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GROVE–SHIOHAMA TYPE SPHERE THEOREM IN FINSLER GEOMETRY

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Abstract

From radial curvature geometry's standpoint, we prove a few sphere theorems of the Grove–Shiohama type for certain classes of compact Finsler manifolds.

1. Introduction

Beyond a doubt, one of the most beautiful theorems in global Riemannian geometry is the diameter sphere theorem of Grove and Shiohama [3]. In their proof, Toponogov's comparison theorem (TCT) was very first applied seriously together with the critical point theory, introduced by themselves, of distance functions. That is, if a complete Riemannian manifold X has a critical point, say $q \in X \setminus \{p\}$, of the distance function d_p to a point $p \in X$, then q is the cut point of p. And hence d_p is not differentiable at q. However, they overcame the analytical obstruction by applying the original TCT to the triangle $\triangle(pxy)$ with the interior angle $\angle(pxy) \leq \pi/2$ at x. That is the point, i.e., they took the manifold into their hands by directly drawing segments on it.

Our purpose of this article is *to prove a sphere theorem of the Grove–Shiohama type for a certain class of forward complete Finsler manifolds whose radial flag curvatures are bounded below by* 1. Of course, our major tools to prove it are a TCT for such a class and the critical point theory, more precisely, Gromov's isotopy lemma ([2]). Such a TCT is easily proved by modifying the TCT established in [6] (see Section 2 in this article), and the isotopy lemma holds from a similar argument to the Riemannian case. The fact that, compared with the Riemannian case, there are few theorems on the relationship between the topology and the curvature of a Finsler manifold is the *worthy of note*. E.g., Shen's finiteness theorem ([10]), Rademacher's quarter pinched sphere theorem ([9]), and the finiteness of topological type and a diffeomorphism theorem to Euclidean spaces of the author with Ohta and Tanaka in [7].

To state our sphere theorems of the Grove–Shiohama type in Finsler case, we will introduce several notions in the geometry and radial curvature geometry: Let (M, F, p) denote a pair of a forward complete, connected, n-dimensional C^∞-Finsler manifold (M, F) with a base point $p \in M$, and $d\colon M \times M \to [0, \infty)$ denote the distance function

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induced from F. Remark that the *reversibility* $F(-v) = F(v)$ is not assumed in general, and hence $d(x, y) \neq d(y, x)$ is allowed.

For a local coordinate $(x^i)_{i=1}^n$ of an open subset $\mathcal{O} \subset M$, let $(x^i, v^j)_{i,j=1}^n$ be the coordinate of the tangent bundle $T\mathcal{O}$ over $\mathcal{O}$ such that

$$v := \sum_{j=1}^n v^j \frac{\partial}{\partial x^j}\bigg|_x, \quad x \in \mathcal{O}.$$

For each $v \in T_x M \setminus \{0\}$, the positive-definite $n \times n$ matrix

$$(g_{ij}(v))_{i,j=1}^n := \left(\frac{1}{2}\frac{\partial^2(F^2)}{\partial v^i \partial v^j}(v)\right)_{i,j=1}^n$$

provides us the Riemannian structure g_v of $T_x M$ by

$$g_v\left(\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\bigg|_x, \sum_{j=1}^n b^j \frac{\partial}{\partial x^j}\bigg|_x\right) := \sum_{i,j=1}^n g_{ij}(v) a^i b^j.$$

This is a Riemannian approximation of F in the direction v. For two linearly independent vectors $v, w \in T_x M \setminus \{0\}$, the *flag curvature* is defined by

$$K_M(v, w) := \frac{g_v(R^v(w, v)v, w)}{g_v(v, v)g_v(w, w) - g_v(v, w)^2},$$

where R^v denotes the curvature tensor induced from the Chern connection. Remark that $K_M(v, w)$ depends on the *flag* $\{sv + tw \mid s, t \in \mathbb{R}\}$, and also on the *flag pole* $\{sv \mid s > 0\}$.

Given $v, w \in T_x M \setminus \{0\}$, define the *tangent curvature* by

$$\mathcal{T}_M(v, w) := g_X(D_Y^Y Y(x) - D_Y^X Y(x), X(x)),$$

where the vector fields X, Y are extensions of v, w, and $D_v^w X(x)$ denotes the covariant derivative of X by v with reference vector w. Independence of $\mathcal{T}_M(v, w)$ from the choices of X, Y is easily checked. Note that $\mathcal{T}_M \equiv 0$ if and only if M is of *Berwald type* (see [11, Propositions 7.2.2, 10.1.1]). In Berwald spaces, for any $x, y \in M$, the tangent spaces $(T_x M, F|_{T_x M})$ and $(T_y M, F|_{T_y M})$ are mutually linearly isometric (cf. [1, Chapter 10]). In this sense, $\mathcal{T}_M$ measures the variety of tangent Minkowski normed spaces.

Let $\tilde{M}$ be a complete 2-dimensional Riemannian manifold, which is homeomorphic to $\mathbb{R}^2$ if $\tilde{M}$ is non-compact, or to $\mathbb{S}^2$ if $\tilde{M}$ is compact. Fix a base point $\tilde{p} \in \tilde{M}$. Then, we call the pair $(\tilde{M}, \tilde{p})$ a *model surface of revolution* if its Riemannian metric $d\tilde{s}^2$ is expressed in terms of the geodesic polar coordinate around $\tilde{p}$ as

$$d\tilde{s}^2 = dt^2 + f(t)^2\, d\theta^2, \quad (t, \theta) \in (0, a) \times \mathbb{S}^1_{\tilde{p}},$$

where $0 < a \le \infty$, $f\colon (0,a) \to \mathbb{R}$ denotes a positive smooth function which is extensible to a smooth odd function around 0, and $\mathbb{S}^1_{\tilde{p}} := \{v \in T_{\tilde{p}}\tilde{M} \mid \|v\| = 1\}$. Define the *radial curvature function* $G\colon [0, a) \to \mathbb{R}$ such that $G(t)$ is the Gaussian curvature at $\tilde{\gamma}(t)$, where $\tilde{\gamma}\colon [0, a) \to \tilde{M}$ is any (unit speed) meridian emanating from $\tilde{p}$. Note that f satisfies the differential equation $f'' + Gf = 0$ with initial conditions $f(0) = 0$ and $f'(0) = 1$. It is clear that, if $f(t) = t, \sin t$, and $\sinh t$, then $\tilde{M} = \mathbb{R}^2, \mathbb{S}^2$, and $\mathbb{H}^2(-1)$, respectively. We call $(\tilde{M}, \tilde{p})$ a *von Mangoldt surface* if G is non-increasing on $[0, a)$. A round sphere is the only compact, 'smooth' von Mangoldt surface, i.e., f satisfies $\lim_{t\uparrow a} f'(t) = -1$. If a von Mangoldt surface has the property $a < \infty$ and if it is not a round sphere, then $\lim_{t\uparrow a} f(t) = 0$ and $\lim_{t\uparrow a} f'(t) > -1$. Therefore, such a surface $(\tilde{M}, \tilde{p})$ has a singular point, say $\tilde{q} \in \tilde{M}$, at the maximal distance from $\tilde{p} \in \tilde{M}$ such that $d(\tilde{p}, \tilde{q}) = a$, and hence $\tilde{M}$ is an Alexandrov space. Its shape can be understood as a 'balloon' (See [4, Example 1.2]). On the other hand, paraboloids and 2-sheeted hyperboloids are typical examples of non-compact von Mangoldt surfaces. An atypical example of such a surface is found in [8, Example 1.2].

We say that a Finsler manifold (M, F, p) has the *radial flag curvature bounded below by that of a model surface of revolution* $(\tilde{M}, \tilde{p})$ if, along every unit speed minimal geodesic $\gamma\colon [0, l) \to M$ emanating from p, we have

$$K_M(\dot{\gamma}(t), w) \ge G(t)$$

for all $t \in [0,l)$ and $w \in T_{\gamma(t)}M$ linearly independent to $\dot{\gamma}(t)$. Also, we say that (M, F, p) has the *radial tangent curvature bounded below by a constant* $\delta \in (-\infty, 0]$ if, along every unit speed minimal geodesic $\gamma\colon [0, l) \to M$ emanating from p,

$$\mathcal{T}_M(\dot{\gamma}(t), w) \ge \delta$$

for all $w \in T_{\gamma(t)}M$.

We set $B^+_r(p) := \{x \in M \mid d(p, x) < r\}$,

$$\mathcal{G}_p(x) := \{\dot{\gamma}(l) \in T_xM \mid \gamma \text{ is a minimal geodesic segment from } p \text{ to } x\}, \tag{1.1}$$

where $l := d(p, x)$, $L_{\mathrm{m}}(c) := \int_0^a \max\{F(\dot{c}), F(-\dot{c})\}\, ds$, and $\mathrm{rad}_p := \sup_{x\in M} d(p, x)$.

Now, our main result is stated as follows:

Theorem 1.1. *Let (M, F, p) be a compact connected n-dimensional C^∞-Finsler manifold whose radial flag curvature is bounded below by* 1 *and radial tangent curvature is equal to* 0. *Assume that*

(1) *$F(w)^2 \ge g_v(w, w)$ for all $x \in B^+_{\pi/2}(p)$, $v \in \mathcal{G}_p(x)$, and $w \in T_xM$;*

(2) *$g_v(w, w) \ge F(w)^2$ for all $x \in M \setminus \overline{B^+_{\pi/2}(p)}$, $v \in \mathcal{G}_p(x)$ and $w \in T_xM$;*

(3) *the reverse curve $\bar{c}(s) := c(a-s)$ of c is geodesic and $L_{\mathrm{m}}(c) \le \mathrm{rad}_p$ for all minimal geodesic segments $c\colon [0, a] \to M \setminus \{p\}$.*

If $\mathrm{rad}_p > \pi/2$, then M is homeomorphic to the sphere $\mathbb{S}^n$.

REMARK 1.2. We give here related remarks for Theorem 1.1: Except for $L_{\mathrm{m}}(c) \leq \mathrm{rad}_p$, all conditions in the theorem are sufficient ones that make our TCTs hold (see Corollary 2.8 and Lemma 2.9 in Section 2). The biggest obstruction when we establish TCTs in Finsler geometry is the covariant derivative even though F is reversible. Thanks to the conditions (1) and (2), we can overcome the obstruction, i.e., by the (1) and $f'(t) = \cos t > 0$ on $[0, \pi/2)$, we can transplant the strictly convexness of $B_{\pi/2}(\tilde{p}) \subset \tilde{M} = \mathbb{S}^2$ to $B^+_{\pi/2}(p)$ (See Section 4), where the convexity on $B_{\pi/2}(\tilde{p})$ arises from the positive second fundamental form for $f' > 0$ on $[0, \pi/2)$. As well as, the strictly concaveness of $\tilde{M} \setminus \overline{B_{\pi/2}(\tilde{p})}$ is transplanted to $M \setminus \overline{B^+_{t_0}(p)}$ by the (2) and $f'(t) = \cos t < 0$ on $(\pi/2, \pi]$. Note that one may construct non-Riemannian spaces satisfying (1) and (2) (cf. [7, Example 1.3]). The geodesic property on $\bar{c}$ in the condition (3) and $\mathcal{T}_M(\dot{\gamma}(t), w) = 0$ just only imply $g_{\dot{\gamma}}(D^{\dot{\gamma}}_{\dot{c}} \dot{c}, \dot{\gamma}) = 0$. Note that $D^{\dot{\gamma}}_{\dot{c}} \dot{c} \neq 0$ in general. We can replace $L_{\mathrm{m}}(c) \leq \mathrm{rad}_p$ in (3) with the following weaker assumption:

$$L_{\mathrm{m}}(c) \begin{cases} < \pi & \text{for } c \text{ satisfying } c([0, a]) \cap (M \setminus B^+_{\pi/2}(p)) \neq \emptyset; \\ \leq \mathrm{rad}_p & \text{for } c \text{ emanating from } q \in \partial B^+_{\mathrm{rad}_p}(p) \text{ to any point in } B^+_{\pi/2}(p). \end{cases}$$

Here, note that $\partial B^+_{\mathrm{rad}_p}(p) = \{q\}$ (see Lemma 3.4). Remark that $\mathrm{diam}(M) \leq \pi$ from the Bonnet–Myers theorem ([1, Theorem 7.7.1]).

We can remove the (3) in Theorem 1.1 as follows:

Corollary 1.3. *Let (M, F, p) be a compact connected n-dimensional C^∞-Finsler manifold whose radial flag curvature is bounded below by* 1 *and radial tangent curvature is equal to* 0*. Assume that*
(1) $F(w)^2 \geq g_v(w, w)$ *for all* $x \in B^+_{\pi/2}(p)$, $v \in \mathcal{G}_p(x)$, *and* $w \in T_x M$;
(2) $g_v(w, w) \geq F(w)^2$ *for all* $x \in M \setminus \overline{B^+_{\pi/2}(p)}$, $v \in \mathcal{G}_p(x)$ *and* $w \in T_x M$.
If F is reversible and $\mathrm{diam}(M) = \mathrm{rad}_p > \pi/2$, *then M is homeomorphic to* $\mathbb{S}^n$.

If F is of Berwald type, the geodesic property on $\bar{c}$ (of the (3)) and $\mathcal{T}_M(\dot{\gamma}(t), w) = 0$ in Theorem 1.1 are automatically satisfied. Hence, we have one more corollary:

Corollary 1.4. *Let (M, F, p) be a compact connected n-dimensional C^∞-Berwald space whose radial flag curvature is bounded below by* 1*. Assume that*
(1) $F(w)^2 \geq g_v(w, w)$ *for all* $x \in B^+_{\pi/2}(p)$, $v \in \mathcal{G}_p(x)$, *and* $w \in T_x M$;
(2) $g_v(w, w) \geq F(w)^2$ *for all* $x \in M \setminus \overline{B^+_{\pi/2}(p)}$, $v \in \mathcal{G}_p(x)$ *and* $w \in T_x M$;
(3) $L_{\mathrm{m}}(c) \leq \mathrm{rad}_p$ *for all minimal geodesic segments* $c\colon [0, a] \to M \setminus \{p\}$.
If $\mathrm{rad}_p > \pi/2$, *then M is homeomorphic to* $\mathbb{S}^n$.

2. TCTs

To prove Theorem 1.1, we need Toponogov's comparison theorems (TCT) in Finsler geometry. In [6], we recently established a TCT for a certain class of Finsler manifolds whose radial flag curvatures are bounded below by that of a von Mangoldt surface. In this section, we modify the TCT in the case where a model surface is the unit sphere.

2.1. Angles, triangles, and a counterexample.

Let (M, F, p) be a forward complete, connected C^∞-Finsler manifold with a base point $p \in M$, and denote by d its distance function. It follows from the Hopf–Rinow theorem that the forward completeness guarantees that any two points in M can be joined by a minimal geodesic segment. Owing to $d(x, y) \neq d(y, x)$ generally, we need a distance with the symmetric property to define the 'angles': Define

$$d_{\mathrm{m}}(x, y) := \max\{d(x, y), d(y, x)\}.$$

Since $|d(p, x) - d(p, y)| \le d_{\mathrm{m}}(x, y)$, we may define the angles with respect to d_{m} as follows.

DEFINITION 2.1 (Angles). Let $c \colon [0, a] \to M$ be a unit speed minimal geodesic segment (i.e., $F(\dot c) \equiv 1$) with $p \notin c([0, a])$. The *forward* and the *backward angles* $\overrightarrow{\angle}(pc(s)c(a))$, $\overleftarrow{\angle}(pc(s)c(0)) \in [0, \pi]$ at $c(s)$ are defined via

$$\cos \overrightarrow{\angle}(pc(s)c(a)) := -\lim_{h \downarrow 0} \frac{d(p, c(s+h)) - d(p, c(s))}{d_{\mathrm{m}}(c(s), c(s+h))} \quad \text{for } s \in [0, a),$$
$$\cos \overleftarrow{\angle}(pc(s)c(0)) := \lim_{h \downarrow 0} \frac{d(p, c(s)) - d(p, c(s-h))}{d_{\mathrm{m}}(c(s-h), c(s))} \quad \text{for } s \in (0, a].$$

REMARK 2.2. The limits in Definition 2.1 are as follows:

$$\lim_{h \downarrow 0} \frac{d(p, c(s+h)) - d(p, c(s))}{d_{\mathrm{m}}(c(s), c(s+h))} = \frac{1}{\lambda} \min\{g_v(v, \dot c(s)) \mid v \in \mathcal{G}_p(c(s))\},$$
$$\lim_{h \downarrow 0} \frac{d(p, c(s)) - d(p, c(s-h))}{d_{\mathrm{m}}(c(s-h), c(s))} = \frac{1}{\lambda} \max\{g_v(v, \dot c(s)) \mid v \in \mathcal{G}_p(c(s))\}$$

where $\lambda := \max\{1, F(-\dot c(s))\}$. These are, of course, in $[-1, 1]$ (see [6, Lemma 2.2]).

DEFINITION 2.3 (Forward triangles). For three distinct points $p, x, y \in M$,

$$\triangle(\overrightarrow{px}, \overrightarrow{py}) := (p, x, y; \gamma, \sigma, c)$$

will denote the *forward triangle* consisting of unit speed minimal geodesic segments γ emanating from p to x, σ from p to y, and c from x to y. Then the corresponding

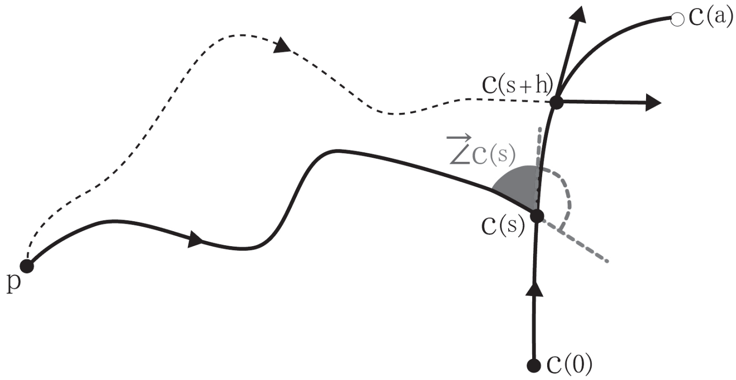

Fig. 1. The forward angle.

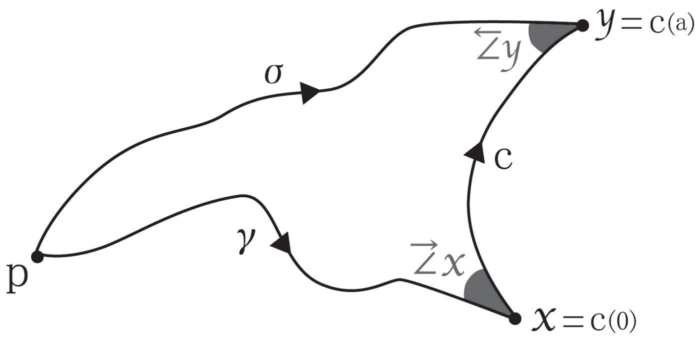

Fig. 2. The forward triangle.

interior angles $\overrightarrow{\angle}x$, $\overleftarrow{\angle}y$ at the vertices x, y are defined by

$$\overrightarrow{\angle}x := \overrightarrow{\angle}(pc(0)c(a)), \quad \overleftarrow{\angle}y := \overleftarrow{\angle}(pc(a)c(0)),$$

respectively, where $a := d(x, y)$.

DEFINITION 2.4 (Comparison triangles). Fix a model surface of revolution $(\tilde{M}, \tilde{p})$. Given a forward triangle $\triangle(\overrightarrow{px}, \overrightarrow{py}) = (p, x, y; \gamma, \sigma, c) \subset M$, a geodesic triangle $\triangle(\tilde{p}\tilde{x}\tilde{y}) \subset \tilde{M}$ is called its *comparison triangle* if

$$\tilde{d}(\tilde{p}, \tilde{x}) = d(p, x), \quad \tilde{d}(\tilde{p}, \tilde{y}) = d(p, y), \quad \tilde{d}(\tilde{x}, \tilde{y}) = L_{\mathrm{m}}(c)$$

hold, where $L_{\mathrm{m}}(c) = \int_0^{d(x,y)} \max\{F(\dot{c}), F(-\dot{c})\}\, ds$.

There are many forward triangles admitting their comparison triangles, but TCT *does not* always hold for all of them:

EXAMPLE 2.5 ([5]). For an even number q, let M be $\mathbb{R}^2$ with the l^q-norm. Then, M is Minkowskian. Take a forward triangle $\triangle(\overrightarrow{px}, \overrightarrow{py}) \subset M$, where $p := (0, 0)$, $x := (1, 0)$, $y := (0, 1) \in M$, and let $c(t) := (1 - t, t)$ denote the side of $\triangle(\overrightarrow{px}, \overrightarrow{py})$ joining x to y. Assume that q is sufficiently large. Then, we observe that both angles $\overrightarrow{\angle} x$ and $\overleftarrow{\angle} y$ are nearly 0, respectively. We are able to think of $(\mathbb{R}^2, \tilde{p})$ as a reference surface for M, because flag curvature $K_M \equiv 0$. It is clear that $\triangle(\overrightarrow{px}, \overrightarrow{py})$ admits its comparison triangle $\triangle(\tilde{p}\tilde{x}\tilde{y}) \subset \mathbb{R}^2$. Since $\triangle(\overrightarrow{px}, \overrightarrow{py})$ is nearly equilateral, $\triangle(\tilde{p}\tilde{x}\tilde{y})$ is too. Hence, $\overrightarrow{\angle} x < \angle\tilde{x}$ and $\overleftarrow{\angle} y < \angle\tilde{y}$ hold. Therefore, TCT does not hold for the $\triangle(\overrightarrow{px}, \overrightarrow{py})$.

2.2. Modified TCTs. From Example 2.5, we understand that some strong conditions are demanded to establish a TCT in Finsler geometry. Taking this into account, we have the following:

Theorem 2.6 ([6, Theorem 1.2]). *Assume that (M, F, p) is a forward complete, connected C^∞-Finsler manifold whose radial flag curvature is bounded below by that of a von Mangoldt surface $(\tilde{M}, \tilde{p})$ satisfying $f'(\rho) = 0$ and $G(\rho) \neq 0$ for unique $\rho \in (0, \infty)$. Let $\triangle(\overrightarrow{px}, \overrightarrow{py}) = (p, x, y; \gamma, \sigma, c) \subset M$ be a forward triangle satisfying that, for some open neighborhood $\mathcal{N}(c)$ of c,*
(1) *$c([0, d(x, y)]) \subset M \setminus \overline{B^+_\rho(p)}$;*
(2) *$g_v(w, w) \geq F(w)^2$ for all $z \in \mathcal{N}(c)$, $v \in \mathcal{G}_p(z)$ and $w \in T_zM$;*
(3) *$\mathcal{T}_M(v, w) = 0$ for all $z \in \mathcal{N}(c)$, $v \in \mathcal{G}_p(z)$ and $w \in T_zM$, and the reverse curve $\bar{c}(s) := c(d(x, y) - s)$ of c is also geodesic.*
If such $\triangle(\overrightarrow{px}, \overrightarrow{py})$ admits a comparison triangle $\triangle(\tilde{p}\tilde{x}\tilde{y}) \subset \tilde{M}$, then we have $\overrightarrow{\angle} x \geq \angle\tilde{x}$ and $\overleftarrow{\angle} y \geq \angle\tilde{y}$.

REMARK 2.7. The application of Theorem 2.6 will be referred to [7]. Here, we proved the finite topological type and a diffeomorphism theorem to $\mathbb{R}^n$.

Corollary 2.8. *Assume that (M, F, p) is a compact connected C^∞-Finsler manifold whose radial flag curvature is bounded below by 1. Let $\triangle(\overrightarrow{px}, \overrightarrow{py}) = (p, x, y; \gamma, \sigma, c) \subset M$ be a forward triangle satisfying that, for some open neighborhood $\mathcal{N}(c)$ of c,*
(1) *$c([0, d(x, y)]) \subset M \setminus \overline{B^+_{\pi/2}(p)}$;*
(2) *$g_v(w, w) \geq F(w)^2$ for all $z \in \mathcal{N}(c)$, $v \in \mathcal{G}_p(z)$ and $w \in T_zM$;*
(3) *$\mathcal{T}_M(v, w) = 0$ for all $z \in \mathcal{N}(c)$, $v \in \mathcal{G}_p(z)$ and $w \in T_zM$, and the reverse curve $\bar{c}(s) := c(d(x, y) - s)$ of c is also geodesic.*

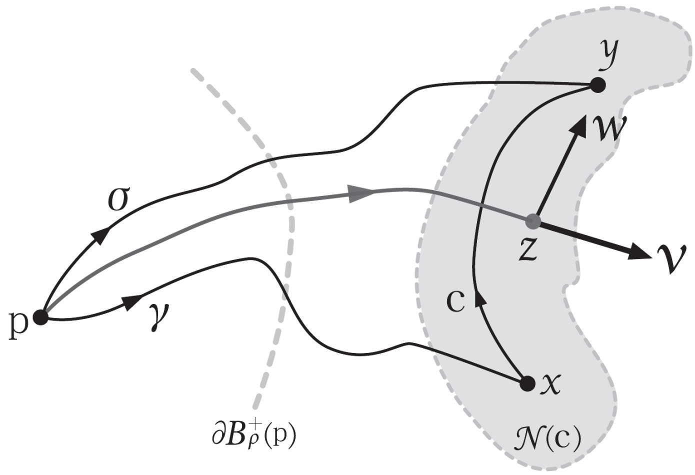

Fig. 3. The forward triangle of TCT.

If such $\triangle(\overrightarrow{px}, \overrightarrow{py})$ admits a comparison triangle $\triangle(\tilde{p}\tilde{x}\tilde{y})$ in $(\mathbb{S}^2, \tilde{p})$, then we have $\overrightarrow{\angle} x \geq \angle\tilde{x}$ and $\overleftarrow{\angle} y \geq \angle\tilde{y}$. Here, $(\mathbb{S}^2, \tilde{p})$ denotes the unit sphere, i.e., its Riemannian metric $d\tilde{s}^2$ is expressed as $d\tilde{s}^2 = dt^2 + f(t)^2 d\theta^2$, $(t, \theta) \in (0, \pi) \times \mathbb{S}^1_{\tilde{p}}$, such that $f(t) = \sin t$.

Proof. In Theorem 2.6, $f'(t) < 0$ on (ρ, ∞), because $f'(\rho) = 0$ and $G(\rho) \neq 0$ for unique $\rho \in (0, \infty)$. Hence, the corollary is immediate from Theorem 2.6, since $f'(t) = \cos t < 0$ on $(\pi/2, \pi)$ and $f'(\pi/2) = 0$ for unique $\pi/2 \in (0, \pi)$. □

Lemma 2.9. *Assume that (M, F, p) is a compact connected C^∞-Finsler manifold whose radial flag curvature is bounded below by 1. Let $\triangle(\overrightarrow{px}, \overrightarrow{py}) = (p, x, y; \gamma, \sigma, c) \subset M$ be a forward triangle satisfying that, for some open neighborhood $\mathcal{N}(c)$ of c,*
(1) $c([0, d(x, y)]) \subset B^+_{\pi/2}(p) \setminus \{p\}$;
(2) *$F(w)^2 \geq g_v(w, w)$ for all $z \in \mathcal{N}(c)$, $v \in \mathcal{G}_p(z)$ and $w \in T_zM$;*
(3) *$\mathcal{T}_M(v, w) = 0$ for all $z \in \mathcal{N}(c)$, $v \in \mathcal{G}_p(z)$ and $w \in T_zM$, and the reverse curve $\bar{c}(s) := c(d(x, y) - s)$ of c is also geodesic.*
If such $\triangle(\overrightarrow{px}, \overrightarrow{py})$ admits a comparison triangle $\triangle(\tilde{p}\tilde{x}\tilde{y})$ in $(\mathbb{S}^2, \tilde{p})$, then we have $\overrightarrow{\angle} x \geq \angle\tilde{x}$ and $\overleftarrow{\angle} y \geq \angle\tilde{y}$.

Proof. Set $\lambda := \max\{F(w), F(-w)\}$. The assumption (2) yields $\lambda^2 \geq g_v(w, w)$. Hence, one can prove this lemma by the almost similar argument as that in [6]. See Section 4 in this article for a detailed explanation of that. □

3. Proof of Theorem 1.1

Let (M, F, p) be the same as that in Theorem 1.1. Hence, our model surface as a reference surface is the unit sphere $(\mathbb{S}^2, \tilde{p})$, i.e., its Riemannian metric $d\tilde{s}^2$ is expressed as $d\tilde{s}^2 = dt^2 + f(t)^2 d\theta^2$, $(t, \theta) \in (0, \pi) \times \mathbb{S}^1_{\tilde{p}}$, such that $f(t) = \sin t$.

Lemma 3.1. *The set $\mathbb{S}^2 \setminus B_t(\tilde{p})$ is strictly convex for all $t \in (\pi/2, \pi)$, i.e., for any distinct two points $\tilde{x}, \tilde{y} \in \partial B_t(\tilde{p})$ and minimal geodesic segment $\tilde{c}\colon [0, a] \to \mathbb{S}^2$ between them, we have $\tilde{c}((0, a)) \subset \mathbb{S}^2 \setminus \overline{B_t(\tilde{p})}$, where $a := \tilde{d}(\tilde{x}, \tilde{y})$.*

Proof. Use the second variation formula. □

Hereafter, by the Bonnet–Myers theorem ([1, Theorem 7.7.1]), we may assume, without loss of generality,

$$\mathrm{rad}_p < \pi. \tag{3.1}$$

Lemma 3.2 (Key lemma)**.** *For any distinct two points $x, y \in M \setminus \overline{B^+_{\pi/2}(p)}$, then*

$$c([0, d(x, y)]) \cap \partial B^+_{\pi/2}(p) = \emptyset$$

holds for all minimal geodesic segments c emanating from x to y. In particular, the set $M \setminus B^+_{\pi/2}(p)$ is convex.

Proof. Suppose that $c([0, d(x, y)]) \cap \partial B^+_{\pi/2}(p) \neq \emptyset$ for some minimal geodesic segment c emanating from x to y. Then, we consider five cases:

CASE 1: Assume that there exist $s_0, s_1, s_2 \in [0, d(x, y))$ with $0 \le s_0 < s_1 < s_2$ such that

$$c([s_0, s_1)) \subset M \setminus \overline{B^+_{\pi/2}(p)}, \quad c([s_1, s_2]) \subset \partial B^+_{\pi/2}(p).$$

For sufficiently small $\varepsilon > 0$ with $\varepsilon < s_1 - s_0$, take the forward triangle $\triangle(\overrightarrow{pc(s_0)}, \overrightarrow{pc(s_1 - \varepsilon)}) \subset M$. Note that $c([s_0, s_1 - \varepsilon]) \subset M \setminus \overline{B^+_{\pi/2}(p)}$. Since $d(p, c(s_0)) > \pi/2$ and $d(p, c(s_1 - \varepsilon)) > \pi/2$, we have, by the assumption and (3.1), that

$$\begin{aligned} |d(p, c(s_0)) - d(p, c(s_1 - \varepsilon))| &\le d_{\mathrm{m}}(c(s_0), c(s_1 - \varepsilon)) \\ &\le L_{\mathrm{m}}(c) < \pi < d(p, c(s_0)) + d(p, c(s_1 - \varepsilon)), \end{aligned}$$

and hence $\triangle(\overrightarrow{pc(s_0)}, \overrightarrow{pc(s_1 - \varepsilon)})$ admits a comparison triangle $\triangle(\tilde{p}\widetilde{c(s_0)}\widetilde{c(s_1 - \varepsilon)}) \subset \mathbb{S}^2$.

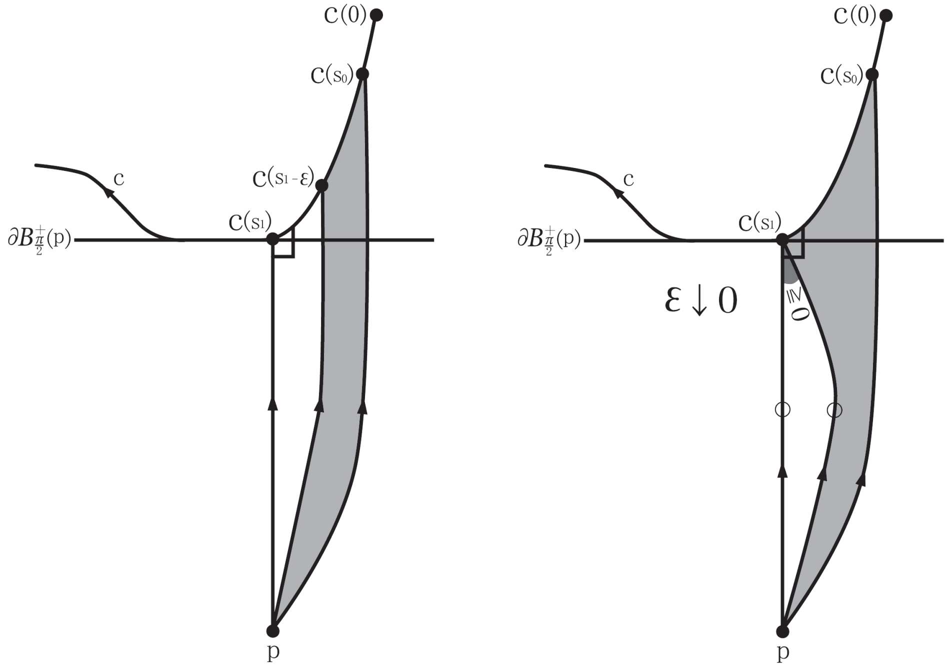

Fig. 4. The limit argument.

By Corollary 2.8, we have $\overleftarrow{\angle}(pc(s_1-\varepsilon)c(s_0)) \geq \angle\widetilde{c(s_1-\varepsilon)}$. It follows from [13, Proposition 2.1] (see Fig. 4[1]) that

$$\frac{\pi}{2} = \overleftarrow{\angle}(pc(s_1)c(s_0)) \geq \lim_{\varepsilon\downarrow 0} \overleftarrow{\angle}(pc(s_1-\varepsilon)c(s_0)) \geq \angle\widetilde{c(s_1)}.$$

Set $\triangle(\tilde{p}\widetilde{c(s_0)}\widetilde{c(s_1)}) := \lim_{\varepsilon\downarrow 0}\triangle(\tilde{p}\widetilde{c(s_0)}\widetilde{c(s_1-\varepsilon)})$, and let $\tilde{\mu}\colon [0, \tilde{d}(\widetilde{c(s_0)}, \widetilde{c(s_1)})] \to \mathbb{S}^2$ denote the side of $\triangle(\tilde{p}\widetilde{c(s_0)}\widetilde{c(s_1)})$ joining $\widetilde{c(s_0)}$ to $\widetilde{c(s_1)}$. If $\angle\widetilde{c(s_1)} = \pi/2$, then

$$\tilde{\mu}([0, \tilde{d}(\widetilde{c(s_0)}, \widetilde{c(s_1)})]) \subset \partial B_{\pi/2}(\tilde{p})$$

because $\partial B_{\pi/2}(\tilde{p})$ is geodesic. This contradicts $\tilde{d}(\tilde{p}, \widetilde{c(s_0)}) > \pi/2$. If $\angle\widetilde{c(s_1)} < \pi/2$, then there exists $a \in (0, \tilde{d}(\widetilde{c(s_0)}, \widetilde{c(s_1)}))$ such that $\tilde{\mu}(a) \in \partial B_{\pi/2}(\tilde{p})$. This contradicts the structure of the cut locus of $\mathbb{S}^2$ because $\angle(\tilde{\mu}(a)\tilde{p}\widetilde{c(s_1)}) < \pi$ and $\partial B_{\pi/2}(\tilde{p})$ is geodesic.

CASE 2: Assume that there exist $s_3, s_4, s_5 \in (0, d(x, y)]$ with $0 < s_3 < s_4 < s_5$ such that

$$c([s_3, s_4]) \subset \partial B^+_{\pi/2}(p), \quad c((s_4, s_5]) \subset M \setminus \overline{B^+_{\pi/2}(p)}.$$

[1]In the right picture of Fig. 4, the circle marks on the segments emanating from p to $c(s_1)$ mean that the segments have the same length.

Consider the forward triangle $\triangle(\overrightarrow{pc(s_4+\varepsilon)}, \overrightarrow{pc(s_5)}) \subset M$, where $\varepsilon > 0$ is sufficiently small with $\varepsilon < s_5 - s_4$. Applying the similar limit argument in Case 1 to $\triangle(\overrightarrow{pc(s_4+\varepsilon)}, \overrightarrow{pc(s_5)})$, we have the triangle $\triangle(\tilde{p}\widetilde{c(s_4)}\widetilde{c(s_5)}) := \lim_{\varepsilon\downarrow 0} \triangle(\tilde{p}\widetilde{c(s_4+\varepsilon)}\widetilde{c(s_5)})$ satisfying $\angle\widetilde{c(s_4)} \le \pi/2$. The angle condition yields the same contradiction as that in Case 1.

CASE 3: Assume that there exist $s_0, s_1, s_2 \in [0, d(x,y)]$ with $s_0 < s_1 < s_2$ such that

$$c([0, d(x, y)]) \cap \partial B^+_{\pi/2}(p) = \{c(s_1)\}, \quad c((s_0, s_1)), c((s_1, s_2)) \subset M \setminus \overline{B^+_{\pi/2}(p)}.$$

Then, we get a contradiction from the same argument as Case 1, or Case 2.

CASE 4: Assume that there exist $s_0, s_1 \in (0, d(x, y))$ with $s_0 < s_1$ such that

$$c((s_0, s_1)) \subset B^+_{\pi/2}(p) \setminus \{p\}, \quad c(s_0), c(s_1) \in \partial B^+_{\pi/2}(p),$$

and that

$$\overrightarrow{\angle}(pc(s_0)c(s_1)) < \frac{\pi}{2}, \quad \overleftarrow{\angle}(pc(s_1)c(s_0)) < \frac{\pi}{2}. \tag{3.2}$$

Take a subdivision $r_0 := s_0 < r_1 < \cdots < r_{k-1} < r_k := s_1$ of $[s_0, s_1]$ such that $\triangle(\overrightarrow{pc(r_{i-1})}, \overrightarrow{pc(r_i)})$ admits a comparison triangle $\tilde{\triangle}^i := \triangle(\tilde{p}\widetilde{c(r_{i-1})}\widetilde{c(r_i)}) \subset \mathbb{S}^2$ for each $i = 1,2,\ldots,k$. Applying Lemma 2.9 to $\triangle(\overrightarrow{pc(r_{i-1})}, \overrightarrow{pc(r_i)})$, but for each $i = 2, 3, \ldots, k-1$, we have

$$\overrightarrow{\angle}c(r_{i-1}) \ge \angle(\tilde{p}\widetilde{c(r_{i-1})}\widetilde{c(r_i)}), \quad \overleftarrow{\angle}c(r_i) \ge \angle(\tilde{p}\widetilde{c(r_i)}\widetilde{c(r_{i-1})}). \tag{3.3}$$

For sufficiently small $\varepsilon, \delta > 0$ with $\varepsilon < r_1 - r_0$ and $\delta < r_k - r_{k-1}$, take two forward triangles $\triangle(\overrightarrow{pc(r_0+\varepsilon)}, \overrightarrow{pc(r_1)})$, $\triangle(\overrightarrow{pc(r_{k-1})}, \overrightarrow{pc(r_k-\delta)}) \subset M$. Note that these two triangles admit their comparison triangles

$$\tilde{\triangle}_\varepsilon := \triangle(\tilde{p}\widetilde{c(r_0+\varepsilon)}\widetilde{c(r_1)}), \quad \tilde{\triangle}_\delta := \triangle(\tilde{p}\widetilde{c(r_{k-1})}\widetilde{c(r_k-\delta)}) \subset \mathbb{S}^2,$$

respectively. Without loss of generality, we may assume $\tilde{\triangle}^1 = \lim_{\varepsilon\downarrow 0}\tilde{\triangle}_\varepsilon$ and $\tilde{\triangle}^k = \lim_{\delta\downarrow 0}\tilde{\triangle}_\delta$ because $\lim_{\varepsilon\downarrow 0}\tilde{\triangle}_\varepsilon$ and $\lim_{\delta\downarrow 0}\tilde{\triangle}_\delta$ are isometric to $\tilde{\triangle}^1$ and $\tilde{\triangle}^k$, respectively. By Lemma 2.9, $\overrightarrow{\angle}c(r_0+\varepsilon) \ge \angle(\tilde{p}\widetilde{c(r_0+\varepsilon)}\widetilde{c(r_1)})$, $\overleftarrow{\angle}c(r_1) \ge \angle(\tilde{p}\widetilde{c(r_1)}\widetilde{c(r_0+\varepsilon)})$, and that $\overrightarrow{\angle}c(r_{k-1}) \ge \angle(\tilde{p}\widetilde{c(r_{k-1})}\widetilde{c(r_k-\delta)})$, $\overleftarrow{\angle}c(r_k-\delta) \ge \angle(\tilde{p}\widetilde{c(r_k-\delta)}\widetilde{c(r_{k-1})})$. Hence, it follows from (3.2) and [13, Proposition 2.1] that

$$\frac{\pi}{2} > \overrightarrow{\angle}c(r_0) \ge \lim_{\varepsilon\downarrow 0}\overrightarrow{\angle}c(r_0+\varepsilon) \ge \angle(\tilde{p}\widetilde{c(r_0)}\widetilde{c(r_1)}), \quad \overleftarrow{\angle}c(r_1) \ge \angle(\tilde{p}\widetilde{c(r_1)}\widetilde{c(r_0)}), \tag{3.4}$$

and that

$$\begin{aligned} &\overrightarrow{\angle}c(r_{k-1}) \ge \angle(\tilde{p}\widetilde{c(r_{k-1})}\widetilde{c(r_k)}), \\ &\frac{\pi}{2} > \overleftarrow{\angle}c(r_k) \ge \lim_{\delta\downarrow 0}\overleftarrow{\angle}c(r_k-\delta) \ge \angle(\tilde{p}\widetilde{c(r_k)}\widetilde{c(r_{k-1})}). \end{aligned} \tag{3.5}$$

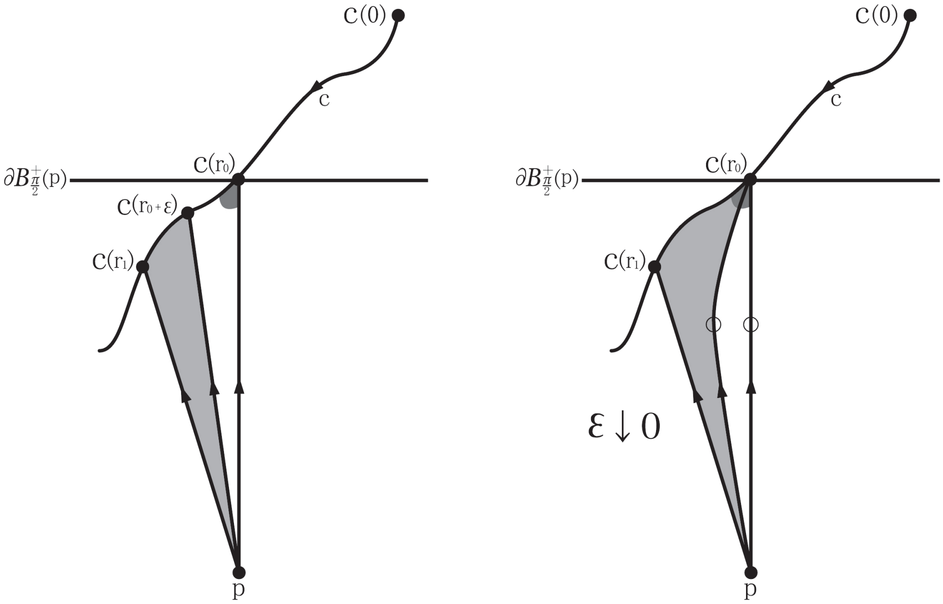

Fig. 5. The limit argument in (3.4).

Starting from $\tilde{\Delta}^1$, we inductively draw a geodesic triangle $\tilde{\Delta}^{i+1} \subset \mathbb{S}^2$ which is adjacent to $\tilde{\Delta}^i$ so as to have a common side $\tilde{p}\widetilde{c(r_i)}$, where $0 := \theta(\widetilde{c(r_0)}) \leq \theta(\widetilde{c(r_1)}) \leq \cdots \leq \theta(\widetilde{c(r_k)})$. Since $\overleftarrow{\angle} c(r_i) + \overrightarrow{\angle} c(r_i) \leq \pi$ for each $i = 1, 2, \ldots, k-1$, we obtain, by (3.3), (3.4), (3.5),

$$\angle(\tilde{p}\widetilde{c(r_i)}\widetilde{c(r_{i-1})}) + \angle(\tilde{p}\widetilde{c(r_i)}\widetilde{c(r_{i+1})}) \leq \pi. \tag{3.6}$$

Let $\hat{\xi}\colon [0, L_{\mathrm{m}}(c|_{[s_0,s_1]})] \to \mathbb{S}^2$ denote the broken geodesic segment consisting of minimal geodesic segments from $\widetilde{c(r_{i-1})}$ to $\widetilde{c(r_i)}$, $i = 1, 2, \ldots, k$. Set $\hat{\xi}(s) := (t(\hat{\xi}(s)), \theta(\hat{\xi}(s)))$. By (3.6), we have the unit speed, but not necessarily minimal at this moment, geodesic $\tilde{\eta}\colon [0, a] \to \mathbb{S}^2$ emanating from $\widetilde{c(r_0)}$ to $\widetilde{c(r_k)}$ and passing under $\hat{\xi}([0, L_{\mathrm{m}}(c|_{[s_0,s_1]})])$, i.e., $\theta(\tilde{\eta}) \in [0, \theta(\widetilde{c(r_k)})]$ on $[0, a]$ and $t(\hat{\xi}(s)) > t(\tilde{\eta}(u))$ for all $(s, u) \in (0, L_{\mathrm{m}}(c|_{[s_0,s_1]})) \times (0, a)$ with $\theta(\hat{\xi}(s)) = \theta(\tilde{\eta}(u))$ (see Fig. 6). Since $a \leq L_{\mathrm{m}}(c|_{[s_0,s_1]}) < \pi$ by the assumption and (3.1), $\tilde{\eta}$ is minimal with $\angle(\tilde{\eta}(0)\, \tilde{p}\, \tilde{\eta}(a)) < \pi$. This contradicts the structure of the cut locus of $\mathbb{S}^2$ because $\partial B_{\pi/2}(\tilde{p})$ is geodesic.

CASE 5: Assume that c is passing through p. Take a sequence $\{c_i\colon [0, l_i] \to M \setminus \{p\}\}_{i\in\mathbb{N}}$ of minimal geodesic segments c_i emanating from $x = c_i(0)$ convergent to c. Applying the same argument as that in Case 4 to each c_i for sufficiently large i, we get a contradiction. Note that $\lim_{i\to\infty} L_{\mathrm{m}}(c_i) = L_{\mathrm{m}}(c) \leq \mathrm{rad}_p$, but $x, y \in M \setminus \overline{B^+_{\pi/2}(p)}$.

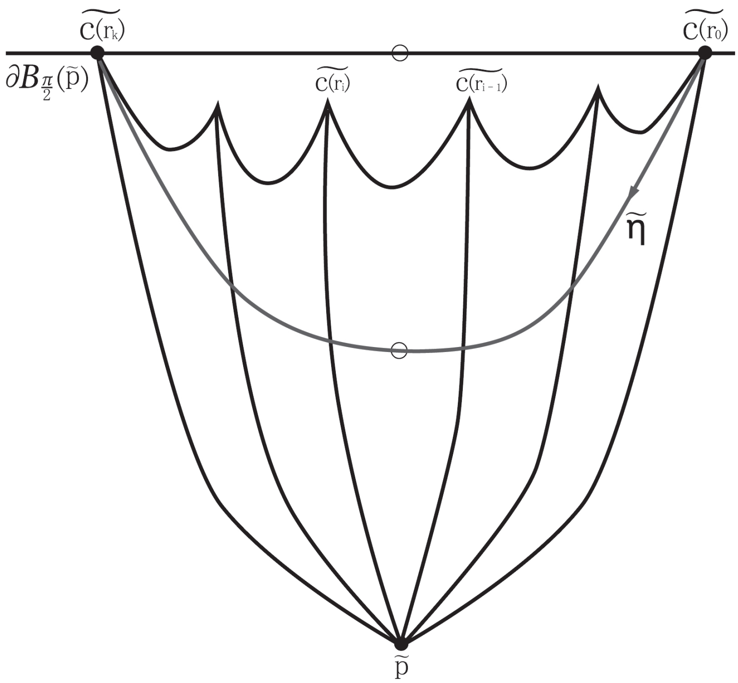

Fig. 6. The segment $\tilde{\eta}$.

Therefore, $c([0, d(x, y)]) \cap \partial B^+_{\pi/2}(p) = \emptyset$ holds for all minimal geodesic segments c emanating from x to y. The second assertion is clear from the first assertion. $\square$

Corollary 3.3. *Let $\triangle(\overrightarrow{px}, \overrightarrow{py}) = (p, x, y; \gamma, \sigma, c) \subset M$ be a forward triangle for $x \in M \setminus \overline{B^+_{\pi/2}(p)}$ and $y \in B^+_{\pi/2}(p)$ satisfying*

$$\{z\} := c((0, a)) \cap \partial B^+_{\pi/2}(p) \neq \emptyset,$$

where $a := d(x, y)$. Then, the forward triangle $\triangle(\overrightarrow{px}, \overrightarrow{pz}) \subset M$ admits its comparison triangle $\triangle(\tilde{p}\tilde{x}\tilde{z}) \subset \mathbb{S}^2$ such that $\overrightarrow{\angle} x \geq \angle\tilde{x}$ and $\overleftarrow{\angle} z \geq \angle\tilde{z}$. Additionally, if the forward triangle $\triangle(\overrightarrow{pz}, \overrightarrow{py}) \subset M$ admits its comparison triangle $\triangle(\tilde{p}\tilde{z}\tilde{y}) \subset \mathbb{S}^2$, then $\overrightarrow{\angle} z \geq \angle\tilde{z}$ and $\overleftarrow{\angle} y \geq \angle\tilde{y}$.

Proof. Apply the same limit argument in the proof of Lemma 3.2 to forward triangles. $\square$

Lemma 3.4. *The function $d(p, \cdot)$ attains its maximum at a unique point $q \in M$. In particular, $M \setminus B^+_{\pi/2}(p)$ is a topological disk.*

Proof. Suppose that there exist two distinct points $x, y \in \partial B^+_{\mathrm{rad}_p}(p)$. Then, the forward triangle $\triangle(\overrightarrow{px}, \overrightarrow{py}) \subset M$ admits its comparison triangle $\triangle(\tilde{p}\tilde{x}\tilde{y}) \subset \mathbb{S}^2$. Let $c\colon [0, d(x, y)] \to M$ and $\tilde{c}\colon [0, L_{\mathrm{m}}(c)] \to \mathbb{S}^2$ be sides of $\triangle(\overrightarrow{px}, \overrightarrow{py})$ and $\triangle(\tilde{p}\tilde{x}\tilde{y})$ emanating from x to y and from $\tilde{x}$ to $\tilde{y}$, respectively. By Corollary 2.8 and Lemma 3.1, $d(p, c(s)) > \mathrm{rad}_p$ holds for all $s \in (0, d(x, y))$. This contradicts the definition of rad_p. The second assertion follows from the uniqueness and Lemma 3.2. $\square$

DEFINITION 3.5. We say that a point $x \in M$ is a (*forward*) *critical point* for $p \in M$ if, for every $w \in T_x M \setminus \{0\}$, there exists $v \in \mathcal{G}_p(x)$ such that $g_v(v, w) \le 0$. Here see (1.1) for the definition of $\mathcal{G}_p(x)$.

By similar arguments to the Riemannian case, we have Gromov's isotopy lemma [2]:

Lemma 3.6. *Given $0 < r_1 < r_2 \le \infty$, if $\overline{B^+_{r_2}(p)} \setminus B^+_{r_1}(p)$ has no critical point for $p \in M$, then $\overline{B^+_{r_2}(p)} \setminus B^+_{r_1}(p)$ is homeomorphic to $\partial B^+_{r_1}(p) \times [r_1, r_2]$.*

Lemma 3.7. *There are no critical point for p in $\overline{B^+_{\pi/2}(p)} \setminus \{p\}$. In particular, $\overline{B^+_{\pi/2}(p)}$ is a topological disk.*

Proof. Since $M \setminus B^+_{\pi/2}(p)$ is convex (Lemma 3.2), $\partial B^+_{\pi/2}(p)$ has no critical point for p. Suppose that there exists a critical point $x \in B^+_{\pi/2}(p) \setminus \{p\}$ for p. Let $q \in M$ be the same as that in Lemma 3.4 such that $d(p, q) = \mathrm{rad}_p$, and $c\colon [0, a] \to M$ a unit speed minimal geodesic segment emanating from q to x, where $a := d(q, x)$. Then, $c([0, a]) \cap \partial B^+_{\pi/2}(p) \ne \emptyset$. From the cases in the proof of Lemma 3.2, it is sufficient to consider the case where $c((0, a)) \cap \partial B^+_{\pi/2}(p)$ is one point, say

$$\{q_1\} := c((0, a)) \cap \partial B^+_{\pi/2}(p).$$

Since both $q = c(0)$ and $x = c(a)$ are critical points for p, we have

$$\overrightarrow{\angle}(pc(0)c(a)) \le \frac{\pi}{2}, \quad \overleftarrow{\angle}(pc(a)c(0)) \le \frac{\pi}{2}. \tag{3.7}$$

Note that c does not pass through p, because, by the definition of critical points, there exist at least two minimal segments emanating from p to x and c is minimal. Now, take a subdivision $s_0 := 0 < s_1 < \cdots < s_{k-1} < s_k := a$ of $[0, a]$ such that $c(s_1) = q_1 \in \partial B^+_{\pi/2}(p)$ and that $\triangle(\overrightarrow{pc(s_{i-1})}, \overrightarrow{pc(s_i)})$ admits a comparison triangle $\tilde{\triangle}^i := \triangle(\tilde{p}\widetilde{c(s_{i-1})}\widetilde{c(s_i)}) \subset \mathbb{S}^2$ for each $i = 2, 3, \ldots, k$. By Corollary 3.3, $\triangle(\overrightarrow{pc(s_0)}, \overrightarrow{pc(s_1)})$ admits its a comparison triangle $\tilde{\triangle}^1 := \triangle(\tilde{p}\widetilde{c(s_0)}\widetilde{c(s_1)}) \subset \mathbb{S}^2$ satisfying

$$\overrightarrow{\angle} c(s_0) \ge \angle(\tilde{p}\widetilde{c(s_0)}\widetilde{c(s_1)}), \quad \overleftarrow{\angle} c(s_1) \ge \angle(\tilde{p}\widetilde{c(s_1)}\widetilde{c(s_0)}). \tag{3.8}$$

Moreover, by Corollary 3.3 again, we have

$$\overrightarrow{\angle} c(s_1) \geq \angle(\tilde{p}\widetilde{c(s_1)}\widetilde{c(s_2)}), \quad \overleftarrow{\angle} c(s_2) \geq \angle(\tilde{p}\widetilde{c(s_2)}\widetilde{c(s_1)}). \tag{3.9}$$

In particular, by (3.7) and (3.8), we have

$$\angle(\tilde{p}\widetilde{c(s_0)}\widetilde{c(s_1)}) \leq \frac{\pi}{2}. \tag{3.10}$$

Applying Lemma 2.9 to $\triangle(\overrightarrow{pc(s_{i-1})}, \overrightarrow{pc(s_i)})$ for each $i = 3, 4, \dots, k$,

$$\overrightarrow{\angle} c(s_{i-1}) \geq \angle(\tilde{p}\widetilde{c(s_{i-1})}\widetilde{c(s_i)}), \quad \overleftarrow{\angle} c(s_i) \geq \angle(\tilde{p}\widetilde{c(s_i)}\widetilde{c(s_{i-1})}). \tag{3.11}$$

In particular, by (3.7) and (3.11), we have

$$\angle(\tilde{p}\widetilde{c(s_k)}\widetilde{c(s_{k-1})}) \leq \frac{\pi}{2}. \tag{3.12}$$

Starting from $\tilde{\triangle}^1$, we inductively draw a geodesic triangle $\tilde{\triangle}^{i+1} \subset \mathbb{S}^2$ which is adjacent to $\tilde{\triangle}^i$ so as to have a common side $\tilde{p}\widetilde{c(s_i)}$, where $0 := \theta(\widetilde{c(s_0)}) \leq \theta(\widetilde{c(s_1)}) \leq \cdots \leq \theta(\widetilde{c(s_k)})$. Since $\overleftarrow{\angle} c(s_i) + \overrightarrow{\angle} c(s_i) \leq \pi$ for each $i = 1, 2, \dots, k-1$, we obtain, by (3.11), (3.8), (3.9),

$$\angle(\tilde{p}\widetilde{c(s_i)}\widetilde{c(s_{i-1})}) + \angle(\tilde{p}\widetilde{c(s_i)}\widetilde{c(s_{i+1})}) \leq \pi. \tag{3.13}$$

Let $\hat{\xi}\colon [0, L_{\mathrm{m}}(c)] \to \mathbb{S}^2$ denote the broken geodesic segment consisting of minimal geodesic segments from $\widetilde{c(s_{i-1})}$ to $\widetilde{c(s_i)}$, $i = 1, 2, \dots, k$. Set $\hat{\xi}(r) := (t(\hat{\xi}(r)), \theta(\hat{\xi}(r)))$. By (3.13), we have the unit speed geodesic $\tilde{\eta}\colon [0, b] \to \mathbb{S}^2$ emanating from $\tilde{\eta}(0) = \widetilde{c(s_0)}$ to $\tilde{\eta}(b) = \widetilde{c(s_k)}$ and passing under $\hat{\xi}([0, L_{\mathrm{m}}(c)])$, i.e., $\theta(\tilde{\eta}) \in [0, \theta(\widetilde{c(s_k)})]$ on $[0, b]$ and $t(\hat{\xi}(r)) > t(\tilde{\eta}(u))$ for all $(r, u) \in (0, L_{\mathrm{m}}(c)) \times (0, b)$ with $\theta(\hat{\xi}(r)) = \theta(\tilde{\eta}(u))$. Since $b \leq L_{\mathrm{m}}(c) < \pi$ by (3.1), $\tilde{\eta}$ is minimal with $\angle\big(\dot{\tilde{\gamma}}(0), \dot{\tilde{\sigma}}(0)\big) < \pi$, where $\tilde{\gamma}$ and $\tilde{\sigma}$ denote minimal geodesic segments (i.e., sub-meridians) emanating from $\tilde{p}$ to $\widetilde{c(s_0)}$ and from p to $\widetilde{c(s_k)}$, respectively. Since $\tilde{\eta}$ lives under $\hat{\xi}([0, L_{\mathrm{m}}(c)])$, we have, by (3.12) and (3.10),

$$\angle\big(\dot{\tilde{\eta}}(0), -\dot{\tilde{\gamma}}(\tilde{d}(\tilde{p}, \widetilde{c(s_0)}))\big) \leq \frac{\pi}{2}, \quad \angle\big(\dot{\tilde{\eta}}(b), \dot{\tilde{\sigma}}(\tilde{d}(\tilde{p}, \widetilde{c(s_k)}))\big) \leq \frac{\pi}{2}. \tag{3.14}$$

Since $\tilde{d}(\tilde{p}, \widetilde{c(s_0)}) > \pi/2 > \tilde{d}(\tilde{p}, \widetilde{c(s_k)})$, there exists $b_0 \in (0, b)$ such that $\tilde{\eta}(b_0) \in \partial B_{\pi/2}(\tilde{p})$. Let $\bar{\eta}\colon [0, \pi] \to \mathbb{S}^2$ be an extension of $\tilde{\eta}$ to the antipodal point $\widetilde{c(s_0)}_\pi$ to $\widetilde{c(s_0)} = \tilde{\eta}(0)$, and set $\bar{\eta}(u) := (t(u), \theta(u))$. Since $\mathbb{S}^2 \setminus B_u(\bar{\eta}(0))$ is strictly convex for all $u \in (\pi/2, \pi)$ (by the same proof of Lemma 3.1), $\angle(\dot{\bar{\eta}}(\mathrm{rad}_p), (\partial/\partial t)|_{\bar{\eta}(\mathrm{rad}_p)}) > \pi/2$ holds. This implies

$$t'(\mathrm{rad}_p) < 0, \tag{3.15}$$

where note that $\bar{\eta}$ emanates from $\widetilde{c(s_0)}$ to $\widetilde{c(s_0)}_\pi$. Since $\bar{\eta}((b_0, \mathrm{rad}_p)) \subset B_{\pi/2}(\tilde{p})$ and $f'(t) = \cos t > 0$ on $(0, \pi/2)$, it follows from [12, (7.1.15)] that

$$t''(u) = f(t(u)) f'(t(u)) \theta'(t(u))^2 > 0 \tag{3.16}$$

holds on (b_0, rad_p). Hence, by (3.15) and (3.16), $t(u)$ is decreasing on $[b_0, \mathrm{rad}_p]$. Since $b \in (b_0, \mathrm{rad}_p)$,

$$\angle\big(\dot{\bar{\eta}}(b), \dot{\bar{\sigma}}(t(b))\big) > \frac{\pi}{2}.$$

This contradicts the right inequality in (3.14). Therefore, $\overline{B^+_{\pi/2}(p)} \setminus \{p\}$ has no critical point for p. By Lemma 3.6, $\overline{B^+_{\pi/2}(p)}$ is a topological disk. □

By Lemmas 3.4 and 3.7, M is homeomorphic to $\mathbb{S}^n$. □

4. Appendix: Proof of Lemma 2.9

Let (M, F, p) be a forward complete, connected C^∞-Finsler manifold with a base point $p \in M$, and let d denote its distance function and $\mathrm{Cut}(p)$ the cut locus of p. Set $B^-_r(x) := \{y \in M \mid d(y, x) < r\}$. Take a point $q \in M \setminus (\mathrm{Cut}(p) \cup \{p\})$ and small $r > 0$ such that $B^-_{2r}(q) \cap (\mathrm{Cut}(p) \cup \{p\}) = \emptyset$ and that $B^\pm_r(q) := B^+_r(q) \cap B^-_r(q)$ is geodesically convex (i.e., any minimal geodesic joining two points in $B^\pm_r(q)$ is contained in $B^\pm_r(q)$). Given a unit speed minimal geodesic segment $c\colon (-\varepsilon, \varepsilon) \to B^\pm_r(q)$, we consider the C^∞-variation

$$\varphi(t, s) := \exp_p\left(\frac{t}{l} \exp_p^{-1}(c(s))\right), \quad (t, s) \in [0, l] \times (-\varepsilon, \varepsilon),$$

where $l := d(p, c(0))$. Since $x := c(0) \notin \mathrm{Cut}(p)$, there is a unique minimal geodesic segment $\gamma\colon [0, l] \to M$ emanating from p to x. By setting $J(t) := (\partial\varphi/\partial s)(t, 0)$, we get the Jacobi field J along γ with $J(0) = 0$ and $J(l) = \dot{c}(0)$. Note that $J(t) \neq 0$ on $(0, l]$ from the minimality of γ, and that

$$J^\perp(t) := J(t) - \frac{g_{\dot{\gamma}(l)}(\dot{\gamma}(l), \dot{c}(0))}{l} t\dot{\gamma}(t), \quad t \in [0, l], \tag{4.1}$$

is the $g_{\dot{\gamma}}$-orthogonal component $J^\perp(t)$ to $\dot{\gamma}(t)$ (see [6, Lemma 3.2]). Moreover, since γ is unique, it follows from the proof of [6, Lemma 2.2] that

$$-\cos \overrightarrow{\angle}(pxc(\varepsilon)) = \cos \overleftarrow{\angle}(pxc(-\varepsilon)) = \lambda^{-1} g_{\dot{\gamma}(l)}(\dot{\gamma}(l), \dot{c}(0)), \tag{4.2}$$

where $\lambda := \max\{1, F(-\dot{c}(0))\}$. Hence, $\pi - \overrightarrow{\angle}(pxc(\varepsilon)) = \overleftarrow{\angle}(pxc(-\varepsilon))$. In the following discussion, we set

$$\omega := \pi - \overrightarrow{\angle}(pxc(\varepsilon)) = \overleftarrow{\angle}(pxc(-\varepsilon)). \tag{4.3}$$

Hereafter, we assume that the radial flag curvature of (M, F, p) is bounded below by 1. Hence, its model surface is the unit sphere $(\mathbb{S}^2, \tilde{p})$ with its metric $d\tilde{s}^2 = dt^2 + f(t)^2\, d\theta^2$, $(t, \theta) \in (0, \pi) \times \mathbb{S}^1_{\tilde{p}}$, such that $f(t) = \sin t$. For small $\delta > 0$ with $\delta < 1$, we set

$$f_\delta(t) := \frac{1}{\sqrt{1-\delta}} \sin(\sqrt{1-\delta}\,t)$$

on $[0, \pi/\sqrt{1-\delta}]$. Then, f_δ satisfies $f_\delta'' + (1-\delta) f_\delta = 0$ with $f_\delta(0) = 0$, $f_\delta'(0) = 1$. Thus, we have a new sphere $(\mathbb{S}^2_\delta, \tilde{o})$ with the metric $d\tilde{s}^2_\delta = dt^2 + f_\delta(t)^2 d\theta^2$ on $(0, \pi/\sqrt{1-\delta}) \times \mathbb{S}^1_{\tilde{o}}$. Since the curvature $1-\delta$ of $(\mathbb{S}^2_\delta, \tilde{o})$ is less than 1, we may also employ $(\mathbb{S}^2_\delta, \tilde{o})$ as a reference surface for M.

Let c, $x = c(0)$, γ and $l = d(p, x)$ be the same in the above. Fix a point $\tilde{x} \in \mathbb{S}^2_\delta$ with $\tilde{d}_\delta(\tilde{o}, \tilde{x}) = l$, where $\tilde{d}_\delta$ denotes the distance function induced from $d\tilde{s}^2_\delta$. Let $\tilde{\gamma}\colon [0, l] \to \mathbb{S}^2_\delta$ be the minimal geodesic segment from $\tilde{o}$ to $\tilde{x}$, and take a unit parallel vector field $\tilde{E}$ along $\tilde{\gamma}$ orthogonal to $\dot{\tilde{\gamma}}$. Define the Jacobi field $\tilde{X}$ along $\tilde{\gamma}$ by

$$\tilde{X}(t) := \frac{1}{f_\delta(l)} f_\delta(t) \tilde{E}(t). \tag{4.4}$$

Lemma 4.1 ([6, Lemma 3.4]). *For any Jacobi field X along γ which is $g_{\dot{\gamma}}$-orthogonal to $\dot{\gamma}$ and satisfies $X(0) = 0$ and $g_{\dot{\gamma}(l)}(X(l), X(l)) = 1$, we have*

$$\tilde{I}_l(\tilde{X}, \tilde{X}) \geq I_l(X, X) + \frac{\delta}{f_\delta(l)^2} \int_0^l f_\delta(t)^2\, dt.$$

Here, I_l and $\tilde{I}_l$ denote the index forms with respect to $\gamma|_{[0,l]}$ and $\tilde{\gamma}|_{[0,l]}$, respectively.

Fix a geodesic $\tilde{c}\colon (-\varepsilon, \varepsilon) \to \mathbb{S}^2_\delta$ with $\tilde{c}(0) = \tilde{x}$ such that

$$\angle\big(\dot{\tilde{\gamma}}(l), \dot{\tilde{c}}(0)\big) = \omega, \quad \big\|\dot{\tilde{c}}\big\| = \lambda := \max\{1, F(-\dot{c}(0))\},$$

where ω is as that in (4.3). Consider the geodesic variation

$$\tilde{\varphi}(t, s) := \exp_{\tilde{o}}\left(\frac{t}{l} \exp_{\tilde{o}}^{-1}(\tilde{c}(s))\right), \quad (t, s) \in [0, l] \times (-\varepsilon, \varepsilon).$$

By setting $\tilde{J}(t) := (\partial\tilde{\varphi}/\partial s)(t, 0)$, we get the Jacobi field $\tilde{J}$ along $\tilde{\gamma}$ with $\tilde{J}(0) = 0$ and $\tilde{J}(l) = \dot{\tilde{c}}(0)$. And the Jacobi field

$$\tilde{J}^\perp(t) := \tilde{J}(t) - \frac{\big\langle \dot{\tilde{\gamma}}(l), \dot{\tilde{c}}(0)\big\rangle}{l} t\dot{\tilde{\gamma}}(t)$$

along $\tilde{\gamma}$ is orthogonal to $\dot{\tilde{\gamma}}(t)$ on $[0, l]$.

Lemma 4.2. *Assume that*

(1) $B^-_{2r}(q) \subset B^+_{\pi/2}(p)$;

(2) $F(v)^2 \geq g_{\dot\gamma(l)}(v, v)$ *for all* $v \in T_x M$.

If $\omega \in (0, \pi)$, *then there exists* $\delta_1 := \delta_1(f, r) > 0$ *such that, for any* $\delta \in (0, \delta_1)$,

$$\tilde I_l(\tilde J^\perp, \tilde J^\perp) - I_l(J^\perp, J^\perp) \geq \delta C_1 g_{\dot\gamma(l)}(J^\perp(l), J^\perp(l)) > 0$$

holds, where $C_1 := 1/(2f(l_0)^2)\int_0^{l_0} f(t)^2\,dt$ *and* $l_0 := d(p, q)$.

Proof. By the assumption (2) in this lemma,

$$\lambda^2 \geq g_{\dot\gamma(l)}(\dot c(0), \dot c(0)). \tag{4.5}$$

Indeed, (4.5) is immediate in the case where $\lambda = 1$. If $\lambda = F(-\dot c(0))$, then

$$1 \geq g_{\dot\gamma(l)}\left(\frac{-\dot c(0)}{F(-\dot c(0))}, \frac{-\dot c(0)}{F(-\dot c(0))}\right) = \frac{1}{F(-\dot c(0))^2} g_{\dot\gamma(l)}(\dot c(0), \dot c(0)).$$

By (4.2) and (4.3), $g_{\dot\gamma(l)}(\dot\gamma(l), \dot c(0)) = \lambda\cos\omega$. Then, $\tilde J^\perp(l) = \pm\lambda\sin\omega\cdot\tilde X(l)$ holds, where $\tilde X$ is the same as that in (4.4). Since both $\tilde J^\perp$ and $\tilde X$ are Jacobi fields on $\mathbb{S}^2_\delta$, $\tilde J^\perp(t) = \pm\lambda\sin\omega\cdot\tilde X(t)$ on $[0, l]$. Hence

$$\tilde I_l(\tilde J^\perp, \tilde J^\perp) = (\lambda\sin\omega)^2 \tilde I_l(\tilde X, \tilde X). \tag{4.6}$$

On the other hand, it follows from (4.1) and (4.5) that

$$g_{\dot\gamma(l)}(J^\perp(l), J^\perp(l)) = g_{\dot\gamma(l)}(\dot c(0), \dot c(0)) - (\lambda\cos\omega)^2 \leq (\lambda\sin\omega)^2.$$

Then, we get a constant $a := (\lambda\sin\omega)^2 - g_{\dot\gamma(l)}(J^\perp(l), J^\perp(l)) \geq 0$. Since $g_{\dot\gamma(l)}(J^\perp(l), J^\perp(l)) > 0$ for $\omega \in (0, \pi)$, we have, by Lemma 4.1,

$$\tilde I_l(\tilde X, \tilde X) \geq \frac{I_l(J^\perp, J^\perp)}{g_{\dot\gamma(l)}(J^\perp(l), J^\perp(l))} + \frac{\delta}{f_\delta(l)^2}\int_0^l f_\delta(t)^2\,dt,$$

hence

$$\begin{aligned} -I_l(J^\perp, J^\perp) &\geq -g_{\dot\gamma(l)}(J^\perp(l), J^\perp(l))\left\{\tilde I_l(\tilde X, \tilde X) - \frac{\delta}{f_\delta(l)^2}\int_0^l f_\delta(t)^2\,dt\right\} \\ &= \{a - (\lambda\sin\omega)^2\}\tilde I_l(\tilde X, \tilde X) + \frac{\delta\cdot g_{\dot\gamma(l)}(J^\perp(l), J^\perp(l))}{f_\delta(l)^2}\int_0^l f_\delta(t)^2\,dt. \end{aligned} \tag{4.7}$$

By (4.6) and (4.7),

$$\begin{aligned} \tilde I_l(\tilde J^\perp, \tilde J^\perp) - I_l(J^\perp, J^\perp) &\geq a\tilde I_l(\tilde X, \tilde X) + \frac{\delta\cdot g_{\dot\gamma(l)}(J^\perp(l), J^\perp(l))}{f_\delta(l)^2}\int_0^l f_\delta(t)^2 \\ &\geq \frac{\delta\cdot g_{\dot\gamma(l)}(J^\perp(l), J^\perp(l))}{f_\delta(l)^2}\int_0^l f_\delta(t)^2, \end{aligned}$$

where note that $a \geq 0$, and that $\tilde{I}_l(\tilde{X}, \tilde{X}) = \sqrt{1-\delta}/\tan(\sqrt{1-\delta}l) > 0$, because $l < \pi/2 < \pi/2\sqrt{1-\delta}$ by the assumption (1) in this lemma. Since $|l - l_0| \leq \max\{d(q, x), d(x, q)\} < r$, and since $l, l_0 < \pi/2$ (from the (1)), taking smaller $\delta_1(f, r) > 0$ if necessary, we get the desired assertion in this lemma for all $\delta \in (0, \delta_1)$. $\square$

Lemma 4.3. *Assume that*
(1) $B^-_{2r}(q) \subset B^+_{\pi/2}(p)$;
(2) $F(v)^2 \geq g_{\dot{\gamma}(l)}(v, v)$ *for all* $v \in T_x M$;
(3) $\mathcal{T}_M(\dot{\gamma}(l), \dot{c}(0)) = 0$.
For each $\delta \in (0, \delta_1)$, $\theta \in (0, \pi/2)$, *if* $\omega \in [\theta, \pi - \theta]$, *then there exists* $\varepsilon' := \varepsilon'(M, l, f, \varepsilon, \delta, \theta) \in (0, \varepsilon)$ *such that* $L(s) \leq \tilde{L}(s)$ *holds for all* $s \in [-\varepsilon', \varepsilon']$. *Here,* $L(s) := d(p, c(s))$ *and* $\tilde{L}(s) := \tilde{d}_\delta(\tilde{o}, \tilde{c}(s))$.

Proof. We will state the outline of the proof, since the proof is very similar to [6, Lemma 3.6] thanks to Lemma 4.2. Set $\mathcal{R}(s) := L(s) - \{L(0) + L'(0)s + L''(0)s^2/2\}$. Then, there exists $C_2 := C_2(M, l) > 0$ such that

$$\begin{aligned} L(s) &= L(0) + L'(0)s + \frac{1}{2}L''(0)s^2 + \mathcal{R}(s) \\ &\leq l + s\lambda\cos\omega + \frac{s^2}{2}I_l(J^\perp, J^\perp) + C_2|s|^3. \end{aligned}$$

Note that $L'(0) = \lambda\cos\omega$ and $L''(0) = I_l(J^\perp, J^\perp)$ hold by [6, Lemma 3.3], (4.2), (4.3), and the assumption (3) in this lemma. Similarly,

$$\tilde{L}(s) \geq l + s\lambda\cos\omega + \frac{s^2}{2}\tilde{I}_l(\tilde{J}^\perp, \tilde{J}^\perp) - C_3|s|^3$$

for some $C_3 := C_3(f, l) > 0$ and all $s \in (-\varepsilon, \varepsilon)$. Since $g_{\dot{\gamma}(l)}(J^\perp(l), J^\perp(l)) > 0$ for all $\omega \in [\theta, \pi - \theta]$, there exists $C_4 := C_4(M, \theta) > 0$ such that $g_{\dot{\gamma}(l)}(J^\perp(l), J^\perp(l)) > C_4 > 0$. From Lemma 4.2, $\tilde{L}(s) - L(s) \geq s^2\{\delta C_1 C_4 - 2(C_2 + C_3)s\}/2$ holds. Therefore, we get $L(s) \leq \tilde{L}(s)$ for all $s \in [-\varepsilon', \varepsilon']$, if $\varepsilon' := \min\{\varepsilon, \delta C_1 C_4/2(C_2 + C_3)\}$. $\square$

Thanks to Lemma 4.3 and the structure of $\mathbb{S}^2_\delta$, we may prove Lemma 2.9 by the same arguments in Sections 4, 5, and 6 in [6]. $\square$

REMARK 4.4. Although we do not consider cases of $\omega = 0$, or π in Lemma 4.3, Lemma 2.9 holds in cases of $\overrightarrow{\angle} x = \pi$, $\overleftarrow{\angle} y = 0$, or $\overrightarrow{\angle} x = 0$, $\overleftarrow{\angle} y = \pi$ because the reverse curve $\bar{c}$ of the geodesic segment c is geodesic.

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