we rewrite the inequality (3.5) as

$$(3.8) \quad \int_{r_0}^{\infty} \frac{\psi_r w}{r} dr \leq \epsilon \|w_r\|_{L^2}^2 + C \|w\|_{L^2}^{2/3} (1+t)^{-4/3} \log^{4/3}(2+t).$$



Applying the inequality (3.8), the third term of the right-hand side of (3.4) is estimated as

$$\begin{aligned}
 \int_{r_0}^{\infty} \frac{\psi w}{r^2} dr &\leq \|w\|_{L^\infty} \int_{r_0}^{\infty} \frac{\psi}{r^2} dr \\
 &\leq \|w\|_{L^\infty} \int_{r_0}^{\infty} -\left(\frac{\psi}{r}\right)_r + \frac{\psi_r}{r} dr \\
 &= \|w\|_{L^\infty} \int_{r_0}^{\infty} \frac{\psi_r}{r} dr \\
 &\leq \epsilon \|w_r\|_{L^2}^2 + C_\epsilon \|w\|_{L^2}^{2/3} (1+t)^{-4/3} \log^{4/3}(2+t).
 \end{aligned}
 \tag{3.9}$$

Now, we estimate the rightmost term of (3.4).

$$\begin{aligned}
 \int_{r_0}^{\infty} -\phi \psi_r w dr &\leq \|w\|_{L^\infty} \int_{r_0}^{\infty} \phi \psi_r dr \\
 &= \|w\|_{L^\infty} \left( \int_{r_0}^t \phi \psi_r dr + \int_t^{\infty} \phi \psi_r dr \right).
 \end{aligned}
 \tag{3.10}$$

We define new symbols  $I_3$  and  $I_4$  as

$$I_3 := \int_{r_0}^t \phi \psi_r dr, \quad I_4 := \int_t^{\infty} \phi \psi_r dr.$$

Using the estimate of the stationary wave  $\phi$  derived in Lemma 2.1 and the rarefaction wave  $\psi$  derived in Lemma 2.2-3), we estimate  $I_3$  and  $I_4$  as

$$\begin{aligned}
 I_3 &\leq C \|w\|_{L^2}^{1/2} \|w_r\|_{L^2}^{1/2} (1+t)^{-1} \int_{r_0}^t |\phi| dr \\
 &\leq \epsilon \|w_r\|_{L^2}^2 + C_\epsilon \|w\|_{L^2}^{2/3} (1+t)^{-4/3} \log^{4/3}(2+t),
 \end{aligned}
 \tag{3.11}$$

$$\begin{aligned}
 I_4 &\leq C \|w\|_{L^2}^{1/2} \|w_r\|_{L^2}^{1/2} (1+t)^{-1} \int_t^{\infty} -\phi \psi_r dr \\
 &\leq C \|w\|_{L^2}^{1/2} \|w_r\|_{L^2}^{1/2} (1+t)^{-1} \int_t^{\infty} \phi_r \psi dr \\
 &\leq \epsilon \|w_r\|_{L^2}^2 + C_\epsilon \|w\|_{L^2}^{2/3} (1+t)^{-4/3},
 \end{aligned}
 \tag{3.12}$$

where we use Sobolev’s embedding lemma and Young’s inequality. On the other hand, using integration by parts and noting  $\phi_r < 0$ , we estimate

$$\int_{r_0}^{\infty} -\phi_r \psi w dr \leq \|w\|_{L^\infty} \int_{r_0}^{\infty} -\phi_r \psi dr = \|w\|_{L^\infty} \int_{r_0}^{\infty} \phi \psi_r dr,
 \tag{3.13}$$

and the rightmost term is the same as (3.10). Then a part of the right-hand side of

(3.4) is estimated as

$$(3.14) \quad \left| \int_{r_0}^{\infty} \frac{\psi_r w}{r} - \frac{\psi w}{r^2} - (\phi \psi)_r w \, dr \right| \\ \leq \epsilon \|w_r\|_{L^2}^2 + C_\epsilon \|w\|_{L^2}^{2/3} (1+t)^{-4/3} \log^{4/3}(2+t).$$

Using Lemma 3.1, substituting (3.14) into (3.4) and integrating in terms of  $t$  over  $[0, t]$ , and then using Gronwall's inequality, we have the basic estimate

$$(3.15) \quad \|w\|_{L^2}^2 + \int_0^t \|\sqrt{\psi_r} w(\tau)\|_{L^2}^2 + \|w_r(\tau)\|_{L^2}^2 + \left\| \frac{w(\tau)}{r} \right\|_{L^2}^2 d\tau \leq C(\|w_0\|_{L^2}^2 + 1).$$

Next, we proceed to the higher-order estimate. Multiplying (2.6) by  $-w_{rr}$  and integrating, we obtain

$$(3.16) \quad \left( \int_{r_0}^{\infty} \frac{1}{2} w_r^2 \, dr \right)_t + \mu \int_{r_0}^{\infty} w_{rr}^2 \, dr \\ = \int_{r_0}^{\infty} \left( -\frac{1}{2} (w^2 + 2\Phi w)_r - (\psi \phi)_r + \mu(n-1) \left( \frac{w + \psi}{r} \right)_r \right) (-w_{rr}) \, dr.$$

We estimate the first term of the right-hand side of (3.16) as

$$(3.17) \quad \int_{r_0}^{\infty} \frac{1}{2} (w^2 + 2\Phi w)_r w_{rr} \, dr \\ \leq \epsilon \|w_{rr}\|_{L^2}^2 + C_\epsilon (\|w_r\|_{L^2}^2 + \|\sqrt{\psi_r} w\|_{L^2}^2 + \|\sqrt{|\phi_r|} w\|_{L^2}^2) \\ \leq \epsilon \|w_{rr}\|_{L^2}^2 + C_\epsilon \left( \|w_r\|_{L^2}^2 + \|\sqrt{\psi_r} w\|_{L^2}^2 + \left\| \frac{w}{r} \right\|_{L^2}^2 \right),$$

where we use Lemma 2.1 and the maximum principle, that is,  $\sup_{r_0 < r} |w(t, r)| < C$ , ( $0 < t < T$ ) as in the previous research [1]. Using decay property of stationary solution  $\phi$  in Lemma 2.1 and rarefaction wave  $\psi$  in Lemma 2.2, we estimate the second term of the right-hand side of (3.16) as

$$(3.18) \quad \int_{r_0}^{\infty} (\psi \phi)_r w_{rr} \, dr \leq \epsilon \|w_{rr}\|_{L^2}^2 + C_\epsilon \int_{r_0}^{\infty} (\phi_r \psi)^2 + (\phi \psi_r)^2 \, dr \\ \leq \epsilon \|w_{rr}\|_{L^2}^2 + C_\epsilon (1+t)^{-2},$$

where we use the estimate

$$\begin{aligned}
 \int_{r_0}^{\infty} (\phi_r \psi)^2 dr &\leq C \int_{r_0}^{\infty} \frac{\psi^2}{(1+r)^4} dr \\
 &= C \left( -\frac{1}{3} \left[ \frac{\psi^2}{(1+r)^3} \right]_{r_0}^{\infty} + \frac{2}{3} \int_{r_0}^{\infty} \frac{\psi \psi_r}{(1+r)^3} dr \right) \\
 &\leq C \|\psi_r\|_{L^\infty} \int_{r_0}^{\infty} \frac{\psi}{(1+r)^3} dr \\
 (3.19) \qquad &= C \|\psi_r\|_{L^\infty} \left( -\frac{1}{2} \left[ \frac{\psi^2}{(1+r)^2} \right]_{r_0}^{\infty} + \int_{r_0}^{\infty} \frac{\psi_r}{2(1+r)^2} dr \right) \\
 &\leq C \|\psi_r\|_{L^\infty}^2 \int_{r_0}^{\infty} \frac{1}{(1+r)^2} dr, \quad \text{and} \\
 \int_{r_0}^{\infty} (\phi \psi_r)^2 dr &\leq C \int_{r_0}^{\infty} \frac{\psi_r^2}{(1+r)^2} dr.
 \end{aligned}$$

Applying the same strategy as (3.19), we can estimate the third term of the right-hand side of (3.16) as

$$\begin{aligned}
 \int_{r_0}^{\infty} \left( \frac{w + \psi}{r} \right)_r w_{rr} dr &\leq \epsilon \|w_{rr}\|_{L^2}^2 + C_\epsilon \int_{r_0}^{\infty} \left\{ \left( \frac{w + \psi}{r} \right)_r \right\}^2 dr \\
 (3.20) \qquad &\leq \epsilon \|w_{rr}\|_{L^2}^2 + C_\epsilon \left( \|w_r\|_{L^2}^2 + \left\| \frac{w}{r} \right\|_{L^2}^2 + (1+t)^{-2} \right).
 \end{aligned}$$

Integrating (3.16) over  $[0, t]$ , substituting (3.17) through (3.20) into the resultant equality, and using the basic estimate (3.15), we obtain

$$(3.21) \qquad \|w_r\|_{L^2}^2 + \mu \int_0^t \|w_{rr}(\tau)\|_{L^2}^2 d\tau \leq C(\|w_0\|_{H^1}^2 + 1).$$

Combining (3.15) and (3.21), we have the desired estimate (3.2). □

#### 4. Concluding remarks

In this section, we explain how to predict the asymptotic state. We take a part of term from the problem (1.2), and make equality as

$$(4.1) \qquad v_t + vv_r = -\mu(n-1) \frac{v}{r^2}.$$

We consider the characteristic equation of (4.1):

$$(4.2) \qquad \frac{dt}{ds} = 1, \quad \frac{dr}{ds} = v, \quad \frac{dv}{ds} = -\mu(n-1) \frac{v}{r^2}.$$

Let  $\bar{r} := r(0)$  and define  $A := v_0(\bar{r}) - \mu(n-1)/\bar{r}$ ; then we can solve (4.2) explicitly and find the relation between  $t$  and  $r$  as

$$t = \frac{r - \bar{r}}{A} - \frac{\mu(n-1)}{A^2} \log \left| \frac{\mu(n-1) + Ar}{\mu(n-1) + A\bar{r}} \right|, \quad \text{for } A \neq 0,$$

$$t = \frac{1}{2\mu(n-1)}(r - \bar{r})^2, \quad \text{for } A = 0.$$

By direct calculation, we find that if  $v_- > 0$ , there exists a monotonically decreasing stationary wave around the boundary. On the other hand, if  $v_- = 0$ , there exists no stationary wave around the boundary. From this observation, we anticipate that the asymptotic state of the solution is the superposition of rarefaction wave which connects 0 to  $v_+$ , and stationary wave which connects  $v_-$  to 0.

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