EQUIVARIANT HYPERBOLIC DIFFEOMORPHISMS AND REPRESENTATION COVERINGS

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Abstract

Let G be a compact Lie group and X be a compact smooth G-manifold with finitely many G-fixed points. We show that if X admits a G-equivariant hyperbolic diffeomorphism having a certain convergence property, there exists an open covering of X indexed by the G-fixed points so that each open set is G-invariant and G-equivariantly diffeomorphic to the tangential G-representation at the corresponding G-fixed point. We also show that the converse is also true in case of holomorphic torus actions.

1. Introduction

It is known that there is a certain similarity between algebraic $\mathbb{C}^*$-actions on smooth complex projective varieties and smooth $\mathbb{R}$-actions on compact Riemannian manifolds arising from the negative gradient flows for Morse functions (see, for example, [4, 2.4]). The first similarity is that the both actions induce decompositions of acted spaces via affine spaces, that is, the Białynicki-Birula decompositions and the Morse decompositions.

On the other hand, a theorem of Białynicki-Birula [2] states that the existence of an algebraic $\mathbb{C}^*$-action on a smooth complex projective variety implies the existence of an open covering of the variety so that each open set is isomorphic to the affine space having the same dimension and is invariant under the $\mathbb{C}^*$-action. So, roughly speaking, if a smooth complex projective variety has an algebraic $\mathbb{C}^*$-action, it is covered by representation spaces of $\mathbb{C}^*$.

The aim of the present paper is to give a Morse theoretic counterpart of the above result of Białynicki-Birula. More generally, we consider a compact smooth manifold having an invariant hyperbolic diffeomorphism satisfying a certain convergence condition. Let us explain our result precisely.

Let G be a compact Lie group and X be a compact smooth G-manifold with finitely many G-fixed points. Let $\varphi\colon X \to X$ be a G-equivariant hyperbolic diffeomorphism of X. Then we say that the hyperbolic diffeomorphism φ satisfying *convergence condition* if the following three conditions are satisfied:

(C1) the G-fixed point set coincides with the fixed point set of φ,

(C2) for each G-fixed point p, the intersection of the corresponding stable and unstable manifolds is the singleton $\{p\}$,

(C3) for any $x \in X$, the sequence $\varphi^n(x)$ converges to a G-fixed point when n tends to ∞ or $-\infty$.

Note that if φ is the hyperbolic diffeomorphism arising from the negative gradient flow of a G-invariant Morse function relative to a G-invariant Riemannian metric, it satisfies the convergence condition. In particular, for a compact torus T, every compact Hamiltonian T-space with finitely many T-fixed points admits such a hyperbolic diffeomorphism.

We next make the following definition:

DEFINITION.   An open covering $(U_p \mid p \in X^G)$ of X indexed by G-fixed points is called a *G-representation covering* of X if each open set U_p is invariant under the G-action and is G-equivariantly diffeomorphic to the tangential G-representation T_pX.

Under the above terminology, we will prove the following theorem:

Theorem 1.1.   *Assume that the compact smooth G-manifold X has a G-equivariant hyperbolic diffeomorphism satisfying the convergence condition. Then X admits a G-representation covering.*

Theorem 1.1 implies the following results:

Corollary 1.2.   *If X has a G-invariant Morse function, there exists a G-representation covering of X.*

Corollary 1.3.   *Every compact Hamiltonian T-space with finitely many T-fixed points admits a T-representation covering.*

Concerning Theorem 1.1, a natural question arises: Does the converse of Theorem 1.1 hold? The following second theorem gives a partial affirmative answer, that is, the converse is also true in case of holomorphic torus actions:

Theorem 1.4.   *Let T be a compact torus. Assume that a compact complex manifold X with holomorphic T-action admits a T-representation covering as a holomorphic chart. Then it admits a T-equivariant hyperbolic diffeomorphism satisfying the convergence condition.*

We explain the contents of the present paper. In Section 2 we recall some basic definitions concerning hyperbolic diffeomorphisms and hyperbolic dynamical systems. In Section 3, we prove Theorem 1.1. In Section 4, we give a method to construct a

hyperbolic dynamical system from torus actions and invariant almost complex structures. Finally, in Section 5 we prove Theorem 1.4.

2. Hyperbolic dynamical systems

In this section we recall some fundamental definitions concerning hyperbolic dynamical systems and hyperbolic diffeomorphisms. For the details, we refer [3], [11].

Let X be a compact manifold and ξ be a smooth vector field on X. Let $\varphi \colon \mathbb{R} \times X \to X$ be the corresponding dynamical system. So the curve $\{\varphi(s, x)\}_{s \in \mathbb{R}}$ gives the flow of ξ whose initial value is x.

Let p be a fixed point of φ. Recall that the differential $(d\xi)_p \colon T_p X \to T_p X$ is an $\mathbb{R}$-linear transformation defined as follow: for a local frame $\xi_1, \ldots, \xi_n \in \Gamma(TU)$ ($n = \dim X$) on an open neighborhood U of p, the representation matrix of $(d\xi)_p$ with respect to the ordered base $(\xi_1)_p, \ldots, (\xi_n)_p$ is given by $[(\xi_i)_p f_j]_{i,j}$ where $f_i \colon U \to \mathbb{R}$ are C^∞-function satisfying $\xi|_U = \sum_{1 \le i \le n} f_i \xi_i$. It is straightforward to check that the definition is independent of choice of local frame.

DEFINITION. (1) A fixed point p of φ is called a *hyperbolic fixed point* of φ if the differential $(d\xi)_p$ has no complex eigenvalues whose real parts are zero.
(2) We say that ξ is a *hyperbolic vector field* if all fixed points of φ are hyperbolic.

Let ξ be a hyperbolic vector field and p be a hyperbolic fixed point. The *stable* and *unstable manifold* of p are defined by

$$W^s(p) := \left\{ x \in X \ \middle| \ \lim_{s \to \infty} \varphi(s, x) = p \right\}$$

and

$$W^u(p) := \left\{ x \in X \ \middle| \ \lim_{s \to -\infty} \varphi(s, x) = p \right\}$$

respectively.

We next explain the notion of hyperbolic diffeomorphisms. Let $\varphi \colon X \to X$ be a diffeomorphism of X.

DEFINITION. (1) A fixed point p of φ is called *hyperbolic fixed point* of φ if the differential $(d\varphi)_p \colon T_p X \to T_p X$ has no complex eigenvalues whose absolute values are 1.
(2) We say that φ is a *hyperbolic diffeomorphism* of X if all fixed points of φ are hyperbolic.

Let φ be a hyperbolic diffeomorphism of X and p be a hyperbolic fixed point. The *stable* and *unstable manifold* of p are defined by

$$W^s(p) := \left\{ x \in X \,\middle|\, \lim_{n \to \infty} \varphi^n(x) = p \right\}$$

and

$$W^u(p) := \left\{ x \in X \,\middle|\, \lim_{n \to -\infty} \varphi^n(x) = p \right\}$$

respectively. The *stable manifold theorem* states that they are injectively immersed sub-manifold of X.

We summarize the facts which will be needed in the later sections:

Lemma 2.1. *Let ξ be a vector field on X and $\varphi \colon \mathbb{R} \times X \to X$ be the correspond-ing dynamical system.*
(1) ξ is hyperbolic if and only if the diffeomorphism $\varphi_s \colon X \to X$ defined by $\varphi_s(x) := \varphi(s, x)$ is hyperbolic for all $s \in \mathbb{R} \setminus \{0\}$.
(2) If the limits $\lim_{s \to \infty} \varphi(s, x)$ and $\lim_{s \to -\infty} \varphi(s, x)$ exist for all $x \in X$, then the lim-its $\lim_{n \to \infty} \varphi_s^n(x) = p$ and $\lim_{n \to -\infty} \varphi_s^n(x) = p$ exist for all $s \in \mathbb{R}$, $x \in X$. Also, the corresponding stable and unstable manifolds coincide.

Proof.   For (1), we refer [3, p. 113, Lemma 4.19]. (2) is clear.   □

3.   Representation covering

Let G be a compact Lie group and X be a compact smooth G-manifold with fi-nitely many G-fixed points. We introduce our main subject in this paper, that is, the notion of G-representation coverings of X:

DEFINITION.   We say that a smooth G-manifold X has a *G-representation cover-ing* if there exists an open covering $(U_p | p \in X^G)$ of X indexed by G-fixed points such that
(1) each U_p is G-invariant,
(2) $U_p \cong T_p X$ as G-manifolds.

We give three examples of smooth G-manifolds admitting representation coverings.

EXAMPLE.   Let

$$X = S^{2n} = \{(z_1, \ldots, z_n, s) \in \mathbb{C}^n \times \mathbb{R} \mid |z_1|^2 + \cdots + |z_n|^2 + s^2 = 1\}$$

be the $2n$-dimensional sphere. We define an action of $T = (S^1)^n$ on S^{2n} by

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n, s) = (t_1 z_1, \ldots, t_n z_n, s).$$

Then one can construct a T-representation covering of the pair (S^{2n}, T) as follow:

Let $S^{2n} = U_0 \cup U_\infty$, $U_0 = \{s \neq 1\}$, $U_\infty = \{s \neq -1\}$ be the standard T-invariant open covering.

Then the maps

$$\varphi_0 \colon U_0 \to T_{(0,\ldots,0,1)}S^{2n},$$

$$\varphi_\infty \colon U_\infty \to T_{(0,\ldots,0,-1)}S^{2n}$$

given by

$$\varphi_0(z_1, \ldots, z_n, s) = \left(\frac{\overline{z_1}}{1-s}, \ldots, \frac{\overline{z_n}}{1-s} \right),$$

$$\varphi_\infty(z_1, \ldots, z_n, s) = \left(\frac{z_1}{1+s}, \ldots, \frac{z_n}{1+s} \right)$$

are T-equivariant. Note that the former is the composition of the stereographic projection and the componentwise complex conjugation $(z_1, \ldots, z_n) \mapsto (\overline{z_1}, \ldots, \overline{z_n})$.

EXAMPLE. Let

$$X = \mathbb{C}P^n = \{[z_0 : z_1 : \cdots : z_n] \mid (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{(0, 0, \ldots, 0)\}\}$$

be the complex projective space of complex dimension n. We define an action of $T = (S^1)^n$ by

$$(t_1, \ldots, t_n) \cdot [z_0 : z_1 : \cdots : z_n] = [z_0 : t_1 z_1 : \cdots : t_n z_n].$$

Then one can construct a T-representation covering of $\mathbb{C}P^n$ as follow:
Let U_i be an open set of $\mathbb{C}P^n$ defined by

$$U_i := \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{C}P^n \mid z_i \neq 0\} \quad (0 \leq i \leq n).$$

Then the open covering $(U_i \mid 0 \leq i \leq n)$ gives a T-representation covering. The same construction is possible for arbitrary toric manifold, i.e., complete non-singular toric variety with the Hausdorff topology (see [5]).

EXAMPLE. Let G be a connected reductive algebraic group over $\mathbb{C}$ and B be a Borel subgroup of G. We take X and T as the flag manifold G/B and the compact real form of the maximal torus of G contained in B, respectively.

Then one can construct a T-representation covering of the pair $(G/B, T)$ as follow: Let $X(w_0)$ be the Bruhat cell corresponding to the longest element w_0 of W. We set $U_w := w w_0^{-1} X(w_0)$ ($w \in W$). Then [12, p. 149, 8.5.1. Proposition (ii)] and [12, p. 152, 8.5.10. Excercise (1)] imply that the open sets $\{U_w \mid w \in W\}$ of X gives an

open covering of the flag manifold X and each U_w is T-equivariantly diffeomorphic to

$$\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-w.\alpha}.$$

Here, Δ^+ is the set of positive roots corresponding to the Borel subgroup B and $\mathfrak{g}_{-w\alpha}$ is the root space for the root $-w.\alpha$.

To state our result, we make the following definition:

DEFINITION. We say that a G-equivariant hyperbolic diffeomorphism

$$\varphi \colon X \to X$$

has *convergence property* if it satisfies the following three properties:
(C1) $X^G \subset \mathrm{Fix}(\varphi)$,
(C2) $W^u(p) \cap W^s(p) = \{p\}$ for all $p \in X^G$,
(C3) the limits $\lim_{n \to \infty} \varphi^n(x)$ and $\lim_{n \to -\infty} \varphi^n(x)$ converge to some G-fixed points of X for all $x \in X$.

The following is our main theorem in this section:

Theorem 3.1. *If X admits a G-equivariant hyperbolic diffeomorphism having convergence property, then there exists a G-representation covering of X.*

As a corollary one finds that the existence of a G-representation covering gives an obstruction for the existence of G-invariant Morse functions:

Corollary 3.2. *If X admits a G-invariant Morse function, there exists a G-representation covering of X.*

Proof. *Take a G-invariant Riemannian metric on X. Let $\tilde{\varphi} \colon \mathbb{R} \times X \to X$ be the hyperbolic dynamical system associated to the negative gradient vector field of the G-invariant Morse function relative to the G-invariant Riemannian metric. Then the hyperbolic diffeomorphism $\varphi \colon X \to X$ defined by $\varphi(x) := \tilde{\varphi}(1, x)$ gives a G-equivariant hyperbolic diffeomorphism having convergence property.* ☐

Corollary 3.2 implies the following. Let T be a compact torus.

Corollary 3.3. *Every compact Hamiltonian T-manifold X with finitely many T-fixed points admits a T-representation covering.*

Proof. *Let $\Phi \colon X \to (\mathrm{Lie}(T))^*$ be the moment map. Then for a generic $a \in \mathrm{Lie}(T)$, the contraction $\Phi_a \colon X \to \mathbb{R}$, $\Phi_a(x) := (\Phi(x))(a)$ give a T-invariant Morse function as well known (see for example [10]).* ☐

REMARK. It is known a similarity between algebraic $\mathbb{C}^*$-actions on smooth complex projective varieties and the $\mathbb{R}$-actions arising from negative gradient flows associated to Morse functions on compact manifolds (for example, see [4, 2.4]). A theorem of Białynicki-Birula [2] states that every smooth complex projective variety with an algebraic $\mathbb{C}^*$-action has a Zariski open covering so that each Zariski open set is $\mathbb{C}^*$-invariant and is isomorphic to an algebraic $\mathbb{C}^*$-representation. From this point of view, Theorem 3.1 (or Corollary 3.2) can be thought as a Morse theoretic counterpart of the above theorem of Białynicki-Birula.

The rest of this section is devoted to the proof of Theorem 3.1. We begin by the following lemma:

Lemma 3.4. *Let E be a Banach space, U be an open neighborhood of $0 \in E$ and $f : U \to E$ be a C^∞-map such that $f(0) = 0$, $(df)_0 = \mathrm{id}_E$. If a positive real number $r > 0$ satisfies that by measuring the operator norm, $|\mathrm{id}_E - (df)_x| < 1/2$ for any $x \in B_{2r}(0)$, then the restriction of f induces a C^∞-diffeomorphism*

$$f^{-1}(B_{r/2}(0)) \cap B_r(0) \to B_{r/2}(0).$$

Proof. This follows from the proof of the inverse mapping theorem given in [8]. □

Let N be a G-manifold (not necessary compact), p be a G-fixed point of N and $\varphi : N \to N$ be a G-equivariant hyperbolic diffeomorphism of N having the following properties:
- $N^G = \mathrm{Fix}(\varphi) = \{p\}$,
- p is the global attractor, i.e., $N = W^s(p)$.

We fix a G-invariant Riemannian metric of N and denote by $\|\cdot\|$ the corresponding fiberwise norm of TN. For a positive real number $r > 0$, we denote by $B_r(0)$ the open ball in T_pN which is centered at 0 and have radius r with respect to the norm $\|\cdot\|$. We note that since the norm $\|\cdot\|$ is G-invariant, each $B_r(0)$ is invariant under the G-action on T_pN. The set $\{B_r(0) \mid r > 0\}$ forms a fundamental neighborhood system of $0 \in T_pN$.

By focusing on the exponential map with respect to the G-invariant Riemannian metric, one can find a positive real number $R > 0$, a T-invariant open neighborhood U of p and a G-equivariant diffeomorphism
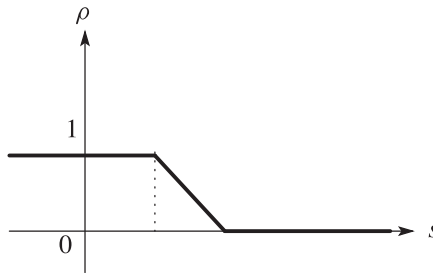
$$\psi : U \to B_R(0) \quad (\subset T_pN).$$

We define a G-equivariant diffeomorphism

$$g \colon \psi(U \cap \varphi^{-1}(U)) \to \psi(U \cap \varphi(U))$$

by

$$g := \psi|_{U \cap \varphi(U)} \circ \varphi|_{U \cap \varphi^{-1}(U)} \circ \psi^{-1}|_{\psi(U \cap \varphi^{-1}(U))}.$$

In the rest of this section we set $E := T_p N$ for simplicity. We note that the vector space E can be viewed as a G-representation, i.e., the tangential representation at p. Let us define a C^∞-map $u \colon E \to E$ by $u(\xi) = \rho(\|\xi\|)\xi$ where ρ is a C^∞-function $\rho \colon \mathbb{R} \to \mathbb{R}$ depicted by the following picture:

We note that since the fiberwise norm $\|\cdot\|$ is G-invariant and since G acts on E as a representation, the C^∞-map u is G-equivariant.

Let A be the differential $(dg)_0 \colon E \to E$ of g at 0 and we set

$$\Phi := A^{-1} \circ g \colon \psi(U \cap \varphi^{-1}(U)) \to E.$$

Since G acts on E as a G-representation, both of A and Φ are G-equivariant. In the rest of this section, we set $V := \psi(U \cap \varphi^{-1}(U))$ for simplicity.

What we first want to show is that the germ of the map $g \colon \psi(U \cap \varphi^{-1}(U)) \to E$ at 0 coincides with the germ of a G-equivariant diffeomorphism $h \colon E \to E$ at 0, that is, the maps coincide on an open neighborhood of 0.

For the proof of Theorem 3.1, we need to go back to the proof of the so-called stable manifold theorem. We should remark that our arguments in Lemmas 3.5 and 3.6 are greatly influenced on the paper of Abbondandolo–Majer [1].

Under the above notations, we next show the following lemma:

Lemma 3.5. *There exist positive real numbers $r > 0$, $\delta > 0$ and C^∞-maps $X \colon [0, 1] \times B_{r/2}(0) \to E$, $Y \colon [0, 1] \times B_\delta(0) \to E$ satisfying the following conditions*:
(1) $Y([0, 1] \times B_\delta(0)) \subset B_{r/2}(0)$,
(2) $Y(0, \xi) = \xi$ *and* $(\partial Y / \partial t)(t, \xi) = X(t, Y(t, \xi))$ *for all* $(t, \xi) \in [0, 1] \times B_\delta(0)$,
(3) $Y(1, \xi) = \Phi(\xi)$ *for all* $\xi \in B_\delta(0)$,

(4) *X and Y are G-equivariant with respect to the second components,*

(5) $X(t, 0) = 0, (D_2 X_t)_0 = 0$ *for each $t \in [0, 1]$. Here X_t is given by $X_t(\xi) = X(t, \xi)$.*

Proof.   We follow up [1, Lemma 4.8]. We define a C^∞-map $Y \colon [0, 1] \times V \to E$ by $Y(t, \xi) = t\Phi(\xi) + (1 - t)\xi$ and set $Y_t(\xi) := Y(t, \xi)$. Note that since the map Φ is G-equivariant and G acts on E as a real G-representation, the map Y_t is G-equivariant for each $t \in [0, 1]$.

Let us consider a continuous map $\beta \colon [0, 1] \times V \to \mathbb{R}$ given by $\beta(t, \xi) := |\mathrm{id}_E - (dY_t)_\xi|$. Since $(dY_t)_0 = \mathrm{id}_E$ for any $t \in [0, 1]$, one finds that the set $\beta^{-1}(\{s \in \mathbb{R} \mid s < 1/2\})$ is an open neighborhood of $[0, 1] \times \{0\}$ in $[0, 1] \times V$. In particular, for each $t \in [0, 1]$, we can choose an open neighborhood U_t of t in $[0, 1]$ and a positive real number $r_t > 0$ so that $U_t \times B_{2r_t}(0) \subset \beta^{-1}(\{s \in \mathbb{R} \mid s < 1/2\})$. Thanks to the compactness of the closed interval $[0, 1]$, there exist finitely many real numbers $t_1, \ldots, t_n \in [0, 1]$ so that $[0, 1]$ is covered by $\{U_{t_i} \mid 1 \le i \le n\}$. We define a positive real number $r > 0$ by $r := \min_{1 \le i \le n} r_{t_i}$. Then one can easily deduce that $[0, 1] \times B_{2r}(0) \subset \beta^{-1}(\{s \in \mathbb{R} \mid s < 1/2\})$.

Therefore we have that $|\mathrm{id}_E - (dY_t)_\xi| < 1/2$ for all $(t, \xi) \in [0, 1] \times B_{2r}(0)$. Thus by Lemma 3.4, we can conclude that there exists a positive real number $r > 0$ which is independent of the parameter $t \in [0, 1]$ so that for each $t \in [0, 1]$, the restriction of Y_t induces a C^∞-diffeomorphism

$$Y_t^{-1}(B_{r/2}(0)) \cap B_r(0) \to B_{r/2}(0).$$

We can choose the real number r so that $B_r(0) \subset V$ by taking r smaller if necessary.

Denote by $\overline{Y}_t$ the above C^∞-diffeomorphism and define a C^∞-map $H \colon [0, 1] \times B_{r/2}(0) \to E$ by $H(t, \xi) := \overline{Y}_t^{-1}(\xi)$. It is easy to check that H is T-equivariant with respect to the second component.

We next claim that

$$\bigcap_{t \in [0,1]} Y_t^{-1}(B_{r/2}(0)) = Y_0^{-1}(B_{r/2}(0)) \cap Y_1^{-1}(B_{r/2}(0)).$$

Clearly the LHS is contained in the RHS. If ξ belongs to the RHS, one has $\xi = Y_0(\xi) \in B_{r/2}(0)$ and $\Phi(\xi) = Y_1(\xi) \in B_{r/2}(0)$. Thus by the convexity of the open ball $B_{r/2}(0)$, we have that $Y_t(\xi) = t\Phi(\xi) + (1 - t)\xi \in B_{r/2}(0)$ for all $t \in [0, 1]$, as desired.

As a consequence, there exists a positive real number $\delta > 0$ so that

$$B_\delta(0) \subset \bigcap_{t \in [0,1]} Y_t^{-1}(B_{r/2}(0)) \cap B_r(0) \cap V.$$

Then each $\overline{Y}_t$ induces an injection $B_\delta(0) \to B_{r/2}(0)$. In particular, for each $(t, \xi) \in [0, 1] \times B_\delta(0)$ we have $Y(t, \xi) \in B_{r/2}(0)$ and have $H(t, Y(t, \xi)) = \xi$.

Since we have chosen $r > 0$ so that $B_r(0) \subset V$, the image of $H\colon [0,1] \times B_\delta(0) \to E$ is contained in V. This allows us to define a C^∞-map $X\colon [0,1] \times B_{r/2}(0) \to E$ by $X(t, \xi) = \Phi(H(t, \xi)) - H(t, \xi)$. It is clear that X is G-equivariant with respect to the second component.

In the above arguments, we have constructed two C^∞-maps

$$X\colon [0,1] \times B_{r/2}(0) \to E, \quad Y\colon [0,1] \times B_\delta(0) \to E.$$

It is straightforward to check that the positive real numbers r, δ and the above two maps X, Y satisfy our desired properties. $\qquad\square$

Lemma 3.6. *There exists a positive real number $\epsilon > 0$ and a G-equivariant C^∞-diffeomorphism $h\colon E \to E$ satisfying the following conditions*:
(1) $B_\epsilon(0) \subset V \cap B_R(0)$,
(2) $h|_{B_\epsilon(0)} = g|_{B_\epsilon(0)}$,
(3) *for each $x \in N$, there exists a non-negative integer $n(x)$ so that $h^{n(x)}(x) \in B_\epsilon(0)$.*

Proof. We follow up [1, Proposition 4.7]. Let $X\colon [0,1] \times B_{r/2}(0) \to E$ and $Y\colon [0,1] \times B_\delta(0) \to E$ as in Lemma 3.5.

Since the function ρ is identity on an open neighborhood of $0 \in \mathbb{R}$ and has compact support, there exist two positive real numbers $0 < s_0 < r_0$ so that $u|_{B_{s_0}(0)} = \mathrm{id}_{B_{s_0}(0)}$ and $u(E) \subset B_{r_0}(0)$. By the condition (5) in Lemma 3.5 and the Tayler's formula, the argument presented in Lemma 3.5 shows that there exist positive real numbers $\epsilon_1, r_1 > 0$ so that $e^{\epsilon_1 r_0/s_0}\|A\| < 1$ and $|X(t, \xi)| \leq \epsilon_1 |\xi|$ for all $(t, \xi) \in [0,1] \times B_{r_1}(0)$. Let us define $\tilde{u}\colon E \to E$ and $\tilde{X}\colon [0,1] \times E \to E$ by

$$\tilde{u}(\xi) = \frac{r_1}{r_0} u\left(\frac{r_0}{r_1}\xi\right), \quad \tilde{X}(t, \xi) = X(t, \tilde{u}(\xi)).$$

Note that $\tilde{u}|_{B_s(0)} = \mathrm{id}_{B_s(0)}$ for $s := (r_1 s_0)/r_0$, and $\tilde{u}(E) \subset B_{r_1}(0)$. Also note that $\tilde{u}$ is G-equivariant and $\tilde{X}$ is G-equivariant with respect to the second component.

We claim that $|\tilde{X}(t, \xi)| \leq \epsilon_1 (r_0/s_0)|\xi|$ for all $(t, \xi) \in [0,1] \times E$. If $\xi \in B_s(0)$, the estimate holds since $X(t, \xi) = \tilde{X}(t, \xi)$. If $\xi \notin B_s(0)$, we have that $|\tilde{X}(t, \xi)| \leq \epsilon_1 |\tilde{u}| \leq \epsilon_1 r_1 \leq \epsilon_1 (r_1/s)|\xi| = \epsilon_1 (r_0/s_0)|\xi|$. As a consequence, for each $c > 0$, $\tilde{X}(t, \xi)$ is bounded velocity on $[0,1] \times B_c(0)$ in sense of [6, p. 178].

Thus, [6, p. 179, 1.1. Theorem] implies that there exists a unique diffeotopy $\tilde{Y}\colon [0,1] \times E \to E$. So we have that $(\partial \tilde{Y}/\partial t)(t, \xi) = \tilde{X}(t, \tilde{Y}(t, \xi))$ and $\tilde{Y}(0, \xi) = \xi$. Thanks to the G-equivariancy of $\tilde{X}$, the map $g^{-1} \cdot \tilde{Y}(t, g \cdot \xi)$ is also the solution of this equation for each $g \in G$. Thus one finds that $\tilde{Y}$ is G-equivariant with respect to the second component.

We next show that the estimate $|\tilde{Y}(t,\xi)| \leq e^{\epsilon_1(r_0/s_0)t}|\xi|$ holds for all $(t,\xi) \in [0,1] \times E$. From the defining equation of $\tilde{Y}(t, \xi)$, we have

$$\tilde{Y}(t, \xi) - \xi = \tilde{Y}(t, \xi) - \tilde{Y}(0, \xi) = \int_0^t \frac{\partial \tilde{Y}}{\partial s}(s, \xi)\, ds = \int_0^t \tilde{X}(s, \tilde{Y}(s, \xi))\, ds.$$

From this, one also has

$$\begin{aligned}
|\tilde{Y}(t, \xi)| &= \left| \xi + \int_0^t \tilde{X}(s, \tilde{Y}(s, \xi))\, ds \right| \\
&\leq |\xi| + \int_0^t |\tilde{X}(s, \tilde{Y}(s, \xi))|\, ds \\
&\leq |\xi| + \int_0^t \epsilon_1(r_0/s_0)|\tilde{Y}(s, \xi)|\, ds.
\end{aligned}$$

By applying Gronwall's integral inequality, we obtain the desired inequality.

Moreover, since X and $\tilde{X}$ coincide locally, the argument presented in Lemma 3.11 shows that there exists a positive real number $\epsilon > 0$ so that Y and $\tilde{Y}$ coincide on $[0,1] \times B_\epsilon(0)$, and $B_\epsilon(0) \subset V \cap B_R(0) \cap B_\delta(0)$.

Let us define a diffeomorphism $\tilde{\Phi} \colon E \to E$ by $\tilde{\Phi}(\xi) := \tilde{Y}(1, \xi)$. Then we claim that a map $h \colon E \to E$ defined by $h := A \circ \tilde{\Phi}$ is the desired diffeomorphism. Note that h is clearly G-equivariant.

The condition (1) is clear. The condition (2) holds since $Y(1, \xi) = \Phi(\xi)$ for all $\xi \in B_\delta(0)$ and Y and $\tilde{Y}$ coincide locally. Finally, since

$$|h(\xi)| \leq \|A\||\tilde{\Phi}(\xi)| \leq \|A\|e^{\epsilon_1(r_0/s_0)}|\xi|$$

and $\|A\|e^{\epsilon_1(r_0/s_0)} < 1$, the condition (3) holds.

The proof is now complete. $\qquad\square$

Proposition 3.7. *There exists a G-equivariant C^∞-diffeomorphism $T_p N \to N$.*

Proof. This follows from a standard argument and Lemma 3.12. We describe the proof to check the G-equivariancy.

Take a G-equivariant C^∞-diffeomorphism $h \colon E \to E$ as in Lemma 3.6. Let us define a map $\phi \colon N \to E$ as follow. For a point $x \in N$, one can find a non-negative integer n so that $\varphi^n(x) \in \psi^{-1}(B_\epsilon(0))$. By using the integer n, we define $\phi(x) := h^{-n}(\psi(\varphi^n(x)))$. We claim that the value $\phi(x)$ is independent of the integer n whenever $\varphi^n(x) \in \psi^{-1}(B_\epsilon(0))$. To prove the claim, we first note that $\psi(\varphi^n(x)) \in B_\epsilon(0)$. Since we have chosen the diffeomorphism h so that $h|_{B_\epsilon(0)} = g|_{B_\epsilon(0)}$, one obtains that $h(\psi(\varphi^n(x))) = g(\psi(\varphi^n(x)))$.

Henceforth we have that

$$h^{-n}(\psi(\varphi^n(x))) = h^{-(n+1)}(h(\psi(\varphi^n(x))))$$
$$= h^{-(n+1)}(g(\psi(\varphi^n(x))))$$
$$= h^{-(n+1)}((\psi \circ \varphi \circ \psi^{-1})(\psi(\varphi^n(x))))$$
$$= h^{-(n+1)}(\psi(\varphi^{n+1}(x))).$$

This completes the proof of the well-definedness of ϕ. Note that ϕ is clearly G-equivariant.

We next construct the inverse of ϕ. For $\xi \in E$, one can find a non-negative integer n so that $h^n(\xi) \in B_\epsilon(0)$. Then we define $\overline{\phi}: E \to N$ by $\overline{\phi}(\xi) := \varphi^{-n}(\psi^{-1}(h^n(\xi)))$. The same argument shows that the value $\overline{\phi}(\xi)$ is independent of the integer n whenever $h^n(\xi) \in B_\epsilon(0)$.

Since $\overline{\phi}$ clearly gives the inverse of ϕ, the proof is now complete. $\qquad\square$

We go to the next step. Let W be a finite dimensional real G-representation and $\pi: E \to W$ be a smooth G-equivariant vector bundle over W. Let $\nabla: \Gamma(TW) \times \Gamma(E) \to \Gamma(E)$ be a G-equivariant covariant derivative, that is, a covariant derivative having the property that

$$(\nabla_{g \cdot \xi}(g \cdot s))(g \cdot x) = g \cdot ((\nabla_\xi s)(x))$$

for all $g \in G$, $\xi \in \Gamma(TW)$, $s \in \Gamma(E)$, $x \in W$. Note that such a covariant derivative exists since for a fixed covariant derivative ∇, one can construct a G-equivariant covariant derivative $\tilde{\nabla}$ by averaging as follow:

$$(\tilde{\nabla}_\xi s)(x) := \int_{g \in G} [g^{-1} \cdot \nabla_{g \cdot \xi}(g \cdot s)](x) \, dg.$$

Here dg is the Haar measure of G normalized so that $\int_{g \in G} dg = 1$.

For a point $x \in W$, we define a curve $c_x: [0, 1] \to W$ by $c_x(t) := (1 - t)x$. We denote by $P_{c_x}: \pi^{-1}(x) \to \pi^{-1}(0)$ the parallel transport associated to the curve c_x and the covariant derivative ∇. Let $\Gamma(c_x)$ be the set of smooth sections of E along the curve c_x. Note that an element $g \in G$ induces a map $g_*: \Gamma(c_x) \to \Gamma(c_{g \cdot x})$ defined by $(g_* s)(t) := g \cdot (s(t))$.

Recall from elementary differential geometry that there exists an $\mathbb{R}$-linear transformation

$$\frac{D}{dt}: \Gamma(c_x) \to \Gamma(c_x)$$

which is characterized by the following two properties:

(1) $(D/dt)(fs) = (df/dt)s + f(Ds/dt)$ $(f \in C^{\infty}([0, 1]),\ s \in \Gamma(c_x))$,
(2) if $s \in \Gamma(c_x)$ is given by $s(t) = \tilde{s}(c_x(t))$ $(t \in (a, b))$ for some $\tilde{s} \in \Gamma(E)$ and two real numbers $0 \leq a < b \leq 1$, then we have

$$\frac{Ds}{dt}(t) = \nabla_{c_x'(t)}\tilde{s} \quad (t \in (a, b)).$$

Lemma 3.8. *The diagram*

$$
\begin{array}{ccc}
\Gamma(c_x) & \xrightarrow{\ D/dt\ } & \Gamma(c_x) \\
g_* \downarrow & & \downarrow g_* \\
\Gamma(c_{g\cdot x}) & \xrightarrow[\ D/dt\ ]{} & \Gamma(c_{g\cdot x})
\end{array}
$$

commutes for all $g \in G$, $x \in W$.

Proof. Take $s \in \Gamma(c_x)$ and $t_0 \in [0, 1]$. Then by considering a local frame of E around $c_x(t_0)$, one can take a global section $\tilde{s} \in \Gamma(E)$ and two real numbers $0 \leq a < b \leq 1$ so that $s(t) = \tilde{s}(c_x(t))$ for all $t \in (a, b)$.

Then one has that

$$\left(g_* \frac{Ds}{dt}\right)(t_0) = g \cdot (\nabla_{c_x'(t_0)}\tilde{s})$$

$$= \nabla_{c_{g\cdot x}'(t_0)}(g \cdot \tilde{s}) \quad \text{(as } \nabla \text{ is } G\text{-equivariant).}$$

On the other hand, since

$$(g \cdot \tilde{s})(c_{g\cdot x}(t)) = g \cdot (\tilde{s}(g^{-1} \cdot (c_{g\cdot x}(t)))) = g \cdot (\tilde{s}(c_x(t))) = g \cdot (s(t)) = (g_*s)(t),$$

we have that

$$\nabla_{c_{g\cdot x}'(t_0)}(g \cdot \tilde{s}) = \frac{D(g_*s)}{dt}(t_0).$$

This completes the proof. $\qquad\square$

Corollary 3.9. *We have that $P_{c_{g\cdot x}}(g \cdot v) = g \cdot (P_{c_x}(v))$ for all $g \in G$, $x \in W$, $v \in \pi^{-1}(x)$.*

Proof. Let $s \in \Gamma(c_x)$ be the parallel section along to the curve c_x (so one has $Ds/dt = 0$).

By the definition of the parallel transport, we have that $s(0) = v$ and $P_{c_x}(v) = s(1)$.

Then the curve $g_* s \in \Gamma(c_{g \cdot x})$ satisfies that

$$(g_* s)(0) = g \cdot (s(0)) = g \cdot v$$

and

$$\frac{D(g_* s)}{dt} = g_* \frac{Ds}{dt} = 0.$$

by Lemma 3.8.

Thus we have that

$$P_{c_{g \cdot x}}(g \cdot v) = (g_* s)(1) = g \cdot (s(1)) = g \cdot (P_{c_x}(v))$$

as desired. $\qquad\square$

Proposition 3.10. *E is isomorphic to $W \times \pi^{-1}(0)$ as a G-equivariant smooth vector bundle over W. Here the G-action on $W \times \pi^{-1}(0)$ is given by the diagonal one.*

Proof. We define a map $\Psi \colon E \to W \times \pi^{-1}(0)$ by $\Psi(u) := (\pi(u), P_{c_{\pi(u)}}(u))$. Then the map Ψ gives an isomorphism of smooth vector bundles and is G-equivariant by Lemma 3.9. $\qquad\square$

We are now in the position to prove Theorem 3.1.

Proof of Theorem 3.1. By the assumption (C2), $W^s(p)$ and $W^u(p)$ are embedded G-invariant submanifold of X (see, for example, [3, p. 115, Lemma 4.20]). Thus one can apply G-equivariant tubular neighborhood theorem [7, p. 178, Theorem 4.8] to $W^s(p)$, and has a G-invariant open neighborhood U_p of $W^s(p)$ in X which is G-equivariantly diffeomorphic to the normal bundle

$$\nu_p := TX|_{W^s(p)}/TW^s(p)$$

of $W^s(p)$ in X.

On the other hand, we have the following series of G-equivariant diffeomorphisms:

$$\nu_p \cong TX|_{W^s(p)}/TW^s(p)$$
$$\cong W^s(p) \times (T_p X/T_p W^s(p)) \quad \text{(as Proposition 3.7 and 3.10)}$$
$$\cong W^s(p) \times T_p W^u(p) \quad \text{(as } T_p X = T_p W^s(p) \oplus T_p W^u(p))$$
$$\cong T_p W^s(p) \times T_p W^u(p) \quad \text{(as Proposition 3.7)}$$
$$\cong T_p X.$$

Since X is decomposed into stable manifolds $W^s(p)$ by the assumption (C3), the family $(U_p \mid p \in X^G)$ of G-stable open sets gives a desired G-representation covering of X. $\qquad\square$

4. Topological generators and hyperbolicity

In the previous section, we have shown that if a smooth G-manifold admits a certain equivariant hyperbolic diffeomorphism, it also admits a G-representation covering.

Then the following natural question arise: "Does the converse of Theorem 3.1 hold?". Unfortunately, the author do not have any satisfactory answer to the question.

The aim of this and the next section is to give a partial affirmative answer for the above question. Roughly speaking, we will show that the converse is also true in case of holomorphic torus actions.

In this section we give a method to construct hyperbolic dynamical systems from torus actions with invariant almost complex structures.

Let $T = (S^1)^r$ be the compact torus of rank r and X be a smooth $2n$-dimensional T-manifold with an almost complex structure J compatible with the T-action (the term "compatible" means that for each $t \in T$, $x \in X$, the induced map $t_* \colon T_x X \to T_{t \cdot x} X$ is $\mathbb{C}$-linear with respect to J_x and $J_{t \cdot x}$).

To construct a hyperbolic dynamical system from the torus action, we focus on a topological generator t_0 of T (by the definition, the cyclic group $\{t_0^k \mid k \in \mathbb{Z}\}$ generated by t_0 is a dense subset of the compact torus T). We also take an element a_0 of the Lie algebra $\mathfrak{t}$ of T so that $\exp(a_0) = t_0$. Let us denote by ξ_0 the fundamental vector field associated to a_0. By the definition we have

$$(\xi_0)_x \phi = \left. \frac{d}{ds} \phi(\exp(sa_0) \cdot x) \right|_{s=0}$$

for all smooth function $\phi \colon X \to \mathbb{R}$ and $x \in X$.

We set $\xi_0^J := -J\xi_0$, and denote by $\mathrm{zero}(\xi_0)$ and $\mathrm{zero}(\xi_0^J)$ the set of zero points of ξ_0 and ξ_0^J respectively.

Lemma 4.1. *We have the following*:

$$\mathrm{zero}(\xi_0^J) = \mathrm{zero}(\xi_0) = X^T.$$

Proof. The first equality and the inclusion $\mathrm{zero}(\xi_0) \supset X^T$ are clear. To see the inverse inclusion $\mathrm{zero}(\xi_0) \subset X^T$, let $p \in \mathrm{zero}\,(\xi_0)$. Since t_0 is a topological generator of T, it is enough to show that $t_0 \cdot p = p$.

Assume that $t_0 \cdot p \neq p$. Then there exists a smooth function $\phi \colon X \to \mathbb{R}$ which separates $t_0 \cdot p$ and p, that is, $\phi(t_0 \cdot p) \neq \phi(p)$.

We define a smooth function $\phi_0 \colon \mathbb{R} \to \mathbb{R}$ by

$$\phi_0(s) = \phi(\exp(sa_0) \cdot p).$$

We also define a diffeomorphism $\rho_0 \colon X \to X$ by $\rho_0(x) = (\exp(s_0 a_0)) \cdot x$. Then we have

$$
\begin{aligned}
\left.\frac{d}{ds}\phi_0(s)\right|_{s=s_0} &= \left.\frac{d}{ds}\phi(\exp(sa_0) \cdot p)\right|_{s=s_0} \\
&= \left.\frac{d}{ds}\phi(\exp((s + s_0)a_0) \cdot p)\right|_{s=0} \\
&= \left.\frac{d}{ds}(\phi \circ \rho_0)(\exp(sa_0) \cdot p)\right|_{s=0} \\
&= (\xi_0)_p(\phi \circ \rho_0) \\
&= 0
\end{aligned}
$$

for all $s_0 \in \mathbb{R}$. Thus the function ϕ_0 is a constant function. In particular, by taking the values at $s = 0$ and $s = 1$, we have $\phi(t_0 \cdot p) = \phi(p)$. This is a contradiction. $\qquad\square$

REMARK 4.2. The following seems to be well-known:

$$
\text{if } H^{\mathrm{odd}}(X) = \{0\}, \text{ we have that } X^T \neq \emptyset.
$$

In fact, one can give a short proof of this fact using our vector field ξ_0: Assume that $X^T = \emptyset$. Then $\mathrm{zero}(\xi_0) = \emptyset$ by Lemma 4.1. Henceforth [9, PROPERTY 9.7] implies that the Euler class of X vanishes. So the Euler characteristic of X also vanishes. This is a contradiction.

Let us denote by $\varphi^J \colon \mathbb{R} \times X \to X$ the dynamical system corresponding to ξ_0^J.

We take a Riemannian metric g of X which is T-invariant and J-Hermitian (note that for a T-invariant metric g_0, the metric

$$
g(u, v) := \frac{g_0(u, v) + g_0(Ju, Jv)}{2} \quad (u, v \in T_x X,\ x \in X)
$$

has the required property since we assumed that J is compatible with the T-action). For a T-fixed point p of X, let $\exp_p \colon T_p X \to X$ be the exponential map associated to the Levi-Civita connection of g. This is T-invariant and the differential at the origin of $T_p X$ is the identity map under the natural identification $T_0(T_p X) = T_p X$. Thus there exists a positive real number δ so that the restriction

$$
\exp_p|_{B_\delta(0)} \colon B_\delta(0) \to \exp_p(B_\delta(0))
$$

gives a T-equivariant diffeomorphism (here $B_\delta(0)$ is the open ball whose center is 0 and radius is δ in $T_p X$ with respect to the metric g). We set $U_p := \exp_p(B_\delta(0))$.

Moreover, by composing the diffeomorphism and the T-equivariant diffeomorphism

$$
T_p X \to B_\delta(0), \quad u \mapsto \frac{\delta u}{\sqrt{1 + \|u\|^2}},
$$

we obtain a T-equivariant diffeomorphism $\varphi_p \colon T_pX \to U_p$.

Let

$$T_pX = \bigoplus_{i=1}^{n} (T_pX)^{\alpha_{p,i}}$$

be the irreducible decomposition of the T-representation space T_pX. Here $\alpha_{p,i} \colon T \to S^1$ is a weight of the T-representation and

$$(T_pX)^{\alpha_{p,i}} = \{u \in T_pX \mid t \cdot u = \alpha_{p,i}(t)u \ (t \in T)\}$$

is the corresponding weight space.

We fix a $\mathbb{C}$-base $u_{p,1}, \ldots, u_{p,n}$ of T_pX so that $u_{p,i} \in (T_pX)^{\alpha_{p,i}}$ for all i. Then we have a local coordinate system $(U_p; x_1, y_1, \ldots, x_n, y_n)$ around p defined by

$$\varphi_p^{-1}(x) = \sum_{i=1}^{n} (x_i(x) + \sqrt{-1}\, y_i(x)) u_{p,i}.$$

Lemma 4.3. *Assume that X^T is non-empty and finite. Then $\varphi^J \colon \mathbb{R} \times X \to X$ is a hyperbolic dynamical system on X.*

Proof.   We set

$$\xi_0|_{U_p} = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x^i} + \sum_{i=1}^{n} g_i \frac{\partial}{\partial y^i} \quad (f_i, g_i \in C^\infty(U_p)).$$

So one has

$$\xi_0^J|_{U_p} = -\sum_{i=1}^{n} f_i J \frac{\partial}{\partial x^i} - \sum_{i=1}^{n} g_i J \frac{\partial}{\partial y^i}.$$

By the definition of the coordinate functions $x_1, y_1, \ldots, x_n, y_n$ and T-equivariancy of $\varphi_p \colon T_pX \to U_p$, we have

$$
\begin{aligned}
f_i(x) &= (\xi_0)_x x_i \\
&= \frac{d}{ds} x_i(\exp(sa_0) \cdot x)\Big|_{s=0} \\
&= \frac{d}{ds}\left(\mathbb{R}\text{-coefficient of } u_{p,i} \text{ in } \exp(sa_0) \cdot \left(\sum_{i=1}^{n}(x_i(x) + \sqrt{-1}\, y_i(x))u_{p,i}\right)\right)\Bigg|_{s=0} \\
&= \frac{d}{ds}(x_i(x)\,\mathrm{Re}\,\alpha_{p,i}(\exp(sa_0)) - y_i(x)\,\mathrm{Im}\,\alpha_{p,i}(\exp(sa_0)))\Big|_{s=0} \\
&= x_i(x)\,\mathrm{Re}\,d\alpha_{p,i}(a_0) - y_i(x)\,\mathrm{Im}\,d\alpha_{p,i}(a_0) \\
&= -y_i(x)\,\mathrm{Im}\ d\alpha_{p,i}(a_0)
\end{aligned}
$$

for all $x \in U_p$. A similar calculation shows that

$$g_i(x) = x_i(x) \operatorname{Im}\, d\alpha_{p,i}(a_0).$$

Thus we obtain that

$$\xi_0^J|_{U_p} = \sum_{i=1}^{n} y_i \operatorname{Im} d\alpha_{p,i}(a_0) J \frac{\partial}{\partial x^i} - \sum_{i=1}^{n} x_i \operatorname{Im} d\alpha_{p,i}(a_0) J \frac{\partial}{\partial y^i}.$$

By using the local frame

$$\left\{ J \frac{\partial}{\partial x^1}, J \frac{\partial}{\partial y^1}, \ldots, J \frac{\partial}{\partial x^n}, J \frac{\partial}{\partial y^n} \right\},$$

to compute the eigenvalues of $(d\xi_0^J)_p \colon T_p X \to T_p X$, one finds that the representation matrix is given by

$$\operatorname{diag}(a_1, a_1, \ldots, a_n, a_n) \quad (a_i = \operatorname{Im} d\alpha_{p,i}(a_0)).$$

Let us assume that φ^J is not hyperbolic. Then the $\mathbb{R}$-linear transformation $(d\xi_0^J)_p$ has a complex eigenvalue whose real part is zero. Hence we have $a_i = 0$ for some $i \in \{1,\ldots,n\}$. Since $d\alpha_{p,i}$ is $\sqrt{-1}\mathbb{R}$-valued, we also obtain that $d\alpha_{p,i}(a_0) = 0$. This implies that $\alpha_{p,i}(t_0) = \alpha_{p,i}(\exp(a_0)) = \exp(d\alpha_{p,i}(a_0)) = 1$. So $\alpha_{p,i}$ is identically equal to 1 since t_0 is a topological generator of T. This implies that $(T_p X)^{\alpha_{p,i}}$ is the trivial irreducible T-representation. Since $\exp_p \colon B_\delta(0) \to U_p$ is a T-equivariant diffeomorphism, we have $X^T \supset \exp_p(B_\delta(0) \cap (T_p X)^{\alpha_{p,i}})$. This is a contradiction since $\dim X^T = 0$ and $\dim B_\delta(0) \cap (T_p X)^{\alpha_{p,i}} = 2$. $\qquad\square$

Lemma 4.4. $\varphi_s^J(t \cdot x) = t \cdot \varphi_s^J(x)$ *for all* $s \in \mathbb{R}$, $t \in T$, $x \in X$.

Proof. Since the almost complex structure J is compatible with the T-action, we have

$$\frac{d}{ds} \varphi_s^J(t \cdot x) = t_* \frac{d}{ds} \varphi_s^J(x) = -t_*(J\xi_0)_{\varphi_s^J(x)} = -J_{t \cdot \varphi_s^J(x)}(\xi_0)_{t \cdot \varphi_s^J(x)} = (\xi_0^J)_{t \cdot \varphi_s^J(x)}.$$

This completes the proof. $\qquad\square$

5. Existence of equivariant hyperbolic diffeomorphisms

Let X be a compact complex T-manifold with finitely many T-fixed points admitting a T-representation covering $(U_p \mid p \in X^T)$. We say that the T-representation covering is *holomorphic* if $U_p \cong T_p X$ as complex T-manifolds.

In the rest of this section, we assume that the T-representation covering is holomorphic.

Let p be a T-fixed point of X and $(U_p; x^1, y^1, \ldots, x^n, y^n)$ be the coordinate neighborhood induced from the corresponding complex T-representation as in Section 4.

Using the coordinate on U_p, we define a Riemannian metric $g^{(p)}$ on U_p as follow:

$$g^{(p)}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = 0 \quad (1 \leq i, j \leq n),$$

$$g^{(p)}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij} e^{-|x^i|}, \quad g^{(p)}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \delta_{ij} e^{-|y^i|} \quad (1 \leq i, j \leq n).$$

Here the symbol δ_{ij} stands for the Kronecker's delta.

Let $(\rho^{(p)} \mid p \in X^T)$ be a partition of unity associated to the open covering $(U_p \mid p \in X^T)$. Then we define a Riemannian metric g on X by

$$g := \sum_{p \in X^T} \rho^{(p)} g^{(p)}.$$

We set

$$\|u\|^{(p)} := \sqrt{g^{(p)}(u, u)}, \quad \|v\| := \sqrt{g(v, v)}$$

for $p \in X^T$, $u \in TU_p$, $v \in TX$.

Proposition 5.1. $\|\xi_0\|_{\varphi_s^J(x)}$ converges to 0 when s tends to ∞ for any $x \in X$.

Proof. One can take a T-fixed point p so that x is contained in U_p.
Then the same calculation in the proof of Lemma 4.3 shows that

$$\xi_0^J|_{U_p} = \sum_{i=1}^n a_i x^i \frac{\partial}{\partial x^i} + \sum_{i=1}^n a_i y^i \frac{\partial}{\partial y^i}$$

in the coordinate system $(U_p; x^1, y^1, \ldots, x^n, y^n)$.
Henceforth the flow $\varphi_s^J(x)$ is expressed as follow:

$$\varphi_s(x) = (p_1 e^{-a_1 s}, q_1 e^{-a_1 s}, \ldots, p_n e^{-a_n s}, q_n e^{-a_n s}).$$

Here $(p_1, q_1, \ldots, p_n, q_n) \in \mathbb{R}^{2n}$ is the coordinate of the point x in the coordinate neighborhood $(U_p, ; x^1, y^1, \ldots, x^n, y^n)$.

As a consequence, we have that

$$(\|\xi_0^J\|_{\varphi_s^J(x)}^{(p)})^2 = \sum_{i=1}^n a_i^2 p_i^2 e^{-2a_i s - |p_i| e^{-a_i s}} + \sum_{i=1}^n a_i^2 q_i^2 e^{-2a_i s - |q_i| e^{-a_i s}}.$$

We claim that when $s \to \infty$, the term $a_i^2 p_i^2 e^{-2a_i s - |p_i| e^{-a_i s}}$ converges to 0. This is because that

- if $p_i = 0$, the claim is obvious,
- if $p_i \neq 0$ and $a_i > 0$, the claim is also obvious,
- if $p_i \neq 0$ and $a_i < 0$, the claim follows from the fact that

$$-2a_i s - |p_i| e^{-a_i s} \to -\infty$$

when $s \to \infty$.

The same consideration shows that the term $a_i^2 q_i^2 e^{-2a_i s - |q_i| e^{-a_i s}}$ also converges to 0 when $s \to \infty$. Thus we have that $\|\xi_0^J\|_{\varphi_s^J(x)}^{(p)} \to 0$ when $s \to \infty$.

Since $\rho^{(p)}$ is bounded, we also have $\|\xi_0^J\|_{\varphi_s^J(x)} \to 0$ when $s \to \infty$. $\qquad\square$

The following is our main theorem in this section:

Theorem 5.2. *There exists a T-invariant hyperbolic diffeomorphism $\varphi \colon X \to X$ satisfying the convergence property.*

Proof.   We define a diffeomorphism $\varphi \colon X \to X$ by $\varphi(x) = \varphi_1^J(x)$. By Lemmas 2.1, 4.3 and 4.4, φ is a T-equivariant hyperbolic diffeomorphism. The local expression of the vector field ξ_0^J on U_p presented in Proposition 5.1 shows that each open set U_p is stable under the induced $\mathbb{R}$-action. So the expression of the flow presented in Proposition 5.1 shows that φ satisfies the condition (C1), (C2).

We next prove that φ^J satisfies the condition (C3).

We first consider the case that $s \to \infty$. Let x be a point of X. Since X is sequentially compact, there exists a real sequence $\{s_n\}_{n=1}^\infty$ having the following two properties:
- $\{s_n\}_{n=1}^\infty$ is strictly increasing and is divergent to ∞.
- $\varphi_{s_n}^J(x)$ converges to a point p of X when n tends to ∞.

By Proposition 5.1 we have

$$\|\xi_0^J\|_p = \|\xi_0^J\|_{\lim_{n\to\infty} \varphi_{s_n}^J(x)} = \lim_{n\to\infty} \|\xi_0^J\|_{\varphi_{s_n}^J(x)} = 0.$$

Henceforth p is a T-fixed point of X by Lemma 4.1. We show that the flow $\varphi_s^J(x)$ converges to the T-fixed point p when s tends to ∞. We proceed proof by contradiction.

Assume that the flow $\varphi_s^J(x)$ does not converge to the T-fixed point p. Since X^T is finite, one can take a compact neighborhood K of p so that $K \cap X^T = \{p\}$. Then there exists an open set U of X having the following two properties:
- $p \in U \subset K$.
- There exists a strictly increasing sequence $\{s_n'\}_{n=1}^\infty$ so that s_n' is divergent to ∞ and $\varphi_{s_n'}^J(x)$ is in $K \setminus U$ for all $n \geq 1$.

Since X is sequentially compact, by taking a subsequence if necessary, one may assume that the sequence $\{\varphi_{s_n'}^J(x)\}_{n=1}^\infty$ converges to a point p' of X. Then the argument used in the proof that p is a T-fixed point also implies that p' is a T-fixed point of X.

However, since $\varphi^J_{s^n_n}(x)$ is in the closed set $K \setminus U$ for all $n \geq 1$, the T-fixed point p' must be contained in K and differs from p. This contradicts to the fact that $K \cap X^T = \{p\}$. This completes the proof that the limit $\lim_{s \to \infty} \varphi^J_s(x)$ converges to some T-fixed point of X.

The convergence of $\lim_{s \to -\infty} \varphi^J_s(x)$ follows from the above case by changing the element a_0 to $-a_0$.

The proof is now complete. $\qquad\square$

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