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ABUNDANCE THEOREM FOR SEMI LOG CANONICAL SURFACES IN POSITIVE CHARACTERISTIC

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Abstract

We prove the abundance theorem for semi log canonical surfaces in positive characteristic.

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0. Introduction

A semi log canonical (for short, slc) n-fold is a generalization of log canonical (for short, lc) n-folds. In this paper, we prove the abundance theorem for slc surfaces in positive characteristic. We use the same definition of slc varieties as the one of [13].

Theorem 0.1. *Let (X, Δ) be a projective slc surface over an algebraically closed field of positive characteristic. If $K_X + \Delta$ is nef, then $K_X + \Delta$ is semi-ample.*

Let us briefly review the history of the semi log canonical varieties in characteristic zero. The notion of semi log canonical singularities is introduced in [15] for a moduli problem. The abundance theorem for slc surfaces is proved in [1] and [11]. [4] generalizes this result to dimension three. Moreover, [4] shows that the abundance theorem for slc n-folds follows from the two parts:
(1) The abundance theorem for lc n-folds.
(2) The finiteness theorem of the pluri-canonical representation for $(n-1)$-folds.
[6] shows that (2) holds for each $n \in \mathbb{Z}_{>0}$. If $n = 3$, then (1) follows from [12]. If $n \geq 4$, then (1) is an open problem. For a recent development of the theory of slc

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varieties in characteristic zero, see [5], [6], [8] and [9]. For related topics, see [10], [2] and [16].

In this paper, we use the strategy of [4]. Hence, we must prove (1) and (2) in the case where $n = 2$ and $\operatorname{char} k > 0$. In this case, (1) is a known result by [7]. It is not difficult to prove (2). However, [4] uses many fundamental results based on the minimal model theory and the Kawamata–Viehweg vanishing theorem. We can freely use the minimal model theory for surfaces in positive characteristic by [20] (cf. [7], [14]). Although there exist counter-examples to the Kodaira vanishing theorem in positive characteristic ([19]), we can use some weaker vanishing theorems obtained in [21] and [22] (cf. [14]).

In characteristic two, some new phenomena happen. For example, in characteristic zero, the Whitney umbrella $\{x^2 = yz^2\} \subset \mathbb{A}^3$ is a typical example of slc surfaces (cf. [1, Definition 12.2.1]). In characteristic two, this is slc but not normal crossing in codimension one. Moreover, [4] uses the following fact: if a field extension L/K satisfies $[L : K] = 2$ and its characteristic is zero, then L/K is a Galois extension. But, in characteristic two, this field extension L/K may be purely inseparable. Thus, some proofs are more complicated.

0.2 (Overview of contents). In Section 1, we summarize the notations. The normalization of an slc surface is an lc surface. Therefore, we should investigate lc surfaces. Every lc surface is birational to a dlt surface. Thus, in Section 2, we consider a dlt surface (X, Δ). More precisely, we consider $\llcorner \Delta \lrcorner$ because $\llcorner \Delta \lrcorner$ has the patching data of the normalization. In Section 3, we calculate the normalization of nodal singularities. In Section 4, we prove the main theorem. In Section 5, we summarize fundamental results on dlt surfaces. These results may be well-known but the author can not find a good reference.

1. Notations

We will not distinguish the notations invertible sheaves and Cartier divisors. For example, we will write $L + M$ for invertible sheaves L and M.

Throughout this paper except for Section 3, we work over an algebraically closed field k of positive characteristic and let $\operatorname{char} k =: p$.

In this paper, a *variety* means a pure dimensional reduced scheme which is separated and of finite type over k. A *curve* or a *surface* means a variety whose dimension is one or two, respectively. Note that varieties, curves and surfaces may be reducible.

Let X be a noetherian reduced scheme and let $X = \bigcup X_i$ be the irreducible decomposition. Let $Y_i \to X_i$ be the normalization of X_i. Then we define the *normalization* of X by $\coprod Y_i \to \coprod X_i \to X$. We say X is *normal* if the normalization morphism is an isomorphism.

Let X be a variety. We say Δ is a $\mathbb{Q}$-*divisor* on X if Δ is a finite sum $\Delta = \sum \delta_i \Delta_i$ where $\delta_i \in \mathbb{Q}$ and Δ_i is an irreducible and reduced closed subscheme of codimension one which is not contained in the singular locus Sing(X). Note that, in this case, the local ring $\mathcal{O}_{X,\Delta_i}$ is a discrete valuation ring.

We will freely use the notation and terminology in [13]. In the definition in [13, Definition 2.8], for a pair (X, Δ), Δ is not necessarily effective. But, in this paper, we assume Δ is an effective $\mathbb{Q}$-divisor. For a reducible normal variety X and an effective $\mathbb{Q}$-divisor Δ, we say (X, Δ) is lc (resp. dlt, klt) if each irreducible component is lc (resp. dlt, klt).

For the definition of (nodes and) slc varieties, see Definition 3.1 and Definition 4.1. These definitions are the same as [13, 1.41, 5.10].

2. Boundaries of dlt surfaces

In this section, we investigate dlt surfaces. First, we consider the case of curves. The main result of this section is Proposition 2.8. Proposition 2.8 is the surface version of Proposition 2.1.

Proposition 2.1. *Let (X, Δ) be an irreducible lc curve. Let $f : X \to R$ be a projective surjective morphism such that $f_*\mathcal{O}_X = \mathcal{O}_R$. Assume that $S := \llcorner\Delta\lrcorner \neq 0$ and let $T := f(S)$. If $K_X + \Delta \equiv_f 0$, then one of the following assertions holds.*
(1) $f_*\mathcal{O}_S = \mathcal{O}_T$.
(2) $f_*\mathcal{O}_S \neq \mathcal{O}_T$. $X \simeq \mathbb{P}^1$ *and* $\dim R = 0$. *Moreover,* $\Delta = S$ *and* S *is two distinct points.*

Proof. If $\dim R = 1$, then we see $X \simeq R$ and we obtain (1). We may assume $\dim R = 0$. Since $\deg(K_X + \Delta) = 0$ and $\llcorner\Delta\lrcorner \neq 0$, we see $X \simeq \mathbb{P}^1$ and S has at most two points. If S is one point, then we obtain (1). □

In the above proposition, (1) is a good case. Hence, we classify the other case (2) as above. For this, we want sufficient conditions for $f_*\mathcal{O}_S = \mathcal{O}_T$.

We use the following vanishing theorem for rational surfaces essentially established in [22].

Proposition 2.2. *Let (X, B) be a projective irreducible klt surface such that X is a rational surface. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor such that $D-(K_X+B)$ is nef and big. Then, $H^1(X, D) = 0$.*

Proof. We can find a birational morphism $f : Y \to X$ from a smooth projective surface Y and finitely many prime divisors $\{F_j\}_{j\in J}$ on Y such that
(1) $\mathrm{Ex}(f) \subset \mathrm{Supp}\big(\sum_{j\in J} F_j\big)$.
(2) $f^{-1}(B) \cup \sum_{j\in J} F_i$ is a simple normal crossing divisor.
(3) $f^*(D-(K_X+B)) - \sum \delta_j F_j$ is ample for some $0 < \delta_j \ll 1$.
(4) $\ulcorner f^*(D-(K_X+B)) - \sum \delta_j F_j \urcorner = \ulcorner f^*(D-(K_X+B))\urcorner$.

We define E by $K_Y = f^*(K_X + B) + E$. Since (X, B) is klt, $\ulcorner E \urcorner$ is effective and f-exceptional. Thus, we obtain $\mathcal{O}_X(D) = f_*\mathcal{O}_Y(\ulcorner f^*D + E\urcorner)$. Therefore, by the Leray spectral sequence, we obtain

$$0 \to H^1(X, \mathcal{O}_X(D)) \to H^1(Y, \mathcal{O}_Y(\ulcorner f^*D + E\urcorner)).$$

Then, the assertion holds by

$$H^1(Y, \mathcal{O}_Y(\ulcorner f^*D + E\urcorner)) = H^1\Big(Y, K_Y + \ulcorner f^*(D - (K_X + B)) - \sum \delta_j F_j\urcorner\Big) = 0,$$

where the last equation follows from [22, Theorem 1.4]. □

Proposition 2.3. *Let $f: X \to Y$ be a projective surjective morphism between irreducible normal varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume the following conditions.*
(1) *(X, Δ) is a $\mathbb{Q}$-factorial lc surface such that $(X, \{\Delta\})$ is klt.*
(2) *$S := \llcorner \Delta \lrcorner \neq 0$ and let $T := f(S)$.*
(3) *$-(K_X + \Delta)$ is f-nef and f-big.*
Then, $f_*\mathcal{O}_S = \mathcal{O}_T$. In particular, for every $y \in Y$, $S \cap f^{-1}(y)$ is connected or an empty set.

Proof. STEP 1. In this step, we assume $\dim Y \geq 1$ and we prove the assertion. Consider the exact sequence:

$$0 \to \mathcal{O}_X(-\llcorner \Delta \lrcorner) \to \mathcal{O}_X \to \mathcal{O}_{\llcorner \Delta \lrcorner} \to 0.$$

Take the push-forward by f:

$$0 \to f_*\mathcal{O}_X(-\llcorner \Delta \lrcorner) \to \mathcal{O}_Y \to f_*\mathcal{O}_{\llcorner \Delta \lrcorner} \to R^1 f_*\mathcal{O}_X(-\llcorner \Delta \lrcorner).$$

It is sufficient to prove that the last term $R^1 f_*\mathcal{O}_X(-\llcorner \Delta \lrcorner)$ vanishes. Since

$$-\llcorner \Delta \lrcorner = K_X + \{\Delta\} - (K_X + \Delta),$$

we have $R^1 f_*\mathcal{O}_X(-\llcorner \Delta \lrcorner) = 0$ by [21, Theorem 2.12].

STEP 2. In this step, we assume $\dim Y = 0$ and we prove the assertion. It is sufficient to prove that S is connected. Since rational surfaces satisfy the Kawamata–Viehweg vanishing theorem by Proposition 2.2, we can apply the same argument as Step 1. Thus we may assume that X is not rational. We can run a $(K_X + \Delta)$-MMP by [20, Theorem 6.8]. Then we have

$$h: X \xrightarrow{q} X' \xrightarrow{h'} R$$

where $q: X \to X'$ is a composition of extremal birational contractions and $h': X' \to R$ is a Mori fiber space.

We prove $\dim R = 1$. Let $\mu\colon X'' \to X'$ be a resolution and $X'' \to Q$ be a ruled surface structure. We may assume that Q is not rational. Note that X' has at worst rational singularities, because $(X, \{\Delta\})$ is klt and $R^1 q_* \mathcal{O}_X = 0$ (cf. [21, Theorem 2.12]). Therefore each μ-exceptional curves goes to one point by $X'' \to Q$. This $X'' \to Q$ factors through X'. In particular, there exists a surjection $X' \to Q$ to a smooth projective curve. This means $\rho(X') \geq 2$. Therefore we see $\dim R \neq 0$.

Hence we may assume $\dim R = 1$. Note that $-(K_{X'} + \Delta')$ is nef and big. Assume that $-(K_{X'} + \Delta')$ is ample. Since $\rho(X') = 2$ by [20, Theorem 6.8 (4) (b)], X' has the two $(K_{X'} + \Delta')$-negative extremal rays. Since these extremal rays are spanned by rational curves (cf. [20, Proposition 4.6]), R is a rational curve. If $X'' \to X'$ is a resolution, then $X'' \to R$ is a ruled surface structure. This means that X'' is rational. This case is excluded. Therefore we may assume that $-(K_{X'} + \Delta')$ curve C' on X' such that $(K_{X'} + \Delta') \cdot C' = 0$. This implies $C'^2 < 0$ and $h'(C'') = R$. Moreover we see

$$0 = (K_{X'} + \Delta') \cdot C' \geq (K_{X'} + C') \cdot C'.$$

If $0 > (K_{X'} + C') \cdot C'$, then $C' \simeq \mathbb{P}^1$ by [20, Theorem 5.3]. This case is excluded. Thus the above inequality is an equality. In particular, we have $C' \subset \llcorner \Delta' \lrcorner$. Let $C \subset X$ be the proper transform of C'. Then C satisfies $h(C) = R$ and $C \subset \llcorner \Delta \lrcorner$. We can apply Step 1 of this proof to $h\colon X \to R$ because $-(K_X + \Delta)$ is h-nef and h-big. Then, $S \cap h^{-1}(r)$ is connected for every $r \in R$. This and $h(C) = R$ imply that S is connected. $\square$

Lemma 2.4. *Let*

$$f\colon X \xrightarrow{q} X' \xrightarrow{f'} R$$

be projective morphisms between normal varieties such that q is birational and $f'_* \mathcal{O}_{X'} = \mathcal{O}_R$. Assume the following conditions.

(1) *(X, Δ) is a $\mathbb{Q}$-factorial lc surface such that $(X, \{\Delta\})$ is klt.*

(2) *$\mathrm{Ex}(q) =: E$ is an irreducible curve.*

(3) *$-(K_X + \Delta)$ is q-nef.*

(4) *$\llcorner \Delta \lrcorner$ is q-nef.*

Then, for every $r \in R$, the number of connected components of $\llcorner \Delta \lrcorner \cap f^{-1}(r)$ is equal to the number of connected components of $\llcorner q_* \Delta \lrcorner \cap f'^{-1}(r)$.

Proof. Let $q(E) =: x'_0$ and $f'(x'_0) =: r_0$. If $E \cap \mathrm{Supp} \llcorner \Delta \lrcorner = \emptyset$, then the assertion is clear. Thus, we may assume $E \cap \mathrm{Supp} \llcorner \Delta \lrcorner \neq \emptyset$.

We claim $q(\mathrm{Supp} \llcorner \Delta \lrcorner) = \mathrm{Supp} \llcorner q_* \Delta \lrcorner$. The inclusion $q(\mathrm{Supp} \llcorner \Delta \lrcorner) \supset \mathrm{Supp} \llcorner q_* \Delta \lrcorner$ is clear. Then, it is enough to show $q(E) \in \mathrm{Supp} \llcorner q_* \Delta \lrcorner$. If $E \not\subset \mathrm{Supp} \llcorner \Delta \lrcorner$, then $E \cap \mathrm{Supp} \llcorner \Delta \lrcorner \neq \emptyset$ implies $q(E) \in \mathrm{Supp} \llcorner q_* \Delta \lrcorner$. On the other hand, if $E \subset \mathrm{Supp} \llcorner \Delta \lrcorner$, then the q-nefness implies that there exists a prime component $C \neq E$ of $\llcorner \Delta \lrcorner$ with $C \cap E \neq 0$. We see

$$q(E) \in q(C) \subset \mathrm{Supp} \llcorner q_* \Delta \lrcorner.$$

In each case, we obtain the claim.

For every $r \in R$, we obtain

$$\begin{aligned} q(\mathrm{Supp}\llcorner\Delta\lrcorner \cap f^{-1}(r)) &= q(\mathrm{Supp}\llcorner\Delta\lrcorner \cap q^{-1}(f'^{-1}(r))) \\ &= q(\mathrm{Supp}\llcorner\Delta\lrcorner) \cap f'^{-1}(r) \\ &= \mathrm{Supp}\llcorner q_*\Delta\lrcorner \cap f'^{-1}(r). \end{aligned}$$

Assume that the numbers of connected components are different. Then there exist at least two connected components X_1 and X_2 of $\mathrm{Supp}\llcorner\Delta\lrcorner \cap f^{-1}(r_0)$ such that $x_0' \in q(X_1)$ and $x_0' \in q(X_2)$. We take the intersection

$$\mathrm{Supp}\llcorner\Delta\lrcorner \cap f^{-1}(r_0) = X_1 \amalg X_2 \amalg \cdots$$

with $q^{-1}(x_0')$ and we obtain the following equation

$$\mathrm{Supp}\llcorner\Delta\lrcorner \cap q^{-1}(x_0') = (X_1 \cap q^{-1}(x_0')) \amalg (X_2 \cap q^{-1}(x_0')) \amalg \cdots.$$

Thus, in order to derive a contradiction, it is sufficient to prove that $\mathrm{Supp}\llcorner\Delta\lrcorner \cap q^{-1}(x_0')$ is connected. Since $-(K_X+\Delta)$ is q-nef and q-big, we can apply Proposition 2.3. Thus $\mathrm{Supp}\llcorner\Delta\lrcorner \cap q^{-1}(x_0')$ is connected. □

Proposition 2.5. *Let $f\colon X \to Y$ be a projective surjective morphism between irreducible normal varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume the following conditions.*
(1) *(X, Δ) is a $\mathbb{Q}$-factorial lc surface such that $(X, \{\Delta\})$ is klt.*
(2) *$S := \llcorner\Delta\lrcorner \neq 0$ and let $T := f(S)$.*
(3) *$K_X + \Delta \equiv_f 0$.*
(4) *$T = f(S) \subsetneq Y$.*
Then, $f_*\mathcal{O}_S = \mathcal{O}_T$. In particular, for every $y \in Y$, $S \cap f^{-1}(y)$ is connected or an empty set.

Proof. By (4), we have $\dim Y \neq 0$. If $\dim Y = 2$, then the assertion follows from Proposition 2.3. Thus we may assume $\dim Y = 1$. It is sufficient to prove that $\mathcal{O}_Y = f_*\mathcal{O}_X \to f_*\mathcal{O}_S$ is surjective. Since the problem is local, by shrinking Y, we may assume that $f(S) = P \in Y$. If S is connected, then $f_*\mathcal{O}_S \simeq \mathcal{O}_P$ and $\mathcal{O}_Y \to f_*\mathcal{O}_S$ is surjective. Therefore, it is sufficient to prove that S is connected. We define a reduced divisor D by

$$S + D = \mathrm{Supp}(f^*P).$$

If $D = 0$, then S is connected since $S = \mathrm{Supp}(f^*P)$. Therefore, we assume that $D \neq 0$. Then, there exists an irreducible curve $E \subset \mathrm{Supp}\, D$ such that $E \cap S \neq 0$. We see $(K_X + \{\Delta\}) \cdot E < 0$. Thus, we obtain a birational morphism $q : X \to X'$ such that

$\mathrm{Ex}(q) = E$. Let $\Delta' := q_*\Delta$. By Lemma 2.4, if $\mathrm{Supp}\llcorner\Delta'\lrcorner$ is connected, then so is $\mathrm{Supp}\llcorner\Delta\lrcorner$. We can repeat this argument and we obtain a projective morphisms

$$f\colon X \xrightarrow{\tilde{q}} X'' \xrightarrow{f''} Y$$

where $\tilde{q}$ is a birational morphism such that $\mathrm{Ex}(\tilde{q}) = \mathrm{Supp}\, D$. Let $\Delta'' := \tilde{q}_*\Delta$. It is sufficient to show that $\mathrm{Supp}\llcorner\Delta''\lrcorner$ is connected. This follows from $\mathrm{Supp}\llcorner\Delta''\lrcorner = \mathrm{Supp}(f''^* P)$. □

In Proposition 2.8, the most complicated case is the Mori fiber space to a curve. Thus we investigate this case in the following lemma.

Lemma 2.6. *Let $f'\colon X' \to R$ be a projective surjective morphism between normal varieties such that $f'_*\mathcal{O}_{X'} = \mathcal{O}_R$. Assume the following conditions.*
(1) *(X', Δ') is a $\mathbb{Q}$-factorial lc surface such that $(X', \{\Delta'\})$ is klt.*
(2) $S' := \llcorner\Delta'\lrcorner \neq 0$.
(3) $K_{X'} + \Delta' \equiv_{f'} 0$.
(4) *There is a $(K_{X'} + \{\Delta'\})$-negative extremal contraction $g'\colon X' \to V$ over R such that* $\dim V = 1$.
Then the g'-horizontal part $(S')^h$ of S' satisfies one of the following assertions.
(a) *$(S')^h = S'_1$, which is a prime divisor, and $[K(S'_1) : K(V)] = 2$.*
(b) *$(S')^h = S'_1$, which is a prime divisor, and $[K(S'_1) : K(V)] = 1$.*
(c) *$(S')^h = S'_1 + S'_2$, where each S'_i is a prime divisor, and $[K(S'_i) : K(V)] = 1$.*
Furthermore, there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D_V on V such that $K_X + \Delta = g'^*(D_V)$.
In the case (b), $f'_*\mathcal{O}_{S'} = \mathcal{O}_{f'(S')}$.

Proof. The assumption (3) means $K_{X'} + \Delta' \equiv_{g'} 0$. Thus, by (4), $\llcorner\Delta'\lrcorner$ is g'-ample. We see $(S')^h \neq 0$.

We prove that general fibers of $g'\colon X' \to V$ are $\mathbb{P}^1$. The dimension of every fiber is one. Since $\dim V = 1$ and $f'_*\mathcal{O}_{X'} = \mathcal{O}_V$, the field extension $K(X')/K(V)$ is algebraically closed and separable (cf. [3, Lemma 7.2]). Therefore general fibers are geometrically integral. Let F be a general fiber of g', that is, F is a fiber which is a proper integral curve such that $F \cap \mathrm{Sing}(X) = \emptyset$. The adjunction formula implies

$$(K_{X'} + F) \cdot F = K_{X'} \cdot F = -\Delta' \cdot F \leq -(S')^h \cdot F < 0.$$

This means $F \simeq \mathbb{P}^1$.

By $(K_{X'} + F) \cdot F = -2$, we have $(S')^h \cdot F \leq 2$ for a general fiber F. Therefore one of (a), (b) and (c) holds. By the abundance theorem ([20, Theorem 18.4]), we see $K_{X'} + \Delta' \sim_{\mathbb{Q}, g'} 0$. This means $m(K_{X'} + \Delta') = g'^*(D)$ for some integer m and some $\mathbb{Z}$-divisor D on V. We define a $\mathbb{Q}$-divisor D_V by $D = mD_V$.

Assume (b) and let us prove $f'_*\mathcal{O}_{S'} = \mathcal{O}_{f'(S')}$. Since $\dim V = 1$, we have $\dim R = 0$ or $\dim R = 1$. Assume $\dim R = 0$. It is sufficient to prove that S' is connected. This holds because all of the fibers of g' are irreducible and $(S')^h \neq 0$. Assume $\dim R = 1$. Then, we see $f'(S') = V \simeq R$. Since S'_1 and R are birational, the morphism $f|_{S'_1}\colon S'_1 \to R$ is an isomorphism. We can write

$$S' = S'_1 + F_1 + \cdots + F_r$$

where each F_i is the reduced subscheme whose support is a fiber of g'.

We prove $f'_*\mathcal{O}_{S'} = \mathcal{O}_R$ by the induction on r. If $r = 0$, then the assertion follows from $S'_1 \simeq R$. Assume $r > 0$. Consider the exact sequence:

$$0 \to \mathcal{O}_{S'} \to \mathcal{O}_{S'-F_r} \oplus \mathcal{O}_{F_r} \to \mathcal{O}_{(S'-F_r)\cap F_r} \to 0.$$

The last map defined by the difference. Note that the last term is the scheme-theoretic intersection. It is easy to see that $(S' - F_r) \cap F_r \simeq S'_1 \cap F_r$. Then $(S' - F_r) \cap F_r$ is reduced because $S'_1 \simeq R$. Consider the push-forward of the above exact sequence:

$$0 \to f'_*\mathcal{O}_{S'} \to f'_*\mathcal{O}_{S'-F_r} \oplus f'_*\mathcal{O}_{F_r} \to f'_*\mathcal{O}_{(S'-F_r)\cap F_r} \to R^1 f'_*\mathcal{O}_{S'}.$$

We see $R^1 f'_*\mathcal{O}_{X'} = 0$ by [21, Theorem 2.12]. This implies $R^1 f'_*\mathcal{O}_{S'} = 0$. Since F_r and $(S' - F_r) \cap F_r$ are reduced, we have $f'_*\mathcal{O}_{F_r} \simeq f'_*\mathcal{O}_{(S'-F_r)\cap F_r} \simeq \mathcal{O}_{f'(F_r)}$. This means $f'_*\mathcal{O}_{S'} \to f'_*\mathcal{O}_{S'-F_r}$ is an isomorphism. By the induction hypothesis, we obtain $f'_*\mathcal{O}_{S'} \simeq f'_*\mathcal{O}_{S'-F_r} \simeq \mathcal{O}_R$. □

REMARK 2.7. In the last argument in the above proof, we use the following fact. Let A be a ring and let M, N, L and P are A-modules. Assume the exact sequence

$$0 \to M \xrightarrow{(\varphi_1,\varphi_2)} N \oplus L \xrightarrow{\psi-\theta} P \to 0.$$

If $\theta\colon L \to P$ is an isomorphism, then $\varphi_1\colon M \to N$ is also an isomorphism.

We can prove the following main result in this section.

Proposition 2.8. *Let (X, Δ) be an irreducible dlt surface. Let $f\colon X \to R$ be a projective surjective morphism such that $f_*\mathcal{O}_X = \mathcal{O}_R$. Assume that $S := \llcorner\Delta\lrcorner \neq 0$ and let $T := f(S)$. If $K_X + \Delta \equiv_f 0$, then one of the following assertions holds.*
(1) $f_*\mathcal{O}_S = \mathcal{O}_T$.
(2) $f_*\mathcal{O}_S \neq \mathcal{O}_T$. *There exist a projective surjective R-morphism $g\colon X \to V$ to a smooth curve V and a $\mathbb{Q}$-divisor D_V on V such that $g_*\mathcal{O}_X = \mathcal{O}_V$ and that $K_X + \Delta = g^*(D_V)$ as $\mathbb{Q}$-divisors. Every connected component of S intersects the g-horizontal part S^h of S. Moreover, the g-horizontal part S^h satisfies one of the following assertions.*
(2.1s) $S^h = S_1$, *which is a prime divisor, and* $[K(S_1) : K(V)] = 2$. *This field extension is separable.*

(2.1i) $S^h = S_1$, *which is a prime divisor, and* $[K(S_1) : K(V)] = 2$. *This field extension is purely inseparable.*
(2.2) $S^h = S_1 + S_2$, *where* S_i *is a prime divisor, and* $g|_{S_i} \colon S_i \to V$ *is an isomorphism for* $i = 1, 2$.

Proof. If f is birational, then Proposition 2.5 implies (1). Thus we may assume that $\dim R < \dim X$. We run a $(K_X + \{\Delta\})$-MMP on X over R. The end result is a proper birational morphism $q \colon X \to X'$ over R. Let $f' \colon X' \to R$ be the induced morphism. Since $K_X + \Delta \equiv_f 0$, we obtain $K_{X'} + \Delta' \equiv_{f'} 0$ where $\Delta' := q_*\Delta$. Let $S' := \llcorner \Delta' \lrcorner$. Then it is easy to see that (X', Δ') is a $\mathbb{Q}$-factorial lc pair and $(X, \{\Delta'\})$ is klt.

STEP 1. Assume that $(X', \{\Delta'\})$ is a minimal model over R. Then $K_{X'} + \{\Delta'\}$ is f'-nef and $K_{X'} + \Delta' \equiv_{f'} 0$. So $-\llcorner \Delta' \lrcorner$ is f'-nef. If $\dim R = 0$, then $\llcorner \Delta' \lrcorner = 0$ because X' is projective. Lemma 2.4 implies $\llcorner \Delta \lrcorner = 0$. This case is excluded. Assume $\dim R = 1$. Since $-\llcorner \Delta' \lrcorner$ is f'-nef, we see $f'(\llcorner \Delta' \lrcorner) \subsetneq R$. Therefore, by Proposition 2.5, we obtain (1).

STEP 2. Assume that there exists a Mori fiber space structure $g' \colon X' \to V$ over R. Let

$$g \colon X \xrightarrow{q} X' \xrightarrow{g'} V.$$

Then $-(K_{X'} + \{\Delta'\})$ is g'-ample. Note that, if $\dim V = 1$, then we can apply Lemma 2.6 and every connected component of S intersects S^h by Lemma 2.4.

First, assume that $\dim R = 0$. If $\llcorner \Delta' \lrcorner$ is connected, then we have (1) by Lemma 2.4. Thus we may assume that $\llcorner \Delta' \lrcorner$ is not connected.

We show $\dim V = 1$. Assume $\dim V = 0$. Then $\llcorner \Delta' \lrcorner$ is ample. Thus its suitable multiple is an effective ample Cartier divisor. This must be connected by the Serre vanishing theorem. This case is excluded.

Thus we can apply Lemma 2.6. Since all of the fibers of the Mori fiber space $g' \colon X' \to V$ are irreducible, we see $\llcorner \Delta' \lrcorner = S'_1 + S'_2$. This implies (2.2).

Second, assume that $\dim R = 1$. Then we have $\dim V = 1$. Note that $T = R \simeq V$. We can apply Lemma 2.6. Thus we obtain (a), (b) or (c) of Lemma 2.6. If (a) or (c) holds, then (2) holds. Thus we may assume that (b) of Lemma 2.6 holds. We have $f'_*\mathcal{O}_{S'} = \mathcal{O}_T$. Lemma 2.4 implies $q(S) = S'$. By Proposition 2.3, we have $f_*\mathcal{O}_S = f'_*\mathcal{O}_{S'} = \mathcal{O}_T$. □

Example 2.9. *Let* $\operatorname{char} k = 2$. *Then, there exists a projective dlt surface* (X, Δ) *and smooth projective curve* R *which satisfy* Proposition 2.8 (2.1i).

CONSTRUCTION. Let $X_0 := \mathbb{A}^2$ and let $C_0 := \{(x, y) \in \mathbb{A}^2 \mid x = y^2\}$. Note that the restriction of the first projection to C_0 is purely inseparable of degree two. Let $X_0 \subset X := \mathbb{P}^1 \times \mathbb{P}^1$ be the natural open immersion and let C be the closure of C_0 in X. Let $g \colon X \to \mathbb{P}^1 =: V =: R$ be the first projection. It is easy to see that C is smooth and $K_X + C \sim g^*\mathcal{O}_{\mathbb{P}^1}(-1)$. Thus, we see that $(X, \Delta := C)$ is dlt and that (X, Δ) satisfies Proposition 2.8 (2.1i).

3. Normalization of nodes

In this section, we calculate the normalization of nodal singularities to reduce problems for slc varieties to ones for dlt varieties. The main theorem of this section is Theorem 3.7. In this section, we do not work over a field and we treat noetherian or excellent schemes.

First we recall the definition of the nodal singularities in the sense of [13, 1.41].

DEFINITION 3.1. Let $(R, \mathfrak{m})$ be a noetherian local ring. We say R has a *node* (or R is *nodal*) if there exists an isomorphism $R \simeq S/(f)$ where $(S, \mathfrak{l})$ is a two-dimensional regular local ring such that $f \in \mathfrak{l}^2$ and that f is not a square in $\mathfrak{l}^2/\mathfrak{l}^3$.

We mainly use the following notations.

NOTATION 3.2. Let $(R, \mathfrak{m})$ be a nodal noetherian local ring. By definition, we can write $R \simeq S/(f)$ where $(S, \mathfrak{l})$ is a two-dimensional regular local ring such that $f \in \mathfrak{l}^2$ and that f is not a square in $\mathfrak{l}^2/\mathfrak{l}^3$. Take a generator $\mathfrak{l} = (x, y)$. We can write

$$f = ax^2 + bxy + cy^2 + g$$

where $a, b, c \in \{0\} \cup S^\times$ and $g \in \mathfrak{l}^3$. We set $\bar{x} := x + (f) \in R/(f)$ and $\bar{y} := y + (f) \in R/(f)$.

REMARK 3.3. We use the same notations as Notation 3.2. We show that we may assume

$$c \in S^\times$$

by replacing a generator $\{x, y\}$ of $\mathfrak{l}$. If $c \in S^\times$, then there is nothing to show. If $a \in S^\times$, then we exchange x and y. Since $a, c \in \{0\} \cup S^\times$, we assume $a = c = 0$. By $f \notin \mathfrak{l}^3$, we see $b \notin \mathfrak{l}$, that is, $b \in S^\times$. Taking another generator $X := x - y$, $Y := y$ of $\mathfrak{l} = (x, y) = (X, Y)$, we obtain

$$\begin{aligned} f &= bxy + g \\ &= b(X+Y)Y + g \\ &= bXY + bY^2 + g. \end{aligned}$$

By $b \in S^\times$, we may assume $c \in S^\times$.

We calculate the normalization of nodes. We divide the proof into the following two cases: R is an integral domain or not. In Lemma 3.4, we treat the case where R is not an integral domain. In Lemma 3.5, we treat the case where R is an integral domain.

Lemma 3.4. *Let $(R, \mathfrak{m})$ be a nodal noetherian local ring. We use the same notations as* Notation 3.2. *Assume that R is not an integral domain. Then the following assertions hold.*
(1) *f has a decomposition $f = l_1 l_2$ with $l_1, l_2 \in S$ which satisfies the following properties.*
- *$l_1 S \neq l_2 S$.*
- *For each i, $l_i \in \mathfrak{l} \setminus \mathfrak{l}^2$.*
- *For each i, l_i is a prime element of S, that is, $l_i S$ is a prime ideal.*

(2) *l_1 and l_2 satisfies $\mathfrak{l} = (l_1, l_2)$.*
(3) *For each i, $S/(l_i)$ is regular.*
(4) *The natural homomorphism*

$$\nu\colon R = S/(f) = S/(l_1 l_2) \to S/(l_1) \times S/(l_2) =: T$$

is the normalization.
(5) *$\mathfrak{m}$ is the conductor of the normalization $\nu\colon R \hookrightarrow T$, that is,*

$$\mathfrak{m} = \{r \in R \mid rT \subset R\}.$$

(6) *The normalization $\nu\colon R \hookrightarrow T$ induces*

$$\theta\colon k(\mathfrak{m}) = R/\mathfrak{m} \to T/\mathfrak{m}T \simeq k(\mathfrak{m}) \times k(\mathfrak{m}),$$

where $p_i \circ \theta$ is the identity map for the projection p_i to the i-th factor.

Proof. (1) Since S is a unique factorization domain, we obtain a decomposition of f into prime elements:

$$f = u l_1^{n_1} \cdots l_r^{n_r}$$

where $u \in S^\times$, $n_i \in \mathbb{Z}_{>0}$ and l_i is a prime element of S. In particular, $l_i \in \mathfrak{l}$. Then, $f \notin \mathfrak{l}^3$ implies $n_1 + \cdots + n_r \leq 2$. Since $n_1 + \cdots + n_r = 1$ implies that R is an integral domain, we see $n_1 + \cdots + n_r = 2$. Thus, we obtain one of the following two cases: $f = u l_1^2$ or $f = u l_1 l_2$ where $l_1 S \neq l_2 S$. By $f \notin \mathfrak{l}^3$ and $l_i \in \mathfrak{l}$, we see $l_i \notin \mathfrak{l}^2$. Then, it is enough to show that the case $f = u l_1^2$ does not occur. Suppose $f = u l_1^2$. We can write

$$l_1 = \alpha x + \beta y + h$$

where $\alpha, \beta \in \{0\} \cup S^\times$ and $h \in \mathfrak{l}^2$. We obtain

$$f = u l_1^2 = u(\alpha x + \beta y + h)^2 = u(\alpha x + \beta y)^2 + (\text{an element of } \mathfrak{l}^3).$$

By replacing f with $u^{-1}f$, this contradicts the definition of nodes: Definition 3.1.

(2) Since R is nodal, (l_1, l_2) generates $\mathfrak{l}/\mathfrak{l}^2$. Then Nakayama's lemma implies the assertion.

(3) The assertion follows from (2).

(4) The assertion follows from (3).

(5) Let $I \subset R$ be the conductor. The inclusion $\mathfrak{m} \supset I$ is clear. We show the inverse inclusion $(l_1, l_2) = \mathfrak{m} \subset I$. By the symmetry, it suffices to prove $l_1 \in I$. Take $\xi = (s_1 + (l_1), s_2 + (l_2)) \in S/(l_1) \times S/(l_2) = T$. Then, we obtain $l_1\xi = (0 + (l_1), l_1 s_2 + (l_2))$. Therefore, $l_1\xi = \nu(l_1 s_2)$. This is what we want to show.

(6) By $\nu(l_1 + l_2) = (l_2 + (l_1), l_1 + (l_2))$, we see $\mathfrak{m}T = \mathfrak{m}/(l_1) \times \mathfrak{m}/(l_2)$. This implies the assertion. □

Lemma 3.5. *Let $(R, \mathfrak{m})$ be a nodal noetherian local ring. We use the same notations as* Notation 3.2. *Suppose $c \in S^{\times}$ (cf.* Remark 3.3). *Assume that R is an integral domain. Consider the following natural injective ring homomorphism*

$$\varphi\colon R \hookrightarrow R\left[\frac{\bar{y}}{\bar{x}}\right] =: T.$$

Then the following assertions hold.

(1) *The ring homomorphism $\theta\colon S[y/x]/(f/x^2) \to R[\bar{y}/\bar{x}] = T$, $y/x \mapsto \bar{y}/\bar{x}$ is an isomorphism.*

(2) *T is a regular ring.*

(3) *One of the following assertions holds.*

(a) *$T/\mathfrak{m}T \simeq k(\mathfrak{m}) \times k(\mathfrak{m})$ and the composition homomorphism*

$$k(\mathfrak{m}) = R/\mathfrak{m} \to T/\mathfrak{m}T \simeq k(\mathfrak{m}) \times k(\mathfrak{m}) \xrightarrow{p_i} k(\mathfrak{m})$$

is the identity map for $i = 1, 2$ where p_i is the projection to the i-th factor.

(b) *$T/\mathfrak{m}T$ is a field and the natural homomorphism*

$$k(\mathfrak{m}) = R/\mathfrak{m} \to T/\mathfrak{m}T$$

is a field extension with $[T/\mathfrak{m}T : k(\mathfrak{m})] = 2$.

(4) *The equation $(\bar{y}/\bar{x})^2 + r_1\bar{y}/\bar{x} + r_2 = 0$ holds in $R[\bar{y}/\bar{x}] = T$ for some $r_1, r_2 \in R$. In particular, T is a finitely generated R-module.*

(5) *T is the integral closure of R in the quotient field $K(R)$.*

(6) *The maximal ideal $\mathfrak{m}$ is the conductor of the normalization, that is, $\mathfrak{m} = \{r \in R \mid rT \subset R\}$.*

Proof. We use the same notations as Notation 3.2.

(1) Set $z := y/x \in K(S)$. Let us check $f/x^2 \in S[y/x] = S[z]$. Since $f \in \mathfrak{l}^2 = (x, y)^2$, we can write $f = \alpha x^2 + \beta xy + \gamma y^2$ for some $\alpha, \beta, \gamma \in S$. Then we see

$f/x^2 \in S[z]$ by the following calculation:

$$f = \alpha x^2 + \beta xy + \gamma y^2 = \alpha x^2 + \beta x(xz) + \gamma (xz)^2 = x^2(\alpha + \beta z + \gamma z^2).$$

Consider the natural homomorphism

$$\theta\colon S\left[\frac{y}{x}\right] \Big/ (f/x^2) \to R\left[\frac{\bar{y}}{\bar{x}}\right], \qquad \frac{y}{x} \mapsto \frac{\bar{y}}{\bar{x}}.$$

We prove that θ is an isomorphism. For the time being, we show this assuming the following two assertions.

(A) The S-algebra homomorphism $S/(f) \to S[y/x]/(f/x^2)$ is injective.

(B) $S[y/x]/(f/x^2)$ is an integral domain.

Consider the following commutative diagram of S-algebras:

$$\begin{array}{ccc} S/(f) & = & R \\ \downarrow {\scriptstyle \text{injective}} & & \downarrow {\scriptstyle \text{injective}} \\ S\left[\frac{y}{x}\right] \Big/ (f/x^2) & \xrightarrow{\ \theta\ } & R\left[\frac{\bar{y}}{\bar{x}}\right]. \end{array}$$

Note that $R[\bar{y}/\bar{x}] \subset K(R) = K(S/(f)) \subset K(S[y/x]/(f/x^2))$. All of the four rings in the above diagram are contained in the quotient field $K(S[y/x]/(f/x^2))$. In $K(S[y/x]/(f/x^2))$, the element $y/x + (f/x^2) \in S[y/x]/(f/x^2)$ is the same as $\bar{y}/\bar{x} \in R[\bar{y}/\bar{x}]$. Therefore we obtain

$$S\left[\frac{y}{x}\right] \Big/ (f/x^2) = R\left[\frac{\bar{y}}{\bar{x}}\right].$$

(A) We show that the natural map $S/(f) \to S[y/x]/(f/x^2)$ is injective. For this, consider the following natural composition map

$$\psi\colon S \to S\left[\frac{y}{x}\right] \to S\left[\frac{y}{x}\right] \Big/ (f/x^2)$$

and we show $\mathrm{Ker}(\psi) = fS$. The inclusion $\mathrm{Ker}(\psi) \supset fS$ is obvious. Let us prove the inverse inclusion $\mathrm{Ker}(\psi) \subset fS$. Take an element $s \in S$ such that $\psi(s) = 0$, that is, $s \in (f/x^2)S[y/x]$. We have

$$s = \frac{f}{x^2}\left(t_0 + t_1 \frac{y}{x} + \cdots + t_m \frac{y^m}{x^m}\right)$$

where $t_i \in S$. Let us show that we can assume $m = 0$. Assume $m \geq 1$. Moreover assume $t_m \in xS$, that is, $t_m = x\tilde{t}_m$ with $\tilde{t}_m \in S$. Then, by the following calculation:

$$t_m \frac{y^m}{x^m} = x\tilde{t}_m \frac{y^m}{x^m} = y\tilde{t}_m \frac{y^{m-1}}{x^{m-1}},$$

we obtain another expression: $s = (f/x^2)(t_0 + \cdots + t_{m-2}(y/x)^{m-2} + t'_{m-1}y^{m-1}/x^{m-1})$ for some $t'_{m-1} \in S$. Thus, we assume $m \geq 1$ and $t_m \notin xS$. Taking the multiplication with x^{m+2}, we obtain

$$sx^{m+2} = f(t_0x^m + \cdots + t_{m-1}xy^{m-1} + t_my^m).$$

This implies $ft_my^m \in xS$. But, both the elements t_m and y are not in xS. Since xS is a prime ideal, we obtain $f \in xS$. Then we can write $f = xg$ with $g \in S$. $f \in \mathfrak{l}^2$ implies $g \in \mathfrak{l}$. Therefore, f is not a prime element, which contradicts that R is an integral domain. Therefore, we may assume $m = 0$ and we obtain

$$s = \frac{f}{x^2}t_0.$$

Since $f \notin xS$, we see $t_0 \in xS$. Repeating this, we see $t_0 \in x^2S$, which implies $s \in fS$. This is what we want to show.

(B) First we prove that $S[y/x]$ is a unique factorization domain. We see that $xS[y/x]$ is a prime ideal because

$$S\left[\frac{y}{x}\right] \Big/ xS\left[\frac{y}{x}\right] \simeq S[Z]/(x, xZ - y) \simeq (S/(x, y))[Z]$$

is an integral domain. By Nagata's criterion ([18, Lemma 1]), $S[y/x]$ is a unique factorization domain if so is

$$\left(S\left[\frac{y}{x}\right]\right)\left[\frac{1}{x}\right] = S\left[\frac{1}{x}\right].$$

This ring $S[1/x]$ is a unique factorization domain because so is S.

We show that $S[y/x]/(f/x^2)$ is an integral domain. Since $S[y/x]$ is a unique factorization domain, let us check that f/x^2 is a prime element. Assume that there exists a decomposition

$$\frac{f}{x^2} = \left(s_0 + s_1\frac{y}{x} + \cdots + s_k\frac{y^k}{x^k}\right)\left(t_0 + t_1\frac{y}{x} + \cdots + t_l\frac{y^l}{x^l}\right)$$

where $s_i, t_j \in S$ and both the factors in the right hand side are not in $(S[y/x])^\times$. We may assume that, if $k \geq 1$ (resp. $l \geq 1$), then s_k (resp. t_l) is not in xS. We show that $k = 0$ or $l = 0$ holds. Assume $k \geq 1$ and $l \geq 1$. We consider the following two cases:

$k = l = 1$ and $k + l \geq 3$. If $k = l = 1$, then we obtain $f = (s_0x + s_1y)(t_0x + t_1y)$. This contradicts that f is a prime element. If $k + l \geq 3$, then taking the multiplication with x^{k+l}, we see $s_k t_l y^{k+l} \in xS$. By $s_k \notin xS$ and $t_l \notin xS$, we have $y^{k+l} \in xS$, which is a contradiction. Therefore, $k = 0$ or $l = 0$ holds. By the symmetry, we may assume $l = 0$ and we obtain

$$\frac{f}{x^2} = \left(s_0 + s_1\frac{y}{x} + \cdots + s_k\frac{y^k}{x^k}\right)t_0.$$

If $k \geq 1$ and $t_0 \in xS$, then, for $t_0 = xt_0'$, we obtain another expression: $f/x^2 = (xs_0 + \cdots + xs_{k-1}(y/x)^{k-1} + ys_k(y/x)^{k-1})t_0'$. Thus we may assume that $k = 0$ or $t_0 \notin xS$ holds. If $k = 0$, then we obtain the following contradiction: $f = x^2s_0t_0$. Assume $t_0 \notin xS$. Taking the multiplication with x^k, we see $k \leq 2$. This implies $f = (s_0x^2 + s_1xy + s_2y^2)t_0$. Since $s_0x^2 + s_1xy + s_2y^2 \in \mathfrak{m}$ and $f \in S$ is a prime element, we have $t_0 \in S^\times \subset (S[y/x])^\times$. This is a contradiction.

(2) Set $z = y/x$. First, we calculate the ring $(S[y/x]/(f/x^2))/(\bar{x})$. The element f/x^2 can be written

$$\begin{aligned}\frac{f}{x^2} &= \frac{ax^2 + bxy + cy^2 + g}{x^2} = \frac{ax^2 + bx(xz) + c(xz)^2 + x^3\tilde{g}}{x^2}\\ &= a + bz + cz^2 + x\tilde{g}\end{aligned}$$

for some $\tilde{g} \in S[z]$. Here, since $(S, (x, y))$ is a regular local ring, we can check that the homomorphism

$$S[Z]/(xZ - y) \to S\left[\frac{y}{x}\right], \quad Z \mapsto \frac{y}{x}$$

is an isomorphism. Then, we see

$$\begin{aligned}\left(S\left[\frac{y}{x}\right]\Big/(f/x^2)\right)\Big/(\bar{x}) &\simeq S\left[\frac{y}{x}\right]\Big/((f/x^2) + (x))\\ &\simeq S[Z]/(xZ - y, a + bZ + cZ^2, x)\\ &\simeq k(\mathfrak{m})[Z]/(\bar{a} + \bar{b}Z + \bar{c}Z^2).\end{aligned}$$

Fix a maximal ideal $\mathfrak{n}$ of $S[y/x]/(f/x^2)$ and we show that the local ring $(S[y/x]/(f/x^2))_{\mathfrak{n}}$ is regular.

We show $\bar{x} \in \mathfrak{n}$. Assume $\bar{x} \notin \mathfrak{n}$. Then $\mathfrak{n}$ corresponds to a maximal ideal of $R[\bar{y}/\bar{x}][1/\bar{x}] = R[1/\bar{x}]$, that is, $\mathfrak{n} = (\mathfrak{n}R[\bar{y}/\bar{x}][1/\bar{x}]) \cap R[\bar{y}/\bar{x}]$. Since $(R, \mathfrak{m})$ is one dimensional local integral domain and $\bar{x} \in \mathfrak{m}$, $R[1/\bar{x}]$ is a field. It implies $\mathfrak{n} = (0)$. Then $S[y/x]/(f/x^2)$ is a field. On the other hand, by the above isomorphism

$$\left(S\left[\frac{y}{x}\right]\Big/(f/x^2)\right)\Big/(\bar{x}) \simeq k(\mathfrak{m})[Z]/(\bar{a} + \bar{b}Z + \bar{c}Z^2)$$

and $\bar{c} \neq 0$, there exists a non-zero ideal $(\bar{x})$ of $S[y/x]/(f/x^2)$. Thus $S[y/x]/(f/x^2)$ is not a field and we obtain a contradiction.

Therefore, $\bar{x} \in \mathfrak{n}$. To show that the local ring $(S[y/x]/(f/x^2))_{\mathfrak{n}}$ is regular, it is enough to prove that the ring

$$\left(S\left[\frac{y}{x}\right]\Big/(f/x^2)\right)\Big/(\bar{x}) \simeq k(\mathfrak{m})[Z]/(\bar{a} + \bar{b}Z + \bar{c}Z^2)$$

is regular. If $\bar{a} + \bar{b}Z + \bar{c}Z^2$ is irreducible over $k(\mathfrak{m})$, then the ring $k(\mathfrak{m})[Z]/(\bar{a} + \bar{b}Z + \bar{c}Z^2)$ is a field. Assume that $\bar{a}+\bar{b}Z+\bar{c}Z^2$ is not irreducible over $k(\mathfrak{m})$. We have $\bar{c} \neq 0$. There are $\alpha, \beta \in R$ such that

$$\bar{a} + \bar{b}Z + \bar{c}Z^2 = \bar{c}(Z + \bar{\alpha})(Z + \bar{\beta}).$$

Since R is nodal, we see $\bar{\alpha} \neq \bar{\beta}$. Therefore,

$$\left(S\left[\frac{y}{x}\right]/(f/x^2)\right)/(\bar{x}) \simeq k(\mathfrak{m})[Z]/(\bar{a} + \bar{b}Z + \bar{c}Z^2) \simeq k(\mathfrak{m}) \times k(\mathfrak{m}).$$

This is what we want to show.

(3) Let us calculate $T/\mathfrak{m}T$. By

$$\mathfrak{m}T = \mathfrak{m}R\left[\frac{\bar{y}}{\bar{x}}\right] = (\bar{x}, \bar{y})R\left[\frac{\bar{y}}{\bar{x}}\right] = \bar{x}R\left[\frac{\bar{y}}{\bar{x}}\right],$$

we obtain $T/\mathfrak{m}T \simeq (S[y/x]/(f/x^2))/(\bar{x})$. By the proof of (2), we obtain

$$\left(S\left[\frac{y}{x}\right]\Big/(f/x^2)\right)\Big/(\bar{x}) \simeq k(\mathfrak{m})[Z]/(\bar{a} + \bar{b}Z + \bar{c}Z^2).$$

If $\bar{a} + \bar{b}Z + \bar{c}Z^2$ is irreducible, then we obtain (b). Assume that $\bar{a} + \bar{b}Z + \bar{c}Z^2$ is not irreducible. Then, we can write

$$\bar{a} + \bar{b}Z + \bar{c}Z^2 = \bar{c}(Z + \bar{\alpha})(Z + \bar{\beta}).$$

Since R is nodal, we see $\bar{\alpha} \neq \bar{\beta}$. This implies (a).

(4) By Notation 3.2, we have

$$f = ax^2 + bxy + cy^2 + g$$

where $a, b, c \in \{0\} \cup S^{\times}$ and $g \in \mathfrak{l}^3 = (x, y)^3$. Moreover, we have $c \in S^{\times}$. For some $\alpha, \beta, \gamma, \delta \in S$, we obtain

$$f = ax^2 + bxy + cy^2 + \alpha x^3 + \beta x^2 y + \gamma xy^2 + \delta y^3,$$

which implies

$$\frac{f}{x^2} = a + b\frac{y}{x} + c\left(\frac{y}{x}\right)^2 + \alpha x + \beta y + \gamma y\frac{y}{x} + \delta y\left(\frac{y}{x}\right)^2$$
$$= (c + \delta y)\left(\frac{y}{x}\right)^2 + (b + \gamma y)\frac{y}{x} + (a + \alpha x + \beta y).$$

By $c \in S^\times$ and $\delta y \in \mathfrak{l}$, we see $c + \delta y \in S^\times$. Therefore the assertion follows from (1).

(5) The assertion follows from (2) and (4).

(6) Let $I := \{r \in R \mid rT \subset R\}$ be the conductor ideal. By this definition, I is an ideal of R. Note that I is also an ideal of T. Since $R \neq T$, we obtain $1 \notin I$. In particular, $I \subset \mathfrak{m}$. Let us show $I \supset \mathfrak{m}$. By (4), we obtain

$$T = R\left[\frac{\bar{y}}{\bar{x}}\right] = R + R\frac{\bar{y}}{\bar{x}}.$$

This implies $\bar{x}T \subset R$. Thus, $\bar{x} \in I$. Since I is an ideal of $T = R[\bar{y}/\bar{x}]$, we see $\bar{y} = \bar{x}\bar{y}/\bar{x} \in I$. Therefore, $I \supset \bar{x}R + \bar{y}R = \mathfrak{m}$. $\square$

We say a scheme X is *excellent* if X is covered by open affine schemes whose corresponding rings are excellent.

Combining Lemma 3.4 and Lemma 3.5, we obtain the following result.

Proposition 3.6. *Let X be a quasi-compact excellent reduced scheme and let η be a scheme-theoretic point whose local ring $\mathcal{O}_{X,\eta}$ is nodal. Let $S := \overline{\{\eta\}}$ be the reduced scheme. Let $\nu\colon Y \to X$ be the normalization, $D \subset X$ the closed subscheme defined by the conductor and $C \subset Y$ its scheme-theoretic inverse image*:

$$\begin{array}{ccc} C := \nu^{-1}(D) & \xrightarrow[\text{immersion}]{\text{closed}} & Y \\ \downarrow & & \downarrow \nu \\ D & \xrightarrow[\text{immersion}]{\text{closed}} & X. \end{array}$$

Then, there exists an open subset $\eta \in X' \subset X$ which satisfies the following properties.

(0) *Set $Y' := \nu^{-1}(X')$, $D' := D \cap X'$, $C' := C \cap Y'$ and $S' := S \cap X'$.*

(1) *D' is reduced and $S' = D'$. In particular, D' is an integral scheme.*

(2) *$\nu|_{C'}\colon C' \to D'$ satisfies one of the following conditions*

- *$C' \simeq D'_1 \amalg D'_2$ with $D'_i \simeq D$ and each morphism*

$$D'_i \hookrightarrow C' \xrightarrow{\nu|_{C'}} D'$$

are isomorphism.

- *C' is an integral scheme and the field extension $K(C')/K(D')$ satisfies $[K(C') : K(D')] = 2$.*

Proof. We may assume $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $D = \operatorname{Spec} A/I$ and $C = \operatorname{Spec} B/J$ where $I = J$. Let $S_\eta := A \setminus \eta$ where we consider η as a prime ideal of A. There are the following two cases.
(α) $\mathcal{O}_{X,\eta} = A_\eta = S_\eta^{-1}A$ is not an integral domain.
(β) $\mathcal{O}_{X,\eta} = A_\eta = S_\eta^{-1}A$ is an integral domain.

(α) Assume that $S_\eta^{-1}A$ is not an integral domain. We can apply Lemma 3.4 to $S_\eta^{-1}A$. Then, by shrinking $\eta \in \operatorname{Spec} A$, we obtain the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & A/\mathfrak{p}_1 \times A/\mathfrak{p}_2 \\ \downarrow & & \downarrow \\ S_\eta^{-1}A & \longrightarrow & S_\eta^{-1}(A/\mathfrak{p}_1) \times S_\eta^{-1}(A/\mathfrak{p}_2), \end{array}$$

where $(0) = \mathfrak{p}_1 \cap \mathfrak{p}_2$. Since A is excellent, for each i, the regular locus U_i of $\operatorname{Spec} A/\mathfrak{p}_i$ forms an open subset of $\operatorname{Spec} A/\mathfrak{p}_i$. Since $S_\eta^{-1}(A/\mathfrak{p}_i)$ is regular, we obtain $\eta \in U_i$. Therefore, by shrinking $\eta \in \operatorname{Spec} A$, we may assume that each $A/\mathfrak{p}_i$ is regular. In particular, the homomorphism $A \to A/\mathfrak{p}_1 \times A/\mathfrak{p}_2$ coincides with the normalization. Since $S_\eta^{-1}(A/I)$ is reduced and A is noetherian, we may assume that A/I is reduced by shrinking $\operatorname{Spec} A$. This implies (1). We show (2). We have the induced homomorphism

$$\theta_i\colon A/I \to (A/(I + \mathfrak{p}_1)) \times (A/(I + \mathfrak{p}_2)) \to A/(I + \mathfrak{p}_i),$$

where the latter map is the projection to the i-th factor. By Lemma 3.4, $S_\eta^{-1}\theta_i$ is an isomorphism. Since $X = \operatorname{Spec} A$ is noetherian and the kernel and the cokernel of θ is a finitely generated A-modules, we obtain the assertion.

(β) Assume that $S_\eta^{-1}A$ is an integral domain. We can apply Lemma 3.5 to $S_\eta^{-1}A$. We obtain the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\nu} & B \\ \downarrow & & \downarrow \\ S_\eta^{-1}A & \longrightarrow & S_\eta^{-1}B. \end{array}$$

By Lemma 3.5, $S_\eta^{-1}(A/I)$ is reduced. This implies (1). By Lemma 3.5, there are the following two cases:
(a) $S_\eta^{-1}(B/J) \simeq S_\eta^{-1}(A/I) \times S_\eta^{-1}(A/I)$ and the composition homomorphism

$$S_\eta^{-1}(A/I) \to S_\eta^{-1}(B/J) \simeq S_\eta^{-1}(A/I) \times S_\eta^{-1}(A/I) \xrightarrow{p_i} S_\eta^{-1}(A/I)$$

is the identity map for $i = 1, 2$ where p_i is the projection to the i-th factor.

(b) $S_\eta^{-1}(B/J)$ is a field and the natural homomorphism

$$S_\eta^{-1}(A/I) \to S_\eta^{-1}(B/J)$$

is a field extension with $[S_\eta^{-1}(B/J) : S_\eta^{-1}(A/I)] = 2$.
For each case, we obtain (2) by a similar argument to (α). $\square$

The following theorem is the main result in this section.

Theorem 3.7. *Let k be a field. Let X be a pure-dimensional reduced separated scheme of finite type over k. Assume that X is S_2 and, for every codimension one scheme-theoretic point $\eta \in X$, the local ring $\mathcal{O}_{X,\eta}$ is regular or nodal. Let $\nu\colon Y \to X$ be the normalization, $D \subset X$ the closed subscheme defined by the conductor and $C \subset Y$ its scheme-theoretic inverse image*:

$$\begin{array}{ccc} C := \nu^{-1}(D) & \xrightarrow[\textit{immersion}]{\textit{closed}} & Y \\ \Big\downarrow {\scriptstyle \nu|_C} & & \Big\downarrow {\scriptstyle \nu} \\ D & \xrightarrow[\textit{immersion}]{\textit{closed}} & X. \end{array}$$

Let L be an invertible sheaf on X and fix $s \in H^0(Y, \nu^* L^{\otimes 2})$. Let $C = \bigcup C_i$ be the irreducible decomposition where each C_i is an integral scheme. Assume the following conditions.
(1) *The equation $g^*(s|_{C_j}) = s|_{C_i}$ holds for every birational map $g\colon C_i \dashrightarrow C_j$ such that $\nu|_{C_i} = \nu|_{C_j} \circ g$ holds as rational maps. Note that $g^*(s|_{C_j}) = s|_{C_i}$ means that there exist non-empty open subsets $C_i' \subset C_i$, $C_j' \subset C_j$ and an isomorphism $g'\colon C_i' \to C_j'$ induced by g such that $g'^*(s|_{C_j'}) = s|_{C_i'}$.*
(2) *For every i, there exists $t_i \in H^0(C_i, \nu^* L)$ such that $s|_{C_i} = t_i^{\otimes 2}$.*
Then there exists an element $u \in H^0(X, L^{\otimes 2})$ such that $\nu^* u = s$.

Proof. Consider the exact sequence:

$$0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_Y \oplus \mathcal{O}_D \to \nu_* \mathcal{O}_C \to 0,$$

which implies

$$0 \to H^0(X, L^{\otimes 2}) \to H^0(Y, \nu^* L^{\otimes 2}) \oplus H^0(D, L^{\otimes 2}|_D) \to H^0(C, \nu^* L^{\otimes 2}|_C).$$

It suffices to show that there exists $t \in H^0(D, L^{\otimes 2}|_D)$ such that $(\nu|_C)^* t = s|_C$. Since X is S_2, we can replace X with arbitrary open subscheme X' with $\operatorname{codim}_X(X \setminus X') \geq 2$. Thus, we may assume that C and D are regular and of pure codimension one. We can apply Proposition 3.6. Then, by replacing X with its open subscheme, $C \to D$ satisfies one of the following properties.

(a) C is two copies of D, that is, $C \simeq D \amalg D$.
(b) $C \to D$ is a finite surjective morphism between integral schemes such that $[K(C) : K(D)] = 2$ and that $K(C)/K(D)$ is separable.
(c) $C \to D$ is a finite surjective morphism between integral schemes such that $[K(C) : K(D)] = 2$ and that $K(C)/K(D)$ is purely inseparable.
If (a) or (b) holds, then the condition (1) implies that $s|_C$ descends to D. If (c) holds, then the condition (2) implies that $s|_C$ descends to D. □

REMARK 3.8. By the above proof, if the characteristic of k is not equal to 2, then we can drop the second condition (2) in Theorem 3.7.

4. Abundance theorem for slc surfaces

The following definition of slc varieties is the same as Definition-Lemma 5.10 in [13]. For more details, see also [13, 1.41, 5.1, 5.9, 5.10]. Moreover, we define sdlt varieties.

DEFINITION 4.1. Let X be a variety. Assume that X is S_2 and that X is regular or nodal in codimension one. Let Δ be an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\nu\colon Y \to X$ be the normalization and we define Δ_Y by $K_Y + \Delta_Y = \nu^*(K_X + \Delta)$. We say (X, Δ) is *slc* if (Y, Δ_Y) is lc. We say (X, Δ) is *sdlt* variety if (Y, Δ_Y) is dlt and every irreducible component of X is normal.

REMARK 4.2. (1) Note that *sdlt* in Definition 4.1 and *semi-dlt* in the sense of [13, Definition 5.19] are different. There is an sdlt variety which is not semi-dlt (see the example after [13, Definition 5.19]).
(2) In characteristic zero, semi-dlt varieties are sdlt by [13, Definition 5.20]. In positive characteristic, we do not know whether the notions of semi-dlt and sdlt have some relations.

We recall the B-birational maps introduced in [4].

DEFINITION 4.3. Let (X, Δ_X) and (Y, Δ_Y) be lc varieties (may be reducible). We say $\sigma\colon (X, \Delta_X) \dashrightarrow (Y, \Delta_Y)$ is a *B-birational map* if $\sigma\colon X \dashrightarrow Y$ is a birational map and there exist proper birational morphisms $\alpha\colon W \to X$ and $\beta\colon W \to Y$ from a normal variety W such that $\beta = \sigma \circ \alpha$ and $\alpha^*(K_X + \Delta_X) = \beta^*(K_Y + \Delta_Y)$. Note that B-birational maps may permute the irreducible components. We define

$$\mathrm{Aut}(X, \Delta_X) := \{\sigma \in \mathrm{Aut}(X) \mid K_X + \Delta_X = \sigma^*(K_X + \Delta_X)\}.$$

To obtain sections on slc varieties, we consider the following sections on sdlt varieties. The idea is very similar to the admissible sections in [4].

DEFINITION 4.4. Let (X,Δ) be an n-dimensional projective sdlt variety with $n \leq 2$. Let $X = \bigcup X_i$ be the irreducible decomposition and let $\nu\colon \coprod X_i \to X$ be the normalization. We define Δ_i by $K_{X_i} + \Delta_i = (\nu^*(K_X + \Delta))|_{X_i}$. Note that (X_i, Δ_i) is dlt. Let m be a positive integer such that $m(K_X + \Delta)$ is Cartier. We define *B-invariant sections* and *separably gluable sections* as follows.
(1) We say $s \in H^0(X, m(K_X + \Delta))$ is *B-invariant* if $g^*(s|_{X_j}) = s|_{X_i}$ for every B-birational map $g\colon (X_i, \Delta_i) \dashrightarrow (X_j, \Delta_j)$.
(2) We say $s \in H^0(X, m(K_X + \Delta))$ is *separably gluable* if $s|_{\coprod_i \llcorner \Delta_i \lrcorner}$ is B-invariant.
We define vector subspaces

$$BI(X, m(K_X + \Delta)) := \{s \text{ is } B\text{-invariant}\} \subset H^0(X, m(K_X + \Delta)),$$
$$SG(X, m(K_X + \Delta)) := \{\text{s is separably gluable}\} \subset H^0(X, m(K_X + \Delta)).$$

Moreover, we define

$$BI^{(2)}(X, 2m(K_X + \Delta)) := \{t^2 \mid t \in BI(X, m(K_X + \Delta))\},$$
$$G(X, 2m(K_X + \Delta)) := \left\{ s \;\middle|\; s|_{\coprod_i \llcorner \Delta_i \lrcorner} \in BI^{(2)}\left(\coprod_i \llcorner \Delta_i \lrcorner, 2m(K_X + \Delta)|_{\coprod_i \llcorner \Delta_i \lrcorner}\right)\right\}.$$

We say $s \in H^0(X, 2m(K_X + \Delta))$ is *gluable* if $s \in G(X, 2m(K_X + \Delta))$.

REMARK 4.5. In characteristic $p \neq 2$, we do not need $BI^{(2)}(X, 2m(K_X + \Delta))$ and $G(X, 2m(K_X + \Delta))$. For more details, see Remark 3.8 and the proof of Proposition 4.9.

The following lemma teaches us that, in order to obtain sections on an slc surface, we should consider gluable sections on a dlt surface.

Lemma 4.6. *Let (X, Δ) be a projective slc surface. Let $\nu\colon Y \to X$ be the normalization and let $K_Y + \Delta_Y := \nu^*(K_X + \Delta)$. Let $\mu\colon (Z, \Delta_Z) \to (Y, \Delta_Y)$ be a birational morphism from a projective dlt surface (Z, Δ_Z) such that $K_Z + \Delta_Z = \mu^*(K_Y + \Delta_Y)$. Then the following assertions hold. If $s \in G(Z, 2m(K_Z + \Delta_Z))$, then $s = \mu^*\nu^* t$ for some $t \in H^0(X, 2m(K_X + \Delta))$.*

Proof. The assertion holds by Theorem 3.7. □

We summarize the basic properties of B-invariant sections and (separably) gluable sections.

Lemma 4.7. *Let (X, Δ) be an n-dimensional projective sdlt variety with $n \leq 2$. Let m be a positive integer such that $m(K_X + \Delta)$ is Cartier.*
(1) If $s \in BI(X, m(K_X + \Delta))$, then $s^2 \in BI^{(2)}(X, 2m(K_X + \Delta))$.
(2) If $t \in BI^{(2)}(X, 2m(K_X + \Delta))$, then $t \in BI(X, 2m(K_X + \Delta))$.

(3) *The vector space* $BI(X, m(K_X + \Delta))$ *generates* $\mathcal{O}_X(m(K_X + \Delta))$ *if and only if* $BI^{(2)}(X, 2m(K_X + \Delta))$ *generates* $\mathcal{O}_X(2m(K_X + \Delta))$.
(4) *If* $s \in SG(X, m(K_X + \Delta))$, *then* $s^2 \in G(X, 2m(K_X + \Delta))$.
(5) *If* $t \in G(X, 2m(K_X + \Delta))$, *then* $t \in SG(X, 2m(K_X + \Delta))$.
(6) *If the vector space* $SG(X, m(K_X + \Delta))$ *generates* $\mathcal{O}_X(m(K_X + \Delta))$, *then* $G(X, 2m(K_X + \Delta))$ *generates* $\mathcal{O}_X(2m(K_X + \Delta))$.
(7) *Assume that* X *is normal and let* $S := \llcorner \Delta \lrcorner \neq 0$. *If the map*

$$SG(X, m(K_X + \Delta)) \to BI(S, m(K_X + \Delta)|_S)$$

is surjective, then so is the map

$$G(X, 2m(K_X + \Delta)) \to BI^{(2)}(S, 2m(K_X + \Delta)|_S).$$

Proof. (1), (2), (3) These assertions follow from the definition.
(4) The assertion follows from $(\nu^* s^2)|_{\coprod \llcorner \Delta_i \lrcorner} = ((\nu^* s)|_{\coprod \llcorner \Delta_i \lrcorner})^2$.
(5) The assertions follows from (2).
(6), (7) The assertions follow from (4). □

Lemma 4.8. *Let* (X, Δ) *be a proper lc curve or a proper lc surface such that* $K_X + \Delta$ *is semi-ample and* $S := \llcorner \Delta \lrcorner \neq 0$. *Let* $f := \varphi_{|k(K_X+\Delta)|} \colon X \to R$ *be a surjective morphism to a projective variety* R *such that* $f_*\mathcal{O}_X = \mathcal{O}_R$. *Let* $T := f(S)$. *Assume the following conditions.*
(a) $f_*\mathcal{O}_S = \mathcal{O}_T$.
(b) *There exist sections* $\{s_i\}_{i=1}^q \subset H^0(S, m(K_X + \Delta)|_S)$ *without common zeros for some* m.
Then, for some $r > 0$, *there exist sections* $\{u_i\}_{i=1}^l \subset H^0(X, rm(K_X + \Delta))$ *which satisfy the following conditions.*
(1) $u_i|_S = s_i^r$ *for* $1 \leq i \leq q$ *and* $u_i|_S = 0$ *for* $q + 1 \leq i \leq l$.
(2) $\{u_i\}_{i=1}^l$ *have no common zeros.*

Proof. There is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H on R such that $K_X + \Delta \sim_{\mathbb{Q}} f^*H$. For $r \gg 0$, we have the following commutative diagram.

$$\begin{array}{ccc} H^0(X, rm(K_X + \Delta)) & \longrightarrow & H^0(S, rm(K_X + \Delta)|_S) \\ \uparrow \simeq & & \uparrow \simeq \\ H^0(R, rmH) & \xrightarrow{\text{surjection}} & H^0(T, rmH|_T) \end{array}$$

Let $u_1, \ldots, u_q \in H^0(X, rm(K_X + \Delta))$ be lifts of $s_1^r, \ldots, s_q^r$ and let us consider the following corresponding sections.

$$\begin{array}{ccc} u_i & \longrightarrow & s_i^r \\ \uparrow & & \uparrow \\ u_i' & \longrightarrow & s_i' \end{array}$$

We may assume that r is so large that $I_T \otimes \mathcal{O}_R(rmH)$ is generated by global sections where I_T is the corresponding ideal to the closed subscheme T. Let $t'_{q+1}, \dots, t'_l$ be the basis of $H^0(R, I_T \otimes \mathcal{O}_R(rmH))$ and let $u'_{q+1}, \dots, u'_l$ be its image to $H^0(R, rmH)$. Then $u'_1, \dots, u'_l$ have no common zeros. Thus the corresponding sections $u_1, \dots, u_l$ satisfy the desired properties. □

The following proposition is the key to prove the abundance theorem for slc surfaces.

Proposition 4.9. *Let (X, Δ) be a projective dlt surface such that $S := \llcorner \Delta \lrcorner \neq 0$. Let m be a sufficiently large and divisible integer such that $m \in 2\mathbb{Z}_{>0}$. If $K_X + \Delta$ is nef, then the following assertions hold.*
(a) *The following map is surjective*:

$$G(X, 2m(K_X + \Delta)) \to BI^{(2)}(S, 2m(K_X + \Delta)|_S).$$

(b) *Assume that $BI(S, m(K_X + \Delta)|_S)$ generates $\mathcal{O}_S(m(K_X + \Delta)|_S)$. Then $G(X, 2m(K_X + \Delta))$ generates $\mathcal{O}_X(2m(K_X + \Delta))$.*

Proof. We may assume that X is irreducible. By the abundance theorem (cf. [7]), we obtain $f := \varphi_{|k(K_X+\Delta)|}: X \to R$ such that $f_*\mathcal{O}_X = \mathcal{O}_R$. Let $f(S) =: T$. Then (1) or (2) holds.
(1) $f_*\mathcal{O}_S = \mathcal{O}_T$.
(2) $f_*\mathcal{O}_S \neq \mathcal{O}_T$.

(1) Assume $f_*\mathcal{O}_S = \mathcal{O}_T$. By the diagram of the proof of Lemma 4.8, the map

$$H^0(X, m(K_X + \Delta)) \to H^0(S, m(K_X + \Delta)|_S)$$

is surjective. Thus the map

$$SG(X, m(K_X + \Delta)) \to BI(S, m(K_X + \Delta)|_S)$$

is also surjective. Thus assertion (a) follows from Lemma 4.7 (7). We prove (b). Since $BI(S, m(K_X + \Delta)|_S)$ generates $\mathcal{O}_S(m(K_X + \Delta)|_S)$, $SG(X, m(K_X + \Delta))$ also generates $\mathcal{O}_X(m(K_X + \Delta))$ by Lemma 4.8. The assertion follows from Lemma 4.7 (6).

(2) Assume $f_*\mathcal{O}_S \neq \mathcal{O}_T$. We can apply Proposition 2.8 and we obtain Proposition 2.8 (2). Then, we have projective morphisms

$$f\colon X \xrightarrow{g} V \to R$$

where V is a smooth projective curve.

CASE (2.1s). Assume Proposition 2.8 (2.1s) holds. By Lemma 4.7 (7), it is sufficient to prove (a)′ and (b)′.
(a)′ The following map is surjective:

$$SG(X, m(K_X + \Delta)) \to BI(S, m(K_X + \Delta)|_S).$$

(b)′ Assume that $BI(S, m(K_X + \Delta)|_S)$ generates $\mathcal{O}_S(m(K_X + \Delta)|_S)$. Then $SG(X, m(K_X + \Delta))$ generates $\mathcal{O}_X(m(K_X + \Delta))$.
First we prove (a)′. Note that there is a Galois involution $\iota\colon S_1 \to S_1$ and ι is B-birational. Let $s \in BI(S, m(K_X + \Delta)|_S)$. Since s is B-invariant, this section s is invariant for ι. Thus $s|_{S_1}$ is the pull-back of a section $t \in H^0(V, m(D_V))$. Let $u := g^*t \in H^0(X, m(K_X + \Delta))$. We prove that $u|_S = s$. Let $S = \bigcup S_i$ be the irreducible decomposition. Since S is reduced, we obtain the exact sequence:

$$0 \to \mathcal{O}_S \to \bigoplus_i \mathcal{O}_{S_i}.$$

Therefore it is sufficient to prove that $u|_{S_i} = s|_{S_i}$ for every i. For $i = 1$, this is clear by the construction. Thus we may assume that S_i is g-vertical. We take a proper birational morphism $\lambda\colon X'' \to X$ in Lemma 5.10. Let $g''\colon X'' \xrightarrow{\lambda} X \xrightarrow{g} V$. Note that $\lambda_*\mathcal{O}_{S''} = \mathcal{O}_S$ by Lemma 5.10 where $S'' := \llcorner \Delta'' \lrcorner$. Thus it is sufficient to prove that $u''|_{S''_i} = s''|_{S''_i}$ where $u'' := \lambda^* u$, $s'' := \lambda^* s$ and S''_i is an irreducible component of S'' such that g''-vertical. Let S''_1 be the proper transform of S_1. Assume $S''_1 \cap S''_i \neq \emptyset$. Note that, since (X'', Δ'') is dlt, the scheme-theoretic intersection $S''_1 \cap S''_i$ is reduced. Hence, Lemma 5.10 implies $g''_*\mathcal{O}_{S''_i} \simeq g''_*\mathcal{O}_{S''_1 \cap S''_i}$. Since $m(K_{X''} + \Delta'')$ is the pull-back of mD_V, this means

$$H^0(S''_i, m(K_{X''} + \Delta'')|_{S''_i}) \simeq H^0(S''_1 \cap S''_i, m(K_{X''} + \Delta'')|_{S''_1 \cap S''_i}).$$

By $u''|_{S''_1} = s''|_{S''_1}$, we have $u''|_{S''_1 \cap S''_i} = s''|_{S''_1 \cap S''_i}$. Therefore, by the above isomorphism, we see $u''|_{S''_i} = s''|_{S''_i}$. If S''_j satisfies $S''_j \cap S''_i \neq \emptyset$ for $S''_1 \cap S''_i \neq \emptyset$, then $u''|_{S''_j} = s''|_{S''_j}$ by the same argument as above. By the inductive argument, if a vertical irreducible component S''_j is contained in a connected component of S'' which intersects S''_1, then $u''|_{S''_j} = s''|_{S''_j}$. By Lemma 2.4 and Proposition 2.8, every vertical irreducible component S''_i satisfies this property. Therefore, we see $u \in SG(X, m(K_X + \Delta))$ such that $u|_S = s$.

Second, we prove (b)′. We prove that $SG(X, m(K_X + \Delta))$ generates $\mathcal{O}_X(m(K_X + \Delta))$. Let $s_1, \ldots, s_r \in BI(S, m(K_X + \Delta)|_S)$ be a basis and let $u_1, \ldots, u_r \in SG(X, m(K_X +$

$\Delta)$) be their lifts. Let $t_1, \dots, t_r \in H^0(V, mD_V)$ be the corresponding sections. Since $BI(S, m(K_X+\Delta)|_S)$ generates $\mathcal{O}_S(m(K_X+\Delta)|_S)$ and $S \to V$ is surjective, $t_1, \dots, t_r$ have no common zeros. Thus the corresponding sections $u_1, \dots, u_r$ generates $\mathcal{O}_X(m(K_X+\Delta))$.

CASE (2.2). Assume Proposition 2.8 (2.2) holds. It is sufficient to prove the above assertions (a)′ and (b)′.

We prove (a)′. Note that there is a B-birational morphism $\iota\colon S_2 \to S_1$ obtained by $S_2 \simeq V \simeq S_1$. Let $s \in BI(S, m(K_X+\Delta)|_S)$. Since s is B-invariant, we see $\iota^*(s|_{S_1}) = s|_{S_2}$. Since $S_1 \simeq V$, $s|_{S_1}$ is the pull-back of a section $t \in H^0(V, mD_V)$. Let $u := g^*t \in H^0(X, m(K_X+\Delta))$. We would like to prove that $u|_S = s$. It is sufficient to prove that $u|_{S_i} = s|_{S_i}$ for every irreducible component S_i of S. By the same argument as (2.1s), it is sufficient to prove this equality only for $i = 1, 2$. It is clear in the case where $i = 1$. Since $\iota^*(u|_{S_1}) = u|_{S_2}$, it is also clear in the case where $i = 2$. The assertion (b) holds by the same argument as (2.1s).

Case (2.1i). Assume Proposition 2.8 (2.1i) holds. We see $p = \operatorname{char} k = 2$.

We prove (a). Let $s \in BI^{(2)}(S, 2m(K_X+\Delta)|_S)$. Then we have $s = \tilde{s}^2$ where $\tilde{s} \in BI(S, m(K_X+\Delta)|_S)$. Note that $g|_{S_1}\colon S_1 \to V$ is the relative Frobenius morphism. Thus the absolute Frobenius morphism $F\colon S_1 \to S_1$ factors through V:

$$F\colon S_1 \xrightarrow{g|_{S_1}} V \xrightarrow{G} S_1.$$

Note that G is a non-k-linear isomorphism as schemes and that, for an invertible sheaf L on V,

$$G^*(g|_{S_1})^*L \simeq G^*(g|_{S_1})^*G^*(G^{-1})^*L \simeq G^*F^*(G^{-1})^*L \simeq L^{\otimes 2}.$$

We show $\mathcal{O}_V(2mD_V) \simeq G^*\mathcal{O}_{S_1}(m(K_X+\Delta)|_{S_1})$. Since $m \in 2\mathbb{Z}$, we can write $m = 2m'$ where $m' \in \mathbb{Z}$. First, we see

$$(g|_{S_1})^*\mathcal{O}_V(2m'D_V) \simeq \mathcal{O}_{S_1}(2m'(K_X+\Delta)|_{S_1}) \simeq (g|_{S_1})^*G^*\mathcal{O}_{S_1}(m'(K_X+\Delta)|_{S_1}).$$

Then, for an invertible sheaf

$$M := (G^{-1})^*\mathcal{O}_V(2m'D_V) \otimes \mathcal{O}_{S_1}(-m'(K_X+\Delta)|_{S_1}),$$

we obtain $F^*M = (g|_{S_1})^*G^*M \simeq \mathcal{O}_{S_1}$. This implies

$$\begin{aligned}\mathcal{O}_V(2mD_V) &\simeq G^*(g|_{S_1})^*\mathcal{O}_V(2m'D_V)\\ &\simeq G^*F^*(G^{-1})^*\mathcal{O}_V(2m'D_V)\\ &\simeq G^*F^*\mathcal{O}_{S_1}(m'(K_X+\Delta)|_{S_1})\\ &\simeq G^*\mathcal{O}_{S_1}(m(K_X+\Delta)|_{S_1}).\end{aligned}$$

Therefore, the section s is the pull-back of

$$t := G^*\tilde{s} \in H^0(V, 2mD_V).$$

Let $u := g^*t \in H^0(X, 2m(K_X + \Delta))$. Then, by the same argument as (2.2s), we see $u|_S = s$. This means $u \in G(X, 2m(K_X + \Delta))$.

We prove (b), that is, we prove that $G(X, 2m(K_X + \Delta))$ generates $\mathcal{O}_X(2m(K_X + \Delta))$. Let $s_1,\dots,s_r \in BI^{(2)}(S, 2m(K_X + \Delta)|_S)$ be a basis and let $u_1,\dots,u_r \in G(X, 2m(K_X + \Delta))$ be their lifts. Let $t_1, \dots, t_r \in H^0(V, 2mD_V)$ be the corresponding sections. Here, $BI^{(2)}(S, 2m(K_X + \Delta)|_S)$ generates $\mathcal{O}_S(m(K_X + \Delta)|_S)$ by Lemma 4.7 (3). Thus, since $S \to V$ is surjective, $t_1, \dots, t_r$ have no common zeros. Thus the corresponding sections $u_1, \dots, u_r$ generates $\mathcal{O}_X(2m(K_X + \Delta))$. □

In order to construct B-invariant sections, we consider the following finiteness theorem.

Theorem 4.10. *Let (C, Δ) be a projective lc curve and let m be a positive integer such that $m(K_C + \Delta)$ is Cartier. Then $\rho_m(\mathrm{Aut}(C, \Delta))$ is a finite group where ρ_m is a group homomorphism defined by*

$$\rho_m\colon\ \mathrm{Aut}(C, \Delta) \to \mathrm{Aut}(H^0(C, m(K_C + \Delta))),$$
$$\sigma \mapsto (s \mapsto \sigma^* s).$$

Proof. We may assume that C is irreducible. If the genus $g(C) \geq 2$, then $\mathrm{Aut}(C)$ is a finite group. Therefore, $\rho_m(\mathrm{Aut}(C, \Delta))$ is a finite group since $\mathrm{Aut}(C, \Delta) \subset \mathrm{Aut}(C)$.

If $g(C) = 1$ and $\Delta \neq 0$, then $\mathrm{Aut}(C, \ulcorner\Delta\urcorner)$ is a quasi-projective scheme and $H^0(C, T_C \otimes \mathcal{O}_C(-\ulcorner\Delta\urcorner)) = 0$. Therefore, $\mathrm{Aut}(C, \ulcorner\Delta\urcorner)$ is a finite group. Thus, $\rho_m(\mathrm{Aut}(C, \Delta))$ is a finite group because $\mathrm{Aut}(C, \Delta) \subset \mathrm{Aut}(C, \ulcorner\Delta\urcorner)$.

Assume that $g(C) = 1$ and $\Delta = 0$. Let $0 \in C$ be the origin of the elliptic curve C. Then $T_{-\sigma(0)} \circ \sigma \in \mathrm{Aut}(C, [0])$ for any $\sigma \in \mathrm{Aut}(C)$, where $T_{-\sigma(0)}$ is the translation of C by $-\sigma(0)$. Note that $H^0(C, \mathcal{O}_C(K_C)) \simeq k$ is spanned by a translation invariant 1-form on C and that $\mathrm{Aut}(C, [0])$ is a finite group. Therefore, $\rho_1(\mathrm{Aut}(C))$ is a finite group. Since $\rho_m = \rho_1^{\otimes m}$, $\rho_m(\mathrm{Aut}(C))$ is finite for every $m > 0$.

Finally, we assume that $C = \mathbb{P}^1$. If $|\mathrm{Supp}\,\Delta| \geq 3$, then $\mathrm{Aut}(C, \Delta)$ is a finite group. If $\deg(K_C + \Delta) < 0$, then there is nothing to prove. Therefore, we can reduce the problem to the case when $\Delta = \llcorner\Delta\lrcorner = \{\text{two points}\}$. In this case, we can easily check that $\rho_m(\mathrm{Aut}(C, \Delta))$ is finite for every $m > 0$. Moreover, $\rho_m(\mathrm{Aut}(C, \Delta))$ is trivial if m is an even positive integer. □

The following proposition shows that the assumption of (b) in Proposition 4.9 holds.

Proposition 4.11. *Let (X, Δ) be a projective lc curve. If $K_X + \Delta$ is nef, then $BI(X, m'(K_X + \Delta))$ generates $\mathcal{O}_X(m'(K_X + \Delta))$ for some integer $m' > 0$.*

Proof. We see that $H^0(X, m(K_X + \Delta))$ generates $\mathcal{O}_X(m(K_X + \Delta))$ for some integer $m > 0$. Let $G := \rho_m(\mathrm{Aut}(X, \Delta))$. Note that this group is finite by Theorem 4.10. Let

$N := |G|$ and let $G = \{g_1, \ldots, g_N\}$. For $1 \le i \le N$, let σ_i be the N-variable elementary symmetric polynomial of degree i. If $s \in H^0(X, m(K_X + \Delta))$, then

$$(\sigma_i(g_1^* s, \ldots, g_N^* s))^{N!/i} \in BI(X, N!\, m(K_X + \Delta)).$$

Since

$$\bigcap_{j=1}^{N} \{g_j^* s = 0\} = \bigcap_{i=1}^{N} \{\sigma_i(g_1^* s, \ldots, g_N^* s) = 0\},$$

$BI(X, N!\, m(K_X + \Delta))$ generates $\mathcal{O}_X(N!\, m(K_X + \Delta))$. □

Let us prove the main theorem of this paper.

Theorem 4.12. *Let (X, Δ) be a projective slc surface. If $K_X + \Delta$ is nef, then $K_X + \Delta$ is semi-ample.*

Proof. Let $\nu\colon Y \to X$ be the normalization and we define Δ_Y by $K_Y + \Delta_Y = \nu^*(K_X + \Delta)$. There exists a birational morphism $\mu\colon Z \to Y$ from a projective dlt surface (Z, Δ_Z) where $K_Z + \Delta_Z = \mu^*(K_Y + \Delta_Y)$. By Lemma 4.6, it is sufficient to prove that $G(Z, m_0(K_Z + \Delta_Z))$ generates $\mathcal{O}_Z(m_0(K_Z + \Delta_Z))$ for some $m_0 > 0$. This follows from Proposition 4.9 (b) and Proposition 4.11. □

5. Appendix: Fundamental properties of dlt surfaces

We summarize fundamental properties for dlt surfaces. In this section, we assume that all surfaces are irreducible. The results in this section may be well-known for experts.

First, we recall the definition of dlt surfaces. It is easy to see that the following definition is equivalent to [13, Definition 2.8] and [17, Definition 2.37].

DEFINITION 5.1. Let X be a normal surface and let Δ be a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $0 \le \Delta \le 1$. Let

$$S(X, \Delta) := \mathrm{Sing}(X) \cup \{x \in \mathrm{Reg}(X) \mid \mathrm{Supp}\,\Delta \text{ is not simple normal crossing at } x\}.$$

We say (X, Δ) is *dlt* if $a(E, X, \Delta) > -1$ for every proper birational morphism $f\colon Y \to X$ and every f-exceptional prime divisor $E \subset Y$ such that $f(E) \in S(X, \Delta)$.

Proposition 5.2. *Let X be a normal surface and let Δ be a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $0 \le \Delta \le 1$. The following assertions are equivalent*:

(1) (X, Δ) *is dlt.*

(2) *There exists a projective birational morphism $\mu\colon X' \to X$ from a smooth surface such that* $\mathrm{Ex}(\mu) \cup \mathrm{Supp}\,\mu^*(\Delta)$ *is a simple normal crossing divisor and each μ-exceptional prime divisor E_i satisfies $a(E_i, X, \Delta) > -1$.*

Proof. Note that $S(X, \Delta)$ is a finite set.

Assume (1), that is, assume that (X, Δ) is dlt. Let $f: Y \to X$ be a log resolution of (X, Δ). Let

$$\mathrm{Ex}(f) := E_1 \amalg \cdots \amalg E_r \amalg F_1 \amalg \cdots \amalg F_s$$

be the decomposition into the connected components where $P_i := f(E_i) \in S(X, \Delta)$ and $Q_j := f(F_j) \notin S(X, \Delta)$. There exists a proper birational morphisms

$$Y \xrightarrow{g} Z \xrightarrow{h} X$$

such that Z is a normal surface and $\mathrm{Ex}(g) = F_1 \sqcup \cdots \sqcup F_s$. Indeed, Z is obtained by glueing the varieties $X \setminus \{P_1, \ldots, P_r\}$ and $Y \setminus (F_1 \sqcup \cdots \sqcup F_s)$. Note that this morphism $h : Z \to X$ is projective because Z is smooth. Thus this morphism satisfies (2).

Assume (2). Let $f: Y \to X$ be a proper birational morphism and let $E \subset Y$ be a prime divisor such that $f(E) \in S(X, \Delta)$. We prove $a(E, X, \Delta) > -1$. We may assume that there exists a proper birational morphism $Y \xrightarrow{f'} X'$ and Y is smooth by replacing Y with a desingularization of a resolution of indeterminacy $Y \dashrightarrow X'$. There are two cases: $\dim f'(E) = 0$ and $\dim f'(E) = 1$. The latter case is clear by (2). Thus we may assume $f'(E)$ is one point. Let $K_{X'} + \Delta' := \mu^*(K_X + \Delta)$. Since $f(E) \in S(X, \Delta)$, there exists an μ-exceptional curve E_i such that $f'(E) \in E_i$. We can write the prime decomposition

$$\Delta' := b_i E_i + \cdots$$

where $b_i < 1$. Then we see that $a(E, X, \Delta) > -1$ since Δ' is simple normal crossing and since the morphism $f': Y \to X'$ is a sequence of blow-ups. □

Proposition 5.3. *Let (X, Δ) be a dlt surface. Then X is $\mathbb{Q}$-factorial.*

Proof. See, for example, [20, Theorem 14.4]. □

Proposition 5.4. *Let (X, Δ) be a dlt surface. If a $\mathbb{Q}$-divisor Δ' satisfies $0 \leq \Delta' \leq \Delta$, then (X, Δ') is dlt.*

Proof. Since X is $\mathbb{Q}$-factorial, the assertion immediately follows from Definition 5.1. □

Proposition 5.5. *Let (X, Δ) be a dlt surface. Then the following assertions are equivalent.*
(1) *(X, Δ) is plt.*
(2) *$\llcorner \Delta \lrcorner$ is smooth.*
(3) *Each connected component of $\llcorner \Delta \lrcorner$ is irreducible.*

Proof. See [17, Proposition 5.51]. Note that the proof of [17, Proposition 5.51] needs the relative Kawamata–Viehweg vanishing theorem for a resolution of singularities $Y \to X$. This follows from [21]. $\square$

Corollary 5.6. *Let (X, Δ) be a dlt surface. Then each prime component of $\llcorner \Delta \lrcorner$ is smooth.*

Proof. Let C be a prime component of $\llcorner \Delta \lrcorner$. Then (X, C) is plt by Proposition 5.5. $\square$

Proposition 5.7. *Let $(X, C + \Delta')$ be a dlt surface where C is a smooth curve in X. Let $(K_X + C + \Delta')|_C =: K_C + \Delta_C$. Then (C, Δ_C) is lc, that is, $0 \le \Delta_C \le 1$.*

Proof. Let $f: Y \to X$ be an arbitrary resolution and let C_Y be the proper transform of C. Let $f^*(K_X + C + \Delta') =: K_Y + C_Y + \Delta_Y$. Note that $C \simeq C_Y$. Consider the following commutative diagram.

$$\begin{array}{ccc} C_Y & \xrightarrow[\text{immersion}]{\text{closed}} & Y \\ {\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[\text{immersion}]{\text{closed}} & X \end{array}$$

We prove that Δ_C is effective. Let f be the minimal resolution. Then Δ_Y is effective and C_Y is not a prime component of Δ_Y. Thus we have $0 \le \Delta_C$ by the adjunction formula.

Let f be a log resolution. Then, by Definition 5.1, we see $\Delta_Y \le 1$. This means $\Delta_C \le 1$. $\square$

Corollary 5.8. *Let (X, Δ) be a dlt surface. Assume $S := \llcorner \Delta \lrcorner \ne 0$ and let $K_S + \Delta_S := (K_X + \Delta)|_S$. Then S is normal crossing and (S, Δ_S) is sdlt.*

Proof. By [17, Theorem 4.15], S is normal crossing. Thus, the assertion follows from Proposition 5.7. $\square$

Proposition 5.9. *Let (X, Δ) be an lc surface. Then there exists a proper birational morphism $h: Z \to X$ from a smooth surface Z such that (Z, Δ_Z) is dlt where Δ_Z is defined by $K_Z + \Delta_Z = h^*(K_X + \Delta)$.*

Proof. Let $f: Y \to X$ be a log resolution of (X, Δ) and let $K_Y + \Delta_Y := f^*(K_X + \Delta)$. Let $\Delta_Y = \Delta_Y^+ - \Delta_Y^-$ where Δ_Y^+ and Δ_Y^- are effective and Δ_Y^+ and Δ_Y^- have no common irreducible components. Since $K_Y \cdot \Delta_Y^- < 0$ and each irreducible component of Δ_Y^- is f-exceptional, there exists a (-1)-curve C such that $C \subset \operatorname{Supp} \Delta_Y^-$. Contract this (-1)-curve $Y \to Y'$. We repeat this procedure and we obtain morphisms

$$f\colon Y \overset{g}{\to} Z \overset{h}{\to} X.$$

Then we see that Z is smooth and $0 \leq \Delta_Z \leq 1$ where $K_Z + \Delta_Z = h^*(K_X + \Delta)$. We prove that (Z, Δ_Z) is dlt. Let $l\colon W \to Z$ be a proper birational morphism and $E \subset W$ be an l-exceptional prime divisor such that $l(E) \in S(Z, \Delta)$. We prove $a(Z, \Delta_Z, E) > -1$. We may assume that W is smooth and $l\colon W \to Z$ factors through Y. We obtain four surfaces:

$$W \overset{p}{\to} Y \overset{g}{\to} Z \overset{h}{\to} X.$$

Note that $p(E) \subset \operatorname{Supp} \Delta_Y^-$. There are two cases:
(0) $\dim p(E) = 0$ and
(0) $\dim p(E) = 1$.

(0) Assume $\dim p(E) = 0$. Note that p is a composition of blow-ups. Since Δ_Y is simple normal crossing and $p(E) \in \operatorname{Supp} \Delta_Y^-$, we obtain $a(Z, \Delta_Z, E) > 0$ by a direct calculation.

(1) Assume $\dim p(E) = 1$. Since $p(E) \subset \operatorname{Supp} \Delta_Y^-$, we obtain the inequality $a(Z, \Delta_Z, E) > 0$. $\square$

Lemma 5.10. *Let (X, Δ) be a dlt surface. Then there exists a proper birational morphism $\lambda\colon X'' \to X$ from a normal surface X'' which satisfies the following properties.*
(1) *For $K_{X''} + \Delta'' := \lambda^*(K_X + \Delta)$, the pair (X'', Δ'') is dlt.*
(2) *If S_i'' and S_j'' are prime components of $\llcorner \Delta'' \lrcorner$ such that $S_i'' \neq S_j''$ and $S_i'' \cap S_j'' \neq \emptyset$, then $S_i'' \cap S_j''$ is one point.*
(3) *$\lambda_*(\llcorner \Delta'' \lrcorner) = \llcorner \Delta \lrcorner$ and $\lambda_* \mathcal{O}_{\llcorner \Delta'' \lrcorner} = \mathcal{O}_{\llcorner \Delta \lrcorner}$.*

Proof. If (X, Δ) satisfies the condition (2), then the assertion is clear. Thus we may assume that there exists prime components S_i and S_j of $\llcorner \Delta \lrcorner$ such that $S_i \neq S_j$ and $S_i \cap S_j$ has at least two points. Let $P \in S_i \cap S_j$. Note that, since (X, Δ) is dlt, $P \in \operatorname{Reg}(X)$ and $\operatorname{Supp} \Delta$ is simple normal crossing at P. Let $\mu\colon Y \to X$ be the blowup at P and let $K_Y + \Delta_Y := \mu^*(K_X + \Delta)$. We apply this argument to (Y, Δ_Y) and we repeat the same procedure. Then, by a direct calculation and Lemma 2.3, we obtain the desired morphism $X'' \to X$. $\square$

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