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HYPERELLIPTIC SURFACES WITH $K^2 < 4\chi - 6$

CARLOS RITO and MARÍA MARTÍ SÁNCHEZ

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Abstract

Let S be a smooth minimal surface of general type with a (rational) pencil of hyperelliptic curves of minimal genus g. We prove that if $K_S^2 < 4\chi(\mathcal{O}_S)-6$, then g is bounded. The surface S is determined by the branch locus of the covering $S \to S/i$, where i is the hyperelliptic involution of S. For $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, we show how to determine the possibilities for this branch curve. As an application, given $g > 4$ and $K_S^2 - 3\chi(\mathcal{O}_S) < -6$, we compute the maximum value for $\chi(\mathcal{O}_S)$. This list of possibilities is sharp.

1. Introduction

For a smooth minimal hyperelliptic surface S of general type, Xiao [8, Theorem 1] has proved that if

$$K_S^2 < \frac{4g}{g+1}(\chi(\mathcal{O}_S) - \epsilon g - 2),$$

where either $\epsilon = 1$ if $\chi(\mathcal{O}_S) > (2g-1)(g+1)+2$, or $\epsilon = 9/8$, then S has a pencil of hyperelliptic curves of genus $\leq g$. This result is not very useful for $g > 4$ and $\chi(\mathcal{O}_S)$ small. For example, in [1] Ashikaga and Konno consider surfaces S of general type with $K_S^2 = 3\chi(\mathcal{O}_S) - 10$. For these surfaces the canonical map is of degree 1 or 2. In the degree 2 case, the canonical image is a ruled surface, thus if S is regular, it has a pencil of hyperelliptic curves. By the above inequality, if $\chi(\mathcal{O}_S) \geq 47$, then S has such a hyperelliptic pencil of curves of genus ≤ 4. But for $\chi(\mathcal{O}_S) \leq 46$ this result gives no information (for $\chi(\mathcal{O}_S) = 46$ the slope formula [7, Theorem 2] implies $g \leq 5 \vee g \geq 9$; we show that in this case S has a hyperelliptic pencil of minimal genus $g \leq 10$ and the cases $g = 9$, $g = 10$ do occur). Ashikaga and Konno study only the case $g \leq 4$ (there is an infinite number of possibilities). Nothing is said for the possibilities with $g \geq 5$ and $\chi(\mathcal{O}_S) \leq 46$. A similar situation occurs in [5].

In this paper we study smooth minimal surfaces S of general type which have a pencil of hyperelliptic curves (by *pencil* we mean a linear system of dimension 1). We say that S has such a pencil of *minimal genus g* if it has a hyperelliptic pencil of genus g and all hyperelliptic pencils of S are of genus $\geq g$. For S such that $K_S^2 < 4\chi(\mathcal{O}_S)-6$,

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we give bounds for the minimal genus g (Theorem 1), improving Xiao's inequality in the cases $g > 4$ and $\chi(\mathcal{O}_S)$ small.

The surface S is the smooth minimal model of a double cover of an Hirzebruch surface $\mathbb{F}_e$ ramified over a curve $\bar{B}$ (which determines S). We prove that if $K_S^2 < 3\chi(\mathcal{O}_S)-6$, then $\bar{B}$ has at most points of multiplicity 8 and we show how to determine the possibilities for $\bar{B}$ (Proposition 2).

As an application, given $g > 4$ and $K_S^2 - 3\chi(\mathcal{O}_S) < -6$, we compute the maximum value for $\chi(\mathcal{O}_S)$; this list of possibilities is sharp (Theorem 3).

The paper is organized as follows. In Section 2 we present the main results of the paper. The hyperelliptic involutions of the fibres of S induce an involution i of S, so in Section 3 we review some general facts on involutions. Since the quotient S/i is a rational surface, a smooth minimal model of S/i is not unique. We make a choice for this minimal model in Section 4 (which is due to Xiao [9]) and we show some consequences of it. Section 5 contains the key result of the paper, which allow us to compute bounds for the minimal genus of the hyperelliptic fibration. We perform a careful analysis of the possibilities for the branch locus of the covering $S \to S/i$ considering the restrictions imposed by the choice of minimal model. Finally this is used in Section 6 to prove the main results, stated in Section 2.

Several calculations are made using a computational algebra system. The respective code lines are available at `http://home.utad.pt/~crito/magma_code.html`.

Notation. We work over the complex numbers; all varieties are assumed to be projective algebraic. A (-2)-*curve* or *nodal curve* A on a surface is a curve isomorphic to $\mathbb{P}^1$ such that $A^2 = -2$. An $(m_1,m_2,\ldots)$-point of a curve, or point of type $(m_1,m_2,\ldots)$, is a singular point of multiplicity m_1, which resolves to a point of multiplicity m_2 after one blow-up, etc. By *double cover* we mean a finite morphism of degree 2. The rest of the notation is standard in algebraic geometry.

2. Main results

Theorem 1. *Let S be a smooth minimal surface of general type with a pencil of hyperelliptic curves of minimal genus g. If $K_S^2 < 4\chi(\mathcal{O}_S)-6$, then g is not greater than*

$$\max\left\{-1+\frac{8\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S)-K_S^2-6},\ 1+\frac{8\chi(\mathcal{O}_S)-16}{4\chi(\mathcal{O}_S)-K_S^2-6},\right.$$
$$\left. 1+\frac{8\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S)-K_S^2-3},\ \frac{3+\sqrt{1+8\chi(\mathcal{O}_S)}}{2}\right\}.$$

Let $B \subset W$ be the branch locus of a double cover $V \to W$, where V and W are smooth surfaces (thus B is also smooth). Let $\rho\colon W \to P$ be the projection of W onto a minimal model and denote by $\bar{B}$ the projection $\rho(B)$.

Suppose that $\bar{B}$ has singular points $x_1, \dots, x_n$ (possibly infinitely near). For each x_i there is an exceptional divisor E_i and a number $r_i \in 2\mathbb{N}$ such that

$$E_i^2 = -1,$$
$$K_W \equiv \rho^*(K_P) + \sum E_i,$$
$$B = \rho^*(\bar{B}) - \sum r_i E_i.$$

Notice that r_i is not the multiplicity of the singular point x_i, it is the multiplicity of the corresponding singularity in the *canonical resolution* (see [2, III. 7]). For example, in the case of a point of type $(2r-1, 2r-1)$ one has $r_1 = 2r-2$ and $r_2 = 2r$.

Since, from Theorem 1, we have a bound for the genus g, we also have a bound for the multiplicities r_i. For the case $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, we prove the result below.

Let N_j be the number of singular points x_i of $\bar{B}$ (possibly infinitely near) such that $r_i = j$.

Proposition 2. *Denote by C_0 and F the negative section and a ruling of the Hirzebruch surface $\mathbb{F}_e$. Let S be a minimal smooth surface of general type with a hyperelliptic pencil of minimal genus $(k-2)/2$. If $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, then S is the smooth minimal model of a double cover $S' \to \mathbb{F}_e$ with branch curve $\bar{B} \equiv kC_0 + (ek/2 + l)F$ such that*:

a) $r_i \leq \min\{8,\ k/2 + 2,\ l - k/2 + 2\}\ \forall i$;

b) $N_4 + N_6 = 15 + K_{S''}^2 - 3\chi(\mathcal{O}_S) - (1/4)(k-10)(l-10)$;

c) $\chi(\mathcal{O}_S) = 1 + (1/4)(k-2)(l-2) - N_4 - 3N_6 - 6N_8$,

where $S'' \to S'$ is the canonical resolution.

Proposition 2 can be used to restrict possibilities for $\bar{B}$. We show the following:

Theorem 3. *Let S be a smooth minimal surface of general type with a hyperelliptic pencil of minimal genus $g > 4$. If $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, then $\chi(\mathcal{O}_S)$ is bounded by the number given in the table below (emptiness means non-existence). All these cases do exist.*

g \ $K^2 - 3\chi$	-7	-8	-9	-10	-11	-12	-13	-14	-15	-16	≤ -17
5	61	56	51	46	41	36	31	26	21	16	
6	49	46	43	40	37	34	27	28		22	
7	42	43	43	35	35	36	28		29	22	
8	44	44	45		36		37		29		
9		45		46			37				
10				46							
≥ 11											

REMARK 4. This result gives three examples where Theorem 1 is almost sharp: in the cases $(g, K^2-3\chi) = (10, -10), (9, -13), (8, -15)$ we have $\chi \le 46, 37, 29$, thus Theorem 1 implies $g \le 11, 10, 9$, respectively (cf. Remark 10).

There is at least one case where Theorem 1 is sharp: a double plane with branch locus a curve of degree 18 with 8 points of multiplicity 6. In this case $\chi = 5$, $K^2 = 8$ and $g = 5$.

3. Involutions

Let S be a smooth minimal surface of general type with a (rational) pencil of hyperelliptic curves. This hyperelliptic structure induces an involution (i.e. an automorphism of order 2) i of S. The quotient S/i is a rational surface.

Since S is minimal of general type, this involution is biregular. The fixed locus of i is the union of a smooth curve R'' (possibly empty) and of $t \ge 0$ isolated points $P_1, \ldots, P_t$. Let $p\colon S \to S/i$ be the projection onto the quotient. The surface S/i has nodes at the points $Q_i := p(P_i)$, $i = 1, \ldots, t$, and is smooth elsewhere. If $R'' \neq \emptyset$, the image via p of R'' is a smooth curve B'' not containing the singular points Q_i, $i = 1, \ldots, t$. Let now $h\colon V \to S$ be the blow-up of S at $P_1, \ldots, P_t$ and set $R' = h^*(R'')$. The involution i induces a biregular involution $\tilde{i}$ on V whose fixed locus is $R := R' + \sum_1^t h^{-1}(P_i)$. The quotient $W := V/\tilde{i}$ is smooth and one has a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & S \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ W & \xrightarrow{g} & S/i \end{array}$$

where $\pi\colon V \to W$ is the projection onto the quotient and $g\colon W \to S/i$ is the minimal desingularization map. Notice that

$$A_i := g^{-1}(Q_i), \quad i = 1, \ldots, t,$$

are (-2)-curves and $\pi^*(A_i) = 2 \cdot h^{-1}(P_i)$.

Set $B' := g^*(B'')$. Since π is a double cover, its branch locus $B' + \sum_1^t A_i$ is *even*, i.e. there is a line bundle L on W such that

$$2L \equiv B := B' + \sum_1^t A_i.$$

4. Choice of minimal model

Part of this section may be found in [9]. We use the notation introduced so far. As above, W is a rational surface, thus either it is isomorphic to $\mathbb{P}^2$ or its minimal model is an Hirzebruch surface $\mathbb{F}_e$.

$(*)$. Blowing-up, if necessary, $\mathbb{P}^2$ at a point, we can suppose that $W \neq \mathbb{P}^2$.

Notice that in this case the map $h\colon V \to S$ is the contraction of two (-1)-curves. With this assumption we do not need to consider the case $W = \mathbb{P}^2$ separately.

Thus there is a birational morphism

$$\rho\colon W \to \mathbb{F}_e.$$

Let $\bar{B} := \rho(B)$ and consider the double cover $S' \to \mathbb{F}_e$ with branch locus $\bar{B}$. If $\bar{B}$ is singular then S' is also singular and S is isomorphic to the minimal smooth resolution of S'.

We can define k and l such that

$$\bar{B} \equiv: kC_0 + \left(\frac{ek}{2} + l\right)F,$$

where C_0 and F are, respectively, the negative section and a ruling of $\mathbb{F}_e$ (thus $C_0^2 = -e$, $C_0F = 1$, $F^2 = 0$). Notice that $\bar{B}^2 = 2kl$ and $K_P\bar{B} = -2k - 2l$.

$(*)$. Among all the possibilities for the map ρ, we choose one satisfying, in this order:

1) the degree k of $\bar{B}$ over a section is minimal;
2) the greatest order of the singularities of $\bar{B}$ is minimal;
3) the number of singularities with greatest order is also minimal.

Recall that a $(2r-1, 2r-1)$ singularity of $\bar{B}$ is a pair (x_j, x_k) such that x_k is infinitely near to x_j and $r_j = 2r - 2$, $r_k = 2r$.

Let

$$r_m := \max\{r_i\}$$

or $r_m := 0$ if $\bar{B}$ is smooth.

By *elementary transformation* over $x_i \in \mathbb{F}_e$ we mean the blow-up of x_i followed by the blow-down of the strict transform of the ruling of $\mathbb{F}_e$ that contains x_i.

The following is a consequence of the two assumptions $(*)$ on the map ρ.

Proposition 5 ([9]). *We have*:

a) *If* $k \equiv 0 \pmod 4$, *then* $r_m \le k/2 + 2$ *and the equality holds only if* x_m *belongs to a singularity* $(k/2+1, k/2+1)$. *In this last case* $l \ge k+2$ *and all the branches of the singularity are tangent to the ruling of* $\mathbb{F}_e$ *that contains it.*

b) *If* $k \equiv 2 \pmod 4$, *then* $r_m \le k/2 + 1$ *and the equality holds only if* x_m *belongs to a singularity* $(k/2, k/2)$. *In this case* $l \ge k$.

In a similar vein:

Proposition 6. *We have that*:

a) *if* $l = k+2$ *and* $k > 8$, *there are at most two* $(k/2+1, k/2+1)$*-points*;

b) $l \geq k/2$ *and* $l \geq k/2 + r_m - 2$;

c) *if* $l = k/2 + r_m - 2$, *then either*:

- $e = 2$, $l = k-2$, *the branch locus* $\bar{B}$ *has a* $(k/2-1, k/2-1)$*-point and all singularities are of multiplicity* $< k/2$, *or*
- *we can suppose* $e = 1$, *the negative section* C_0 *of* $\mathbb{F}_1$ *is contained in* $\bar{B}$, $\bar{B}$ *has a point of multiplicity* r_m *contained in* C_0 *and the remaining singularities are of multiplicity* $< r_m$.

Proof. a) This is due to Borrelli ([3]). Suppose that there are three singularities $(k/2+1, k/2+1)$. The rulings of $\mathbb{F}_e$ through these points are contained in $\bar{B}$ and then $\bar{B}C_0 = l - ek/2 \geq 4$ ($\bar{B}C_0$ is even). This implies $e \leq 1$. Making, if necessary, an elementary transformation over one of these points, we can suppose that $e = 1$.

Let ρ be as above and E_i, E_i', $i = 1,2,3$, be the exceptional divisors corresponding to three singularities $(k/2+1, k/2+1)$ of $\bar{B}$. The general element of the linear system $|\rho^*(4C_0 + 5F) - \sum_1^3(2E_i + 2E_i')|$ is a smooth and irreducible rational curve C such that $CB < k$. This contradicts the choice $(*)$ of the map ρ.

b) If $r_m > k/2$ then the result follows from Proposition 5. Suppose now $r_m \leq k/2$. We have $\bar{B}C_0 \geq -e$, i.e. $l - ek/2 \geq -e$. Therefore if $e \geq 2$, then

$$l \geq k-2 \geq \frac{k}{2} \quad \text{and} \quad l \geq k-2 \geq \frac{k}{2} + r_m - 2.$$

When $e = 0$ we obtain immediately $l \geq k$, by the choice of the map ρ, thus $l \geq k/2 + r_m$.

If $e = 1$ then $\bar{B}C_0 = l - k/2 \geq 0$. Blowing-down C_0 we obtain a singularity of order at most $l - k/2 + 1$, hence the choice of the minimal model implies $r_m \leq l - k/2 + 2$ (notice that the equality happens only if the order of the singularity is $(r_m - 1, r_m - 1)$).

c) Assume that $l = k/2 + r_m - 2$. Proposition 5 implies $r_m \leq k/2$. From $\bar{B}C_0 \geq -e$ we obtain $k/2 + r_m - 2 = l \geq ek/2 - e$, thus either $e = 1$ or $e = 2$ and $r_m = k/2$ (notice that $e = 0$ implies $l \geq k$).

In the case $e = 1$ we can, as in the proof of b), contract the section with self-intersection (-1) to obtain a branch curve in $\mathbb{P}^2$ with at most singularities of type $(l - k/2 + 1, l - k/2 + 1)$.

Suppose now that $e = 2$ and there is a point x_i of multiplicity $k/2$. In this case $\bar{B}C_0 = -2$, hence $x_i \notin C_0$. We make an elementary transformation over x_i to obtain the case $e = 1$ also with $l = k - 2$. □

5. Bound of genus

In this section we prove the key result to establish bounds for the minimal genus of the hyperelliptic fibrations.

From [6] (cf. also [4]), we get the following:

Proposition 7. *Let $S'' \to S'$ be the canonical resolution of a double cover $S' \to \mathbb{F}_e$ with branch locus $\bar{B} \equiv kC_0 + (ek/2 + l)F$. Let S be the minimal model of S'' and $t := K_S^2 - K_{S''}^2$. If S is of general type, then*:
a) $\sum(r_i - 2)(k - r_i - 2) = H$;
b) $2l = G + \sum(r_i - 2)$,
where

$$H = 2k^2 - k(4\chi(\mathcal{O}_S) + t - K_S^2 + 8) + 16\chi(\mathcal{O}_S) + 2t - 2K_S^2$$

and

$$G = -2k + 4\chi(\mathcal{O}_S) + t - K_S^2 + 8.$$

Proof. From [6, Propositions 2 and 3, a)] one gets:
a) $2kl = -48 + 12l + 12k - 8\chi(\mathcal{O}_S) + 4K_S^2 - 4t + \sum(r_i - 2)(r_i - 4)$;
b) $2k + 2l = 8 + 4\chi(\mathcal{O}_S) + t - K_S^2 + \sum(r_i - 2)$.
The result is obtained replacing (a) by (a) + $(6 - k)$(b). □

The motivation for Lemma 8 and Proposition 9 below is the following. Among all the solutions of the equations of Proposition 7, the ones with biggest l correspond to the solutions with singularities of maximal order. This gives an upper bound for l. But we also have a lower bound for l, implied by the assumptions $(*)$ on the map ρ (Propositions 5 and 6). We note that the arguments used in the proofs are mostly formal.

Lemma 8. *Suppose that $k > 8$. With the above notation, we have*
a) *$2l \leq G + H/(k - r_m - 2)$, and*
b) *if r_m is obtained only from singularities of type $(r_m - 1, r_m - 1)$, then*

$$2l \leq G + \frac{H}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)}(2r_m - 6).$$

Proof. a) Proposition 5 implies $r_m \leq k/2 + 2$. If $k - r_m - 2 \leq 0$, we get from $k - 2 \leq r_m \leq k/2 + 2$ that $k \leq 8$. Hence $k - r_m - 2 > 0$ and the statement follows from Proposition 7.

b) By the assumptions, if x_i does not belong to a $(r_m - 1, r_m - 1)$ singularity, we have $r_i < r_m$. Let $n \geq 1$ be the number of singularities of type $(r_m - 1, r_m - 1)$ and $s \geq 0$ be the number of singular points x_j of another type. As seen in Section 4, each singularity (r_m-1, r_m-1) corresponds to two infinitely near singular points x_k, x_{k+1} with $r_k = r_m - 2$, $r_{k+1} = r_m$. Therefore

$$\sum_{i=1}^{2n+s}(r_i-2) = n(2r_m-6) + \sum_{j=1}^{s}(r_j-2),$$

with $r_j < r_m$. Thus from Proposition 7, b) we get

$$2l = G + n(2r_m-6) + \sum_{j=1}^{s}(r_j-2).$$

By Proposition 7, a),

$$H = n((r_m-4)(k-r_m) + (r_m-2)(k-r_m-2)) + \sum_{j=1}^{s}(r_j-2)(k-r_j-2),$$

hence

$$n = \frac{H - \sum_{j=1}^{s}(r_j-2)(k-r_j-2)}{(r_m-4)(k-r_m) + (r_m-2)(k-r_m-2)}$$

and then

$$2l = G + \frac{H - \sum_{j=1}^{s}(r_j-2)(k-r_j-2)}{(r_m-4)(k-r_m) + (r_m-2)(k-r_m-2)}(2r_m-6) + \sum_{j=1}^{s}(r_j-2). \tag{1}$$

Since $r_j < r_m$, $j = 1, \dots, s$,

$$(r_m-4)(k-r_m) + (r_m-2)(k-r_m-2) \le (2r_m-6)(k-r_j-2).$$

This implies

$$\sum_{j=1}^{s}(r_j-2) \le \sum_{j=1}^{s}\frac{(r_j-2)(k-r_j-2)(2r_m-6)}{(r_m-4)(k-r_m) + (r_m-2)(k-r_m-2)}$$

and the result follows from (1). □

The next result will allow us to give bounds for k. Notice that, since $\bar{B}$ is even and $\bar{B}C_0 = l - ek/2$,

$$k \equiv 0 \pmod 4 \Longrightarrow l \equiv 0 \pmod 2.$$

Proposition 9. *In the conditions of* Proposition 7, *suppose that* $k > 8$. *If* $k \equiv 0 \pmod 4$, *one of the following holds*:

a) $r_m = k/2 + 2$, $l = k + 2$ *and*

$$(4\chi(\mathcal{O}_S) + t - K_S^2 - 8)k \leq 16\chi(\mathcal{O}_S) - 16, \quad \textit{with} \quad t \geq 2;$$

b) $r_m = k/2 + 2$, $l \geq k + 4$ *and*

$$(4\chi(\mathcal{O}_S) + t - K_S^2 - 8)k^2 - 16\chi(\mathcal{O}_S)k + 32\chi(\mathcal{O}_S) \leq 0, \quad \textit{with} \quad t \geq 2;$$

c) $r_m = k/2$, $l = k - 2$ *and*

$$\begin{aligned}&(4\chi(\mathcal{O}_S) + t - K_S^2 - 4)k^2 + (-48\chi(\mathcal{O}_S) - 8t + 8K_S^2 + 32)k\\ &+ 160\chi(\mathcal{O}_S) + 16t - 16K_S^2 - 96 \leq 0, \quad \textit{with} \quad t \geq 1,\end{aligned}$$

or

$$(4\chi(\mathcal{O}_S) + t - K_S^2 + 2)k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 8, \quad \textit{with} \quad t \geq 1,$$

or

$$\begin{aligned}&(4\chi(\mathcal{O}_S) + t - K_S^2 - 5)k^2 + (-48\chi(\mathcal{O}_S) - 8t + 8K_S^2 + 44)k\\ &+ 160\chi(\mathcal{O}_S) + 16t - 16K_S^2 - 128 \leq 0, \quad \textit{with} \quad t \geq 2;\end{aligned}$$

d) $r_m = k/2$, $l = k + j$, $j \geq 0$, *and*

$$(4\chi(\mathcal{O}_S) + t - K_S^2 + 8 + 2j - 2n)k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 8n,$$

with $n \leq j + 7$, *where* n *is the number of points* x_i (*possibly infinitely near*) *such that* $r_i = k/2$;

e) $r_m \leq k/2 - 2$ *and*

$$k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)},$$

or

$$(4\chi(\mathcal{O}_S) + t - K_S^2)k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2.$$

If $k \equiv 2 \pmod 4$, *one of the following holds*:

f) $r_m = k/2 + 1$ *and*

$$(4\chi(\mathcal{O}_S) + t - K_S^2 - 2)k \leq 24\chi(\mathcal{O}_S) + 2t - 2K_S^2 - 20, \quad \textit{with} \quad t \geq 1,$$

or

$$\begin{aligned}&(4\chi(\mathcal{O}_S) + t - K_S^2 - 8)k^2 + (-32\chi(\mathcal{O}_S) - 4t + 4K_S^2 + 48)k\\ &+ 80\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 96 \leq 0, \quad \textit{with} \quad t \geq 2;\end{aligned}$$

g) $r_m \leq k/2 - 1$ *and*

$$k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)},$$

or

$$2(4\chi(\mathcal{O}_S) + t - K_S^2 - 6)k \leq 24\chi(\mathcal{O}_S) + 2t - 2K_S^2 - 28.$$

REMARK 10. As noted in Remark 4, there are examples where cases e) and g) fail to be sharp by 1. The reason for not having a sharp result is the following: in these examples we have $r_m = 0$, thus we are using $l \geq k/2 - 2$ in the proof of e) and g). But in fact we have $l \geq k/2$ in these cases, from Proposition 6, b).

The last example referred in Remark 4 shows that case d) with $k = 12$, $j = 0$, $n = 7$ is sharp.

Proof of Proposition 9. Let H, G be as defined in Proposition 7 and let

$$\begin{aligned} P_1(l, r_m, G, H, k) &:= (2l - G)(k - r_m - 2) - H, \\ P_2(l, r_m, G, H, k) &:= (2l - G)((r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)) \\ &\quad - H(2r_m - 6). \end{aligned}$$

From Lemma 8,

$$P_1 \leq 0 \quad \text{and} \quad P_2 \leq 0.$$

a) Let n be the number of $(k/2+1, k/2+1)$ points. From Propositions 5, a) and 6, a), $n = 1$ or 2. From Proposition 7, we have

$$\sum (r_i - 2)(k - r_i - 2) = H' \quad \text{and} \quad 2l = G' + \sum (r_i - 2),$$

where

$$H' = H - n(k/2(k/2-4) + (k/2-2)^2), \quad G' = G + n(k-2)$$

and $r_i \leq k/2$, $\forall i$.

The result follows from

$$P_1(k+2, k/2, G', H', k) \leq 0.$$

Notice that $t \geq 2n$.

b) From Proposition 5, there are at most $(k/2 + 1, k/2 + 1)$ singularities. The inequality

$$P_2(k+4, k/2+2, G, H, k) \leq 0$$

gives the result.

c) Let n be the number of points of multiplicity $k/2$ and m be the number of $(k/2-1, k/2-1)$ singularities. From Proposition 6, c), $n = 0$ or 1.

If $n = 0$, then $r_m = k/2$ implies $m \geq 1$ (thus $t \geq 1$). From

$$P_2(k-2, k/2, G, H, k) \leq 0$$

one gets the first inequality.

Suppose $n = 1$. Notice that, as shown in the proof of Proposition 6, c), the point of multiplicity $k/2$ is obtained from the blow-up of $\mathbb{P}^2$ at a point of type $(k/2-1, k/2-1)$. Hence $t \geq 1$.

Let

$$H' := H - (k/2-2)^2, \quad G' = G + k/2 - 2$$

(we remove the contribution of the point of multiplicity $k/2$).

If $m = 0$, then

$$P_1(k-2, k/2-2, G', H', k) \leq 0$$

implies the second inequality.

If $m > 0$, then

$$P_2(k-2, k/2, G', H', k) \leq 0$$

gives the third inequality. In this case $t \geq 2$.

d) Let $j := l - k$ and let n be the number of points x_i (possibly infinitely near) such that $r_i = k/2$. From Proposition 7, we have

$$\sum (r_i - 2)(k - r_i - 2) = H' \quad \text{and} \quad 2l = G' + \sum (r_i - 2),$$

where

$$H' = H - n(k/2-2)^2, \quad G' = G + n(k/2-2)$$

and $r_i \leq k/2 - 2$, $\forall i$.

The inequality

$$P_1(k+j, k/2-2, G', H', k) \leq 0$$

gives

$$(4\chi(\mathcal{O}_S) + t - K_S^2 + 8 + 2j - 2n)k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 8n.$$

It only remains to show that $n \leq j + 7$.

One can verify, using the double cover formulas (see e.g. [2, V. 22]), that $n \geq j+8$ implies $\chi(\mathcal{O}_S) < 1$, except for $n = 8$, $l = k$ and $n = 10$, $k = 12$, $l = 14$. We *claim* that in these cases $K_S^2 \leq 0$. This is impossible because S is of general type.

Proof of the claim. From the double cover formulas one gets that $\chi(\mathcal{O}_S) \leq 2$ and there is at least a (-2)-curve A contained in the branch curve B, otherwise $K_S^2 \leq 0$. One has

$$B \equiv -\frac{k}{2}K_W + (l-k)\tilde{F} + \sum \left(\frac{k}{2} - r_i\right)E_i,$$

where $\tilde{F}$ is the total transform of F and each E_i is an exceptional divisor with self-intersection -1. Since $AB = -2$, $AK_W = 0$, $l \geq k$ and $r_i \leq k/2$ $\forall i$, we have $AE_i < 0$

for some i such that $r_i < k/2$. The only possibility is the existence of a (3, 3)-point in $\bar{B}$ and $\chi(\mathcal{O}_S) = 1$. But the imposition of such a singularity in the branch locus decreases the self-intersection of the canonical divisor by 1, thus $K_S^2 \leq 0$.

e) From Proposition 6, b), $l \geq k/2 + r_m - 2$. Let

$$f(r_m) := P_1(k/2 + r_m - 2, r_m, G, H, k).$$

We have

$$f(r_m) = -2r_m^2 + br_m + c \leq 0,$$

where

$$b = 4\chi(\mathcal{O}_S) + t - K_S^2 - k + 8$$

and

$$c = k^2 - 10k - 8\chi(\mathcal{O}_S) + 24.$$

Suppose that $c = f(0) > 0$ (i.e. $k > 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}$). Then $f(r_m)$ has exactly one positive root x. One has

$$4x - b = \sqrt{b^2 + 8c}$$

and $k/2 - 2 \geq r_m \geq x$ implies that

$$(4(k/2 - 2) - b)^2 \geq b^2 + 8c.$$

This inequality gives the result.

f) Let n be the number of points of type $(k/2, k/2)$.

If $n = 1$, we proceed as in a), with $l \geq k$.

If $n > 1$, the inequality is given by

$$P_2(k, k/2 + 1, G, H, k) \leq 0.$$

g) It is analogous to the proof of e): in this case the result follows from $k/2-1 \geq r_m \geq x$. □

6. Proof of main results

Proof of Theorem 1. Consider the parabola given by $f(x) = ax^2 + bx + c$, with $a > 0$. If $f(k) \leq 0$, $f(z) \geq 0$ and $z \geq -b/2a$ (the first coordinate of the vertex), then $k \leq z$.

This fact and Proposition 9 imply that, if $K_S^2 < 4\chi(\mathcal{O}_S) - 6$, one of the following holds:

a) $k \leq (16\chi(\mathcal{O}_S) - 16)/(4\chi(\mathcal{O}_S) - K_S^2 - 6)$;

b) $k \leq (16\chi(\mathcal{O}_S))/(4\chi(\mathcal{O}_S) + t - K_S^2 - 8)$, $t \geq 2$;
c) $k \leq 4 + (16\chi(\mathcal{O}_S))/(4\chi(\mathcal{O}_S) + t - K_S^2 - 4)$, $t \geq 1$;
c′) $k \leq 4 + (16\chi(\mathcal{O}_S) - 4)/(4\chi(\mathcal{O}_S) + t - K_S^2 - 5)$, $t \geq 2$;
d) $k \leq 4 + (16\chi(\mathcal{O}_S) - 32)/(4\chi(\mathcal{O}_S) - K_S^2 - 6)$;
e) $k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}$;
e′) $k \leq 4 + (16\chi(\mathcal{O}_S))/(4\chi(\mathcal{O}_S) - K_S^2)$;
f) $k \leq 2 + (16\chi(\mathcal{O}_S) - 16)/(4\chi(\mathcal{O}_S) - K_S^2 - 1)$;
f′) $k \leq 2 + (16\chi(\mathcal{O}_S) - 16)/(4\chi(\mathcal{O}_S) + t - K_S^2 - 8)$, $t \geq 2$;
g) $k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}$;
g′) $k \leq 2 + (16\chi(\mathcal{O}_S) - 16)/(4\chi(\mathcal{O}_S) - K_S^2 - 6)$.

We want to show that k is not greater than

$$\max\left\{\frac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2 - 6}, 4 + \frac{16\chi(\mathcal{O}_S) - 32}{4\chi(\mathcal{O}_S) - K_S^2 - 6}, 4 + \frac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2 - 3}, 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}\right\}.$$

The result follows easily. Just notice that

$$4\chi(\mathcal{O}_S) - K_S^2 - 6 \leq 8 \Longrightarrow 2 + \frac{16\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) - K_S^2 - 6} \leq \frac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2 - 6}$$

and

$$4\chi(\mathcal{O}_S) - K_S^2 - 6 \geq 8 \Longrightarrow 2 + \frac{16\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) - K_S^2 - 6} \leq 4 + \frac{16\chi(\mathcal{O}_S) - 32}{4\chi(\mathcal{O}_S) - K_S^2 - 6}. \qquad \square$$

Proof of Proposition 2. Let (α), (β) be the equations of Proposition 7, a), b), respectively. One has that $[(\alpha) + (k-10)(\beta)]/8$ is equivalent to

$$\frac{1}{8}\sum(r_i - 2)(8 - r_i) = 15 + K_S^2 - t - 3\chi(\mathcal{O}_S) - \frac{1}{4}(k-10)(l-10) \tag{2}$$

and $(\beta) + (2)$ is equivalent to

$$\chi(\mathcal{O}_S) = 1 + \frac{1}{4}(k-2)(l-2) - \frac{1}{8}\sum r_i(r_i - 2). \tag{3}$$

Now it suffices to show that $r_m \leq 8$.

Suppose that $K_S^2 < 3\chi(\mathcal{O}_S) - 6$.

From [8, Theorem 1] one gets that if $\chi(\mathcal{O}_S) \geq 54$, then S has a pencil of hyperelliptic curves of genus ≤ 6. In this case $k \leq 14$, thus $r_m \leq k/2 + 2$ implies $r_m \leq 8$.

From the proof of Theorem 1 we obtain that if $\chi(\mathcal{O}_S) \leq 31$, then one of the possibilities below occur. In all cases $r_m \leq 8$.

a) and b) $k < 16$, $r_m < 8$;
c), c′) and d) $k \leq 18$, $r_m = k/2 \leq 8$;
e) $k \leq 20$, $r_m \leq k/2 - 2 \leq 8$;
e′) $k \leq 16$, $r_m \leq k/2 - 2 \leq 6$;
f) $k \leq 14$, $r_m = k/2 + 1 \leq 8$;
f′) $k \leq 16$, $r_m = k/2 + 1 \leq 8$;
g) $k \leq 18$, $r_m \leq k/2 - 1 \leq 8$;
g′) $k \leq 14$, $r_m \leq k/2 - 1 \leq 6$.

Suppose now that $32 \leq \chi(\mathcal{O}_S) \leq 53$. From Theorem 1 we get that $k \leq 18$ or $k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}$. In this last case $k \leq 24$ and $r_m \leq k/2 - 1$ (see Proposition 9 e), g)). Thus we have $r_m \leq 18/2 + 2$ or $r_m \leq 24/2 - 1$. Since r_m is even, $r_m \leq 10$.

Let N_j be the number of points x_i such that $r_i = j$. We have

$$\sum (r_i - 2) \geq 8N_{10} + 6N_8$$

and, from (2),

$$8N_{10} \geq (k - 10)(l - 10) - 32.$$

Using Proposition 7, b) and the assumption $\chi(\mathcal{O}_S) \geq 32$, this implies

$$2l + 2k \geq 15 + (k - 10)(l - 10) + 6N_8,$$

or equivalently

$$(k - 12)(l - 12) \leq 29 - 6N_8. \tag{4}$$

Suppose $r_m = 10$. Then Propositions 5 and 6 give two possibilities:

- $k = 16$, $l \geq k + 2 = 18$, there is a singularity of type (9, 9) ($N_8 \geq 1$);
- $k \geq 18$, $l \geq k/2 + r_m - 2 \geq 17$.

Both cases contradict (4). We conclude that $r_m \leq 8$. □

Proof of Theorem 3. First we claim that if A is a (-2)-curve contained in the branch curve B, the image $\bar{A}$ of A in $\mathbb{F}_e$ does not intersect a negligible singularity of $\bar{B}$, unless $\bar{A}$ is the negative section of $\mathbb{F}_1$ and the only singularity of $\bar{B}$ is a double point in C_0 (this corresponds to a smooth branch curve in $\mathbb{P}^2$). In fact otherwise there is a (-1)-curve E such that $AE = 1$ or 2. If $AE = 1$, then $A + E$ can be contracted to a smooth point of the branch curve $\bar{B} \subset \mathbb{F}_e$. This is a contradiction because the canonical resolution blows-up only singular points of $\bar{B}$. Suppose $AE = 2$. The inverse image of A is a (-1)-curve which contracts to a smooth point of S. The inverse image of E is then contracted to a curve $\hat{E}$ with arithmetic genus 1 and $\hat{E}^2 = 2$. We obtain from the adjunction formula that $K_S\hat{E} = -2$, which is impossible because S is of general type.

Recall that $t := K_S^2 - K_{S''}^2$. The following holds:

(1) $l \geq k/2$ (Because $l - ek/2 = \bar{B}C_0 \geq -e$ and $\bar{B}C_0$ is even.);
(2) $l = k/2 \iff (t = 2 \wedge N_4 = N_6 = N_8 = 0)$ (In this case $e = 1$ and $\bar{B}C_0 = 0$.);
(3) $l = k/2 + 2 \implies (N_6 = N_8 = 0 \wedge t \geq N_4 \wedge (t = N_4 \vee N_4 > 1))$; (If $N_4 \neq 0$, this corresponds to a branch curve in $\mathbb{P}^2$ with N_4 points of type (3, 3) (see Proposition 6, b), c)).);
(4) $l = k - 2 \wedge t = 0 \implies k/2$ even; (As in (1), $l \geq ek/2 - e$, thus $e \leq 2$. If $e = 2$, $\bar{B}C_0 = -2$ implies $t \geq 1$. Hence $e = 1$ and then l even implies $k/2$ even.);
(5) $l < k - 2 \implies l - k/2$ even; (As in (1), $l \geq ek/2 - e$, thus $e = 1$ and then $l - k/2 = \bar{B}C_0$ is even.)
(6) $t = 1 \wedge N_4 = N_6 = N_8 = 0 \implies l = k - 2$. (If there are only negligible singularities, $t = 1$ is only possible if the negative section of $\mathbb{F}_2$ is an isolated component of the branch locus.)

For given values of $K_S^2 - 3\chi(\mathcal{O}_S)$ and k, we want to choose the solution of the equation given in Proposition 2, b) which maximizes the value of $\chi(\mathcal{O}_S)$, given by the equation in Proposition 2, c). We can assume $N_6 = N_8 = 0$.

It suffices to compute the numerical possibilities for Proposition 2, b), c) which satisfy conditions (1), ..., (6). We note the following: since $k \geq 12$, [8, Theorem 1] implies $\chi(\mathcal{O}_S) \leq 69$, then Theorem 1 gives $k \leq 28$; $l \geq k/2$, $k \geq 12$ and (2) imply $-7 \geq K_S^2 - 3\chi(\mathcal{O}_S) \geq -18 + t + N_4$, thus $K_S^2 - 3\chi(\mathcal{O}_S) \geq -18$, $t \leq 11$ and $N_4 \leq 11$.

A simple algorithm is available at `http://home.utad.pt/~crito/magma_code.html`.

It remains to prove the existence. All cases can be constructed as double covers of $\mathbb{P}^2$, $\mathbb{F}_0$, $\mathbb{F}_1$ or $\mathbb{F}_2$. The table below contains information about l or the degree of the branch curve in $\mathbb{P}^2$ and about the singularities of the branch curve, if any.

g / $K^2 - 3\chi$	-7	-8	-9	-10
5	$\mathbb{F}_0$, $l = 26$	$\mathbb{F}_0$, $l = 24$	$\mathbb{F}_0$, $l = 22$	$\mathbb{F}_1$, $l = 20$
6	$\mathbb{F}_0$, $l = 18$	$\mathbb{F}_1$, $l = 17$	$\mathbb{F}_0$, $l = 16$	$\mathbb{F}_1$, $l = 15$
7	$\mathbb{F}_1$, $l = 14$, (3, 3)	$\mathbb{F}_2$, $l = 14$	$\mathbb{F}_1$, $l = 14$	$\mathbb{F}_1$, $l = 12$, (3, 3)
8	$\mathbb{F}_1$, $l = 13$, (3, 3)	$\mathbb{F}_1$, $l = 13$, (4)	$\mathbb{F}_1$, $l = 13$	
9		$\mathbb{P}^2$, 22, (3, 3)		$\mathbb{F}_1$, $l = 12$
10				$\mathbb{P}^2$, 22

g	-11	-12	-13	-14	-15	-16
5	$\mathbb{F}_0$, $l = 18$	$\mathbb{F}_0$, $l = 16$	$\mathbb{F}_0$, $l = 14$	$\mathbb{F}_0$, $l = 12$	$\mathbb{F}_1$, $l = 10$	$\mathbb{F}_1$, $l = 8$
6	$\mathbb{F}_0$, $l = 14$	$\mathbb{F}_1$, $l = 13$	$\mathbb{F}_1$, $l = 11$, (4)	$\mathbb{F}_1$, $l = 11$		$\mathbb{F}_1$, $l = 9$
7	$\mathbb{F}_1$, $l = 12$, (4)	$\mathbb{F}_1$, $l = 12$	$\mathbb{P}^2$, 18, (3, 3)		$\mathbb{F}_1$, $l = 10$	$\mathbb{P}^2$, 16
8	$\mathbb{P}^2$, 20, (3, 3)		$\mathbb{F}_1$, $l = 11$		$\mathbb{P}^2$, 18	
9			$\mathbb{P}^2$, 20			
10						

Suppose first that S is smooth and is the double cover of an Hirzebruch surface $\mathbb{F}_e$ with branch locus $B \equiv 2L \equiv kC_0 + (ek/2 + l)F$. We get from the double cover formulas (see e.g. [2]) that

$$\chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_{\mathbb{F}_e}) + \frac{1}{2}L(K_{\mathbb{F}_e} + L) = 2 + \frac{1}{4}kl - \frac{1}{2}(k + l)$$

and

$$K_S^2 = 2(K_{\mathbb{F}_e} + L)^2 = 16 - 4(k + l) + kl.$$

Now we compute χ and K^2 for the cases given in the table above taking in account that a 4-uple point in the branch locus decreases K^2 by 2 and χ by 1 and a $(3,3)$-point decreases both K^2 and χ by 1. Notice that $k = 2g + 2$.

Finally if S is a double cover of $\mathbb{P}^2$ with branch locus a smooth curve of degree d, then

$$\chi(\mathcal{O}_S) = 2 + \frac{1}{8}d(d - 6) \quad \text{and} \quad K_S^2 = \frac{1}{2}(d - 6)^2.$$

The result follows by computing χ and K^2 for $d = 16$, 18, 20 and 22. □

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Carlos Rito
Departamento de Matemática
Universidade de Trás-os-Montes e Alto Douro
5001-801 Vila Real
Portugal
e-mail: crito@utad.pt

María Martí Sánchez
Nebrija Universidad
Calle Pirineos 55
28040 Madrid
Spain
e-mail: mmartisanc@gmail.com