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ON GENERA OF LEFSCHETZ FIBRATIONS AND FINITELY PRESENTED GROUPS

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Abstract

It is known that every finitely presented group is the fundamental group of the total space of a Lefschetz fibration. In this paper, we give another proof which improves the result of Korkmaz. In addition, Korkmaz defined the genus of a finitely presented group. We also evaluate upper bounds for genera of some finitely presented groups.

1. Introduction

Gompf [5] proved that every finitely presented group is the fundamental group of a closed symplectic 4-manifold. Donaldson [4] proved that every closed symplectic 4-manifold admits a Lefschetz pencil. By blowing up the base locus of a Lefschetz pencil, we obtain a Lefschetz fibration over S^2. In addition, blowing up does not change the fundamental group of a 4-manifold. Therefore, it immediately follows that every finitely presented group is the fundamental group of the total space of a Lefschetz fibration.

Amoros–Bogomolov–Katzarkov–Pantev [1] and Korkmaz [8] also constructed Lefschetz fibrations whose fundamental groups are a given finitely presented group. In particular, Korkmaz [8] provided explicitly a genus and a monodromy of such a Lefschetz fibration.

Let $F_n = \langle g_1, \dots, g_n \rangle$ be the free group of rank n. For $x \in F_n$, the *syllable length* $l(x)$ of x is defined by

$$l(x) = \min\left\{s \;\middle|\; x = g_{i(1)}^{m(1)} \cdots g_{i(s)}^{m(s)},\ 1 \leq i(j) \leq n,\, m(j) \in \mathbb{Z}\right\}.$$

For a finitely presented group Γ with a presentation $\Gamma = \langle g_1, \dots, g_n \mid r_1, \dots, r_k \rangle$, Korkmaz [8] proved that for any $g \geq 2\left(n + \sum_{1 \leq i \leq k} l(r_i) - k\right)$ there exists a genus-g Lefschetz fibration $f\colon X \to S^2$ such that the fundamental group $\pi_1(X)$ is isomorphic to Γ, providing explicitly a monodromy.

In this paper, we improve this result.

Theorem 1.1. *Let Γ be a finitely presented group with a presentation $\Gamma = \langle g_1, \dots, g_n \mid r_1, \dots, r_k \rangle$, and let $l = \max_{1 \leq i \leq k}\{l(r_i)\}$. Then for any $g \geq 2n + l - 1$, there*

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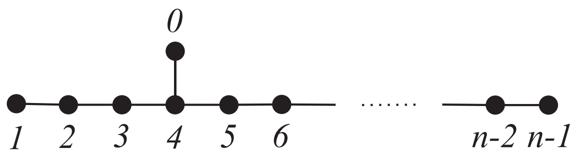


Fig. 1. The Dynkin diagram.

exists a genus-g Lefschetz fibration $f\colon X \to S^2$ *such that the fundamental group* $\pi_1(X)$ *is isomorphic to* Γ.

In this theorem, if $k = 0$, we suppose $l = 1$. We will prove the theorem by providing an explicit monodromy.

In addition, Korkmaz [8] defined the *genus* $g(\Gamma)$ of a finitely presented group Γ to be the minimal genus of a Lefschetz fibration with sections whose fundamental group is isomorphic to Γ. The Lefschetz fibrations constructed in Theorem 1.1 have sections. Hence the definition of the genus of a finitely presented group is well-defined.

We will also prove the following theorem.

Theorem 1.2. (1) *Let* B_n *denote the n-strands braid group. Then for* $n \geq 3$, *we have* $2 \leq g(B_n) \leq 4$.
(2) *Let* $\mathcal{H}_g$ *be the hyperelliptic mapping class group of a closed connected orientable surface of genus* $g \geq 1$. *Then we have* $2 \leq g(\mathcal{H}_g) \leq 4$.
(3) *Let* $\mathcal{M}_{0,n}$ *denote the mapping class group of a sphere with n punctures. Then for* $n \geq 3$, *we have* $2 \leq g(\mathcal{M}_{0,n}) \leq 4$.
(4) *Let* S_n *denote the n-symmetric group. Then for* $n \geq 3$, *we have* $2 \leq g(S_n) \leq 4$.
(5) *Let* $\mathcal{A}_n$ *denote the n-Artin group associated to the Dynkin diagram shown in* Fig. 1. *Then for* $n \geq 6$, *we have* $2 \leq g(\mathcal{A}_n) \leq 5$.
(6) *Let* $n, k \geq 0$ *be integers with* $n + k \geq 3$, *and let* $m_1, \dots, m_k \geq 2$ *be integers. Then we have* $(n + k + 1)/2 \leq g(\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}) \leq n + k + 1$.

2. A Lefschetz fibration and preliminaries

2.1. A Lefschetz fibration and its monodromy.

Here, we review briefly the theory of Lefschetz fibrations.

Let X be a closed connected orientable smooth 4-manifold. A smooth map $f\colon X \to S^2$ is a genus-g *Lefschetz fibration* over S^2 if it satisfies following properties:

- All regular fibers are diffeomorphic to a closed connected oriented surface of genus g.
- Each critical point of f has an orientation-preserving chart on which $f(z_1, z_2) = z_1^2 + z_2^2$ relative to a suitable smooth chart on S^2.
- Each singular fiber contains only one critical point.

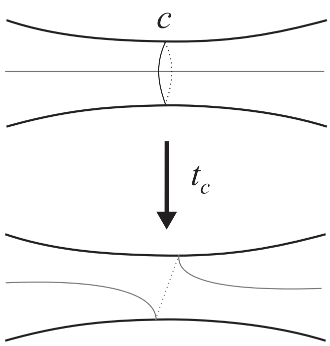


Fig. 2. The right Dehn twist about c.

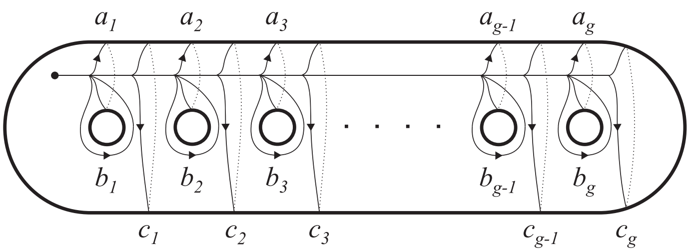


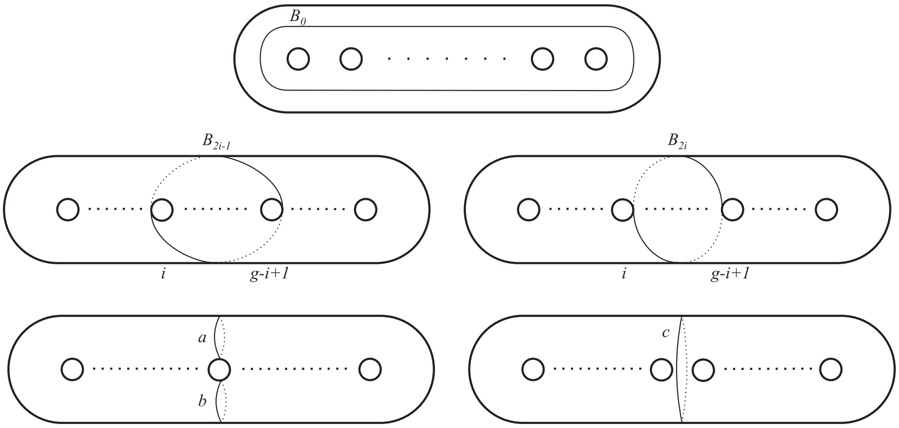
Fig. 3.

- f is *relatively minimal*, that is, no fiber contains an embedded sphere with the self-intersection number -1.

Let $\mathcal{M}_g$ be the mapping class group of a closed connected oriented surface Σ_g of genus g, that is, the group of isotopy classes of orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$. In this paper, for elements x and y of a group, the composition xy means that we first apply x and then y. So for $f, g \in \mathcal{M}_g$, the composition fg means that we first apply f and then g. For a simple closed curve c on Σ_g, let t_c be the isotopy class of the right Dehn twist about c (see Fig. 2). For a genus-g Lefschetz fibration which has n singular fibers, there are simple closed curves $c_1, \ldots, c_n$ on Σ_g, each of which is called the *vanishing cycle*, such that each singular fiber F_i is obtained by collapsing c_i to a point to create a transverse self-intersection, and $t_{c_1} \cdots t_{c_n} = 1$. This equation is called the *monodromy* of a Lefschetz fibration. Conversely, if there are simple closed curves $c_1, \ldots, c_n$ on Σ_g such that $t_{c_1} \cdots t_{c_n} = 1$, then we can construct a genus-g Lefschetz fibration with the monodromy $t_{c_1} \cdots t_{c_n} = 1$.

For a Lefschetz fibration $f: X \to S^2$, a smooth map $s: S^2 \to X$ is a section of f if $f \circ s: S^2 \to S^2$ is the identity map.

For a closed connected orientable surface Σ_g of genus g, let $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ and $c_1, \ldots, c_g$ be loops on Σ_g as shown in Fig. 3. Then the fundamental group $\pi_1(\Sigma_g)$



(a) The case g is odd. (b) The case g is even.

Fig. 4.

of Σ_g has a following presentation

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid r \rangle,$$

where $r = b_g^{-1} \cdots b_1^{-1}(a_1 b_1 a_1^{-1}) \cdots (a_g b_g a_g^{-1})$.

Let $B_0, \dots, B_g$ and a, b, c be simple closed curves on Σ_g as shown in Fig. 4. In this paper, let W denote the following

$$W = \begin{cases} (t_c t_{B_g} \cdots t_{B_0})^2 & \text{when } g \text{ is even,} \\ (t_a^2 t_b^2 t_{B_g} \cdots t_{B_0})^2 & \text{when } g \text{ is odd.} \end{cases}$$

It was shown in [7] that $W = 1$ in the mapping class group $\mathcal{M}_g$ of Σ_g. In addition, the Lefschetz fibration $f_W \colon X_W \to S^2$ with the monodromy $W = 1$ has a section (see [7] and [8]).

2.2. Preliminaries. We now state the way to obtain the presentation of the fundamental group of a Lefschetz fibration with a section. For a group Γ and $\{x_1,\dots,x_n\} \subset \Gamma$, let $\langle x_1, \dots, x_n \rangle$ denote the normal closure of $\{x_1, \dots, x_n\}$ in Γ.

Proposition 2.1 (cf. [6]). *Let $f \colon X \to S^2$ be a genus-g Lefschetz fibration with the monodromy $t_{c_1} \cdots t_{c_n} = 1$. Suppose that f has a section. Then we have*

$$\pi_1(X) \cong \pi_1(\Sigma_g)/\langle c_1, \dots, c_n \rangle,$$

where we regard $c_1, \dots, c_n$ as elements in $\pi_1(\Sigma_g)$.

For $x, y \in \mathcal{M}_g$, let $x^y = y^{-1}xy$. For example, for simple closed curves $c_1, \ldots, c_n$ on Σ_g and $h \in \mathcal{M}_g$, we have $(t_{c_1} \cdots t_{c_n})^h = (h^{-1}t_{c_1}h) \cdots (h^{-1}t_{c_n}h) = t_{(c_1)h} \cdots t_{(c_n)h}$, where $(c_i)h$ means the image of c_i by h.

Proposition 2.2 ([8]). *Let $f: X \to S^2$ be a genus-g Lefschetz fibration with the monodromy $V = t_{c_1} \cdots t_{c_n} = 1$. Suppose that f has a section. Let d be a simple closed curve on Σ_g which intersects some c_i transversely at only one point. Let f': $X' \to S^2$ be the genus-g Lefschetz fibration with the monodromy $VV^{t_d} = 1$. Then we have*

$$\pi_1(X') \cong \pi_1(\Sigma_g)/\langle c_1, \ldots, c_n, d \rangle,$$

where we regard $c_1, \ldots, c_n$ and d as elements in $\pi_1(\Sigma_g)$.

In this paper, we denote the Lefschetz fibration with the monodromy $V = 1$ by $f_V: X_V \to S^2$. For example, in the above proposition, $f = f_V$, $X = X_V$ and $f' = f_{VV^{t_c}}$, $X' = X_{VV^{t_c}}$.

We next state results of Korkmaz [8].

Theorem 2.3 ([8]). (1) *Let Σ_g be a closed connected orientable surface of genus $g \geq 0$. Then we have $g(\pi_1(\Sigma_g)) = g$.*
(2) *Let $m(\Gamma)$ denote the minimal number of generators for Γ. Then we have $m(\Gamma)/2 \leq g(\Gamma)$, with the equality if and only if Γ is isomorphic to $\pi_1(\Sigma_g)$.*
(3) *For the mapping class group $\mathcal{M}_1$ of Σ_1, we have $2 \leq g(\mathcal{M}_1) \leq 4$.*
(4) *Let B_n denote the n-strands braid group. Then for $n \geq 3$, we have $2 \leq g(B_n) \leq 5$.*
(5) *Let $n, k \geq 0$ be integers with $n + k \geq 3$, and let $m_1, \ldots, m_k \geq 2$ be integers. Then we have $(n + k + 1)/2 \leq g(\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}) \leq 2(n + k) + 1$.*

Theorem 1.2 improves Theorem 2.3 (4) and (5).

3. Proof of Theorem 1.1

First of all, we show a proposition used in proofs of Theorem 1.1 and 1.2. For elements x and y in a group, let $[x, y] = xyx^{-1}y^{-1}$. For a real number a, $[a]$ is the maximal integer less than or equal to a.

Proposition 3.1. *Let $f_W: X_W \to S^2$ be the genus-g Lefschetz fibration with the monodromy $W = 1$, where W is as above, and let $a_1, b_1, \ldots, a_g, b_g$ be the generators of $\pi_1(\Sigma_g)$ as shown in* Fig. 3. *Then we have followings*:

(1) (*See* [8].) *Let* $U = WW^{t_{b_1}} \cdots W^{t_{b_g}}$, *then the fundamental group* $\pi_1(X_U)$ *of the Lefschetz fibration* X_U *has the following presentation*

$$\pi_1(X_U) = \begin{cases} \left\langle a_1, b_1, \dots, a_g, b_g \,\middle|\, \begin{array}{l} b_1, \dots, b_g, \\ a_1 a_g, \dots, a_{g/2} a_{(g+2)/2} \end{array} \right\rangle & \text{when } g \text{ is even}, \\ \left\langle a_1, b_1, \dots, a_g, b_g \,\middle|\, \begin{array}{l} b_1, \dots, b_g, \\ a_1 a_g, \dots, a_{(g-1)/2} a_{(g+3)/2}, \\ a_{(g+1)/2} \end{array} \right\rangle & \text{when } g \text{ is odd}, \end{cases}$$

and, the group $\pi_1(X_U)$ *is isomorphic to the free group of rank* $[g/2]$.
(2) *Let* $U' = WW^{t_{b_2}} \cdots W^{t_{b_{g-1}}}$, *then the fundamental group* $\pi_1(X_{U'})$ *of the Lefschetz fibration* $X_{U'}$ *has the following presentation*

$$\pi_1(X_{U'}) = \begin{cases} \left\langle a_1, b_1, \dots, a_g, b_g \,\middle|\, \begin{array}{l} [a_1, b_1], \\ b_2, \dots, b_{g-1}, \\ b_1 b_g, \\ a_1 a_g, \dots, a_{g/2} a_{(g+2)/2} \end{array} \right\rangle & \text{when } g \text{ is even}, \\ \left\langle a_1, b_1, \dots, a_g, b_g \,\middle|\, \begin{array}{l} [a_1, b_1], \\ b_2, \dots, b_{g-1}, \\ b_1 b_g, \\ a_1 a_g, \dots, a_{(g-1)/2} a_{(g+3)/2}, \\ a_{(g+1)/2} \end{array} \right\rangle & \text{when } g \text{ is odd}, \end{cases}$$

and, the group $\pi_1(X_{U'})$ *is isomorphic to the free product of the free group of rank* $([g/2] - 1)$ *with* $\mathbb{Z} \oplus \mathbb{Z}$.

Proof. Simple closed curves $B_0, \dots, B_g$ and a, b, c as shown in Fig. 4 can be described in $\pi_1(\Sigma_g)$, up to conjugation, as follows

- $B_{2k} = a_k b_{k+1} b_{k+2} \cdots b_{g-k-1} b_{g-k} c_{g-k} a_{g-k+1}$, where $0 \leq k \leq g/2$,
- $B_{2k+1} = a_{k+1} b_{k+1} b_{k+2} \cdots b_{g-k-1} b_{g-k} c_{g-k} a_{g-k}$, where $0 \leq k \leq g/2$,
- $a = a_{(g+1)/2}$, $b = c_{(g-1)/2} a_{(g+1)/2}$ and $c = c_{g/2}$,

where let $a_0 = a_{g+1} = 1$. In addition, note that $c_i = b_i^{-1} \cdots b_1^{-1} (a_1 b_1 a_1^{-1}) \cdots (a_i b_i a_i^{-1})$ up to conjugation, for $1 \leq i \leq g$. Since X_W has a section, by Proposition 2.1, we first obtain a presentation of $\pi_1(X_W)$ as follows.

$$\pi_1(X_W)=\begin{cases}\left\langle a_1, b_1, \dots, a_g, b_g \;\middle|\; \begin{array}{l} c_g,\ c_{g/2},\\ a_1a_g, \dots, a_{g/2}a_{(g+2)/2},\\ b_1a_gb_ga_g^{-1}, \dots, b_{g/2}a_{(g+2)/2}b_{(g+2)/2}a_{(g+2)/2}^{-1}\end{array}\right\rangle\\ \text{when } g \text{ is even,}\\ \left\langle a_1, b_1, \dots, a_g, b_g \;\middle|\; \begin{array}{l} c_g,\ a_{(g+1)/2},\ b_{(g+1)/2},\ c_{(g-1)/2},\\ a_1a_g, \dots, a_{(g-1)/2}a_{(g+3)/2},\\ b_1a_gb_ga_g^{-1}, \dots, b_{(g-1)/2}a_{(g+3)/2}b_{(g+3)/2}a_{(g+3)/2}^{-1}\end{array}\right\rangle\\ \text{when } g \text{ is odd.}\end{cases}$$

(We have that $\pi_1(X_W)$ is isomorphic to $\pi_1(\Sigma_{[g/2]})$.) Since each b_i intersects some B_j transversely at only one point, by Proposition 2.2, we obtain the claim. □

REMARK. From Proposition 3.1, we have followings.

- For $n \geq 1$, there are genus-$2n$ and $(2n+1)$ Lefschetz fibrations whose fundamental groups are isomorphic to the free group of rank n.
- For $n \geq 2$, there are genus-$(2n-2)$ and $(2n-1)$ Lefschetz fibrations whose fundamental groups are isomorphic to the free product of the free group of rank $(n-2)$ with $\mathbb{Z} \oplus \mathbb{Z}$.

Let Γ be a finitely presented group with a presentation $\Gamma = \langle g_1, \dots, g_n \mid r_1, \dots, r_k\rangle$ and let $l = \max_{1\leq i\leq k}\{l(r_i)\}$. For $g \geq n + l - 1$ and r_i, we construct a simple closed curve R_i on Σ_g as below.

At first, we construct a simple closed curve R in the case $n = 4$ and $r = g_2g_1g_2^2g_4^{-1}g_3^{-2}$ as an example. Note that $l(r) = 5$. Let x_1, x_2, x_3, x_4, x_5 be loops on Σ_g which are homotopic to a_2, a_1, a_2, a_4 and a_3, respectively, as shown in Fig. 5 (a). Let y_1, y_2, y_3, y_4 be loops on Σ_g which are homotopic to a_5, a_6, a_7, a_8, respectively, and let z_1, z_2, z_3, z_4 be loops on Σ_g which are homotopic to a_5, a_6, a_7, a_8, respectively, as shown in Fig. 5 (a). First we deform Σ_g around $y_1, z_1, \dots, y_4, z_4$ as shown in Fig. 5 (b). Then let D be a subsurface containing y_t and z_t which is surrounded by a simple closed curve on Σ_g as shown in Fig. 5 (b). Next, for $1 \leq t \leq 4$, we move y_t to the right side of x_t in D, and z_t to the left side of x_{t+1} in D, as shown in Fig. 5 (c). Let $\bar{R}$ be the loop as shown in Fig. 6 (a), and let $R = (\bar{R})t_{x_1}^{-1}t_{x_2}^{-1}t_{x_3}^{-2}t_{x_4}t_{x_5}^{2}$, as shown in Fig. 6 (b). Finally, we deform the surface so that $y_1, \dots, y_4$ and $z_1, \dots, z_4$ go back to their original position as shown in Fig. 6 (c).

In general, a loop R_i is constructed as follows. Let $r_i = g_{j(1)}^{m(1)} \cdots g_{j(l(r_i))}^{m(l(r_i))}$. For $1 \leq t \leq l(r_i)$, let x_t be a loop on Σ_g which is homotopic to $a_{j(t)}$. If $j(s) = j(s')$ for some $s < s'$, we put $x_{s'}$ to the right side of x_s. For $1 \leq t \leq l(r_i) - 1$, let y_t and z_t be loops on Σ_g which are homotopic to a_{n+t}, such that z_t is in the right side of y_t.

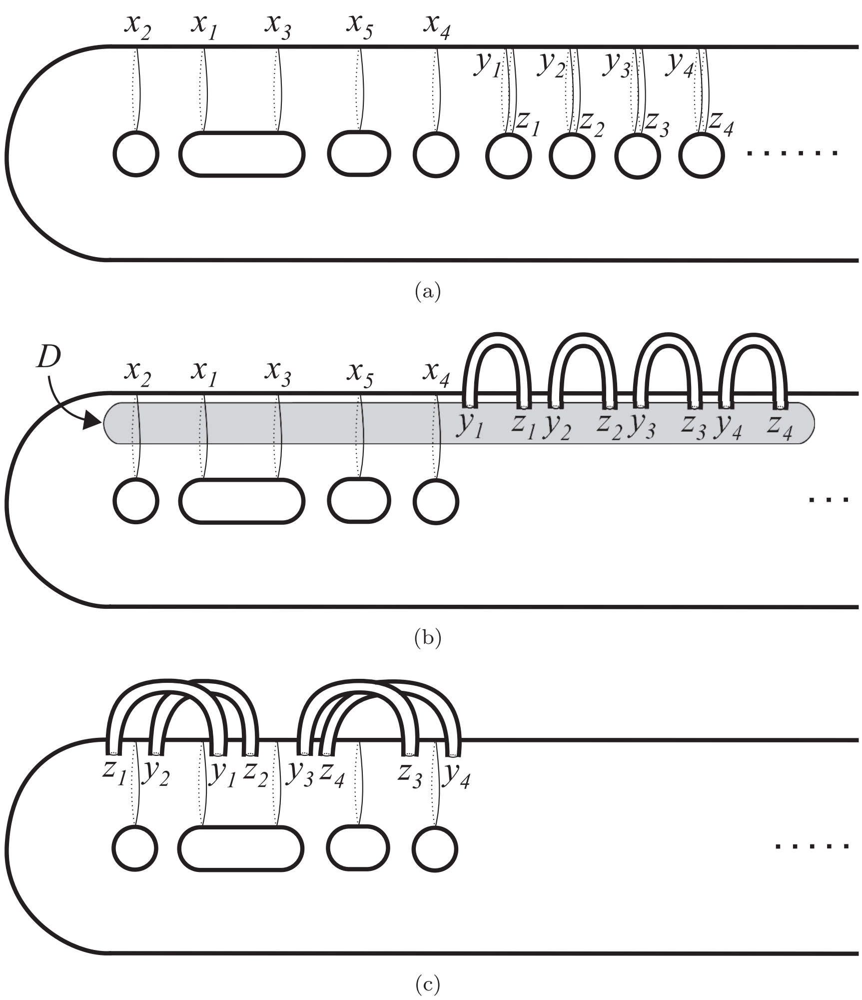
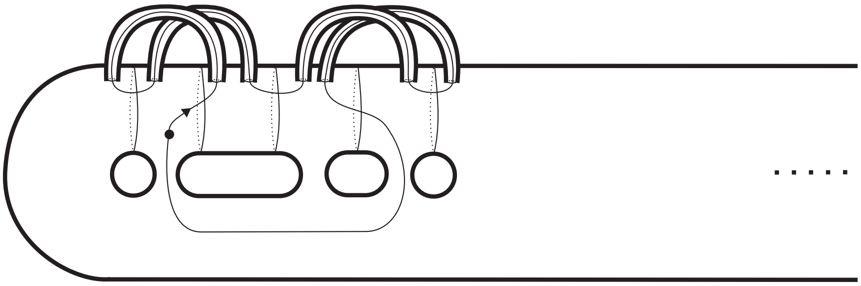
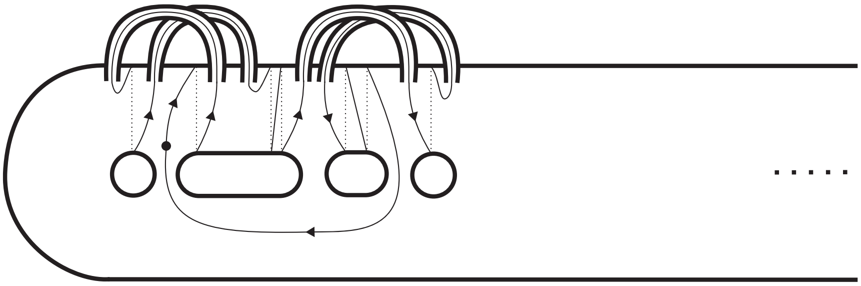


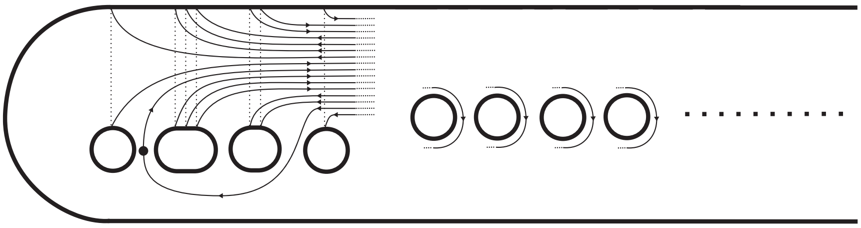
Fig. 5. The loop R in the case $n = 4$, $r = g_2 g_1 g_2^2 g_4^{-1} g_3^{-2}$.



(a)



(b) The loop R.



(c)

Fig. 6. The loop R in the case $n = 4$, $r = g_2 g_1 g_2^2 g_4^{-1} g_3^{-2}$.

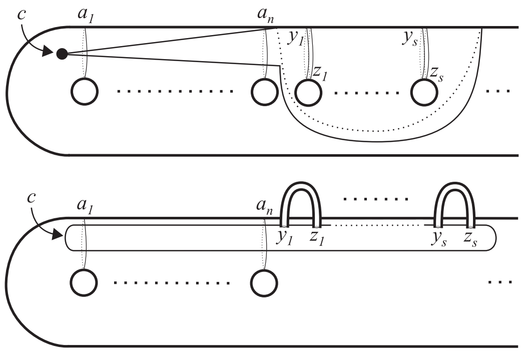


Fig. 7. The loop c where $s = l(r_i) - 1$.

First we deform Σ_g around $y_1, z_1, \dots, y_{l(r_i)-1}, z_{l(r_i)-1}$, similarly to the above example. Let c be a simple closed curve which is described in $\pi_1(\Sigma_g)$ as follows

$$c = (a_{n+1}b_{n+1}a_{n+1}^{-1}) \cdots (a_{n+l(r_i)-1}b_{n+l(r_i)-1}a_{n+l(r_i)-1}^{-1})b_{n+l(r_i)-1}^{-1} \cdots b_{n+1}^{-1},$$

and intersects each of $a_1, \dots, a_n$ at two points, as shown in Fig. 7. Then let D be a subsurface whose boundary is c, and which contains y_t and z_t.

Next we deform D as follows. For $1 \leq t \leq l(r_i) - 1$, we move y_t to just right side of x_t in D, and z_t to just left side of x_{t+1} in D as shown in Fig. 5 (c). We regard that this motion does not affect on loops a_i, b_i and c_i. Hence $x_1, \dots, x_{l(r_i)}$ also do not deform, as shown in Fig. 5 (c).

After that, we define a simple closed curve as shown in Fig. 6 (a). More precisely, we construct arcs L_i and L_i' as follows. The arc L_i is in D. L_i begins from the point at the left side of x_1 on the loop c, crosses $x_1, y_1, z_1, x_2, y_2, z_2, \dots$, in this order, finally crosses $x_{l(r_i)}$, and stops at the right side of $x_{l(r_i)}$ on the loop c. Let L_i' be an arc whose base point is the end point of L_i, end point is the base point of L_i, and which does not intersect the interior of D and loops $a_1, b_1, \dots, a_n, b_n$ and c_n. Note that the surface which is obtained by removing loops c, $a_1, b_1, \dots, a_n, b_n$ and c_n from Σ_g, and which contains L_i' is a disk. Hence the arc L_i' is unique up to homotopy relative to the base point and the end point. Let $L_i \cdot L_i'$ denote the composition of L_i and L_i'.

We now define $R_i = (L_i \cdot L_i')t_{x_1}^{-m(1)} \cdots t_{x_{l(r_i)}}^{-m(l(r_i))}$. Finally, we deform the surface so that $y_1, z_1, \dots, y_{l(r_i)-1}, z_{l(r_i)-1}$ go back to their original position.

Note that the loop R_i is described in $\pi_1(\Sigma_g)$, up to conjugation, as follows:

$$(*) \qquad R_i = \left(\prod_{1 \leq t \leq m(1)} x_{i,1,t}a_{j(1)} \right) \cdots \left(\prod_{1 \leq t \leq m(l(r_i))} x_{i,l(r_i),t}a_{j(l(r_i))} \right) \tilde{L}_i,$$

where $x_{i,s,t}$ is a loop which is some products of $a_{n+1}, b_{n+1}, \dots, a_{l(r_i)-1}, b_{l(r_i)-1}$ and c_{n+1}, and $\tilde{L}_i$ is a loop which is described in $\pi_1(\Sigma_g)$ as follows:

$$\tilde{L}_i = \begin{cases} b^{-1}_{j(l(r_i))} b^{-1}_{j(l(r_i))-1} \cdots b^{-1}_{j(1)+1} b^{-1}_{j(1)} & \text{when } j(1) \leq j(l(r_i)), \\ b_{j(l(r_i))+1} b_{j(l(r_i))} \cdots b_{j(1)} b_{j(1)-1} & \text{when } j(1) > j(l(r_i)). \end{cases}$$

We now prove Theorem 1.1.

Proof of Theorem 1.1. For $g \geq 2n + l - 1$, let V be the following

$$V = U W^{t_{a_{n+1}}} \cdots W^{t_{a_{[g/2]}}},$$

where $U = W W^{t_{b_1}} \cdots W^{t_{b_g}}$. In addition, let V' be the following

$$V' = V V^{t_{R_1}} \cdots V^{t_{R_k}},$$

where R_i is the loop constructed previously. We show that the fundamental group $\pi_1(X_{V'})$ is isomorphic to Γ.

Since each of $b_1, \dots, b_g$ and $a_{n+1}, \dots, a_{[g/2]}$ intersects some B_i transversely at only one point, by Proposition 2.2, we have

$$\begin{aligned}\pi_1(X_V) &= \pi_1(\Sigma_g)/\langle b_1, \dots, b_g, a_{n+1}, \dots, a_{[g/2]}\rangle \\ &= \pi_1(X_U)/\langle a_{n+1}, \dots, a_{[g/2]}\rangle.\end{aligned}$$

In addition, by the presentation of (1) of Proposition 3.1, we have

$$\pi_1(X_U) = \langle a_1, \dots, a_{[g/2]}\rangle.$$

Therefore we have

$$\begin{aligned}\pi_1(X_V) &= \langle a_1, \dots, a_{[g/2]} \mid a_{n+1}, \dots, a_{[g/2]}\rangle \\ &= \langle a_1, \dots, a_n\rangle,\end{aligned}$$

Because of the presentation of $\pi_1(X_U)$ in (1) of Proposition 3.1, we assume $g \geq 2n + l - 1$ in place of $g \geq n + l - 1$.

For any $1 \leq i \leq k$, consider the vanishing cycle $((B_0)t_{a_{n+1}})t_{R_i}$ of $X_{V'}$. Note that $(B_0)t_{a_{n+1}}$ and $(a_{n+1})t_{R_i}$ are described in $\pi_1(\Sigma_g)$, up to conjugation, as follows:

- $(B_0)t_{a_{n+1}} = a_{n+1}(b_1 \cdots b_g)$,
- $(a_{n+1})t_{R_i} = a_{n+1}(z R_i z^{-1})$ for some $z \in \pi_1(\Sigma_g)$.

Then, we have that $((B_0)t_{a_{n+1}})t_{R_i}$ is described in $\pi_1(\Sigma_g)$ as follows:

$$\begin{aligned}((B_0)t_{a_{n+1}})t_{R_i} &= (x \cdot a_{n+1}(b_1 \cdots b_n) \cdot x^{-1})t_{R_i} \\ &= (x)t_{R_i}(a_{n+1})t_{R_i}(b_1 \cdots b_n)t_{R_i}(x^{-1})t_{R_i} \\ &= (x)t_{R_i}(y \cdot a_{n+1}(z R_i z^{-1}) \cdot y^{-1})(w \cdot (B_0)t_{R_i} \cdot w^{-1})((x)t_{R_i})^{-1},\end{aligned}$$

for some elements x, y and w in $\pi_1(\Sigma_g)$. Since $a_{n+1} = (B_0)t_{R_i} = 1$ in $\pi_1(X_{V'})$, we have $R_i = 1$ from $((B_0)t_{a_{n+1}})t_{R_i} = 1$, in $\pi_1(X_{V'})$. For a vanishing cycle c of X_V, if R_i intersects c transversely at s points, then the vanishing cycle $(c)t_{R_i}$ of $X_{V'}$ is described in $\pi_1(\Sigma_g)$, up to conjugation, as follows:

$$(c)t_{R_i} = x_1 R_i^{\varepsilon_1} \cdots x_s R_i^{\varepsilon_s} x_{s+1},$$

where $\varepsilon_j = \pm 1$ and $x_1, \dots, x_{s+1}$ are elements in $\pi_1(\Sigma_g)$ such that $c = x_1 \cdots x_{s+1}$. Since $R_i = 1$ and $c = 1$ in $\pi_1(X_{V'})$, we can delete the relation $(c)t_{R_i} = 1$ of $\pi_1(X_{V'})$. We now define $\hat{r}_i = a_{j(1)}^{m(1)} \cdots a_{j(l(r_i))}^{m(l(r_i))}$ for $r_i = g_{j(1)}^{m(1)} \cdots g_{j(l(r_i))}^{m(l(r_i))}$. Since $x_{i,s,t}$ and $\tilde{L}_i$ in the description $(*)$ of R_i are 1 in $\pi_1(X_{V'})$, the natural epimorphism $\pi_1(\Sigma_g) \twoheadrightarrow \pi_1(X_{V'})$ sends R_i to $\hat{r}_i$. Note that the vanishing cycles of $X_{V'}$ consist of c and $(c)t_{R_i}$ for all vanishing cycles c of X_V and $1 \leq i \leq k$. Therefore, we have

$$\begin{aligned} \pi_1(X_{V'}) &= \langle a_1, \dots, a_n \mid \hat{r}_1, \dots, \hat{r}_k \rangle \\ &\cong \Gamma. \end{aligned}$$

Thus, the proof of Theorem 1.1 is completed. □

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

4.1. Proof of (1) of Theorem 1.2. For $n \geq 2$, let B_n denote the n-strands braid group. The group B_n has a presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ and with relations

- $\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} = 1$, where $1 \leq i < j - 1 \leq n - 2$,
- $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} = 1$, where $1 \leq i \leq n - 2$.

Let $x = \sigma_1$ and $y = \sigma_1 \cdots \sigma_{n-1}$. Then B_n can be presented with generators x, y and with relations

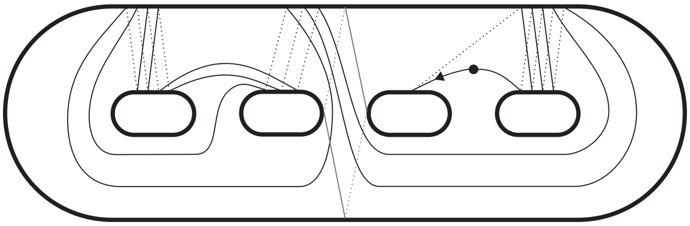
- $xy^k xy^{-k} x^{-1} y^k x^{-1} y^{-k} = 1$, where $2 \leq k \leq n - 2$,
- $xyxy^{-1}xyx^{-1}y^{-1}x^{-1}yx^{-1}y^{-1} = 1$,
- $(xy)^{n-1} y^{-n} = 1$.

A correspondence between the first presentation and the second presentation is given by $\sigma_i = y^{i-1} x y^{1-i}$ for $1 \leq i \leq n - 1$. See [8] for this presentation.

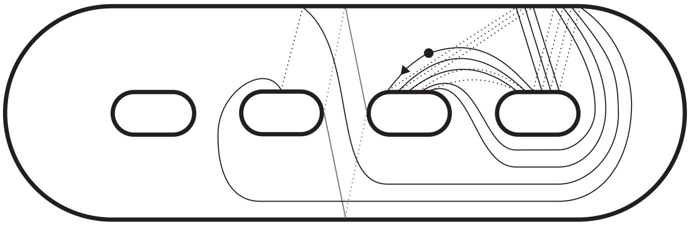
We now prove (1) of Theorem 1.2.

Proof of (1) of Theorem 1.2. For $n \geq 3$, since B_n is generated by two generators x, y, we have $g(B_n) \geq 2$ from (2) of Theorem 2.3 (cf. [8]). Therefore, we prove $g(B_n) \leq 4$ for $n \geq 3$.

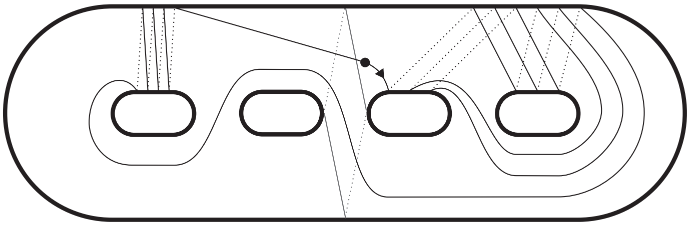
Let $R_{1,k}, R_2$ and $R_{3,n}$ be simple closed curves on Σ_4 as shown in Fig. 8, where $2 \leq k \leq n - 2$. Note that $R_{1,k}$, R_2 and $R_{3,n}$ intersect B_4 transversely at only one point, for $2 \leq k \leq n - 2$. Loops $R_{1,k}$, R_2 and $R_{3,n}$ can be described in $\pi_1(\Sigma_4)$, up to conjugation, as follows



(a) The loop $R_{1,k}$ with $k = 2$.



(b) The loop R_2.



(c) The loop $R_{3,n}$ with $n = 4$.

Fig. 8.

- $R_{1,k} = a_3^{-1}a_4^{-k}(b_3b_4)^{-1}a_2a_1^{-k}(b_1)a_2^{-1}(b_1b_2)^{-1}a_1^ka_2^{-1}(b_3b_4)a_4^k$, where $2 \leq k \leq n-2$,
- $R_2 = a_3^{-1}a_4^{-1}(b_4^{-1})a_3^{-1}a_4a_3^{-1}a_4^{-1}(b_2b_3b_4)^{-1}a_2^{-1}(b_3b_4)a_4a_3a_4^{-1}a_3(b_4)a_4$,
- $R_{3,n} = (a_3^{-1}a_4^{-1}(b_4^{-1}))^{n-1}(b_1b_3)^{-1}a_1^{-n}$.

Let V_1 be the following:

$$V_1 = WW^{t_{b_1}}W^{t_{b_2}}W^{t_{b_3}}W^{t_{b_4}}\left(\prod_{2\leq k\leq n-2}W^{t_{R_{1,k}}}\right)W^{t_{R_2}}W^{t_{R_{3,n}}}.$$

Then, from Proposition 2.2 and (1) of Proposition 3.1, the fundamental group $\pi_1(X_{V_1})$ can be presented with generators a_2, a_1 and with relations

- $a_2a_1^ka_2a_1^{-k}a_2^{-1}a_1^ka_2^{-1}a_1^{-k} = 1$, where $2 \leq k \leq n-2$,
- $a_2a_1a_2a_1^{-1}a_2a_1a_2^{-1}a_1^{-1}a_2^{-1}a_1a_2^{-1}a_1^{-1} = 1$,
- $(a_2a_1)^{n-1}a_1^{-n} = 1$.

Let $a_2 = x$ and $a_1 = y$. Then it follows that $\pi_1(X_{V_1})$ is isomorphic to B_n. Therefore, for $n \geq 3$ we have $g(B_n) \leq 4$.

Thus, the proof of (1) of Theorem 1.2 is completed. □

4.2. Proof of (2) of Theorem 1.2. For $g \geq 1$, let $\mathcal{H}_g$ be the hyperelliptic mapping class group of Σ_g, that is, a subgroup of the mapping class group $\mathcal{M}_g$ which consists of elements commutative with a hyperelliptic involution. It is well known that there is the natural epimorphism $B_{2g+2} \twoheadrightarrow \mathcal{H}_g$. For $g \geq 2$, Birman and Hilden [2] gave a presentation of the group $\mathcal{H}_g$ with generators $\sigma_1, \dots, \sigma_{2g+1}$ and with relations

- $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} = 1$, where $1 \leq i < j-1 \leq 2g$,
- $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = 1$, where $1 \leq i \leq 2g$,
- $(\sigma_1 \cdots \sigma_{2g+1})^{2g+2} = 1$,
- $(\sigma_1 \cdots \sigma_{2g+1}\sigma_{2g+1} \cdots \sigma_1)^2 = 1$,
- $[\sigma_1 \cdots \sigma_{2g+1}\sigma_{2g+1} \cdots \sigma_1, \sigma_1] = 1$.

Similarly to Subsection 4.1, let $x = \sigma_1$ and $y = \sigma_1 \cdots \sigma_{2g+1}$. Then, note that $y^{2g+2} = 1$. We calculate

$$\begin{aligned}\sigma_1 \cdots \sigma_{2g+1}\sigma_{2g+1} \cdots \sigma_1 &= y(y^{2g}xy^{-2g}) \cdots (yxy^{-1})x \\ &= y^{2g+1}(xy^{-1})^{2g}x \\ &= y^{-1}(xy^{-1})^{2g}x \\ &= (y^{-1}x)^{2g+1}.\end{aligned}$$

Then we have $(\sigma_1 \cdots \sigma_{2g+1}\sigma_{2g+1} \cdots \sigma_1)^2 = (y^{-1}x)^{4g+2}$. In addition, we have

$$\begin{aligned}[\sigma_1 \cdots \sigma_{2g+1}\sigma_{2g+1} \cdots \sigma_1, \sigma_1] &= (y^{-1}x)^{2g+1}x(x^{-1}y)^{2g+1}x^{-1} \\ &= (y^{-1}x)^{2g+1}(yx^{-1})^{2g+1}.\end{aligned}$$

Therefore, $\mathcal{H}_g$ can be presented with generators x, y and with relations

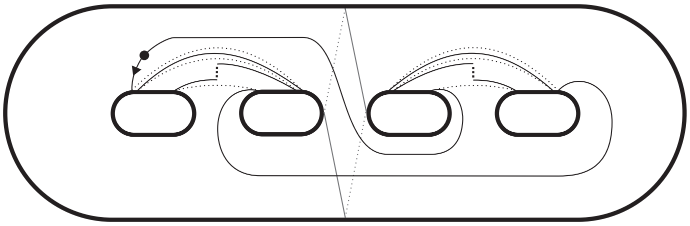


Fig. 9. The loop R_6.

- $xy^kxy^{-k}x^{-1}y^kx^{-1}y^{-k} = 1$, where $2 \le k \le 2g$,
- $xyxy^{-1}xyx^{-1}y^{-1}x^{-1}yx^{-1}y^{-1} = 1$,
- $(xy)^{2g+1}y^{-2g-2} = 1$,
- $y^{2g+2} = 1$,
- $(y^{-1}x)^{4g+2} = 1$,
- $(y^{-1}x)^{2g+1}(yx^{-1})^{2g+1} = 1$.

We now prove (2) of Theorem 1.2.

Proof of (2) of Theorem 1.2. For $g \ge 2$, since $\mathcal{H}_g$ is generated by two generators x, y, we have $g(\mathcal{H}_g) \ge 2$ from (2) of Theorem 2.3 (cf. [8]). Therefore, we prove $g(\mathcal{H}_g) \le 4$ for $g \ge 2$.

Let R_4, R_5 and R_6 be simple closed curves on Σ_4 described in $\pi_1(\Sigma_4)$, up to conjugation, as follows

- $R_4 = a_1^{2g+2}(b_1^{-1})$,
- $R_5 = (a_1^{-1}a_2)^{4g+2}(b_1^{-1})$,
- $R_6 = (a_1^{-1}a_2)^{2g+1}(b_2b_3b_4)(a_4^{-1}a_3)^{2g+1}(b_3^{-1})$.

For the loop R_6, see Fig. 9. Note that R_4, R_5 and R_6 intersect B_2, B_1 and B_4 transversely at only one point, respectively. Let V_2 be the following:

$$V_2 = WW^{t_{b_1}}W^{t_{b_2}}W^{t_{b_3}}W^{t_{b_4}}\left(\prod_{2\le k\le 2g} W^{t_{R_{1,k}}}\right)W^{t_{R_2}}W^{t_{R_{3,2g+2}}}W^{t_{R_4}}W^{t_{R_5}}W^{t_{R_6}}.$$

Then, from Proposition 2.2 and (1) of Proposition 3.1, the fundamental group $\pi_1(X_{V_2})$ can be presented with generators a_2, a_1 and with relations

- $a_2a_1^ka_2a_1^{-k}a_2^{-1}a_1^ka_2^{-1}a_1^{-k} = 1$, where $2 \le k \le 2g$,
- $a_2a_1a_2a_1^{-1}a_2a_1a_2^{-1}a_1^{-1}a_2^{-1}a_1a_2^{-1}a_1^{-1} = 1$,
- $(a_2a_1)^{2g+1}a_1^{-2g-2} = 1$,
- $a_1^{2g+2} = 1$,
- $(a_1^{-1}a_2)^{4g+2} = 1$,
- $(a_1^{-1}a_2)^{2g+1}(a_1a_2^{-1})^{2g+1} = 1$.

Let $a_2 = x$ and $a_1 = y$. Then it follows that $\pi_1(X_{V_2})$ is isomorphic to $\mathcal{H}_g$. Therefore, for $g \ge 2$ we have $g(\mathcal{H}_g) \le 4$. In particular, since the group $\mathcal{H}_1$ is isomorphic to $\mathcal{M}_1$, we have $2 \le g(\mathcal{H}_1) \le 4$ from (3) of Theorem 2.3 (cf. [8]).

Thus, the proof of (2) of Theorem 1.2 is completed. □

4.3. Proof of (3) of Theorem 1.2. For $n \ge 3$, let $\mathcal{M}_{0,n}$ denote the mapping class group of an n-punctured sphere, that is, the group of isotopy classes of orientation-preserving diffeomorphisms $S^2 \setminus \{p_1, \ldots, p_n\} \to S^2 \setminus \{p_1, \ldots, p_n\}$. Magnus [9] gave a presentation of the group $\mathcal{M}_{0,n}$ with generators $\sigma_1, \ldots, \sigma_{n-1}$ and with relations

- $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} = 1$, where $1 \le i < j-1 \le n-2$,
- $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = 1$, where $1 \le i \le n-2$,
- $(\sigma_1 \cdots \sigma_{n-1})^n = 1$,
- $\sigma_1 \cdots \sigma_{n-1}\sigma_{n-1} \cdots \sigma_1 = 1$.

Similarly to Subsection 4.1 and 4.2, let $x = \sigma_1$ and $y = \sigma_1 \cdots \sigma_{n-1}$. Then $\mathcal{M}_{0,n}$ can be presented with generators x, y and with relations

- $xy^kxy^{-k}x^{-1}y^kx^{-1}y^{-k} = 1$, where $2 \le k \le n-2$,
- $xyxy^{-1}xyx^{-1}y^{-1}x^{-1}yx^{-1}y^{-1} = 1$,
- $(xy)^{n-1}y^{-n} = 1$,
- $y^n = 1$,
- $(y^{-1}x)^{n-1} = 1$.

We now prove (3) of Theorem 1.2.

Proof of (3) of Theorem 1.2. For $n \ge 3$, since $\mathcal{M}_{0,n}$ is generated by two generators x, y, we have $g(\mathcal{M}_{0,n}) \ge 2$ from (2) of Theorem 2.3 (cf. [8]). Therefore, we prove $g(\mathcal{M}_{0,n}) \le 4$ for $n \ge 3$.

Let R_7 and R_8 be simple closed curves on Σ_4 described in $\pi_1(\Sigma_4)$, up to conjugation, as follows

- $R_7 = a_1^n(b_1^{-1})$,
- $R_8 = (a_1^{-1}a_2)^{n-1}(b_1^{-1})$.

Note that R_7 and R_8 intersect B_2 and B_1 transversely at only one point, respectively. Let V_3 be the following:

$$V_3 = V_1 W^{t_{R_7}} W^{t_{R_8}}.$$

Then, from Proposition 2.2 and (1) of Proposition 3.1, the fundamental group $\pi_1(X_{V_3})$ can be presented with generators a_2, a_1 and with relations

- $a_2a_1^ka_2a_1^{-k}a_2^{-1}a_1^ka_2^{-1}a_1^{-k} = 1$, where $2 \le k \le n-2$,
- $a_2a_1a_2a_1^{-1}a_2a_1a_2^{-1}a_1^{-1}a_2^{-1}a_1a_2^{-1}a_1^{-1} = 1$,
- $(a_2a_1)^{n-1}a_1^{-n} = 1$,
- $a_1^n = 1$,
- $(a_1^{-1}a_2)^{n-1} = 1$.

Let $a_2 = x$ and $a_1 = y$. Then it follows that $\pi_1(X_{V_3})$ is isomorphic to $\mathcal{M}_{0,n}$. Therefore, for $n \ge 3$ we have $g(\mathcal{M}_{0,n}) \le 4$.

Thus, the proof of (3) of Theorem 1.2 is completed. □

4.4. Proof of (4) of Theorem 1.2. For $n \geq 3$, let S_n denote the n-symmetric group. It is well known that the group S_n has a presentation with generators $\sigma_1,\dots,\sigma_{n-1}$ and with relations

- $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} = 1$, where $1 \leq i < j-1 \leq n-2$,
- $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = 1$, where $1 \leq i \leq n-2$,
- $\sigma_i^2 = 1$, where $1 \leq i \leq n-1$.

Similarly to Subsection 4.1, let $x = \sigma_1$ and $y = \sigma_1 \cdots \sigma_{n-1}$. Since $\sigma_i = y^{i-1}xy^{1-i}$, $\sigma_i^2 = 1$ if and only if $x^2 = 1$. Therefore S_n can be presented with generators x, y and with relations

- $xy^kxy^{-k}x^{-1}y^kx^{-1}y^{-k} = 1$, where $2 \leq k \leq n-2$,
- $xyxy^{-1}xyx^{-1}y^{-1}x^{-1}yx^{-1}y^{-1} = 1$,
- $(xy)^{n-1}y^{-n} = 1$,
- $x^2 = 1$.

We now prove (4) of Theorem 1.2.

Proof of (4) of Theorem 1.2. For $n \geq 3$, since S_n is generated by two generators x, y, we have $g(S_n) \geq 2$ from (2) of Theorem 2.3 (cf. [8]). Therefore, we prove $g(S_n) \leq 4$ for $n \geq 3$.

Let R_9 be the simple closed curve on Σ_4 described in $\pi_1(\Sigma_4)$, up to conjugation, as follows

- $R_9 = a_2^2(b_2^{-1})$.

Note that R_9 intersects B_4 transversely at only one point. Let V_4 be the following:

$$V_4 = V_1 W^{t_{R_9}}.$$

Then, from Proposition 2.2 and (1) of Proposition 3.1, the fundamental group $\pi_1(X_{V_4})$ can be presented with generators a_2, a_1 and with relations

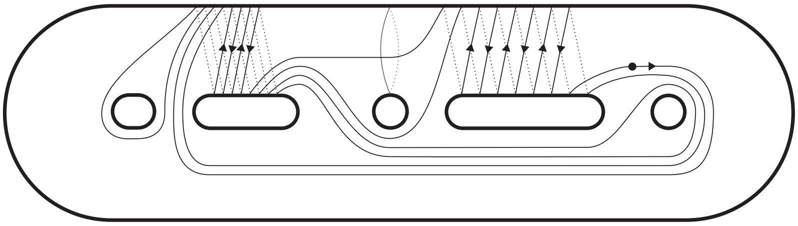
- $a_2a_1^ka_2a_1^{-k}a_2^{-1}a_1^ka_2^{-1}a_1^{-k} = 1$, where $2 \leq k \leq n-2$,
- $a_2a_1a_2a_1^{-1}a_2a_1a_2^{-1}a_1^{-1}a_2^{-1}a_1a_2^{-1}a_1^{-1} = 1$,
- $(a_2a_1)^{n-1}a_1^{-n} = 1$,
- $a_2^2 = 1$.

Let $a_2 = x$ and $a_1 = y$. Then it follows that $\pi_1(X_{V_4})$ is isomorphic to S_n. Therefore, for $n \geq 3$ we have $g(S_n) \leq 4$.

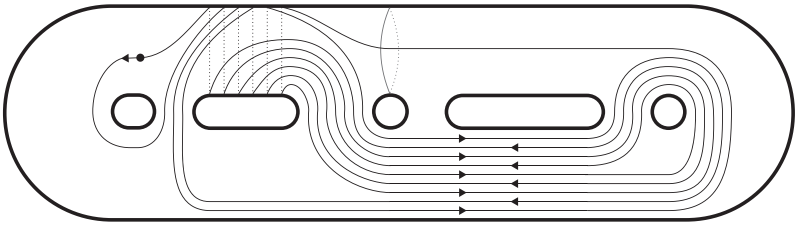
Thus, the proof of (4) of Theorem 1.2 is completed. □

4.5. Proof of (5) of Theorem 1.2. The Artin group is introduced by [3]. For $n \geq 6$, the n-Artin group $\mathcal{A}_n$ associated to the Dynkin diagram shown in Fig. 1 is defined by a presentation with generators $\sigma_1, \dots, \sigma_{n-1}, \tau$ and with relations

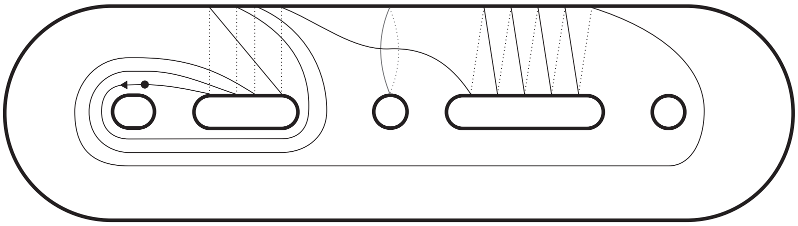
- $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} = 1$, where $1 \leq i < j-1 \leq n-2$,
- $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = 1$, where $1 \leq i \leq n-2$,
- $\sigma_4\tau\sigma_4\tau^{-1}\sigma_4^{-1}\tau^{-1} = 1$,
- $\tau\sigma_i\tau^{-1}\sigma_i^{-1} = 1$, where $1 \leq i \leq n-1$ with $i \neq 4$.



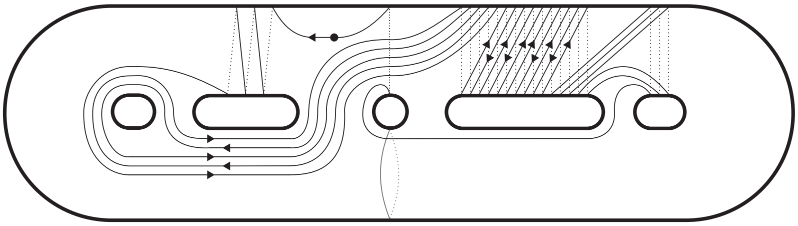
(a) The loop $R_{1,k}$ with $k = 2$.



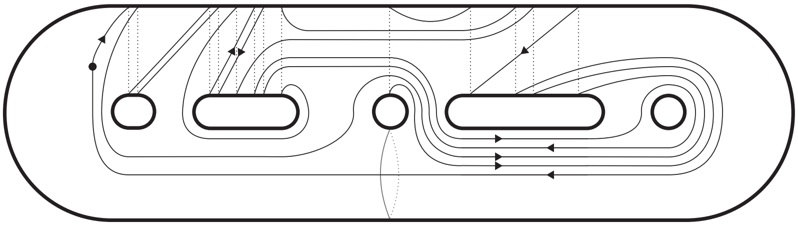
(b) The loop R_2.



(c) The loop R_3 with $n = 3$.



(d) The loop R_4.



(e) The loop $R_{5,i}$ with $i = 3$.

Fig. 10.

It is known that there is the natural epimorphism $\mathcal{A}_{2g+1} \twoheadrightarrow \mathcal{M}_g$. Similarly to Subsection 4.1, let $x = \sigma_1$ and $y = \sigma_1 \cdots \sigma_{n-1}$. In addition, let $z = \tau$. Then the group $\mathcal{A}_n$ can be presented with generators x, y, z and with relations

- $xy^k xy^{-k} x^{-1} y^k x^{-1} y^{-k} = 1$, where $2 \leq k \leq n-2$,
- $xyxy^{-1}xyx^{-1}y^{-1}x^{-1}yx^{-1}y^{-1} = 1$,
- $(xy)^{n-1}y^{-n} = 1$,
- $(y^3xy^{-3})z(y^3xy^{-3})z^{-1}(y^3x^{-1}y^{-3})z^{-1} = 1$,
- $z(y^{i-1}xy^{1-i})z^{-1}(y^{i-1}x^{-1}y^{1-i}) = 1$, where $1 \leq i \leq n-1$ with $i \neq 4$.

We now prove (5) of Theorem 1.2.

Proof of (5) of Theorem 1.2. Since $\mathcal{A}_n$ is generated by three generators x, y and z, we have $g(\mathcal{A}_n) \geq 2$ from (2) of Theorem 2.3 (cf. [8]). Therefore, we prove $g(\mathcal{A}_n) \leq 5$.

Let $R_{1,k}$, R_2, R_3, R_4 and $R_{5,i}$ be simple closed curves on Σ_5 as shown in Fig. 10, where $2 \leq k \leq n-2$ and $2 \leq i \leq n-1$ with $i \neq 4$. Note that we can not consider the loop $R_{5,1}$. Note that $R_{1,k}$, R_2 and R_3 intersect a transversely at only one point, for $2 \leq k \leq n-2$, and that R_4 and $R_{5,i}$ intersect b transversely at only one point, for $2 \leq i \leq n-1$ with $i \neq 4$. Loops $R_{1,k}$, R_2, R_3, R_4 and $R_{5,i}$ can be described in $\pi_1(\Sigma_5)$, up to conjugation, as follows

- $R_{1,k} = b_5^{-1}(b_2b_3b_4)^{-1}a_2^k(b_3b_4)b_5^{-1}(b_3b_4)^{-1}a_2^{-k}(b_2b_3b_4)b_5a_4^{-2k}(b_3^{-1})a_2^{-k}b_1^{-1}a_2^ka_4^{2k}$, where $2 \leq k \leq n-2$,
- $R_2 = b_1a_2(b_3b_4)b_5^{-1}(b_3b_4)^{-1}a_2^{-1}(b_3b_4)b_5^{-1}(b_2b_3b_4)^{-1}a_2(b_3b_4)b_5(b_3b_4)^{-1}a_2^{-1}(b_2b_3b_4) \times b_5a_2(b_3b_4)b_5(b_3b_4)^{-1}a_2^{-1}$,
- $R_3 = (b_1(b_2)a_2)^{n-1}(b_1(b_2b_3b_4)b_5)a_4^{n+2}a_2^2$,
- $R_4 = a_2^3b_1(b_2)a_4^3a_5^{-1}a_4^{-3}(b_2^{-1})b_1(b_2)a_4^3a_5a_4^{-3}(b_2^{-1})b_1^{-1}(b_2)a_4^3a_5(a_3b_3b_4)^{-1}$,
- $R_{5,i} = a_1a_2^{i-1}(b_4)b_5^{-1}(b_4)a_2^{1-i}a_1^{-1}(b_1(b_2b_4)b_5)a_4^{1-i}(a_3b_4)b_5(a_4^{2-i}a_2^{2-i}(b_2))a_2^{-1}a_4^{i-2} \times (b_1(b_2b_3b_4)b_5)^{-1}$, where $2 \leq i \leq n-1$ with $i \neq 4$.

Let V_5 be the following:

$$V_5 = W W^{t_{b_2}} W^{t_{b_3}} W^{t_{b_4}} \left(\prod_{2 \leq k \leq n-2} W^{t_{R_{1,k}}} \right) W^{t_{R_2}} W^{t_{R_3}} W^{t_{R_4}} \left(\prod_{2 \leq i \leq n-1, i \neq 4} W^{t_{R_{5,i}}} \right).$$

Then, from Proposition 2.2 and (2) of Proposition 3.1, the fundamental group $\pi_1(X_{V_5})$ can be presented with generators b_1, a_2, a_1 and with relations

- $b_1a_2^kb_1a_2^{-k}b_1^{-1}a_2^kb_1^{-1}a_2^{-k} = 1$, where $2 \leq k \leq n-2$,
- $b_1a_2b_1a_2^{-1}b_1a_2b_1^{-1}a_2^{-1}b_1^{-1}a_2b_1^{-1}a_2^{-1} = 1$,
- $(b_1a_2)^{n-1}a_2^{-n} = 1$,
- $(a_2^3b_1a_2^{-3})a_1(a_2^3b_1a_2^{-3})a_1^{-1}(a_2^3b_1^{-1}a_2^{-3})a_1^{-1} = 1$,
- $a_1(a_2^{i-1}b_1a_2^{1-i})a_1^{-1}(a_2^{i-1}b_1^{-1}a_2^{1-i}) = 1$, where $2 \leq i \leq n-1$ with $i \neq 4$,
- $a_1b_1a_1^{-1}b_1^{-1}$.

Let $b_1 = x$, $a_2 = y$ and $a_1 = z$. Then $\pi_1(X_{V_5})$ is isomorphic to $\mathcal{A}_n$. Therefore, for $n \geq 6$ we have $g(\mathcal{A}_n) \leq 5$.

Thus, the proof of (5) of Theorem 1.2 is completed. □

4.6. Proof of (6) of Theorem 1.2.

Proof of (6) of Theorem 1.2. Let $n, k \geq 0$ be integers with $n+k \geq 3$.

At first, we consider the case $n+k$ is even. We put $n+k=2r$. Let $A_{i,j}$ and $B_{i,j}$ be simple closed curves on Σ_{n+k+1} as shown in (a) and (b) of Fig. 11, respectively, where $1 \leq i < j \leq r$, and let $C_{i,j}$ be the simple closed curve on Σ_{n+k+1} as shown in (c), (d) and (e) of Fig. 11, where $1 \leq i, j \leq r$. Note that each of $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ intersects a_{r+1} transversely at only one point. Loops $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ can be described in $\pi_1(\Sigma_{n+k+1})$, up to conjugation, as follows

- $A_{i,j} = a_i a_j^{-1} a_{2r-i+2} a_{2r-j+2}^{-1} (c_{r+1}^{-1} b_{r+1}^{-1})$, where $1 \leq i < j \leq r$,
- $B_{i,j} = b_i b_j b_i^{-1} a_{2r-j+2} b_{2r-j+2} a_{2r-j+2}^{-1} (b_{r+1}^{-1} c_r)$, where $1 \leq i < j \leq r$,
- $C_{i,j} = a_i b_j^{-1} a_i^{-1} a_{2r-j+2} b_{2r-j+2}^{-1} a_{2r-j+2}^{-1} (a_{r+1} b_{r+1}^{-1})$, where $1 \leq i, j \leq r$ and $i \neq j$,
- $C_{i,i} = b_i^{-1} a_i b_i a_i^{-1} (b_{r+1}^{-1})$, where $1 \leq i \leq r$.

Let V_6 be the following:

$$V_6 = W\left(\prod_{1\leq i<j\leq r} W^{t_{A_{i,j}}}\right)\left(\prod_{1\leq i<j\leq r} W^{t_{B_{i,j}}}\right)\left(\prod_{1\leq i,j\leq r} W^{t_{C_{i,j}}}\right).$$

Note that we have relations $a_{r+1}=1$, $b_{r+1}=1$, $c_r=1$ and $c_{r+1}=1$ in $\pi_1(X_W)$. In addition, we have the relation $a_{2r-j+2} b_{2r-j+2} a_{2r-j+2}^{-1} = b_j^{-1}$ in $\pi_1(X_W)$ (see the presentation of $\pi_1(X_W)$ in the proof of Proposition 3.1). Then, from Proposition 2.2, the fundamental group $\pi_1(X_{V_6})$ can be presented with generators $a_1, b_1, \ldots, a_r, b_r$ and with relations

- $a_i a_j^{-1} a_i^{-1} a_j$, where $1 \leq i < j \leq r$,
- $b_i b_j b_i^{-1} b_j^{-1}$, where $1 \leq i < j \leq r$,
- $a_i b_j^{-1} a_i^{-1} b_j$, where $1 \leq i, j \leq r$ and $i \neq j$,
- $b_i^{-1} a_i b_i a_i^{-1}$, where $1 \leq i \leq r$.

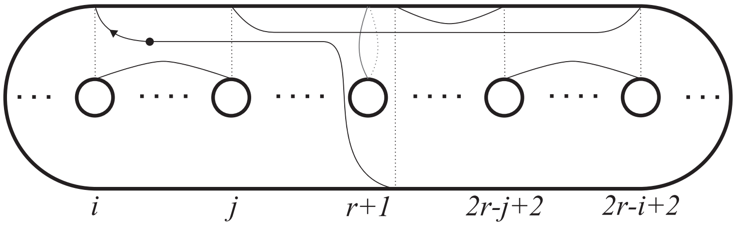
Namely, $\pi_1(X_{V_6})$ is isomorphic to $\mathbb{Z}^{2r}$. We next consider the simple closed curve $R_i^{m_i}$ on Σ_{n+k+1} as shown in Fig. 12, where $1 \leq i \leq 2r$ and $m_i \geq 2$. Note that $R_i^{m_i}$ intersects a_{r+1} transversely at only one point. Loops $R_i^{m_i}$ can be described in $\pi_1(\Sigma_{n+k+1})$, up to conjugation, as follows

- $R_i^{m_i} = a_i^{m_i} (a_{2r-i+2} b_{2r-i+2}^{-1} a_{2r-i+2}^{-1} a_{r+1} b_{r+1}^{-1} b_i^{-1})$, where $1 \leq i \leq r$,
- $R_{r+i}^{m_{r+i}} = b_i^{m_{r+i}} (a_i^{-1} a_{2r-i+2}^{-1} a_{r+1} b_{r+1}^{-1})$, where $1 \leq i \leq r$.

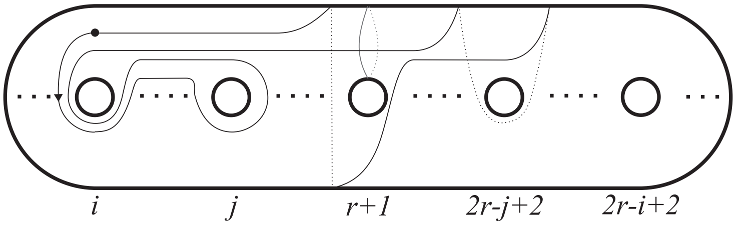
Let V_7 be the following:

$$V_7 = V_6\left(\prod_{1\leq i\leq k} W^{t_{R_i^{m_i}}}\right).$$

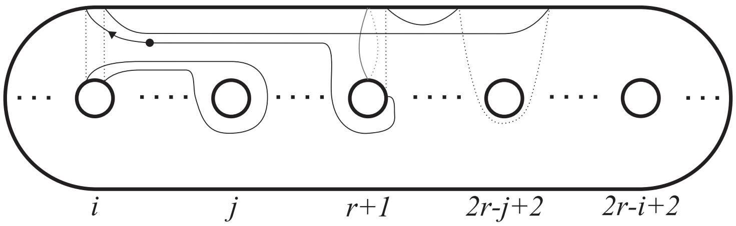
Then, from Proposition 2.2, the fundamental group $\pi_1(X_{V_7})$ is isomorphic to $\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$. Therefore, if $n+k$ is even, we have $g(\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}) \leq n+k+1$.



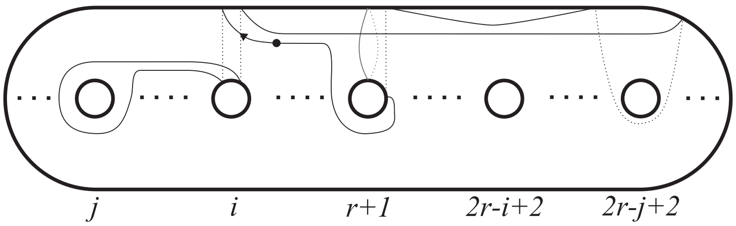
(a) The loop $A_{i,j}$, $1 \leq i < j \leq r$.



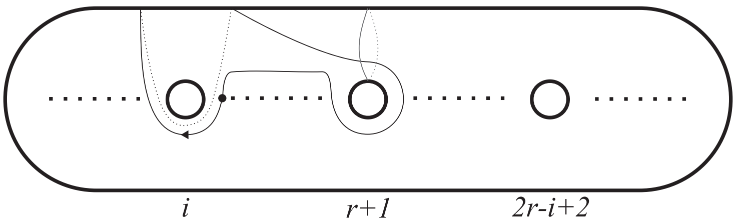
(b) The loop $B_{i,j}$, $1 \leq i < j \leq r$.



(c) The loop $C_{i,j}$, $1 \leq i < j \leq r$.

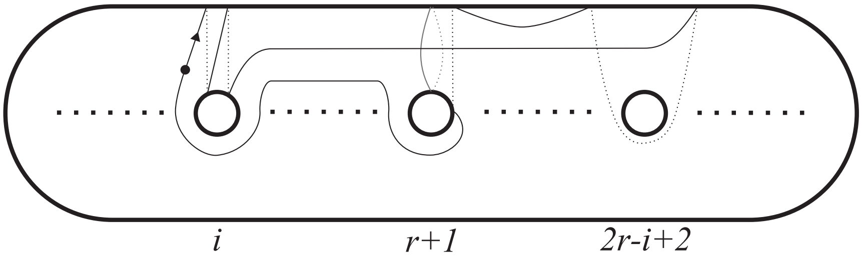


(d) The loop $C_{i,j}$, $1 \leq j < i \leq r$.

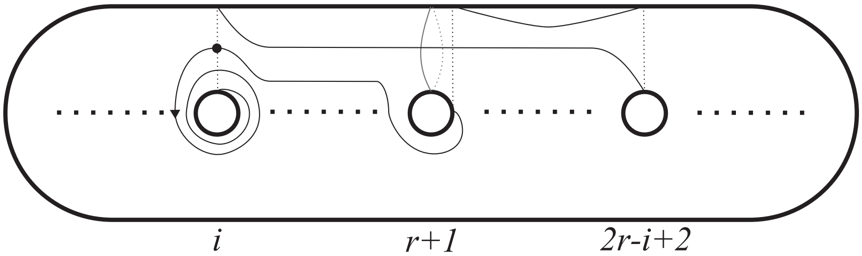


(e) The loop $C_{i,i}$, $1 \leq i \leq r$.

Fig. 11.



(a) The loop $R_i^{m_i}$ with $m_i = 2$, $1 \leq i \leq r$.



(b) The loop $R_{r+i}^{m_{r+i}}$ with $m_{r+i} = 2$, $1 \leq i \leq r$.

Fig. 12.

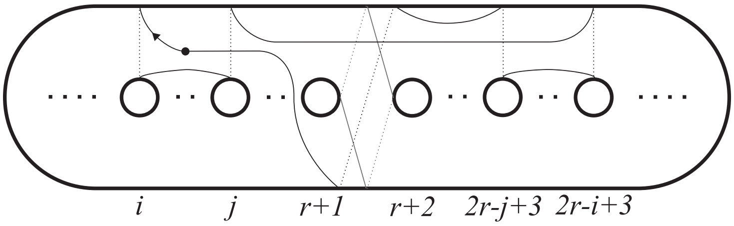
Next, we consider the case $n+k$ is odd. We put $n+k = 2r+1$. Let $A_{i,j}$ and $B_{i,j}$ be simple closed curves on Σ_{n+k+1} as shown in (a) and (b) of Fig. 13, respectively, where $1 \leq i < j \leq r$, and let $C_{i,j}$ be the simple closed curve on Σ_{n+k+1} as shown in (c), (d) and (e) of Fig. 13, where $1 \leq i, j \leq r$. In addition, let $A_{i,r+1}$ and $C_{r+1,i}$ be simple closed curves on Σ_{n+k+1} as shown in (a) and (b) of Fig. 14, where $1 \leq i \leq r$. Note that each of $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ intersects B_{2r+2} transversely at only one point. Loops $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ can be described in $\pi_1(\Sigma_{n+k+1})$, up to conjugation, as follows

- $A_{i,j} = a_i a_j^{-1} a_{2r-i+3} a_{2r-j+3}^{-1} (c_{r+1}^{-1} b_{r+1}^{-1})$, where $1 \leq i < j \leq r$,
- $A_{i,r+1} = a_i a_{r+1}^{-1} (b_{r+2}) a_{2r-i+3} (c_{r+2}) a_{r+1}$, where $1 \leq i \leq r$,
- $B_{i,j} = b_i b_j b_i^{-1} (b_{r+2}) a_{2r-j+3} b_{2r-j+3} a_{2r-j+3}^{-1} (b_{r+2}^{-1} b_{r+1} c_{r+1})$, where $1 \leq i < j \leq r$,
- $C_{i,j} = a_i b_j a_i^{-1} (b_{r+2}) a_{2r-j+3} b_{2r-j+3} a_{2r-j+3}^{-1} (b_{r+2}^{-1} b_{r+1} c_{r+1})$, where $1 \leq i, j \leq r$ and $i \neq j$,
- $C_{i,i} = b_i^{-1} a_i b_i a_i^{-1} (b_{r+1}^{-1})$, where $1 \leq i \leq r$,
- $C_{r+1,i} = a_{r+1} b_i a_{r+1}^{-1} (b_{r+2}) a_{2r-i+3} b_{2r-i+3} a_{2r-i+3}^{-1} (c_{r+2})$, where $1 \leq i \leq r$.

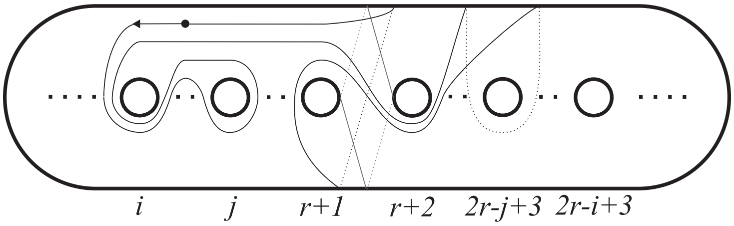
Let V_8 be the following:

$$V_8 = W W^{t_{b_{r+1}}} \left(\prod_{1 \leq i < j \leq r+1} W^{t_{A_{i,j}}} \right) \left(\prod_{1 \leq i < j \leq r} W^{t_{B_{i,j}}} \right) \left(\prod_{1 \leq i \leq r+1, 1 \leq j \leq r} W^{t_{C_{i,j}}} \right).$$

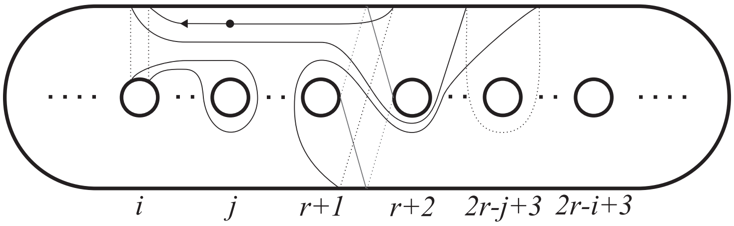
Since b_{r+1} intersects B_{2r+2} transversely at only one point, we have the relation $b_{r+1} = 1$



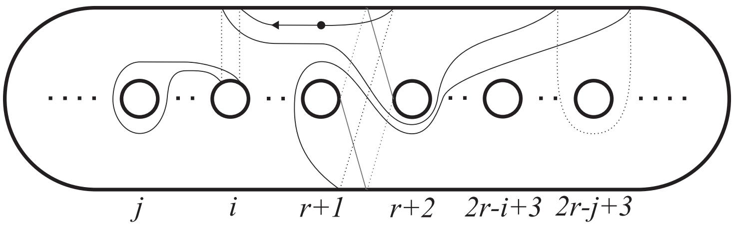
(a) The loop $A_{i,j}$, $1 \leq i < j \leq r$.



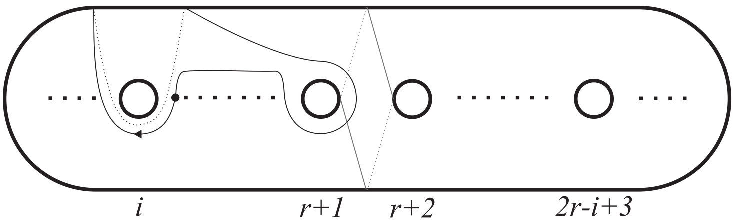
(b) The loop $B_{i,j}$, $1 \leq i < j \leq r$.



(c) The loop $C_{i,j}$, $1 \leq i < j \leq r$.

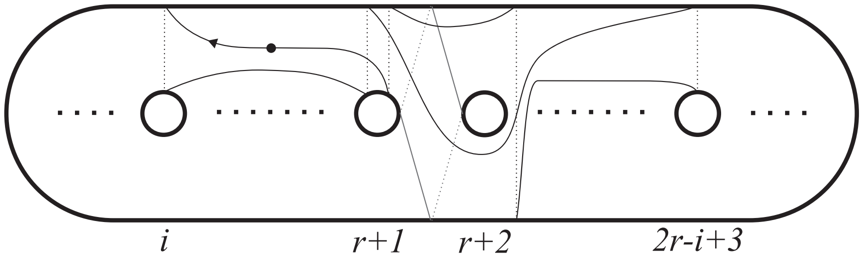


(d) The loop $C_{i,j}$, $1 \leq j < i \leq r$.

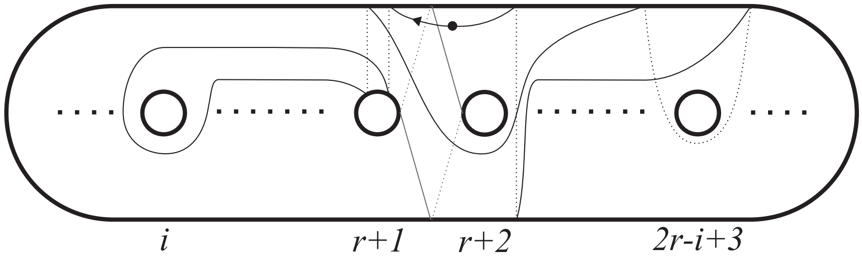


(e) The loop $C_{i,i}$, $1 \leq i \leq r$.

Fig. 13.



(a) The loop $A_{i,r+1}$, $1 \leq i \leq r$.



(b) The loop $C_{r+1,i}$, $1 \leq i \leq r$.

Fig. 14.

in $\pi_1(X_{WW^{t_{b_{r+1}}}})$ from Proposition 2.2. Hence we have relations $b_{r+2} = 1$ and $c_{r+2} = 1$ in $\pi_1(X_{WW^{t_{b_{r+1}}}})$. Then, from Proposition 2.2 and the presentation of $\pi_1(X_W)$ in the proof of Proposition 3.1, the fundamental group $\pi_1(X_{V_8})$ is isomorphic to an abelian generated by $a_1, b_1, \ldots, a_r, b_r$ and a_{r+1}. We next consider the simple closed curve $R_i^{m_i}$ on Σ_{n+k+1} as shown in Fig. 15, where $1 \leq i \leq 2r+1$ and $m_i \geq 2$. Note that $R_i^{m_i}$ intersects B_{2r+2} transversely at only one point. Loops $R_i^{m_i}$ can be described in $\pi_1(\Sigma_{n+k+1})$, up to conjugation, as follows

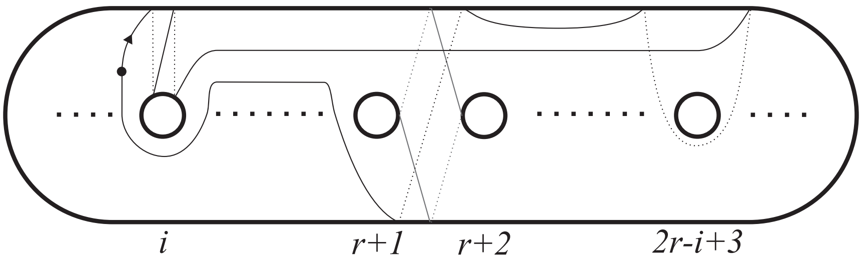
- $R_i^{m_i} = a_i^{m_i}(a_{2r-i+3}b_{2r-i+3}^{-1}a_{2r-i+3}^{-1}c_{r+1}^{-1}b_{r+1}^{-1}b_i^{-1})$, where $1 \leq i \leq r$,
- $R_{r+i}^{m_{r+i}} = b_i^{m_{r+i}}(a_i^{-1}a_{2r-i+3}^{-1}c_{r+1}^{-1}b_{r+1}^{-1})$, where $1 \leq i \leq r$,
- $R_{2r+1}^{m_{2r+1}} = a_{r+1}^{m_{2r+1}}(b_{r+1}^{-1})$.

Let V_9 be the following:

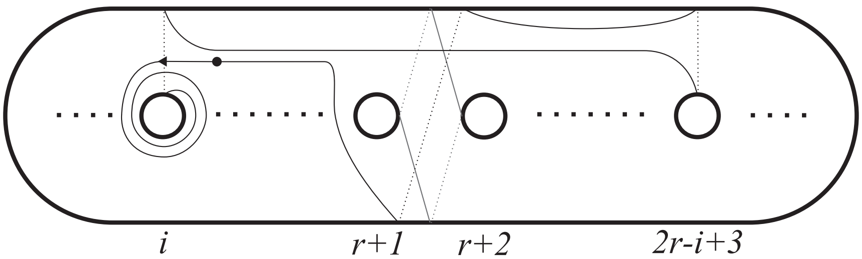
$$V_9 = V_8\left(\prod_{1\leq i\leq k} W^{t_{R_i^{m_i}}}\right).$$

Then, from Proposition 2.2, the fundamental group $\pi_1(X_{V_9})$ is isomorphic to $\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$. Therefore, if $n+k$ is odd, we have $g(\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}) \leq n+k+1$.

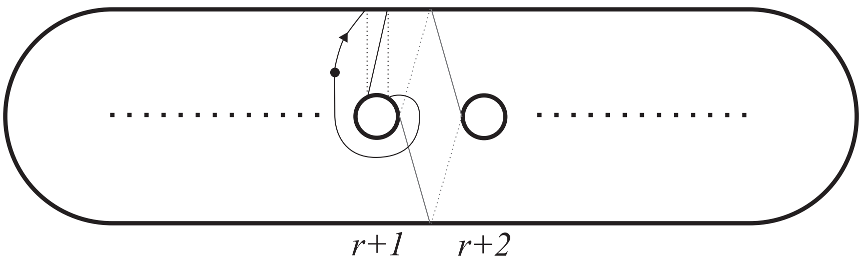
Moreover, it is immediately follows from Theorem 2.3 (2) or (5) (cf. [8]) that $g(\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}) \geq (n+k+1)/2$. Thus, the proof of (6) of Theorem 1.2 is completed. □



(a) The loop $R_i^{m_i}$ with $m_i = 2$, $1 \leq i \leq r$.



(b) The loop $R_{r+i}^{m_{r+i}}$ with $m_{r+i} = 2$, $1 \leq i \leq r$.



(c) The loop $R_{2r+1}^{m_{2r+1}}$ with $m_{2r+1} = 2$.

Fig. 15.

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