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PERSISTENCE AND EXTINCTION OF AN IMPULSIVE STOCHASTIC LOGISTIC MODEL WITH INFINITE DELAY

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Abstract

This paper considers an impulsive stochastic logistic model with infinite delay at the phase space C_g. Firstly, the definition of solution to an impulsive stochastic functional differential equation with infinite delay is established. Based on this definition, we show that our model has a unique global positive solution. Then we establish the sufficient conditions for extinction, nonpersistence in the mean, weak persistence and stochastic permanence of the solution. The threshold between weak persistence and extinction is obtained. In addition, the effects of impulsive perturbation and delay on persistence and extinction are discussed, respectively. Finally, numerical simulations are introduced to support the theoretical analysis results.

1. Introduction

A famous logistic model with infinite delay can be expressed as follows

$$dx(t)/dt = x(t)\left[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0} x(t+\theta)\,d\mu(\theta)\right], \tag{1.1}$$

where $\tau \geq 0$ represents the time delay and $\mu(\theta)$ is a probability measure on $(-\infty,0]$. A further and extensive feature is considered in the model (1.1) or systems similar to (1.1) towards persistence, extinction or other dynamical behavior. Here, we only refer to the references([1], [2], [3], [4], [5], [6], [7]). Particularly, [1] and [7] are good references in this field.

In the real world, population models are always influenced by environmental noises (see e.g. [8], [9], [10], [11], [12], [13], [14], [15]). Moreover, May [10] has revealed the fact that due to environmental noise, the birth rate, competition coefficient and other parameters involved in the system exhibit random fluctuation to a greater or lesser extent. Inspired by works referred above, we estimate the birth rate $r(t)$ and the intraspecific competition coefficient $a(t)$ by an average value with errors which follow a nor-

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mal distribution. In other words, we may substitute the parameters $r(t)$, $-a(t)$ with $r(t) + \sigma_1(t)\dot{\omega}_1(t)$, $-a(t) + \sigma_2(t)\dot{\omega}_2(t)$, respectively. Here, for $i = 1, 2$, $\sigma_i(t)$ is positive continuous bounded function on $\overline{R}_+ = [0, +\infty)$ and $\sigma_i^2(t)$ represents an intensity of the white noise $\dot{\omega}_i$ at t; $(\dot{\omega}_1(t), \dot{\omega}_2(t))$ is a 2-dimensional white noise, namely, $(\omega_1(t), \omega_2(t))$ is a 2-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in\overline{R}_+}$ satisfying the usual conditions. Then we obtain the following stochastic model:

$$
\begin{aligned}
dx(t) = x(t)\left[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0} x(t+\theta)\,d\mu(\theta)\right] dt \\
+ \sigma_1(t)x(t)\,d\omega_1(t) + \sigma_2(t)x^2(t)\,d\omega_2(t).
\end{aligned}
\tag{1.2}
$$

On the other hand, affected by a variety of factors both naturally and artificially, such as earthquake, drought, flooding, fire, crop-dusting, planting, hunting and harvesting, the inner discipline of species or environment often suffers some dispersed changes over a relatively short time interval at the fixed times. In mathematics perspective, such sudden changes could be described by impulses (see e.g. [16], [17], [18], [19], [20], [21]). In this paper, we will study the following impulsive stochastic logistic system with infinite delay

$$
\begin{cases}
dx(t) = x(t)\left[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\displaystyle\int_{-\infty}^{0} x(t+\theta)\,d\mu(\theta)\right] dt \\
\qquad + \sigma_1(t)x(t)\,d\omega_1(t) + \sigma_2(t)x^2(t)\,d\omega_2(t), \quad t \neq t_k,\ K \in N, \\
x(t_k^+) - x(t_k) = h_k x(t_k), \quad k \in N
\end{cases}
\tag{1.3}
$$

where N denotes the set of positive integers, $0 < t_1 < t_2 \cdots$, $\lim_{k\to+\infty} t_k = +\infty$.

Since phase space $BC((-\infty, 0]; R)$ may cause the usual well-posedness questions related to functional equations of unbounded delay ([3], [22], [23]), we let the initial value ξ be positive and belong to the phase space C_g ([3], [22]) which is defined by

$$
C_g = \left\{\varphi \in C((-\infty, 0]; R) \colon \|\varphi\|_{c_g} = \sup_{-\infty < s \le 0} e^{\mathbf{r}s}|\varphi(s)| < +\infty\right\},
$$

where we choose $g(s) = e^{-\mathbf{r}s}$, $\mathbf{r} > 0$. Furthermore, C_g is an admissible Banach space ([3], [23]).

For the system (1.3), some important topics arise naturally.

(Q1) The model (1.3) describes a population dynamics, then it is critical to investigate the persistence and extinction of this model. Moreover, it is also important to obtain the threshold between extinction and persistence for the species.

(Q2) When analyzing population models, permanence is one of the most interesting and important topics. Then under what conditions is the model (1.3) permanent?

(Q3) What are the impacts of impulsive perturbation and delay on the extinction, persistence and permanence of the system (1.3), respectively?

For the model (1.3) we always assume:
(A1): As far as biological meanings are concerned, we consider $1 + h_k > 0$, $k \in N$. When $h_k > 0$, is satisfied, the perturbation turns to be the description process of planting of species and harvesting if not.
(A2): $r(t)$, $a(t)$, $b(t)$ and $c(t)$ are continuous and bounded functions on $\overline{R}_+$ and $\inf_{t\in\overline{R}_+} a(t) > 0$.
(A3): μ satisfies that

$$\mu_{\mathbf{r}} = \int_{-\infty}^{0} e^{-2\mathbf{r}\theta}\, d\mu(\theta) < +\infty.$$

The assumption (A3) above may be satisfied when $\mu(\theta) = e^{k\mathbf{r}\theta}(k > 2)$ for $\theta \le 0$, so there are a large number of these probability measures.

For the simplicity, we define the following notations:

$$f^u = \sup_{t\in R} f(t), \quad f^l = \inf_{t\in R} f(t), \quad \langle x(t)\rangle = \frac{1}{t}\int_0^t x(s)\, ds,$$
$$x_* = \liminf_{t\to+\infty} x(t), \quad x^* = \limsup_{t\to+\infty} x(t), \quad R_+ = (0, +\infty).$$

The following definitions are commonly used and we list them here.

DEFINITION. 1. The population $x(t)$ is said to be extinctive [13] if $\lim_{t\to+\infty} x(t) = 0$.
2. The population $x(t)$ is said to be nonpersistent in the mean (see e.g., Liu and Ma [24]) if $\limsup_{t\to+\infty}\langle x(t)\rangle = 0$.
3. The population $x(t)$ is said to be weakly persistent (see e.g., Hallam and Ma [25]) if $\limsup_{t\to+\infty} x(t) > 0$.
4. The population $x(t)$ is said to be stochastic permanence [13] if for an arbitrary $\varepsilon > 0$, there are constants $\beta > 0$, $\alpha > 0$ such that $\liminf_{t\to+\infty} \mathcal{P}\{x(t) \ge \beta\} \ge 1 - \varepsilon$ and $\liminf_{t\to+\infty} \mathcal{P}\{x(t) \le \alpha\} \ge 1 - \varepsilon$.

The rest of the paper is arranged as follows. In Section 2, we propose a new definition of solution for impulsive stochastic functional differential equations with infinite delay and verify that the model (1.3) has a unique positive global solution. Afterward, sufficient conditions for extinction are established as well as nonpersistence in the mean, weak persistence and stochastic permanence in Section 3. Section 4 devotes to introducing some figures to illustrate the main results. Finally, we end the paper with a series of conclusions and remarks in Section 5.

2. Positive and global solutions

Now let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions. Let $B(t)$ denote a m-dimension standard Brownian motion defined on this probability space.

DEFINITION 1. Considering the following impulsive stochastic functional differential equation with infinite delay:

$$\begin{cases} dX(t) = F(t, X_t)\,dt + G(t, X_t)\,dB(t), & t \neq t_k,\ k \in N, \\ X(t_k^+) - X(t_k) = H_k X(t_k), & k \in N \end{cases} \tag{2.1}$$

where $X_t = \{X(t+\theta)\colon -\infty < \theta \leq 0\}$ can be regarded as C_g-value stochastic process. The initial value $X_0 = \xi = \{\xi(\theta)\colon -\infty < \theta \leq 0\}$ is an $\mathcal{F}_0$-measurable C_g-valued random variable such that $\xi \in \mathcal{M}^2((-\infty, 0]; R^d)$, where $\mathcal{M}^2((-\infty, 0]; R^d)$ is the family of all $\mathcal{F}_0$-measurable, R^d-valued processes $\varphi(t)$, $t \in (-\infty, 0]$ such that $E\int_{-\infty}^0 |\varphi(t)|^2\,dt < +\infty$. An R^d-value stochastic process $X(t)$ defined on R is called a solution of the equation (2.1) with initial data above, if $X(t)$ has the following properties.
(i) $X(t)$ is $\mathcal{F}_t$-adapted and continuous on $(0, t_1)$ and (t_k, t_{k+1}), $k \in N$; $F(t, X_t) \in \mathscr{L}^1(\overline{R}_+; R^d)$ and $G(t, X_t) \in \mathscr{L}^2(\overline{R}_+; R^{d\times m})$, where $\mathscr{L}^k(\overline{R}_+, R^d)$ is all R^d valued measurable $\mathcal{F}_t$-adapted processes $f(t)$ satisfying $\int_0^T |f(t)|\,dt < +\infty$ a.s. (almost surely) for every $T > 0$;
(ii) for each t_k, $k \in N$, $X(t_k^+) = \lim_{t\to t_k^+} X(t)$ and $X(t_k^-) = \lim_{t\to t_k^-} X(t)$ exist and $x(t_k^-) = x(t_k)$ with probability one;
(iii) for almost all $t \in [0, t_1]$, $X(t)$ obeys the integral equation

$$X(t) = \xi(0) + \int_0^t F(s, X_s)\,ds + \int_0^t G(s, X_s)\,dB(s). \tag{2.2}$$

And for almost all $t \in (t_k, t_{k+1}]$, $k \in N$, $X(t)$ obeys the integral equation

$$X(t) = X(t_k^+) + \int_{t_k}^t F(s, X_s\,ds + \int_{t_k}^t G(s, X_s)\,dB(s). \tag{2.3}$$

Moreover, $X(t)$ satisfies the impulsive conditions at each $t = t_k$, $k \in N$ with probability one.

REMARK 2.1. Now let us demonstrate the derivation procedure of Definition 1. First of all, noticing that the impulsive stochastic functional differential equation with infinite delay (2.1) becomes the following stochastic functional differential equation with infinite delay:

$$dX(t) = F(t, X_t)\,dt + g(t, X_t)\,dB(t)$$

on $[0, t_1]$ and each interval $(t_k, t_{k+1}] \in R_+$, $k \in N$. According to the definition of the solutions of stochastic functional differential equations with infinite delay (see e.g. [26], [27]), the condition (i), Equations (2.2) and (2.3) should be satisfied. Second, since there are impulsive perturbations in Equation (2.1), the condition (ii) and the impulsive conditions in (iii) should be satisfied. According to the two facts above, Definition 1 is thus proposed.

Now consider the following stochastic functional differential equation with infinite delay:

$$
\begin{aligned}
dy(t) = y(t)\Bigg[& r(t) - \prod_{0<t_k<t}(1+h_k)a(t)y(t) + \prod_{0<t_k<t-\tau}(1+h_k)b(t)y(t-\tau) \\
& + c(t)\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+h_k)y(t+\theta)\,d\mu(\theta)\Bigg]dt + \sigma_1(t)y(t)\,d\omega_1(t) \\
& + \prod_{0<t_k<t}(1+h_k)\sigma_2(t)y^2(t)\,d\omega_2(t),
\end{aligned}
\tag{2.4}
$$

with the same initial condition as the model (1.3).

Wei ([26], [27]) and Xu ([28], [29]) have proved that, in order for a stochastic functional differential equation with infinite delay to have a unique global solution for any given initial data $\xi \in C_g$, the coefficients of the equation are generally required to satisfy the linear growth condition and the locally Lipschitz condition. The locally Lipschitz condition guarantees that the unique solution exists on $(-\infty, \tau_e)$, where τ_e is the explosion time(see Mao [30]). Clearly, the coefficients of Equation (2.4) satisfy the locally Lipschitz condition, but do not satisfy the linear growth condition.

Lemma 2.1. *Let the assumptions* (A1)–(A3) *hold. In the model* (2.4), *for any given initial condition* $\xi \in C_g$, *there is a unique solution* $x(t)$ *on* $t \in R$ *and the solution will remain in* R_+ *with probability* 1.

Proof. Since the coefficients of Equation 2.4 are locally Lipschitz continuous, for any given initial condition $\xi \in C_g$, there is a unique local solution $y(t)$ on $t \in (-\infty, \tau_e)$, where τ_e is the explosion time. To show this solution is global, we need to show that $\tau_e = +\infty$ a.s. Let $k_0 > 0$ be sufficiently large for

$$\frac{1}{k_0} < \min_{-\infty<\theta\le 0}|\xi(\theta)| \le \max_{-\infty<\theta\le 0}|\xi(\theta)| < k_0.$$

For each integer $k \ge k_0$, we define a stopping time

$$\tau_k = \inf\left\{t \in (-\infty, \tau_e)\colon y(t) \le \frac{1}{k} \text{ or } y(t) \ge k\right\},$$

where throughout this paper we set $\inf \varnothing = +\infty$ (as usual $\varnothing$ denotes the empty set). Clearly, τ_k is increasing as $k \to +\infty$. Set $\tau_{+\infty} = \lim_{k\to+\infty}\tau_k$, whence $\tau_{+\infty} \le \tau_e$ a.s. for all $t \geq 0$. If we can show that $\tau_{+\infty} = +\infty$ a.s., then $\tau_e = +\infty$ a.s. and $x(t) \in R_+$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show is that $\tau_{+\infty} = +\infty$ a.s. Now let us define a C^2-function $V\colon R_+ \to R_+$ by $V(y) = \sqrt{y} - 1 - 0.5 \ln y$. Let $k \geq k_0$ and $T > 0$ be arbitrary. For $0 \leq t \leq \tau_k \wedge T$, applying Itô's formula (see e.g. [30, p. 32], [31], [15]) to $V(y)$, we have

$$\begin{aligned}
dV(y) &= V_y\,dy + V_t\,dt + \frac{1}{2}V_{yy}(dy)^2 \\
&= 0.5(y^{-0.5} - y^{-1})\Bigg[y\Bigg(r(t) - \prod_{0<t_k<t}(1+h_k)a(t)y + \prod_{0<t_k<t-\tau}(1+h_k)b(t)y(t-\tau) \\
&\qquad + c(t)\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+h_k)y(t+\theta)\,d\mu(\theta)\Bigg)\,dt \\
&\qquad + \sigma_1(t)y\,d\omega_1(t) + \prod_{0<t_k<t}(1+h_k)\sigma_2(t)y^2\,d\omega_2(t)\Bigg] \\
&\quad + 0.5[-0.25y^{-1.5} + 0.5y^{-2}]\sigma_1^2(t)y^2\,dt \\
&\quad + 0.5[-0.25y^{-1.5} + 0.5y^{-2}]\Bigg(\prod_{0<t_k<t}(1+h_k)\Bigg)^2\sigma_2^2(t)y^4\,dt \\
&= 0.5r(t)(y^{0.5} - 1)\,dt - 0.5\prod_{0<t_k<t}(1+h_k)a(t)(y^{0.5} - 1)y\,dt \\
&\quad + 0.5\prod_{0<t_k<t-\tau}(1+h_k)b(t)(y^{0.5} - 1)y(t-\tau)\,dt \\
&\quad + 0.5c(t)(y^{0.5} - 1)\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+h_k)y(t+\theta)\,d\mu(\theta)\,dt \\
&\quad + 0.5(-0.25y^{-1.5} + 0.5y^{-2})\sigma_1^2(t)y^2\,dt \\
&\quad + 0.5(-0.25y^{-1.5} + 0.5y^{-2})\Bigg(\prod_{0<t_k<t}(1+h_k)\Bigg)^2\sigma_2^2(t)y^4\,dt \\
&\quad + 0.5(y^{0.5} - 1)\sigma_1(t)\,d\omega_1(t) + 0.5(y^{1.5} - y)\prod_{0<t_k<t}(1+h_k)\sigma_2(t)\,d\omega_2(t) \\
&\leq 0.5r(t)(y^{0.5} - 1)\,dt - 0.5\prod_{0<t_k<t}(1+h_k)a(t)(y^{0.5} - 1)y\,dt \\
&\quad + 0.0625\Bigg(\prod_{0<t_k<t-\tau}(1+h_k)\Bigg)^2 b^2(t)(y^{0.5} - 1)^2\,dt + y^2(t-\tau)\,dt
\end{aligned}$$

$$
\begin{aligned}
&+ 0.0625c^2(t)(y^{0.5}-1)^2\,dt + \left[\int_{-\infty}^{0} \prod_{0<t_k<t+\theta}(1+h_k)y(t+\theta)\,d\mu(\theta)\right]^2 dt\\
&+ 0.5(-0.25y^{-1.5}+0.5y^{-2})\sigma_1^2(t)y^2\,dt\\
&+ 0.5(-0.25y^{-1.5}+0.5y^{-2})\left(\prod_{0<t_k<t}(1+h_k)\right)^2\sigma_2^2(t)y^4\,dt\\
&+ 0.5(y^{0.5}-1)\sigma_1(t)\,d\omega_1(t) + 0.5[y^{1.5}-y]\prod_{0<t_k<t}(1+h_k)\sigma_2(t)y\,d\omega_2(t)\\
\le{}& 0.5r(t)(y^{0.5}-1)\,dt - 0.5\prod_{0<t_k<t}(1+h_k)a(t)(y^{0.5}-1)y\,dt\\
&+ 0.0625\left(\prod_{0<t_k<t-\tau}(1+h_k)\right)^2 b^2(t)(y^{0.5}-1)^2\,dt + y^2(t-\tau)\,dt\\
&+ 0.0625c^2(t)(y^{0.5}-1)^2\,dt + \int_{-\infty}^{0}\left(\prod_{0<t_k<t+\theta}(1+h_k)\right)^2 y^2(t+\theta)\,d\mu(\theta)\,dt\\
&+ 0.5(-0.25y^{-1.5}+0.5y^{-2})\sigma_1^2(t)y^2\,dt\\
&+ 0.5(-0.25y^{-1.5}+0.5y^{-2})\left(\prod_{0<t_k<t}(1+h_k)\right)^2\sigma_2^2(t)y^4\,dt\\
&+ 0.5(y^{0.5}-1)\sigma_1(t)\,d\omega_1(t) + 0.5[y^{1.5}-y]\prod_{0<t_k<t}(1+h_k)\sigma_2(t)y\,d\omega_2(t)\\
={}& \Bigg\{-0.125\left(\prod_{0<t_k<t}(1+h_k)\right)^2\sigma_2^2(t)y^{2.5} + 0.25\left(\prod_{0<t_k<t}(1+h_k)\right)^2\sigma_2^2(t)y^2\\
&\quad - 0.5\prod_{0<t_k<t}(1+h_k)a(t)y^{1.5} + 0.5\prod_{0<t_k<t}(1+h_k)a(t)y\\
&\quad + 0.0625\left(\prod_{0<t_k<t}(1+h_k)\right)^2 b^2(t)y + 0.0625c^2(t)y + 0.5r(t)y^{0.5}\\
&\quad - 0.125c^2(t)y^{0.5} - 0.125\left(\prod_{0<t_k<t}(1+h_k)\right)^2 b^2(t)y^{0.5}\\
&\quad - 0.125\sigma_1^2(t)y^{0.5} + 0.0625\left(\prod_{0<t_k<t-\tau}(1+h_k)\right)^2 b^2(t) - 0.5r(t)\\
&\quad + 0.0625c^2(t) + 0.25\sigma_1^2(t)\Bigg\}\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{-\infty}^{0} \Bigg(\prod_{0<t_k<t+\theta} (1+h_k) \Bigg)^2 y^2(t+\theta)\, d\mu(\theta)\, dt + y^2(t-\tau)\, dt \\
&+ 0.5(y^{0.5}(t)-1)\sigma_1(t)\, d\omega_1(t) + 0.5(y^{1.5}-y) \prod_{0<t_k<t} (1+h_k)\sigma_2(t)\, d\omega_2(t) \\
= {}& F(y)\, dt + \int_{-\infty}^{0} \Bigg(\prod_{0<t_k<t+\theta} (1+h_k) \Bigg)^2 y^2(t+\theta)\, d\mu(\theta)\, dt \\
&- \Bigg(\prod_{0<t_k<t} (1+h_k) \Bigg)^2 y^2\, dt + y^2(t-\tau)\, dt - y^2\, dt + 0.5[y^{0.5}-1]\sigma_1(t)\, d\omega_1(t) \\
&+ 0.5(y^{1.5}-y) \prod_{0<t_k<t} (1+h_k)\sigma_2(t)\, d\omega_2(t),
\end{aligned}
$$

where

$$
\begin{aligned}
F(y) = {}& -0.125 \Bigg(\prod_{0<t_k<t} (1+h_k) \Bigg)^2 \sigma_2^2(t) y^{2.5} \\
&+ \Bigg[1 + 0.25 \Bigg(\prod_{0<t_k<t} (1+h_k) \Bigg)^2 \sigma_2^2(t) + \Bigg(\prod_{0<t_k<t} (1+h_k) \Bigg)^2 \Bigg] y^2 \\
&- 0.5 \prod_{0<t_k<t} (1+h_k) a(t) y^{1.5} + 0.5 \prod_{0<t_k<t} (1+h_k) a(t) y \\
&+ 0.0625 \Bigg(\prod_{0<t_k<t-\tau} (1+h_k) \Bigg)^2 b^2(t) y + 0.0625 c^2(t) y + 0.5 r(t) y^{0.5} \\
&- 0.125 c^2(t) y^{0.5} - 0.125 \Bigg(\prod_{0<t_k<t} (1+h_k) \Bigg)^2 b^2(t) y^{0.5} - 0.125 \sigma_1^2(t) y^{0.5} \\
&+ 0.0625 \Bigg(\prod_{0<t_k<t} (1+h_k) \Bigg)^2 b^2(t) \\
&- 0.5 r(t) + 0.0625 c^2(t) + 0.25 \sigma_1^2(t).
\end{aligned}
$$

Combined with assumption (A2), it is easy to see that $F(y)$ is bounded, say by K, in

R_+. We therefore obtain that

$$\begin{aligned}
&dV(y(t))\\
&\le K\,dt + \int_{-\infty}^{0}\left(\prod_{0<t_k<t+\theta}(1+h_k)\right)^2 y^2(t+\theta)\,d\mu(\theta)\,dt - \left(\prod_{0<t_k<t}(1+h_k)\right)^2 y^2(t)\,dt\\
&\quad + y^2(t-\tau)\,dt - y\,dt + 0.5[y^{0.5}(t)-1]\sigma_1(t)\,d\omega_1(t)\\
&\quad + 0.5[y^{1.5}(t)-y(t)]\sigma_2(t)\,d\omega_2(t).
\end{aligned}$$

Integrating both sides from 0 to t, and then taking expectations, we have

$$\begin{aligned}
EV(y(t)) \le V(y(0)) + Kt + E\int_0^t\int_{-\infty}^{0}&\left(\prod_{0<t_k<s+\theta}(1+h_k)\right)^2 y^2(s+\theta)\,d\mu(\theta)\,ds\\
&- E\int_0^t\left(\prod_{0<t_k<s}(1+h_k)\right)^2 y^2(s)\,ds + E\int_0^t y^2(s-\tau)\,ds \qquad (2.5)\\
&- E\int_0^t y^2(s)\,ds.
\end{aligned}$$

Moreover, we can derive that

(2.6)

$$\begin{aligned}
&\int_0^t\int_{-\infty}^{0}\left(\prod_{0<t_k<s+\theta}(1+h_k)\right)^2 y^2(s+\theta)\,d\mu(\theta)\,ds\\
&= \int_0^t\left[\int_{-\infty}^{-s}\left(\prod_{0<t_k<s+\theta}(1+h_k)\right)^2 y^2(s+\theta)\,d\mu(\theta)\right.\\
&\qquad\left. + \int_{-s}^{0}\left(\prod_{0<t_k<s+\theta}(1+h_k)\right)^2 y^2(s+\theta)\,d\mu(\theta)\right]ds\\
&= \int_0^t ds\int_{-\infty}^{-s} e^{2\mathbf{r}(s+\theta)}y^2(s+\theta)e^{-2\mathbf{r}(s+\theta)}\,d\mu(\theta)\\
&\quad + \int_{-t}^{0} d\mu(\theta)\int_{-\theta}^{t}\left(\prod_{0<t_k<s+\theta}(1+h_k)\right)^2 y^2(s+\theta)\,ds\\
&\le \|\xi\|_{C_g}^2\int_0^t e^{-2\mathbf{r}s}\,ds\int_{-\infty}^{0}e^{-2\mathbf{r}\theta}\,d\mu(\theta) + \int_{-\infty}^{0}d\mu(\theta)\int_0^t\left(\prod_{0<t_k<s}(1+h_k)\right)^2 y^2(s)\,ds\\
&\le \|\xi\|_{C_g}^2\mu_{\mathbf{r}}t + \int_0^t\left(\prod_{0<t_k<s}(1+h_k)\right)^2 y^2(s)\,ds.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_0^t y^2(s-\tau)\,ds &= \int_{-\tau}^{t-\tau} y^2(s)\,ds = \int_{-\tau}^{0} \xi^2(s)\,ds + \int_0^{t-\tau} y^2(s)\,ds\\ &\le \int_{-\tau}^{0} \xi^2(s)\,ds + \int_0^t y^2(s)\,ds.\end{aligned} \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5) leads to

$$EV(y(t)) \le V(y(0)) + Kt + \|\xi\|^2_{C_g}\mu_{\mathbf{r}}t + \int_{-\tau}^{0} \xi^2(s)\,ds. \tag{2.8}$$

Let $t = \tau_k \wedge T$, and we obtain that

$$EV(y(\tau_k \wedge T)) \le V(y(0)) + KT + \|\xi\|^2_{C_g}\mu_{\mathbf{r}}T + \int_{-\tau}^{0} \xi^2(s)\,ds.$$

Note that for every $\omega \in \{\tau_k \le T\}$, $y(\tau_k,\omega)$ equals either k or $1/k$, and hence $V(y(\tau_k,\omega))$ is no less than either

$$\sqrt{k} - 1 - 0.5\log(k)$$

or

$$\sqrt{\frac{1}{k}} - 1 - 0.5\log\left(\frac{1}{k}\right) = \sqrt{\frac{1}{k}} - 1 + 0.5\log(k).$$

Thus,

$$V(y(\tau_k,\omega)) \ge [\sqrt{k} - 1 - 0.5\log(k)] \wedge \left[\sqrt{\frac{1}{k}} - 1 + 0.5\log(k)\right].$$

It then follows from (2.8) that

$$\begin{aligned}&V(y(0)) + KT + \|\xi\|^2_{C_g}\mu_r T + \int_{-\tau}^{0} \xi^2(s)\,ds\\ &\ge E[1_{\{\tau_k \le T\}}(\omega)V(y(\tau_k,\omega))]\\ &\ge \mathcal{P}\{\tau_k \le T\}\left([\sqrt{k} - 1 - 0.5\log(k)] \wedge \left[\sqrt{\frac{1}{k}} - 1 + 0.5\log(k)\right]\right),\end{aligned}$$

where $1_{\{\tau_k \le T\}}$ is the indicator function of $\{\tau_k \le T\}$. Letting $k \to +\infty$ gives

$$\lim_{k\to+\infty} \mathcal{P}\{\tau_k \le T\} = 0$$

and hence

$$\mathcal{P}\{\tau_{+\infty} \le T\} = 0.$$

Since $T > 0$ is arbitrary, we derive

$$\mathcal{P}\{\tau_{+\infty} < +\infty\} = 0.$$

Thus $\mathcal{P}\{\tau_{+\infty} = +\infty\} = 1$ as required. $\square$

Theorem 2.1. *Let the assumptions* (A1)–(A3) *hold. For the model* (1.3), *with any given initial condition* $\xi \in C_g$, *there is a unique solution* $x(t)$ *on* $t \in R$ *and the solution will remain in* R_+ *with probability* 1.

Proof. Now let

$$x(t) = \prod_{0<t_k<t} (1+h_k)y(t),$$

where $y(t)$ is the solution of the system (2.4). We need only to show that $x(t)$ is the solution Equation (1.3). In fact, $x(t)$ is continuous on $(t_k, t_{k+1}) \subset (0, +\infty)$, $k \in N$ and for every $t \neq t_k$,

$$\begin{aligned}
dx(t) &= d\left[\prod_{0<t_k<t} (1+h_k)y(t)\right] = \prod_{0<t_k<t} (1+h_k)\, dy(t) \\
&= \prod_{0<t_k<t} (1+h_k)y(t)\Bigg[r(t) - \prod_{0<t_k<t} (1+h_k)a(t)y(t) + \prod_{0<t_k<t-\tau} (1+h_k)b(t)y(t-\tau) \\
&\qquad + c(t)\int_{-\infty}^{0} \prod_{0<t_k<t+\theta} (1+h_k)y(t+\theta)\, d\mu(\theta)\Bigg]\, dt \\
&\quad + \prod_{0<t_k<t} (1+h_k)\sigma_1(t)y(t)\, d\omega_1(t) + \left(\prod_{0<t_k<t} (1+h_k)\right)^2 \sigma_2(t)y^2(t)\, d\omega_2(t) \\
&= x(t)\left[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0} x(t+\theta)\, d\mu(\theta)\right] dt \\
&\quad + \sigma_1(t)x(t)\, d\omega_1(t) + \sigma_2(t)x^2(t)\, d\omega_2(t).
\end{aligned}$$

Moreover, for every $k \in N$ and $t_k \in [0, +\infty)$,

$$\begin{aligned}
x(t_k^+) &= \lim_{t\to t_k^+} \prod_{0<t_j<t} (1+h_j)y(t) = \prod_{0<t_j\le t_k} (1+h_j)y(t_k^+) \\
&= (1+h_k) \prod_{0<t_j<t_k} (1+h_j)y(t_k) \\
&= (1+h_k)x(t_k).
\end{aligned}$$

In addition,

$$x(t_k^-) = \lim_{t\to t_k^-} \prod_{0<t_j<t} (1+h_j)y(t) = \prod_{0<t_j<t_k} (1+h_j)y(t_k^-)$$
$$= \prod_{0<t_j<t_k} (1+h_j)y(t_k) = x(t_k).$$

Now let us prove the uniqueness of the solution. For $t \in [0, t_1]$, the model (1.3) becomes the following equation:

$$\begin{aligned} dx(t) = x(t)\Big[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0} x(t+\theta)\,d\mu(\theta)\Big]\,dt \\ + \sigma_1(t)x(t)\,d\omega_1(t) + \sigma_2(t)x^2(t)\,d\omega_2(t). \end{aligned} \tag{2.9}$$

Since the coefficients of Equation (2.9) are locally Lipschitz continuous, by the theory of stochastic differential equation (see e.g. Theorem 3.15 in [32, p. 91]), the solution of Equation (2.9) is unique. For $t \in (t_k, t_{k+1}]$, $k \in N$, the model (1.3) becomes:

$$\begin{aligned} dx(t) = x(t)\Big[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0} x(t+\theta)\,d\mu(\theta)\Big]\,dt \\ + \sigma_1(t)x(t)\,d\omega_1(t) + \sigma_2(t)x^2(t)d\omega_2(t). \end{aligned} \tag{2.10}$$

Note that the coefficients of Equation (2.10) are also locally Lipschitz continuous; then the solution of Equation (2.10) is also unique. Consequently, the solution of the model (1.3) is unique. This completes the proof. □

3. Persistence and extinction

In this section, we shall study the persistence and extinction of the model (1.3).

Theorem 3.1. *Let the assumptions* (A1)–(A3) *hold. Suppose that $x(t)$ is a solution of* Equation (1.3); *then*

$$\limsup_{t\to+\infty} t^{-1}\ln x(t) \le \limsup_{t\to+\infty} t^{-1}\Big[\sum_{0<t_k<t} \ln(1+h_k) + \int_0^t f(s)\,ds\Big] = G^*, \quad a.s.,$$

where $f(t) = r(t) - 0.5\sigma_1^2(t)$. Particularly, if $G^* < 0$ and $\inf_{t\in\overline{R}_+}\{a(t) - b(t+\tau) - c^u\} \ge 0$, *then* $\lim_{t\to+\infty} x(t) = 0$ *a.s.*

Proof. The proof is rather technical so we divide it into two cases.

CASE 1. $b(t) \geq 0$ and $c(t) \geq 0$: Applying Itô's formula (see e.g. [30, p. 32]), [31]) to Equation (2.4) leads to

$$\begin{aligned}
d \ln y &= \frac{dy}{y} - \frac{(dy)^2}{2y^2} \\
&= \Bigg[r(t) - \frac{\sigma_1^2(t)}{2} - \prod_{0<t_k<t} (1+h_k) a(t) y + \prod_{0<t_k<t-\tau} (1+h_k) b(t) y(t-\tau) \\
&\quad + c(t) \int_{-\infty}^{0} \prod_{0<t_k<t+\theta} (1+h_k) y(t+\theta)\, d\mu(\theta) \\
&\quad - \frac{\left(\prod_{0<t_k<t}(1+h_k)\right)^2 \sigma_2^2(t) y^2}{2} \Bigg] dt \\
&\quad + \sigma_1(t) d\omega_1(t) + \prod_{0<t_k<t} (1+h_k) \sigma_2(t) y\, d\omega_2(t).
\end{aligned}$$

Integrating both sides from 0 to t, where $t \in [0, t_1]$ or $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots,$ we obtain

$$\begin{aligned}
&\ln y(t) - \ln y(0) \\
&= \int_0^t \Bigg[r(s) - \frac{\sigma_1^2(s)}{2} - \prod_{0<t_k<s} (1+h_k) a(s) y(s) + \prod_{0<t_k<s-\tau} (1+h_k) b(s) y(s-\tau) \\
&\quad + c(s) \int_{-\infty}^{0} \prod_{0<t_k<s+\theta} (1+h_k) y(s+\theta)\, d\mu(\theta) \\
&\quad - \frac{\left(\prod_{0<t_k<t}(1+h_k)\right)^2 \sigma_2^2(s) y^2(s)}{2} \Bigg] ds \\
&\quad + \int_0^t \sigma_1(s)\, d\omega_1(s) + \int_0^t \prod_{0<t_k<s} (1+h_k) \sigma_2(s) y(s)\, d\omega_2(s) \\
&= \int_0^t \Bigg[r(s) - \frac{\sigma_1^2(s)}{2} - a(s)x(s) + b(s)x(s-\tau) \\
&\quad + c(s) \int_{-\infty}^{0} x(s+\theta)\, d\mu(\theta) - \frac{\sigma_2^2(s) x^2(s)}{2} \Bigg] ds \\
&\quad + \int_0^t \sigma_1(s)\, d\omega_1(s) + \int_0^t \sigma_2(s) x(s)\, d\omega_2(s).
\end{aligned} \tag{3.1}$$

On the other hand,

$$\begin{aligned}\int_0^t b(s)x(s-\tau)\,ds &= \int_{-\tau}^{t-\tau} b(s+\tau)x(s)\,ds\\ &= \int_{-\tau}^{0} b(s+\tau)x(s)\,ds + \int_0^{t-\tau} b(s+\tau)x(s)\,ds\\ &\le \int_{-\tau}^{0} b(s+\tau)x(s)\,ds + \int_0^{t} b(s+\tau)x(s)\,ds.\end{aligned}\tag{3.2}$$

Therefore, for $t \in \overline{R}_+$, substituting (3.2) into (3.1) gives

$$\begin{aligned}\ln y(t) - \ln y(0) \le \int_0^t \Big[r(s) - \frac{\sigma_1^2(s)}{2} - (a(s) - b(s+\tau))x(s)\\ + c(s)\int_{-\infty}^0 x(s+\theta)\,d\mu(\theta) - \frac{\sigma_2^2(s)x^2(s)}{2}\Big]\,ds\\ + \int_{-\tau}^0 b(s+\tau)x(s)\,ds + M_1(t) + M_2(t),\end{aligned}\tag{3.3}$$

where $M_1(t) = \int_0^t \sigma_1(s)\,d\omega_1(s)$ and $M_2(t) = \int_0^t \sigma_2(s)x(s)\,d\omega_2(s)$.

By the assumptions (A2) and (A3), we can compute that

$$\begin{aligned}&\int_0^t c(s)\int_{-\infty}^0 x(s+\theta)\,d\mu(\theta)\,ds\\ &= \int_0^t c(s)\Big[\int_{-\infty}^{-s} x(s+\theta)\,d\mu(\theta)\,ds + \int_{-s}^0 x(s+\theta)\,d\mu(\theta)\Big]\,ds\\ &= \int_0^t c(s)\,ds\int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)}x(s+\theta)e^{-\mathbf{r}(s+\theta)}\,d\mu(\theta) + \int_{-t}^0 d\mu(\theta)\int_{-\theta}^t c(s)x(s+\theta)\,ds\\ &\le c^u\|\xi\|_{c_g}\int_0^t e^{-\mathbf{r}s}\,ds\int_{-\infty}^0 e^{-\mathbf{r}\theta}\,d\mu(\theta) + c^u\int_{-\infty}^0 d\mu(\theta)\int_0^t x(s)\,ds\\ &\le c^u\|\xi\|_{c_g}\int_0^t e^{-\mathbf{r}s}\,ds\int_{-\infty}^0 e^{-2\mathbf{r}\theta}\,d\mu(\theta) + c^u\int_{-\infty}^0 d\mu(\theta)\int_0^t x(s)\,ds\\ &\le \frac{1}{\mathbf{r}}c^u\|\xi\|_{c_g}\mu_{\mathbf{r}}(1-e^{-\mathbf{r}t}) + c^u\int_0^t x(s)\,ds.\end{aligned}$$

Consequently,

$$\begin{aligned}\ln y(t) - \ln y(0) \le \int_0^t \Big[r(s) - \frac{\sigma_1^2(s)}{2} - (a(s) - b(s+\tau) - c^u)x(s) - \frac{\sigma_2^2(s)x^2(s)}{2}\Big]\,ds\\ + \frac{1}{\mathbf{r}}c^u\|\xi\|_{c_{\mathbf{r}}}\mu_{\mathbf{r}}(1-e^{-\mathbf{r}t}) + \int_{-\tau}^0 b(s+\tau)x(s)\,ds + M_1(t) + M_2(t).\end{aligned}$$

The quadratic variation of $M_1(t)$ is $\langle M_1\rangle(t) = \int_0^t \sigma_1^2(s)\,ds \le (\sigma_1^u)^2 t$. Making use of the strong law of large numbers for martingales (see e.g. [30] on p. 16) leads to

$$\lim_{t\to+\infty} \frac{M_1(t)}{t} = 0, \quad \text{a.s.} \tag{3.4}$$

The quadratic variation of $M_2(t)$ is $\langle M_2\rangle(t) = \int_0^t \sigma_2^2(s)x^2(s)\,ds$. By virtue of the exponential martingale inequality, for any positive constants T_0, γ and δ, we have

$$\mathcal{P}\left\{\sup_{0\le t\le T_0}\left[M_2(t) - \frac{\gamma}{2}\langle M_2\rangle(t)\right] > \delta\right\} \le e^{-\gamma\delta}.$$

Choose $T_0 = k$, $\gamma = 1$, $\delta = 2\ln k$. Then it follows that

$$\mathcal{P}\left\{\sup_{0\le t\le k}\left[M_2(t) - \frac{1}{2}\langle M_2\rangle(t)\right] > 2\ln k\right\} \le \frac{1}{k^2}.$$

The Borel–Cantelli lemma implies that for almost all $\omega \in \Omega$, there is a random integer $k_0 = k_0(\omega)$ such that for $k \ge k_0$,

$$\sup_{0\le t\le k}\left[M_2(t) - \frac{1}{2}\langle M_2\rangle(t)\right] \le 2\ln k.$$

This is to say

$$M_2(t) \le 2\ln k + \frac{1}{2}\langle M_2\rangle(t) = 2\ln k + \frac{1}{2}\int_0^t \sigma_2^2(s)x^2(s)\,ds$$

for all $0 \le t \le k$, $k \ge k_0$ a.s. Substituting this inequality into (3.3), we can obtain that

$$\begin{aligned}\ln y(t) - \ln y(0) \le{}& \int_{-\tau}^0 b(s+\tau)x(s)\,ds \\ &+ \int_0^t \left[r(s) - \frac{\sigma_1^2(s)}{2} - (a(s) - b(s+\tau) - c^u)x(s)\right] ds \\ &+ 2\ln k + \frac{1}{\mathbf{r}}c^u\|\xi\|_{c_g}\mu_{\mathbf{r}}(1 - e^{-\mathbf{r}t}) + M_1(t)\end{aligned} \tag{3.5}$$

for all $0 \le t \le k$, $k \ge k_0$ a.s. On the other hand, it follows from (3.5) that

$$\begin{aligned}&\sum_{0<t_k<t} \ln(1+h_k) + \ln y(t) - \ln y(0) \\ &\le \sum_{0<t_k<t} \ln(1+h_k) + \int_{-\tau}^0 b(s+\tau)x(s)\,ds \\ &\quad + \int_0^t \left[r(s) - \frac{\sigma_1^2(s)}{2} - (a(s) - b(s+\tau) - c^u)x(s)\right] ds \\ &\quad + 2\ln k + \frac{1}{\mathbf{r}}c^u\|\xi\|_{c_g}\mu_{\mathbf{r}}(1 - e^{-\mathbf{r}t}) + M_1(t)\end{aligned}$$

for all $0 \le t \le k$, $k \ge k_0$ a.s. In other words, we have shown that

$$
\begin{aligned}
\ln x(t) - \ln x(0) \le {} & \sum_{0<t_k<t} \ln(1+h_k) + \int_{-\tau}^{0} b(s+\tau)x(s)\,ds \\
& + \int_0^t \left[r(s) - \frac{\sigma_1^2(s)}{2} - (a(s) - b(s+\tau) - c(s))x(s) \right] ds \\
& + 2\ln k + \frac{1}{\mathbf{r}} c^u \|\xi\|_{c_{\mathbf{r}}} \mu_{\mathbf{r}} (1 - e^{-\mathbf{r}t}) + M_1(t)
\end{aligned}
\tag{3.6}
$$

for all $0 \le t \le k$, $k \ge k_0$ a.s. Therefore, for $k-1 \le t \le k$, $k \ge k_0$, a.s., we have

$$
\begin{aligned}
\ln x(t) - \ln x(0) \le {} & \sum_{0<t_k<t} \ln(1+h_k) + \int_{-\tau}^{0} b(s+\tau)x(s)\,ds + \int_0^t \left[r(s) - \frac{\sigma_1^2(s)}{2} \right] ds \\
& + 2\ln k + \frac{1}{\mathbf{r}} c^u \|\xi\|_{c_g} \mu_{\mathbf{r}} (1 - e^{-\mathbf{r}t}) + M_1(t).
\end{aligned}
$$

Then we have the desired assertion by the assumption (A2) and the equality (3.4).

CASE 2. $b(t) \ge 0$ and $c(t) < 0$; $b(t) < 0$ and $c(t) \ge 0$; $b(t) < 0$ and $c(t) < 0$.

Applying the arguments above and comparison theorem of stochastic differential equations, we can easily draw the conclusion. □

Theorem 3.2. *Let the assumptions* (A1)–(A3) *hold. If* $G^* = 0$ *and* $\inf_{t\in\overline{R}_+}\{a(t) - b(t+\tau) - c^u\} > 0$, *then the population modeled by Equation* (1.3) *is non-persistent in the mean a.s.*

Proof. We only give the proof of case $b(t) \ge 0$ and $c(t) \ge 0$. Making use of comparison theorem of stochastic differential equations, the proof of case $b(t) \ge 0$ and $c(t) < 0$; $b(t) < 0$ and $c(t) \ge 0$; $b(t) < 0$ and $c(t) < 0$ are easily derived, respectively. From $G^* = 0$ and the assumption (A2), for arbitrarily $\varepsilon > 0$, there exists a constant T such that $t^{-1}\big[\sum_{0<t_k<t} \ln(1+h_k) + \int_0^t f(s)\,ds\big] + t^{-1}\int_{-\tau}^{0} b(s+\tau)x(s)\,ds + t^{-1}(1/r)c^u\|\xi\|_{c_g}\mu_{\mathbf{r}}(1-e^{-\mathbf{r}t}) + 2\ln k/t + M_1(t)/t < \varepsilon$ for all $T \le k-1 \le t \le k$, $k \ge k_0$ a.s. Substituting this inequality into (3.6) yields

$$
\begin{aligned}
\ln x(t) - \ln x(0) \le {} & \sum_{0<t_k<t} \ln(1+h_k) + \int_{-\tau}^{0} b(s+\tau)x(s)\,ds + \int_0^t \left[r(s) - \frac{\sigma_1^2(s)}{2} \right] ds \\
& + 2\ln k + \frac{1}{\mathbf{r}} c^u \|\xi\|_{c_g} \mu_{\mathbf{r}} (1 - e^{-\mathbf{r}t}) + M_1(t) \\
< {} & \varepsilon t - \int_0^t (a(s) - b(s+\tau) - c^u)x(s)\,ds
\end{aligned}
$$

for all $T \le k-1 \le t \le k$, $k \ge k_0$ a.s.

Define $h(t) = \int_0^t x(s)\,ds$ and $I = \inf_{t\in\overline{R}_+}[a(t) - b(t+\tau) - c^u]$. The rest of proof is similar to Theorem 3 in [15] and is hence omitted. $\square$

Theorem 3.3. *Let the assumptions* (A1)–(A3) *hold. If* $G^* > 0$ *and* $c(t) \geq 0$, *then the population* $x(t)$ *modeled by* (1.3) *is weakly persistent a.s.*

Proof. If this assertion is not true, let $F = \{\limsup_{t\to+\infty} x(t) = 0\}$ and suppose $P(F) > 0$. In the light of (3.1), we derive

$$\begin{aligned}
&\sum_{0<t_k<t} \ln(1+h_k) + \ln y(t) - \ln y(0)\\
&= \sum_{0<t_k<t} \ln(1+h_k) + \int_0^t \bigg[r(s) - \frac{\sigma_1^2(s)}{2} - a(s)x(s) + b(s)x(s-\tau)\\
&\qquad + c(s)\int_{-\infty}^0 x(s+\theta)\,d\mu(\theta) - \frac{\sigma_2^2(s)x^2(s)}{2}\bigg]\,ds\\
&\quad + M_1(t) + M_2(t),
\end{aligned}$$

which implies

$$\begin{aligned}
&t^{-1}\ln x(t) - t^{-1}\ln x(0)\\
&= t^{-1}\sum_{0<t_k<t} \ln(1+h_k) + t^{-1}\int_0^t \bigg[r(s) - \frac{\sigma_1^2(s)}{2} - a(s)x(s) + b(s)x(s-\tau)\\
&\qquad + c(s)\int_{-\infty}^0 x(s+\theta)\,d\mu(\theta) - \frac{\sigma_2^2(s)x^2(s)}{2}\bigg]\,ds\\
&\quad + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}.
\end{aligned} \tag{3.7}$$

On the other hand, for $\forall\omega \in F$, we have $\lim_{t\to+\infty} x(t,\omega) = 0$. Consequently, by the law of large numbers for local martingales (see e.g. [30, p. 12]), we obtain that $\lim_{t\to+\infty} M_2(t)/t = 0$. Substituting this equality, $c(t) \geq 0$ and (3.4) into (3.7), one can deduce a contradiction

$$\begin{aligned}
0 &\geq \limsup_{t\to+\infty}[t^{-1}\ln x(t,\omega)] = G^* + \limsup_{t\to+\infty} t^{-1}\int_0^t\int_{-\infty}^0 x(s+\theta)\,d\mu(\theta)\,ds\\
&\geq G^* > 0.
\end{aligned}$$

$\square$

REMARK 3.1. Theorems 3.1–3.3 have a direct and fantastic biological explanation. It is obvious to see that the extinction and persistence of population $x(t)$ modeled by (1.3) largely rely on the assumptions (A1)–(A3), G^*, $c(t)$ and $\inf_{t\in\overline{R}_+}\{a(t) - b(t+\tau) - c^u\}$. Under the assumption (A1)–(A3), if $G^* > 0$ and $c(t) \geq 0$, the population $x(t)$ will be weakly persistent; Under the assumptions (A1)–(A3), if $G^* < 0$ and $\inf_{t\in\overline{R}_+}\{a(t) - b(t+\tau) - c^u\} \geq 0$, the population $x(t)$ will be extinct. That is to say, under the assumptions (A1)–(A3), if $\inf_{t\in\overline{R}_+}\{a(t) - b(t+\tau) - c^u\} \geq 0$ and $c(t) \geq 0$

hold, then G^* is the threshold between weak persistence and extinction for the population $x(t)$.

REMARK 3.2. Generally speaking, as the biology implied, Theorem 3.1 reveals that the population probably will go to an end in the worst cases, while Theorem 3.2 shows that the living chances are considerably rare. From Theorem 3.3 we can easily find that the population size is limited to zero with the time permitted, however, the opportunity of the survival of it still exist. This can well explain why the conditions are gradually stronger from Theorem 3.1 to Theorem 3.3.

When it comes to the study of population system, the role of stochastic permanence indicating the eternal existence of the population, can never be ignorant with its theoretical and practical significance. And its importance has catched the eyes of scientists all over the world. So now let us show that $x(t)$ modeled by Equation (1.3) is stochastically permanent in some cases.

Assumption (A4): There are two positive constants m and M such that $m \leq \prod_{0<t_k<t}(1+h_k) \leq M$ for all $t > 0$.

REMARK 3.3. Assumption A4 is easy to be satisfied. For example, if $h_k = e^{(-1)^{k+1}/k} - 1$, then $e^{0.5} < \prod_{0<t_k<t}(1+h_k) < e$ for all $t > 0$. Thus $1 \leq \prod_{0<t_k<t}(1+h_k) \leq e$ for all $t > 0$.

Theorem 3.4. *Let the assumptions* (A1)–(A4) *hold. If* $(r(t) - \sigma_1^2(t)/2)_* > 0$, $b(t) \geq 0$ *and* $c(t) \geq 0$, *then the population* $x(t)$ *represented by Equation* (1.3) *will be stochastic permanence.*

Proof. First, we claim that for arbitrary $\varepsilon > 0$, there is constant $\beta > 0$ such that $\liminf_{t\to+\infty} \mathcal{P}\{x(t) \geq \beta\} \geq 1 - \varepsilon$.

Define $V_1(y) = 1/y^2$ for $y \in R_+$. Applying Itô's formula to Equation 2.4 we can obtain that

$$
\begin{aligned}
dV_1(y) &= -2y^{-3}\,dy + 3y^{-4}(dy)^2 \\
&= 2V_1(y)\Bigg[1.5\Bigg(\prod_{0<t_k<t}(1+h_k)\Bigg)^2 \sigma_2^2(t)y^2 + \prod_{0<t_k<t}(1+h_k)a(t)y - r(t) + 1.5\sigma_1^2(t) \\
&\qquad - \prod_{0<t_k<t-\tau}(1+h_k)b(t)y(t-\tau) \\
&\qquad - c(t)\int_{-\infty}^{0} \prod_{0<t_k<t+\ theta}(1+h_k)y(t+\theta)\,d\mu(\theta)\Bigg]dt \\
&\quad - 2\sigma_1(t)y^{-2}\,d\omega_1(t) - 2\prod_{0<t_k<t}(1+h_k)\sigma_2(t)y^{-1}\,d\omega_2(t).
\end{aligned}
$$

Since $(r(t)-\sigma_1^2(t)/2)_* > 0$, we can choose a sufficient small constant $0<\kappa<1$ such that $(r(t)-\sigma_1^2(t)/2)_* - \kappa(\sigma_1^u)^2 > 0$.

Define

$$V_2(y) = (1 + V_1(y))^\kappa.$$

Making use of Itô's formula again leads to

$$\begin{aligned}
dV_2 &= \kappa(1+V_1(y))^{\kappa-1}\,dV_1 + 0.5\kappa(\kappa-1)(1+V_1(y))^{\kappa-2}(dV_1)^2\\
&= \kappa(1+V_1(y))^{\kappa-2}\Bigg\{(1+V_1(y))2V_1(y)\Bigg[1.5\Bigg(\prod_{0<t_k<t}(1+h_k)\Bigg)^2\sigma_2^2(t)y^2\\
&\qquad\qquad + \prod_{0<t_k<t}(1+h_k)a(t)y - r(t) + 1.5\sigma_1^2(t)\\
&\qquad\qquad - \prod_{0<t_k<t-\tau}(1+h_k)b(t)y(t-\tau)\\
&\qquad\qquad - c(t)\int_{-\infty}^0 \prod_{0<t_k<t+\theta}(1+h_k)y(t+\theta)\,d\mu(\theta)\Bigg]\\
&\qquad + 2\sigma_1^2(t)(\kappa-1)V_1^2(y)\\
&\qquad + 2\Bigg(\prod_{0<t_k<t}(1+h_k)\Bigg)^2\sigma_2^2(t)(\kappa-1)V_1(y)\Bigg\}\,dt\\
&\quad - 2\kappa(1+V_1(y))^{\kappa-1}y^{-2}\sigma_1(t)d\omega_1(t)\\
&\quad - 2\kappa(1+V_1(y))^{\kappa-1}y^{-1}\prod_{0<t_k<t}(1+h_k)\sigma_2(t)d\omega_2(t)\\
&= \kappa(1+V_1(y))^{\kappa-2}\Bigg\{\Bigg(-2r(t)+3\sigma_1^2(t) - 2\prod_{0<t_k<t-\tau}(1+h_k)b(t)y(t-\tau)\\
&\qquad - 2c(t)\int_{-\infty}^0 \prod_{0<t_k<t+\theta}(1+h_k)y(t+\theta)\,d\mu(\theta)\\
&\qquad + 2\sigma_1^2(t)(\kappa-1)\Bigg)V_1^2(y) + 2\prod_{0<t_k<t}(1+h_k)a(t)V_1^{1.5}(y)\\
&\qquad + \Bigg(3\sigma_1^2(t) - 2r(t) - 2\prod_{0<t_k<t-\tau}(1+h_k)b(t)y(t-\tau)\\
&\qquad\quad - 2c(t)\int_{-\infty}^0 \prod_{0<t_k<t+\theta}(1+h_k)y(t+\theta)\,d\mu(\theta)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + (2\kappa+1)\left(\left(\prod_{0<t_k<t}(1+h_k)\right)^2\sigma_2^2(t)\right)V_1(y) \\
&\qquad\qquad + 2\prod_{0<t_k<t}(1+h_k)a(t)V_1^{0.5}(y) + 3\left(\prod_{0<t_k<t}(1+h_k)\right)^2\sigma_2^2(t)\Bigg\}\,dt \\
&\quad - 2\kappa(1+V_1(y))^{\kappa-1}y^{-2}\sigma_1(t)\,d\omega_1(t) \\
&\quad - 2\kappa(1+V_1(y))^{\kappa-1}y^{-1}\prod_{0<t_k<t}(1+h_k)\sigma_2(t)\,d\omega_2(t) \\
&\le \kappa(1+V_1(y))^{\kappa-2}\{(-2r(t)+\sigma_1^2(t)+2\kappa\sigma_1^2(t))V_1^2(y)+2Ma(t)V_1^{1.5}(y) \\
&\quad +(3\sigma_1^2(t)-2r(t)+(2\kappa+1)M^2\sigma_2^2(t))V_1(y)+2Ma(t)V_1^{0.5}(y)+3M^2\sigma_2^2(t)\}\,dt \\
&\quad - 2\kappa(1+V_1(y))^{\kappa-1}y^{-2}\sigma_1(t)\,d\omega_1(t) \\
&\quad - 2\kappa(1+V_1(y))^{\kappa-1}y^{-1}\prod_{0<t_k<t}(1+h_k)\sigma_2(t)\,d\omega_2(t) \\
&\le \kappa(1+V_1(y))^{\kappa-2}\Bigg\{-2\left(\left(r(t)-\frac{\sigma_1^2(t)}{2}\right)_* - \varepsilon - \kappa(\sigma_1^u)^2\right)V_1^2(y)+2a^uMV_1^{1.5}(y) \\
&\qquad\qquad + (3(\sigma_1^u)^2-2r^l+(2\kappa+1)M^2(\sigma_2^u)^2)V_1(y)+2a^uMV_1^{0.5}(y) \\
&\qquad\qquad + 3M^2(\sigma_2^u)^2\Bigg\}\,dt \\
&\quad - 2\kappa(1+V_1(y))^{\kappa-1}y^{-2}\sigma_1(t)\,d\omega_1(t) \\
&\quad - 2\kappa(1+V_1(y))^{\kappa-1}y^{-1}\prod_{0<t_k<t}(1+h_k)\sigma_2(t)\,d\omega_2(t)
\end{aligned}
$$

for sufficiently large $t \ge T$. Now, let $\eta > 0$ be sufficiently small satisfy

$$0 < \eta/\kappa < \left(r(t) - \frac{\sigma_1^2(t)}{2}\right)_* - \kappa(\sigma_1^u)^2 - \varepsilon.$$

Define $V_3(y) = e^{\eta t}V_2(y)$. By virtue of Itô's formula, we derive

$$
\begin{aligned}
dV_3(y) &= \eta e^{\eta t}V_2(y) + e^{\eta t}\,dV_2(y) \\
&\le \kappa e^{\eta t}(1+V_1(y(t)))^{\kappa-2}\Bigg\{\eta(1+V_1(y))^2/\kappa \\
&\qquad - 2\left(\left(r(t)-\frac{\sigma_1^2(t)}{2}\right)_* - \varepsilon - \kappa(\sigma_1^u)^2\right)V_1^2(y) \\
&\qquad + 2a^uMV_1^{1.5}(y) + (3(\sigma_1^u)^2 - 2r_* \\
&\qquad + (2\kappa+1)M^2(\sigma_2^u)^2)V_1(y) + 2a^uMV_1^{0.5}(y)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad + 3M^2(\sigma_2^u)^2\Big\}\,dt\\
&\quad - 2\kappa e^{\eta t}(1+V_1(y))^{\kappa-1}y^{-2}\sigma_1(t)\,d\omega_1(t)\\
&\quad - 2\kappa e^{\eta t}(1+V_1(y))^{\kappa-1}y^{-1}\prod_{0<t_k<t}(1+h_k)\sigma_2(t)\,d\omega_2(t)\\
&\le \kappa e^{\eta t}(1+V_1(y(t))^{\kappa-2}\bigg\{-2\bigg(\bigg(r(t)-\frac{\sigma_1^2(t)}{2}\bigg)_* - \varepsilon - \kappa(\sigma_1^u)^2 - \eta/\kappa\bigg)V_1^2(y)\\
&\qquad + 2a^u M V_1^{1.5}(y) + (3(\sigma_1^u)^2 - 2r_* + (2\kappa+1)M^2(\sigma_2^u)^2\\
&\qquad + 2\eta/\kappa)V_1(y) + 2a^u M V_1^{0.5}(y) + 3M^2(\sigma_2^u)^2 + \eta/\kappa\bigg\}\,dt\\
&\quad - 2\theta e^{\eta t}(1+V_1(y))^{\kappa-1}y^{-2}\sigma_1(t)\,d\omega_1(t)\\
&\quad - 2\kappa e^{\eta t}(1+V_1(y))^{\kappa-1}y^{-1}\prod_{0<t_k<t}(1+h_k)\sigma_2(t)\,d\omega_2(t)\\
&= e^{\eta t}H(y)\,dt - 2\kappa e^{\eta t}(1+V_1(y))^{\kappa-1}y^{-2}\sigma_1(t)d\omega_1(t)\\
&\quad - 2\kappa e^{\eta t}(1+V_1(y))^{\kappa-1}y^{-1}\prod_{0<t_k<t}(1+h_k)\sigma_2(t)\,d\omega_2(t)
\end{aligned}
$$

for $t \ge T$. Note that $H(y)$ is bounded in R_+, namely $H = \sup_{y\in R_+} H(y) < +\infty$. Consequently,

$$
\begin{aligned}
dV_3(y(t)) = He^{\eta t}\,dt &- 2\kappa e^{\eta t}(1+V_1(y(t)))^{\kappa-1}y^{-2}(t)\sigma_1(t)\,d\omega_1(t)\\
&- 2\kappa e^{\eta t}(1+V_1(y))^{\kappa-1}y^{-1}\prod_{0<t_k<t}(1+h_k)\sigma_2(t)\,d\omega_2(t)
\end{aligned}
$$

for sufficiently large t. Integrating both sides of the above inequality and then taking expectations, we have

$$
\begin{aligned}
E[V_3(y(t))] &= E[e^{\eta t}(1+V_1(y(t)))^{\kappa}]\\
&\le e^{\eta T}(1+V_1(y(T)))^{\kappa} + \frac{H}{\eta}(e^{\eta t} - e^{\eta T}).
\end{aligned}
$$

That is to say

$$
\limsup_{t\to+\infty} E[V_1^{\kappa}(y(t))] \le \limsup_{t\to+\infty} E[(1+V_1(y(t)))^{\kappa}] < \frac{H}{\eta}.
$$

In other words, we have just shown that

$$
\limsup_{t\to+\infty} E\left[\frac{1}{y^{2\kappa}(t)}\right] \le \frac{H}{\eta}.
$$

Then

$$\limsup_{t\to+\infty} E[1/x^{2\kappa}(t)] = \limsup_{t\to+\infty}\left[\prod_{0<t_k<t}(1+h_k)\right]^{-2\kappa} E[1/y^{2\kappa}(t)] \le m^{-2\kappa}\frac{H}{\eta} = H_1.$$

So for any $\varepsilon > 0$, set $\beta = \varepsilon^{1/2\kappa}/H_1^{1/2\kappa}$, by Chebyshev's inequality, one can derive that

$$\mathcal{P}\{x(t) < \beta\} = \mathcal{P}\left\{\frac{1}{x^{2\kappa}(t)} > \frac{1}{\beta^{2\kappa}}\right\} \le \frac{E[1/x^{2\kappa}(t)]}{1/\beta^{2\kappa}}.$$

This is to say

$$\limsup_{t\to+\infty} \mathcal{P}\{x(t) < \beta\} \le \beta^{2\kappa} H_1 = \varepsilon.$$

Consequently

$$\liminf_{t\to+\infty} \mathcal{P}\{x(t) \ge \beta\} \ge 1 - \varepsilon.$$

Next, we prove that for arbitrary $\varepsilon > 0$, there are constants $\alpha > 0$ such that $\liminf_{t\to+\infty} \mathcal{P}\{x(t) \le \alpha\} \ge 1 - \varepsilon$.

Let $0 < p < 1$ and choose $\varepsilon_1 \in (0, 2\mathbf{r})$, Applying Itô's formula to Equation (2.4) obtains

$$\begin{aligned}
dy^p(t) &= py^{p-1}(t)\,dy(t) + \frac{1}{2}p(p-1)y^{p-2}(t)(dy(t))^2 \\
&= py^{p-1}(t)\Bigg[\Bigg(y(t)\Bigg(r(t) - \prod_{0<t_k<t}(1+h_k)a(t)y(t) + \prod_{0<t_k<t-\tau}(1+h_k)b(t)y(t-\tau) \\
&\qquad + c(t)\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+h_k)y(t+\theta)\,d\mu(\theta)\Bigg)\Bigg)\,dt \\
&\qquad + \sigma_1(t)y(t)\,d\omega_1(t) + \prod_{0<t_k<t}(1+h_k)\sigma_2(t)y^2(t)\,d\omega_2(t)\Bigg] \\
&\quad + \frac{1}{2}p(p-1)\sigma_1^2(t)y^p(t)\,dt + \frac{1}{2}p(p-1)\left(\prod_{0<t_k<t}(1+h_k)\right)^2\sigma_2^2(t)y^{p+2}(t)\,dt \\
&\le py^{p-1}(t)\Bigg[\Bigg(y(t)\Bigg(r(t) - ma(t)y(t) + Mb(t)y(t-\tau) \\
&\qquad + Mc(t)\int_{-\infty}^{0} y(t+\theta)\,d\mu(\theta)\Bigg)\Bigg)\,dt \\
&\qquad + \sigma_1(t)y(t)\,d\omega_1(t) + \prod_{0<t_k<t}(1+h_k)\sigma_2(t)y^2(t)\,d\omega_2(t)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}p(p-1)\sigma_1^2(t)y^p(t)\,dt+\frac{1}{2}p(p-1)M^2\sigma_2^2(t)y^{p+2}(t)\,dt\\
\le{}&\bigg[r(t)py^p(t)+\frac{p^2M^2b^2(t)y^{2p}(t)}{4}+y^2(t-\tau)+\frac{p^2M^2c^2(t)y^{2p}(t)}{4}\\
&+\int_{-\infty}^0 y^2(t+\theta)\,d\mu(\theta)\bigg]\,dt\\
&+p\sigma_1(t)y^p(t)\,d\omega_1(t)+p\sigma_2(t)\prod_{0<t_k<t}(1+h_k)y^{p+1}(t)\,d\omega_2(t)\\
&-\frac{1}{2}p(1-p)\sigma_1^2(t)y^p(t)\,dt-\frac{1}{2}p(1-p)M^2\sigma_2^2(t)y^{p+2}(t)\,dt\\
={}&F(y(t))\,dt\\
&-\bigg[\varepsilon_1y^p(t)+e^{\varepsilon_1\tau}y^2(t)-y^2(t-\tau)-\int_{-\infty}^0 y^2(t+\theta)\,d\mu(\theta)+\mu_{\mathbf{r}}y^2(t)\bigg]\,dt\\
&+p\sigma_1(t)y^p(t)\,d\omega_1(t)+p\sigma_2(t)\prod_{0<t_k<t}(1+h_k)y^{p+1}(t)\,d\omega_2(t),
\end{aligned}
$$

where

$$
\begin{aligned}
F(y)={}&e^{\varepsilon_1\tau}y^2+\mu_{\mathbf{r}}y^2+(\varepsilon_1+r(t)p)y^p+p^2b^2(t)y^{2p}(t)+p^2c^2(t)y^{2p}\\
&-\frac{1}{2}p(1-p)M^2\sigma_1^2(t)y^p-\frac{1}{2}p(1-p)M^2\sigma_2^2(t)y^{2+p}.
\end{aligned}
$$

From $0<p<1$ and the assumption A2, we have $F(y)$ is bounded in R_+, namely

$$
H_2=\sup_{y\in R_+}F(y)<+\infty.
$$

Therefore we have

$$
\begin{aligned}
dy^p(t)={}&[H_2-\varepsilon_1y^p(t)-e^{\varepsilon_1\tau}y^2(t)+y^2(t-\tau)]\,dt\\
&+\int_{-\infty}^0 y^2(t+\theta)\,d\mu(\theta)\,dt-\mu_{\mathbf{r}}y^2(t)\,dt\\
&+p\sigma_1(t)y^p(t)\,d\omega_1(t)+p\sigma_2(t)\prod_{0<t_k<t}(1+h_k)y^{p+1}(t)\,d\omega_2(t).
\end{aligned}
$$

Once again by the Itô's formula we have

$$
\begin{aligned}
&d[e^{\varepsilon_1t}y^p(t)]\\
&=e^{\varepsilon_1t}[\varepsilon_1y^p(t)dt+dy^p(t)]\\
&\le e^{\varepsilon_1t}\bigg[H_2-e^{\varepsilon_1\tau}y^2(t)+y^2(t-\tau)+\int_{-\infty}^0 y^2(t+\theta)\,d\mu(\theta)\,ds-\mu_{\mathbf{r}}y^2(t)\bigg]\,dt\\
&\quad+e^{\varepsilon_1t}\bigg(p\sigma_1(t)y^p(t)\,d\omega_1(t)+p\sigma_2(t)\prod_{0<t_k<t}(1+h_k)y^{p+1}(t)\,d\omega_2(t)\bigg).
\end{aligned}
$$

Hence we derive that

$$
\begin{aligned}
e^{\varepsilon_1 t} E[y^p(t)] &\le \xi^p(0) + \frac{e^{\varepsilon_1 t} H_2}{\varepsilon_1} - \frac{H_2}{\varepsilon_1} \\
&\quad - E\int_0^t e^{\varepsilon_1 s + \varepsilon_1 \tau} y^2(s)\,ds + E\int_0^t e^{\varepsilon_1 s} y^2(s-\tau)\,ds \\
&\quad + E\int_0^t e^{\varepsilon_1 s}\int_{-\infty}^0 y^2(s+\theta)\,d\mu(\theta)\,ds - E\int_0^t \mu_{\mathbf{r}} e^{\varepsilon_1 s} y^2(s)\,ds \\
&= \xi^p(0) + \frac{e^{\varepsilon_1 t} H_2}{\varepsilon_1} - \frac{H_2}{\varepsilon_1} \\
&\quad - E\int_0^t e^{\varepsilon_1 s + \varepsilon_1 \tau} y^2(s)\,ds + E\int_{-\tau}^{t-\tau} e^{\varepsilon_1 s + \varepsilon_1 \tau} y^2(s)\,ds \\
&\quad + E\int_0^t e^{\varepsilon_1 s}\int_{-\infty}^0 y^2(s+\theta)\,d\mu(\theta)\,ds - E\int_0^t \mu_{\mathbf{r}} e^{\varepsilon_1 s} y^2(s)\,ds \\
&\le \xi^p(0) + \frac{e^{\varepsilon_1 t} H_2}{\varepsilon_1} - \frac{H_2}{\varepsilon_1} + \int_{-\tau}^0 e^{\varepsilon_1 s + \varepsilon_1 \tau} y^2(s)\,ds \\
&\quad + E\int_0^t e^{\varepsilon_1 s}\int_{-\infty}^0 y^2(s+\theta)\,d\mu(\theta)\,ds - E\mu_{\mathbf{r}}\int_0^t e^{\varepsilon_1 s} y^2(s)\,ds.
\end{aligned}
$$

From the assumptions (A1) and (A2), we have

$$
\begin{aligned}
&\int_0^t e^{\varepsilon_1 s}\int_{-\infty}^0 y^2(s+\theta)\,d\mu(\theta)\,ds \\
&= \int_0^t e^{\varepsilon_1 s}\left[\int_{-\infty}^{-s} y^2(s+\theta)\,d\mu(\theta) + \int_{-s}^0 y^2(s+\theta)\,d\mu(\theta)\right] ds \\
&= \int_0^t e^{\varepsilon_1 s}\,ds \int_{-\infty}^{-s} e^{2\mathbf{r}(s+\theta)} y^2(s+\theta) e^{-2\mathbf{r}(s+\theta)}\,d\mu(\theta) + \int_{-t}^0 d\mu(\theta)\int_{-\theta}^t e^{\varepsilon_1 s} y^2(s+\theta)\,ds \\
&= \int_0^t e^{\varepsilon_1 s}\,ds \int_{-\infty}^{-s} e^{2\mathbf{r}(s+\theta)} y^2(s+\theta) e^{-2\mathbf{r}(s+\theta)}\,d\mu(\theta) + \int_{-t}^0 d\mu(\theta)\int_0^{t+\theta} e^{\varepsilon_1 (s-\theta)} y^2(s)\,ds \\
&\le \|\xi\|_{c_g}^2 \int_0^t e^{(\varepsilon_1 - 2\mathbf{r})s}\,ds \int_{-\infty}^0 e^{-2\mathbf{r}\theta}\,d\mu(\theta) + \int_{-\infty}^0 e^{-\varepsilon_1 \theta}\,d\mu(\theta)\int_0^t e^{\varepsilon_1 s} y^2(s)\,ds \\
&\le \|\xi\|_{c_g}^2 \mu_{\mathbf{r}} t + \mu_{\mathbf{r}}\int_0^t e^{\varepsilon_1 s} y^2(s)\,ds.
\end{aligned}
$$

This immediately implies that

$$
\limsup_{t\to+\infty} E[y^p(t)] \le \frac{H_2}{\varepsilon_1}.
$$

Consequently,

$$\limsup_{t\to+\infty} E(x^p(t)) = \limsup_{t\to+\infty}\left[\prod_{0<t_k<t}(1+h_k)\right]^p E(x^p(t)) \le \left[M^p\frac{H_2}{\varepsilon_1}\right] = \alpha.$$

Then the desired assertion follows from the Chebyshev inequality. This completes the whole proof. □

REMARK 3.4. From Theorems 3.1–3.3, we found that the delay has no effect on the persistence and extinction of the stochastic model (1.3) in autonomous case.

REMARK 3.5. The present paper is the first attempt, so far as our knowledge is concerned, to investigate the stochastic population systems with infinite delay and impulsive perturbation at the phase space C_g. In view of

$$G^* = \limsup_{t\to+\infty} t^{-1}\left[\sum_{0<t_k<t}\ln(1+h_k) + \int_0^t (r(t) - 0.5\sigma_1^2(t))\,ds\right]$$

in Theorems 3.1–3.3, we can find that the impulse does not affect the properties including extinction, nonpersistence in the mean, weak persistence and stochastic permanence if the impulsive perturbations are bounded and some changes significantly if not.

4. Examples and numerical simulations

In this section, we shall cite an example to illustrate the analytical findings. For convenience, let the probability measure $\mu(\theta)$ be e^{θ} on $(-\infty, 0]$. Thus the stochastic nonautonomous the model (1.3) will be written as

$$\begin{cases} dx(t) = x(t)\Big[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)e^{-t}\displaystyle\int_{-\infty}^{0} e^{\theta}\xi(\theta)\,d\theta \\ \qquad\qquad + c(t)e^{-t}\displaystyle\int_0^t e^{\theta}x(\theta)\,d\theta\Big]\,dt \\ \qquad\qquad + \sigma_1(t)x(t)\,d\omega_1(t) + \sigma_2(t)x^2(t)\,d\omega_2(t), \quad t \ne t_k,\ K \in N, \\ x(t_k^+) - x(t_k) = h_k x(t_k), \quad k \in N. \end{cases} \tag{4.1}$$

By employing the Euler scheme to discretize this equation, where the integral term is approximated by using the composite $\boldsymbol{\theta}$-rule as a quadrature [33]. Taking $\xi(\theta) = e^{-0.5\theta}$

and $\tau \equiv 0.3$, we can obtain the discrete approximate solution with respect to (4.1):

$$\begin{cases} x_{k+1} = x_k + x_k\Big[r(k\Delta t) - a(k\Delta t)x_k + b(k\Delta t)x_{k-300} + c(k\Delta t)e^{-k\Delta t}\displaystyle\int_{-\infty}^{0} e^{1.5\theta}\, d\theta \\ \qquad\qquad + c(k\Delta t)e^{-k\Delta t}\displaystyle\sum_{j=0}^{k}\omega_j^{(k)}e^{j\Delta t}x_j\Big]\Delta t \\ \qquad + x_k(\Delta B_1)_k + x_k^2(\Delta B_2)_k, \quad t \neq t_k,\ K \in N, \\ x_{k+1} - x_k = h_k x_k, \quad t = t_k,\ k \in N, \end{cases} \tag{4.2}$$

where $(\Delta B_i)_k = B_i((k+1)\Delta t) - B_i(k\Delta t)$, $k = 0,1,2,\ldots$, $i = 1,2$. The general composite $\boldsymbol{\theta}$-rule has weights

$$\{\omega_0^{(k)}, \omega_1^{(k)}, \ldots, \omega_k^{(k)}\} = \{\boldsymbol{\theta}, 1, \ldots, 1-\boldsymbol{\theta}\}, \boldsymbol{\theta} \in [0, 1]$$

and $\sum_{j=0}^{k}\omega_j^{(k)} = k,\ k \geq 0$.

Here, we choose $r(t) = 0.2 + 0.05\sin t$, $a(t) = 0.2 + 0.01\cos t$, $b(t) = 0.03$, $c(t) = 0.06$, $\sigma_2(t) = 0.03$ and step size $\Delta t = 0.001$. In Fig. 1 (a), Fig. 1 (b) and Fig. 1 (c), we consider $\sigma_1^2(t) = 0.5 + 0.1\sin t$. The only difference between conditions of Fig. 1 (a), Fig. 1 (b) and Fig. 1 (c) is that the representation of h_k is different. In Fig. 1 (a), we choose $t_k = 10k$ and $h_k = 0$. Then the conditions of Theorem 3.1 are satisfied. In view of Theorem 3.1, the population $x(t)$ will be extinct. In Fig. 1 (b), we consider $t_k = 10k$ and $h_k = e^{0.5} - 1$. Then the conditions of Theorem 3.2 hold. By virtue of Theorem 3.2, population $x(t)$ will be nonpersistent in the mean. In Fig. 1 (c), we choose $t_k = 10k$ and $h_k = e^{0.7} - 1$, then the conditions of Theorem 3.3 are satisfied. That is to say, the population $x(t)$ will be weakly persistent. In Fig.1(d), we consider $\sigma_1^2(t) = 0.1 + 0.1\sin t$, $t_k = 10k$ and $h_k = e^{(-1)^{k+1}/k} - 1$. Then the conditions of Theorem 3.4 hold, which means that the population $x(t)$ will be stochastic permanence. By comparing Fig. 1 (a)–(c), we can see that the impulsive perturbation can change the properties of the population system significantly.

5. Conclusions and remarks

With the space C_g as the phase space, we investigate the persistence and extinction of an impulsive stochastic logistic model with infinite delay. Sufficient conditions for extinction are established as well as nonpersistence in the mean, weak persistence and stochastic permanence. In addition, the threshold between weak persistence and extinction is obtained.

Some interesting topics deserve our further engagement. One may put forward a more realistic and sophisticated model to integrate the colored noise into the model [32]. Another significant problem is devoted to multidimensional stochastic model with impulsive perturbation and infinite delay, and such investigations are to be done in future.

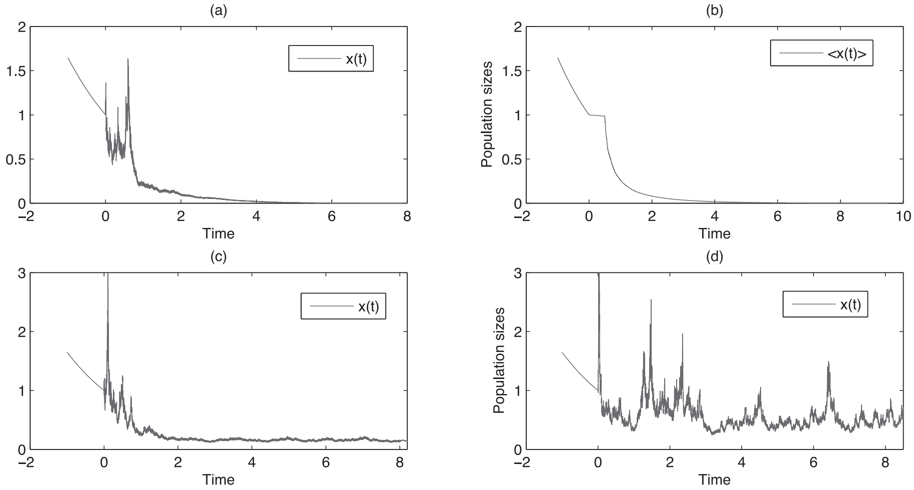

Fig. 1. The horizontal axis and the vertical axis in this and following figures represent the time t and the populations size $x(t)$ (step size $\Delta t = 0.001$).

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