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ON FAMILIES OF COMPLEX CURVES OVER $\mathbb{P}^1$ WITH TWO SINGULAR FIBERS

CHENG GONG, JUN LU and SHENG-LI TAN

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Abstract

Let $f\colon S \to \mathbb{P}^1$ be a family of genus $g \geq 2$ curves with two singular fibers F_1 and F_2. We show that $F_1 = F_2{}^*$ and $F_2 = F_1{}^*$ are dual to each other, S is a ruled surface, the geometric genera of the singular fibers are equal to the irregularity of the surface, and the virtual Mordell–Weil rank of f is zero. We prove also that $c_1^2(S) \leq -2$ if $g = 2$, and $c_1^2(S) \leq -4$ if $g > 2$. As an application, we will classify all such fibrations of genus $g = 2$.

1. Introduction

It is well-known that a non-trivial family $f\colon S \to \mathbb{P}^1$ of complex curves of genus $g \geq 1$ admits at last two singular fibers. If f is non-isotrivial, then the number s of singular fibers is at least 3 ([8]). Furthermore, if f is semistable, then $s \geq 4$ ([8]), or $s \geq 5$ when $g > 1$ ([24]).

A very interesting problem is to classify all families $f\colon S \to \mathbb{P}^1$ with minimal number of singular fibers. Beauville [9] proves that there are exactly 6 families f of semistable elliptic curves with 4 singular fibers, and each family is modular. In [10], U. Schmickler Hirzebruch classified all elliptic fibrations f with two singular fibers F_1 and F_2. She proves that there are 5 such families, and in each family, $F_1 = F_2{}^*$ in Kodaira's notation. (See also [26] for the equations.)

For a fiber $F = f^{-1}(0)$ of genus $g \geq 1$, the dual fiber F^* is defined as follows (see [14], Definition 2.5). Let $\bar{F} = \sum_i n_i C_i$ be the normal-crossing model of F, let $M_F = \operatorname{lcm}\{n_i\}$ be the least common multiple of $\{n_i\}$, and n be any positive integer satisfying $n \equiv -1 \pmod{M_F}$. F^* is just the pullback fiber of F under the base change $t = w^n$. So the dual of F is not unique. When the semistable model of F is smooth, then F^* is unique. Two fibers F_1 and F_2 are said to be dual to each other if $F_1 = F_2^*$ and $F_2 = F_1^*$.

Let $F_1, \ldots, F_s$ be all singular fibers of a fibration $f\colon S \to C$, and let $l_i = l(F_i)$ be the number of irreducible components of F_i. When f has a section, the rank of the Mordell–Weil group of f is denoted by r. We have a formula to compute the rank r

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(see [22], Theorem 3),

$$r = \rho(S) - 2 - \sum_i (l(F_i) - 1),$$

where $\rho(S) = \operatorname{rank} \mathrm{NS}(S)$ is the Picard number of S. When f has no section, r is still defined by the formula above. In the general case, r is called the virtual Mordell–Weil rank of f by Nguyen ([19], Definition 0.2).

The purpose of this paper is to try to classify families $f: S \to \mathbb{P}^1$ of curves of genus $g \geq 2$ with exactly two singular fibers F_1 and F_2. First we need to give a numerical characterization of such families.

Theorem 1.1. *Let $f: S \to \mathbb{P}^1$ be a relatively minimal fibration of genus $g \geq 2$ with two singular fibers F_1 and F_2. Then F_1 and F_2 are dual to each other, i.e., $F_1 = {F_2}^*$ and $F_2 = {F_1}^*$.*
(1) *S is a ruled surface, and the geometric genera of the singular fibers are equal to the irregularity $q(S)$ of S, $g(F_1) = g(F_2) = q(S)$ (see Section 2).*
(2) *The virtual Mordell–Weil rank of f is zero.*
(3) *We have the following inequalities,*

$$c_1^2(S) \leq \begin{cases} -2, & g = 2, \\ -4, & g \geq 3. \end{cases}$$

EXAMPLE 1.1. The equation $y^2 = t(x^{g+1} - t)(x^{g+1} + t)$ defines a family $f: S \to \mathbb{P}^1$ of curves of genus g with two singular fibers.

$$c_1^2(S) = \begin{cases} -2, & g = 2, \\ -4, & g = 3, 4. \end{cases} \quad q(S) = \begin{cases} 0, & g = 2, 4, \\ 1, & g = 3. \end{cases}$$

As an application, we will classify all such fibrations of genus 2.

Theorem 1.2. *Let $f: S \to \mathbb{P}^1$ be a relatively minimal fibration of genus $g = 2$ with two singular fibers F and F^*. Then f is isomorphic to one of the following 11 families.*

No.	$F,\ F^*$ *in* [18]	*Families*	*Monodromies*
1	I*, I*	$y^2 = t(x^5 + ax^4 + bx^3 + cx^2 + x)$	$[I_3]$, $[I_3]$
2	II, II	$y^2 = x^6 + dx^4t + ex^2t^2 + t^3$	$[\phi_3^3]$, $[\phi_3^3]$
3	III, III	$y^2 = x^6 + dx^3t + t^2$	$[\phi_3^2]$, $[\phi_3^4]$,
4	IV, IV	$y^2 = (x^6 + dx^3t + t^2)t$	$[\phi_3^2 I_3]$, $[\phi_3^4 I_3]$
5	V, V*	$y^2 = x^6 + t$	$[\phi_3]$, $[\phi_3^5]$
6	VI, VI	$y^2 = x^5 + dtx^3 + t^2x$	$[\phi_2^2]$, $[\phi_2^6]$
7	VII, VII*	$y^2 = x^5 + xt$	$[\phi_2]$, $[\phi_2^7]$
8	VIII-1, VIII-4	$y^2 = x^5 + t$	$[\phi_1]$, $[\phi_1^9]$
9	VIII-2, VIII-3	$y^2 = t(x^5 + t^2)$	$[\phi_1^3]$, $[\phi_1^7]$
10	IX-1, IX-4	$y^2 = x^5 + t^2$	$[\phi_1^2]$, $[\phi_1^8]$
11	IX-2, IX-3	$y^2 = t(t + x^5)$	$[\phi_1^4]$, $[\phi_1^6]$

where $[\phi_i]$*'s and* $[I_3]$ *are defined in* [13], *satisfying*

$$[\phi_1^k] = [\phi_1^{k+5} I_3], \quad [\phi_2^k] = [\phi_2^{k+4} I_3] = [\phi_2^{3k}], \quad [\phi_3] = [\phi_3^5 I_3], \quad [\phi_3^2] = [\phi_3^4].$$

The duality of the two singular fibers in Theorem 1.1 is a consequence of Matsumoto–Montesinos' theory on the monodromy of degeneration of curves. The proof of (1) and (2) in Theorem 1.1 is based on a new formula and a new inequality on the Hodge number $h^{1,1}(S)$ obtained in [15]. In order to get the optimal upper bounds of the first Chern number $c_1^2(S)$, we use the local-global formula of Kodaira type obtained by the third author. The main part of the proof depends heavily on the classification of singular fibers according to their topological monodromies and Chern numbers.

2. Formulas for the invariants of fibrations

For a relatively minimal fibration $f\colon S \to C$ of genus g over a smooth curve C of genus b, it is convenient to use the relative numerical invariants of the fibration:

$$\begin{aligned} K_f^2 &= c_1^2(S) - 8(g-1)(b-1), \\ e_f &= c_2(S) - 4(g-1)(b-1), \\ \chi_f &= \chi(\mathcal{O}_S) - (g-1)(b-1), \\ q_f &= q(S) - g(C). \end{aligned}$$

We can compute e_f topologically. It is the sum of the topological contributions of the singular fibers:

$$e_f = \sum_F (\chi_{\mathrm{top}}(F) - (2-2g)),$$

where F runs over all singular fibers. The third author [23] gives a new formula for $e_F := \chi_{\mathrm{top}}(F) - (2-2g)$,

$$e_F = 2N_F + \mu_F,$$

where μ_F is the sum of the Milnor numbers of the singular points of F_{red}. $N_F = g - p_a(F_{\mathrm{red}})$ is an integer between 0 and g. $N_F = g$ iff F_{red} is a tree of smooth rational curves, and $N_F = 0$ iff F is reduced or $g = 1$ and F is of type ${}_m\mathrm{I}_n$.

Let $\Gamma_1, \ldots, \Gamma_k$ be all irreducible components of a given fiber F, and $\tilde{\Gamma}_i \to \Gamma_i$ be the normalization of Γ_i. $g(F) := \sum_{i=1}^k g(\tilde{\Gamma}_i)$ is called the *geometric genus* of F. We denote by $\bar{F} = \sigma^* F$ the *normal crossing model* of F, i.e., σ is the blowing-ups of the singular points of F such that $\bar{F} = \sigma^* F$ is a normal crossing divisor. $N_{\bar{F}} := g - p_a(\bar{F}_{\mathrm{red}})$. Note that

$$g \geq p_a(F_{\mathrm{red}}) \geq p_a(\bar{F}_{\mathrm{red}}) \geq g(F) \geq q_f,$$

the last inequality is due to Beauville (see [8], [15]). We get

$$0 \leq N_F \leq N_{\bar{F}} \leq g - q_f. \tag{1}$$

Note that $N_{\bar{F}} = g$, i.e., $p_a(\bar{F}_{\mathrm{red}}) = 0$, if and only if $\bar{F}$ is a tree of smooth rational curves. If F is semistable, then $F = \bar{F}$ and $N_F = 0$.

The relative invariants can be computed by using the modular invariants $\kappa(f)$, $\lambda(f)$ and $\delta(f)$.

$$\begin{cases} K_f^2 = \kappa(f) + \displaystyle\sum_{i=1}^s c_1^2(F_i), \\ e_f = \delta(f) + \displaystyle\sum_{i=1}^s c_2(F_i), \\ \chi_f = \lambda(f) + \displaystyle\sum_{i=1}^s \chi_{F_i}, \end{cases} \tag{2}$$

where $c_1^2(F)$, $c_2(F)$ and χ_F are the Chern numbers of the singular fiber F, which are nonnegative rational numbers, and each of them vanishes if and only if F is semistable (when $g \geq 2$) (see [23], [25] or [14]). So for a semistable fibration f,

$$K_f^2 = \kappa(f), \quad e_f = \delta(f), \quad \chi_f = \lambda(f).$$

If f is isotrivial, then $\kappa(f) = \delta(f) = \lambda(f) = 0$, so

$$
\tag{3}
\begin{cases}
c_1^2(X) = 8(g-1)(g(C)-1) + \sum_{i=1}^{s} c_1^2(F_i), \\
c_2(X) = 4(g-1)(g(C)-1) + \sum_{i=1}^{s} c_2(F_i), \\
\chi(\mathcal{O}_X) = (g-1)(g(C)-1) + \sum_{i=1}^{s} \chi_{F_i}.
\end{cases}
$$

We refer to [25], [26] and [14] for more properties of the Chern numbers $c_1^2(F)$ and $c_2(F)$.

Let $F_1, \ldots, F_{s_1}$ be all singular fibers satisfying $g(F_i) < g$. By [15, Theorem 1.4], we have the following new formula

$$
\tag{4}
\begin{aligned}
2\chi_f = (g-q_f)(2g(C)-2+s_1) - \sum_{i=1}^{s_1}(g(F_i)-q_f) \\
- \left(h^{1,1}(S) - 2g(C)q_f - 2 - \sum_{i=1}^{s}(l(F_i)-1)\right) + \sum_{i=1}^{s_1} N_{\bar{F}_i}
\end{aligned}
$$

and the following inequalities

$$
\tag{5}
\begin{cases}
g(F_i) - q_f \geq 0, \\
N_{\bar{F}_i} \leq g - q_f, \\
h^{1,1}(S) - 2g(C)q_f - 2 - \sum_{i=1}^{s}(l(F_i)-1) \geq 0.
\end{cases}
$$

3. Matsumoto–Montesinos' theory on the degeneration of curves

Let (f, F) be a fiber germ $f\colon S \to \Delta$ whose semistable model is smooth, let μ be a monodromy homeomorphism along a simple closed curve around $p = f(F) = 0 \in \Delta$ in a neighborhood of p, and let $[\mu]$ be the topological monodromy of (f, F), i.e., the conjugacy class of μ in the mapping class group of Riemann surface of genus g. In particular, $[\mu] = [\mathrm{id}]$ iff the central fiber F is smooth [17, Corollary 1.1]. Let $\tilde{F}$ be the d-th root model of F under a local base change of degree d totally ramified over p defined by $w = t^d$ (see [14]). Denote by $[\tilde{\mu}]$ the topological monodromy of the germ of $\tilde{F}$. It is well-known that $[\tilde{\mu}] = [\mu^d]$. If $[\mu^d] = [\mathrm{id}]$, then $[\mu]$ is periodic, which is equivalent to the fact that the semistable model of F must be smooth.

From Matsumoto–Montesinos' theory on degenerated Riemann surfaces [14, 15], one has a bijective map as follows:

$$\Phi\colon \mathcal{A} \to \mathcal{B}, \quad (f, F) \to [\mu],$$

where $\mathcal{A}$ is the set of topologically equivalent classes of fiber germs with smooth semi-stable models, and $\mathcal{B}$ is the set of all conjugacy classes of periodic maps in the mapping class group. Furthermore, from Matsumoto–Montesinos' theory, the periodic topological monodromy is uniquely determined by the dual graph of the *minimal normal-crossing model* ([14, Definition 2.2]) $\bar{F}$ of F.

From Matsumoto–Montesinos' theory, or Xiao's theory on principle components [28], one can see that $\bar{F}$ can be written as follows:

$$\bar{F} = nC_0 + \sum_{i=1}^{s} \Gamma_i, \tag{6}$$

where Γ_i's are disjoint *H-J branches* ([14, Definition 3.4]), and F contains only one principle component C_0 which is a nonsingular curve satisfying $C_0\Gamma_{i,\mathrm{red}} = 1$ for all i.

One can check that the n-th root model of F is smooth, but for any $d < n$, the d-th model of F is not smooth. Thus the order of $[\mu]$ is equal to n.

Let M_F be the least common multiplicity of the coefficients of the irreducible components in the divisor $\bar{F}$. The dual model F^* of F in the sense of [14, Definition 2.5] is just the $(M_F - 1)$-th root model of F. Denote by $[\mu^*]$ the conjugacy class of the monodromy of F^*. Then $[\mu^*] = [\mu^{M_F-1}]$. By definition, n is a factor of M_F. Thus $[\mu^{M_F}] = [\mathrm{id}]$. In particular,

$$[\mu^*] = [\mu^{-1}].$$

From the bijiective map Φ, we see that F^* is determined uniquely by F. As a consequence, our notion F^* coincides with the one defined by using the monodromy (when the semistable model of F is smooth).

Let F^* be the dual model of F. By the definition of F^*, under a base change of degree $n-1$, we gets the minimal normal-crossing model of F^* as follows

$$\overline{F^*} = nC_0^* + \sum_{i=1}^{s} \Gamma_i^*, \tag{7}$$

where Γ_i^*'s as the pull-back of Γ_i's are disjoint H-J branches and $\Gamma_{i,\mathrm{red}}^* C_0^* = 1$.

REMARK 3.1. We refer to [16], [17], [5], [4], [6] for more details of the Matsumoto–Montesinos' theory.

4. Proof of Theorem 1.1

Let $f : S \to \mathbb{P}^1$ be a fibration with two singular fibers F_1 and F_2. In this case, f is isotrivial (see [8]).

Now consider the n-cyclic base change $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ totally ramified over $0 = f(F_1)$ and $\infty = f(F_2)$. Let $\tilde{f} : \tilde{S} \to \mathbb{P}^1$ be the pullback fibration of f under π. It is well-known that $\tilde{f}$ is semistable for some n. Because f is isotrivial, $\tilde{f}$ must be a trivial fibration. Hence there is a generically finite n-cover Π: $\tilde{S} = F \times \mathbb{P}^1 \dashrightarrow S$, which implies that $\kappa(S) = -\infty$.

Let σ_1 (resp. σ_2) be the loop around $0 = f(F_1)$ (resp. $\infty = f(F_2)$) such that $\sigma_1\sigma_2 = 1 \in \pi_1(\mathbb{P}^1 - 0 - \infty)$. Let μ_i be the topological homeomorphism along σ_i. Thus $[\mu_1 \circ \mu_2] = [\mathrm{id}]$, i.e., $[\mu_1] = [\mu_2^{-1}]$. Thus F_2 is the dual model of F_1, $F_2 = {F_1}^*$. By (3),

$$K_f^2 = c_1^2(F_1) + c_1^2({F_1}^*), \quad \chi_f = \chi_{F_1} + \chi_{{F_1}^*}, \quad e_f = c_2(F_1) + c_2({F_1}^*). \tag{8}$$

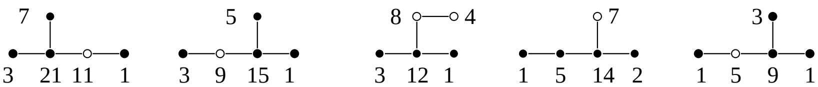
In our case, $s = 2$, $g(C) = 0$, $\chi_f = g - q(S)$. (4) and (5) imply that

$$s_1 = 2, \quad h^{1,1}(S) = l_1 + l_2, \quad g(F_i) = q(S), \quad N_{\bar{F}_i} = g - q(S). \tag{9}$$

So the Mordell–Weil rank $r = 0$.

Now we will prove (3) of Theorem 1.1. Equivalently, we need to prove $K_f^2 = c_1^2(F_1) + c_1^2(F_2) \le 8g - 12$, where $F_2 = {F_1}^*$. In this case, the semistable models of the two singular fibers are smooth. When $g = 2$, according to the classification of Namikawa-Ueno [18], there are exactly 11 pairs $(F_1, {F_1}^*)$ (see Theorem 1.2 or the next section), one can compute directly $c_1^2(S) = -8(g-1) + c_1^2(F_1) + c_1^2(F_2)$ and check directly that $c_1^2(S) \le -2$. So we can assume that $g > 2$.

Note that singular fibers satisfying $c_1^2(F) > 4g - 11/2$ are classified in [14, Theorem 2.1]. There are totally 22 types. But only Types 1, 2, 3, 4 and 6 have nonsingular semistable models, where $g = 6$, 4, 3, 3, and 3, and the Chern numbers $c_1^2(F)$ are 130/7, 54/5, 7, 48/7 and 20/3, respectively. On the other hand, one can compute the dual models F^*. The following is the dual graphes of the normal crossing models of F^* corresponding to the fibers F of Type i, which are trees of smooth rational curves.



Type 1* Type 2* Type 3* Type 4* Type 6*

By a direct computation, $c_1^2(F^*)$ are respectively 73/7, 16/5, 2, 29/7, and 7/3.

If one of F_1 and F_2 satisfies $c_1^2(F_i) > 4g - 11/2$, then the singular fibers are of Type k and Type k^* for some $k = 1, \ldots, 4$, or 6, we can check that $K_f^2 = c_1^2(F_i) + c_1^2({F_i}^*) < 8g - 12$.

If $c_1^2(F_i) \le 4g - 11/2$ for $i = 1, 2$, we need the following lemma whose proof will be given in Section 6.

Lemma 4.1. *There is no fiber F whose semistable model is smooth, and*

$$c_1^2(F) = c_1^2(F^*) = 4g - \frac{11}{2},$$

where F^* is the dual model of F.

Then one of the inequalities $c_1^2(F_1) \leq 4g - 11/2$ and $c_1^2(F_2) \leq 4g - 11/2$ is strict. Hence $K_f^2 = c_1^2(F_1) + c_1^2(F_2) < 8g - 11$, i.e., $K_f^2 \leq 8g - 12$.

5. Proof of Theorem 1.2

5.1. Classification of genus 2 singular fibers. Suppose that $f\colon S \to \mathbb{P}^1$ has exactly two singular fibers F and F^*. From the complete list of genus two singular fibers (see [18]), we can check that there are 11 pairs of fibers (F, F^*) whose semistable models are smooth.

F	$\mathrm{I}^*_{0\text{-}0\text{-}0}$	II	III	IV	V	VI	VII	VIII-1	VIII-2	IX-1	IX-2
c_1^2	2	2	2	3	1	2	1	$\frac{4}{5}$	$\frac{12}{5}$	$\frac{8}{5}$	$\frac{6}{5}$
c_2	10	4	10	9	5	10	5	4	12	8	6
F^*	$\mathrm{I}^*_{0\text{-}0\text{-}0}$	II	III	IV	V*	VI	VII*	VIII-4	VIII-3	IX-4	IX-3
c_1^2	2	2	2	3	3	2	3	$\frac{16}{5}$	$\frac{13}{5}$	$\frac{12}{5}$	$\frac{14}{5}$
c_2	10	4	10	9	15	10	15	16	7	12	14

We have $K_f^2 = c_1^2(F) + c_1^2(F^*)$ and $\chi_f = (1/12)(c_1^2(F) + c_2(F)) + (1/12)(c_1^2(F^*) + c_2(F^*))$.

(1) Type (IV, IV): $K_f^2 = 6$, $\chi_f = 2$, $q(S) = 0$;

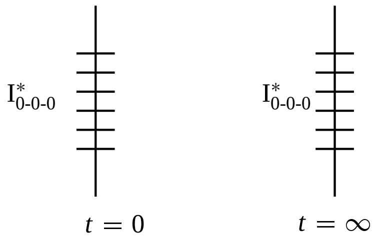
(2) Type (VIII-2, VIII-3): $K_f^2 = 5$, $\chi_f = 2$, $q(S) = 0$;

(3) Type (II, II): $K_f^2 = 4$, $\chi_f = 1$, $q(S) = 1$;

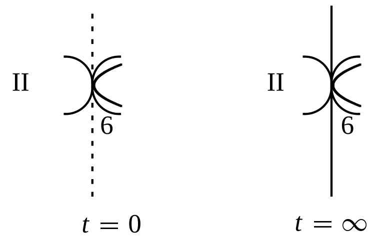
(4) Others: $K_f^2 = 4$, $\chi_f = 2$, $q(S) = 0$.

According to [1, Lemma 1.2] and [29, Lemma 5.1.2], $f\colon S \to \mathbb{P}^1$ is the relatively minimal model of a normalized double cover $\pi\colon \Sigma \to \Sigma_e$ over a Hirzebruch surface $\Sigma_e \to \mathbb{P}^1$ branched along a curve B. Namely, in the process of the canonical resolution, the multiplicities of singular points of the horizontal branch curves are at most 3.

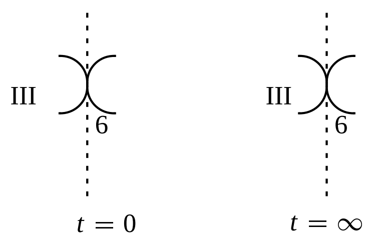
From [11], we know the local structure of the branch curves near the singular fibers. In the following, the dashed line is not contained in the branch locus. The number is the intersection number of the curve with the cental fiber F_0 of the ruling $\Sigma_e \to \mathbb{P}^1$.



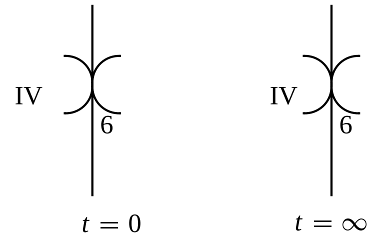
Type $(\mathrm{I}^*_{0\text{-}0\text{-}0},\ \mathrm{I}^*_{0\text{-}0\text{-}0})$



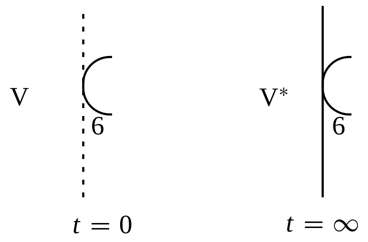
Type (II, II)



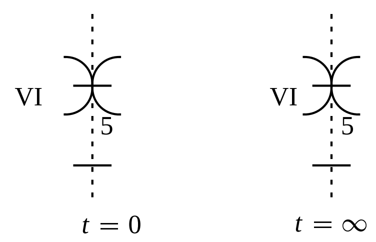
Type (III, III)



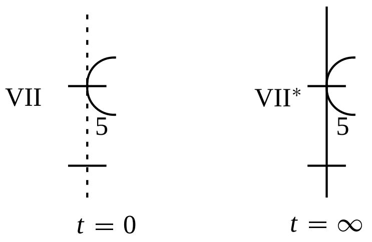
Type (IV, IV)



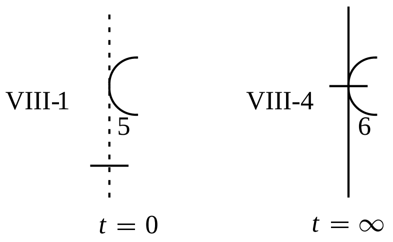
Type (V, V*)



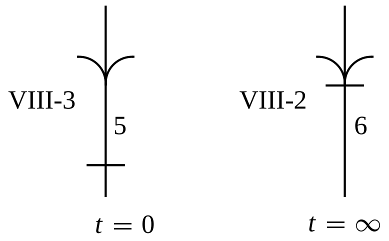
Type (VI, VI)



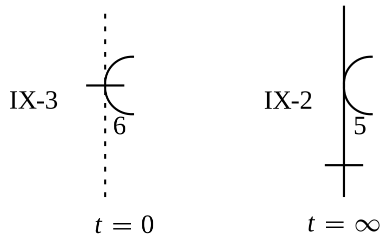
Type (VII, VII*)



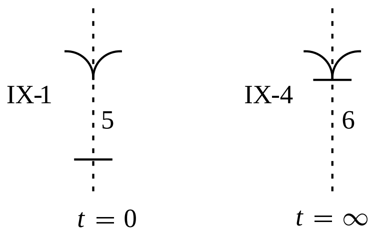
Type (VIII-1, VIII-4)



Type (VIII-2, VIII-3)



Type (IX-2, IX-3)



Type (IX-1, IX-4)

5.2. Determination of the Hirzebruch surfaces. We will determine e of the normalized double cover $\pi\colon \Sigma \to \Sigma_e$ induced by $f\colon S \to \mathbb{P}^1$.

Lemma 5.1. *We have $e = 0$ and $\Sigma_e = \mathbb{P}^1 \times \mathbb{P}^1$.*

$$B \equiv \begin{cases} (6, 2), & \text{if} \quad K_f^2 = 4,\ q(S) = 0, \\ (6, 4), & \text{otherwise.} \end{cases}$$

Proof. From [27, Theorem 2.1], $K_f^2 = 4 = 4g - 4$ and $q(S) = 0$, if and only if S is a double cover over $\mathbb{P}^1 \times \mathbb{P}^1$ ramified over a curve of type (6, 2). It is easy to see that this double cover is normalized.

Now we consider the remaining cases. Suppose that S is a normalized double cover over a Hirzebruch surface Σ_e branched along a curve $B \equiv 6C_0 + 2aF_0$, where C_0 is a section with $C_0^2 = -e$ and F_0 is a fiber of the ruling $\psi\colon \Sigma_e \to \mathbb{P}^1$. Let $K_\psi := K_{\Sigma_e/\mathbb{P}^1} \equiv -2C_0 - eF_0$. Then $K_\psi B = 6e - 4a$, $B^2 = 24a - 36e$.

From the formulae for the invariants of a double cover surface, one has

$$\chi_f = \frac{1}{4}K_\psi B + \frac{1}{8}B^2 - \frac{1}{2}\sum_{i=1}^{k} w_i(w_i - 1) = 2a - 3e - I_P,$$

where $I_P = (1/2)\sum_{i=1}^{k} w_i(w_i - 1)$.

TYPE (II, II). $\chi_f = 1$. By the canonical resolution, we have $I_P = 3$. Thus $a = (3/2)e + 2$ and e is even.

In this case, each singular point of the horizonal part of B is of type $(3 \to 3)$ with a vertical tangent line, i.e., it is topologically equivalent to a singular point defined by $t^3 + x^6 = 0$. In particular, B contains no section and $F_0 \subset B$, so $BC_0 = 2a - 6e \geq F_0C_0 \geq 1$, i.e., $a \geq 3e+1$, it implies $e \leq 2/3$. So $e = 0$ and $a = 2$. Hence $\Sigma_e = \mathbb{P}^1 \times \mathbb{P}^1$ and $B \equiv 6C_0 + 4F_0$ is of type (6, 4).

TYPE (VIII-2, VIII-3). $\chi_f = 2$. By the canonical resolution, we see that $I_P = 2$. Hence $a = \frac{3}{2}e + 2$ and e is even.

If B does not contain C_0, $BC_0 \geq 0$, i.e., $a = (3/2)e + 2 \geq 3e$. Hence $e = 0$ and $B \equiv 6C_0 + 4F_0$.

If B contains C_0, since B is a reduced curve containing two fibers, $B - 2F_0 - C_0$ does not contain C_0. Thus $BC_0 \geq 2F_0C_0 + C_0^2 = 2 - e$, i.e., $2a - 5e \geq 2$. Hence $e = 0$ and $B \equiv 6C_0 + 4F_0$.

TYPE (IV, IV). $\chi_f = 2$ and $I_P = 2$. So $a = (3/2)e + 2$ and e is even. Since B does not contain C_0, $BC_0 \geq 0$, i.e., $a \geq 3e$. Hence $e = 0$ and $B \equiv 6C_0 + 4F_0$. □

5.3. The case when $K_f^2 = 4$. Now we will classify genus 2 fibrations $f: S \to \mathbb{P}^1$ with 2 singular fibers according to the types of the fibers. S is a normalized double cover over $\mathbb{P}^1 \times \mathbb{P}^1$ ramified over a curve B of type (6, 2) or (6, 4). Suppose that B is defined by an algebraic equation $h(x,t) = 0$, then $f: S \to \mathbb{P}^1$ is defined by $y^2 = h(x,t)$.

By a suitable transformation, we can always assume that B has two singular points $(0,0)$ and (x_0, ∞). We claim that $x_0 \neq 0$. Indeed, otherwise, the sum of the intersection numbers of B with the line $x = 0$ would be bigger than 2 or 4. Hence we can also assume that $x_0 = \infty$.

If B is of type (6, 2), then

$$h(x, t) = h_2(x)t^2 + h_1(x)t + h_0(x),$$

where $\deg h_i(x) \leq 6$. In the neighborhood of (∞, ∞), we can use the coordinates $u = 1/x$ and $s = 1/t$. Then h can be written as

$$\bar{h}(u, s) = \bar{h}_2(u) + \bar{h}_1(u)s + \bar{h}_0(u)s^2,$$

where $\bar{h}_i(u) = u^6 h_i(1/u)$.

Because the calculations are similar, we will only do the calculations for several typical types.

TYPE (III, III). In this case, $(0, 0) \in B$ is a singular point of type A_5 with a double tangent line $t^2 = 0$. Hence $h_2(0) \neq 0$, $h_1(0) = h_0(0) = 0$. Since the intersection number of the line $t = 0$ with B at $(0,0)$ is 6, $h_0(x) = ax^6$ ($a \neq 0$). Similarly, (∞, ∞) is a singular point of type A_5 with tangent line $s = 0$, we can see that u^6 divides $\bar{h}_2(u)$. Thus $h_2(x)$ is a nonzero constant c, we can assume that $c = 1$.

The multiplicity of the singular point $(0, 0)$ of B is 2, so x^2 divides $h_1(x)$. If x^3 does not divide $h_1(x)$, then the singular point is analytically isomorphic to $u^2 = v^4$, which is of type A_5, a contradiction. Thus x^3 divides $h_1(x)$. Symmetrically, u^3 divides

$\bar{h}_1(u)$. Hence $h_1(x) = bx^3$. By a linear transformation of x, we have $h(x, t) = t^2 + dx^3t + x^6$.

TYPE (II, II). In this case, B contains the vertical line $t = \infty$ and so

$$h(x, t) = h_3(x)t^3 + h_2(x)t^2 + h_1(x)t + h_0(x), \quad \deg h_i \leq 6.$$

The singular point of B at $(0, 0)$ has multiplicity 3 and admits a triple tangent line $t^3 = 0$. Similarly, $h_0(x) = ax^6$ $(a \neq 0)$, $h_3(0) \neq 0$, $x^2 \mid h_2(x)$ and $x^3 \mid h_1(x)$. We can assume that $a = 1$ by a linear transformation of x.

The local equation of B at (∞, ∞) is as follows.

$$s \cdot (\bar{h}_3(u) + \bar{h}_2(u)s + \bar{h}_1(u)s^2 + \bar{h}_0(u)s^3) = 0.$$

Symmetrically, $u^6 \mid \bar{h}_3(u)$, hence $h_3(x)$ is a nonzero constant, we can assume that this constant is 1 by a linear transformation of t.

If x^4 does not divide $h_1(x)$, then by blowing up at the singular point $(0, 0)$, we can see easily that the strict transform of B is smooth, which contradicts with the fact that the singular point $(B, (0, 0))$ is of type $(3 \to 3)$. Hence $x^4 \mid h_1(x)$.

In the neighborhood of (∞, ∞), we have also

$$u^4 \mid \bar{h}_2, \quad u^2 \mid \bar{h}_1.$$

Thus $h_2(x) = ex^2$ and $h_1(x) = dx^4$, where d and e are constant. $h(x,t) = x^6 + dx^4t + ex^2t^2 + t^3$.

5.4. The case when $K_f^2 > 4$. We will use Ishizaka's method to get the defining equations.

TYPE (IV, IV). The monodromy type of the pair (III, III) is $([\phi_3^3], [\phi_3^3])$, and that of (IV,IV) is $([\phi_3^3 I_3],[\phi_3^3 I_3])$. According to [13, Lemma 1.2], the equation of the branch curve corresponding to Type (IV, IV) is $t \cdot h(x,t) = 0$, where $h(x,t) = 0$ is the equation of the branch curve corresponding to the Type (III, III). Thus the defining equation of the family is $y^2 = t(t^2 + dx^3t + x^6)$.

TYPE (VIII-2, VIII-3). The equation can be obtained from that of Type (IX-1, IX-4) because $([\phi_1^3], [\phi_1^7]) = ([\phi_1^8 I_3], [\phi_1^2 I_3])$. We have $h(x, t) = t(x^5 + t^2)$.

6. Proof of Lemma 4.1

In this section, we use freely the notations used in [14].

Lemma 6.1. *Let F and F^* be written as in* (6) *and* (7)*. Suppose the semistable model of F is smooth. Then*

(1) *F admits at worst one singularity which is not a node.*

(2) *$\beta_F = \beta_F^-$, $\beta_F + \beta_{F^*} = s$, and F or F^* is a nodal curve.* (*See* [14, *Section* 3.1 *and Section* 3.2] *for the definitions of β_F, β_F^-*).

(3) *$K\Gamma_{i,\mathrm{red}} = \mu_{\Gamma_i^*}$ and $K\Gamma_{i,}{}^*{}_{\mathrm{red}} = \mu_{\Gamma_i}$ where μ_Γ is the sum of Milnor's numbers of the singularities of Γ_{red}.*

Proof. (1) Let $p \in F$ be a singular point which is not a node. Consider the minimal partial resolution of (p, F) such that the total transform of F is a normal crossing divisor $\bar{F}$. Let E be the support of the exceptional curves, and let C be a (-1) curve in E. Then the minimality of the resolution implies that C meets in at least 3 points with the other components in $\bar{F}$. So C is exactly the C_0 in (6). If $q \in F$ is another non-nodal singularity, then C_0 lies also in the exceptional set of q, a contradiction.

(2) $\beta_F = \beta_{\bar{F}}$ is obviously a consequence of (6).

Let C_i (resp. C_i^*) be the unique irreducible component of Γ_i (resp. Γ_i^*) meeting with C_0 (resp. C_0^*), and let n_i (resp. n_i^*) be the multiplicity of C_i (resp. C_i^*) in $\bar{F}$ (resp. $\overline{F^*}$). Then $n_i^* = n - n_i$ for all i. Thus one gets $\beta_{\overline{F^*}} = s - \beta_F$ by Lemma 2.1 in [14].

If F has a non-nodal singularity, then C_0 is a (-1)-curve and $s \geq 3$. So $n = \sum_{i=1}^s n_i$ by Zariski's Lemma. Thus $-(C_0^*)^2 = s - 1 \geq 2$. It means that C_0^* is not a (-1)-curve. Hence $F^* = \overline{F^*}$.

(3) Note that $\mu_{\Gamma_i} + 1$ (resp. $\mu_{\Gamma_i^*} + 1$) is the number of irreducible components of Γ_i (resp. Γ_i^*). (3) is directly from [21, p. 222]. □

Let F and F^* be as in (6) and (7). From Lemma 6.1 (2), we can assume that F^* is a nodal curve. $c_1^2(F) = c_1^2(F^*) = 4g - 11/2$ is equivalent to the following equalities.

$$\begin{cases} \dfrac{11}{2} = 4p_a(\bar{F}_{\mathrm{red}}) - F_{\mathrm{red}}^2 + \beta_F + \displaystyle\sum_{i=1}^r m_i(m_i - 2), \\ \dfrac{11}{2} = 4p_a(\overline{F^*}_{\mathrm{red}}) - (F_{\mathrm{red}}^*)^2 + \beta_{F^*}. \end{cases}$$

Every terms in the right hand sides of the above equalities are non-negative. Note that $p_a(\bar{F}_{\mathrm{red}}) = p_a(\bar{F}_{\mathrm{red}}) \leq 1$, $\beta_F + \beta_{F^*} = s \geq 2$ (Lemma 6.1) and $\sum_{i=1}^r m_i(m_i - 2) \leq 5$. So F_{red} has at most one non-nodal singularity p which is of type A_2, A_3, D_4 and $\sum_{i=1}^r m_i(m_i - 2) = 3$ (see [14, Lemma 3.3]). If such p exists, $s = 3$ since all $m_i \leq 3$ and C_0 is a (-1)-curve.

Suppose that $p_a(\bar{F}_{\mathrm{red}}) = 1$. It implies that F is a nodal curves, $F_{\mathrm{red}}^2 = {F_{\mathrm{red}}^*}^2 = -1$ and $\beta_F = \beta_{F^*} = 1/2$ from the above inequalities. Thus $\beta_F + \beta_{F^*} = 1$, a contradiction. Hence $p_a(\bar{F}_{\mathrm{red}}) = 0$, i.e., the dual graphs of $\bar{F}, \overline{F^*}$ are trees of smooth rational curves.

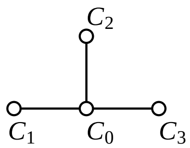
Claim 1. *F is also a nodal curve.*

Suppose that F has a singularity p as above. We get

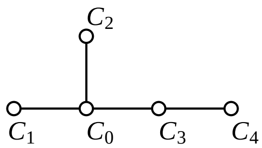
$$\begin{cases} \dfrac{5}{2} = F_{\mathrm{red}}K_S + \beta_F, \\ \dfrac{7}{2} = F^*_{\mathrm{red}}K_S + \beta_{F^*}. \end{cases}$$

Obviously, $2\beta_F \geq 1$ is an odd integer. By Example 3.1 of [14], $\beta_F = 1$ (if p is of type A_2), $1/2 < \beta_F \leq 1$ (if p is of type A_3) or $\beta_F \leq 1$ (if p is of type D_4). So $\beta_F = 1/2$ and p is of type D_4. Thus $F_{\mathrm{red}}K_S = 2$, $\beta_{F^*} = s - \beta_F = \frac{5}{2}$ and $F^*_{red}K_S = 1$. Hence F has at most two components which are not (−2)-curves. It implies that F^* consists of one (−3)-curve and some (−2)-curves. By Lemma 6.1 (3), the dual graph of $\bar{F}$ has two posibilities:

Case A



Case B



Let $e_i = -C_i^2$ and $e_0 = 1$. In Case A, one has

$$\beta_F = \frac{1}{2} = \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3}, \quad F_{\mathrm{red}}K_S = 2 = e_1 + e_2 + e_3 - 9, \quad e_i \geq 3.$$

In Case B,

$$\beta_F = \frac{1}{2} = \frac{1}{e_1} + \frac{1}{e_2} + \frac{e_3}{e_3e_4 - 1}, \quad F_{\mathrm{red}}K_S = 2 = e_1 + e_2 + e_3 + e_4 - 11, \quad e_i \geq 3.$$

By a straightforward computation, one can prove that both cases are impossible.

Therefore F must be a nodal curve and

$$\begin{cases} \dfrac{7}{2} = F_{\mathrm{red}}K_S + \beta_F, \\ \dfrac{7}{2} = F^*_{\mathrm{red}}K_S + \beta_{F^*}. \end{cases}$$

Claim 2. $F_{\mathrm{red}}K_S = F^*_{\mathrm{red}}K_S = 1$.

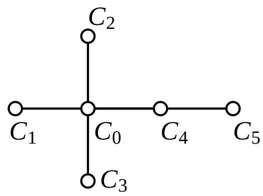
Let $e_0 = -C_0^2$ and $e_0^* = -(C_0^*)^2$. It is obvious that $s = e_0 + e_0^* \geq 4$. Since

$$7 = (F_{\mathrm{red}} + F^*_{\mathrm{red}})K_S + s,$$

$(F_{\mathrm{red}} + F^*_{\mathrm{red}})K_S \leq 3$. Without loss of generality, we assume that $F_{\mathrm{red}}K_S = 1$. Hence $\beta_F = 5/2$ and F consists of one (−3)-curve and some (−2)-curves.

If $F^*_{\mathrm{red}}K_S = 2$, then $s = 4$, $e_0 = e_1 = 2$ and $\beta_{F^*} = 3/2$. By Lemma 6.1 (3), the dual graph of $\overline{F^*}$ is as follows.

Case C



Let $e_i = -C_i^2$. One has $e_i \geq 2$ and

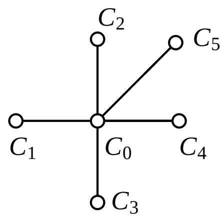
$$\beta_{F^*} = \frac{3}{2} = \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} + \frac{e_4}{e_4e_5 - 1}, \quad F^*_{\mathrm{red}}K_S = 2 = e_1 + e_2 + e_3 + e_4 + e_5 - 10.$$

By a straightforward computation, one can prove that it is impossible. Hence $F^*_{\mathrm{red}}K_S = 1$ and $s = 5$.

Claim 3. *Such F does not exist.*

Without loss of generality, we can assume that $e_0 = 2$ and $e_0^* = 3$. Since $F^*_{\mathrm{red}}K_S = 1$, the irreducible components of F^* are (-2)-curves except for C_0^*. Again by Lemma 6.1 (3), the dual graph of $\bar{F}$ is as follows.

Case D



Hence $\beta_F^* = 1/3 + (1/2) \times 4 \neq 5/2$, a contradiction.

Up to now, we complete the proof of Lemma 4.1. □

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Cheng Gong
School of Mathematical Sciences
Soochow University, Shizi RD 1
Suzhou 215006, Jiangsu
P.R. of China
e-mail: cgong@suda.edu.cn

Jun Lu
Department of Mathematics, and Shanghai Key Laboratory of PMMP
East China Normal University
Dongchuan RD 500
Shanghai 200241
P.R. of China
e-mail: jlu@math.ecnu.edu.cn

Sheng-Li Tan
Department of Mathematics, and Shanghai Key Laboratory of PMMP
East China Normal University
Dongchuan RD 500
Shanghai 200241
P.R. of China
e-mail: sltan@math.ecnu.edu.cn