FORELLI–RUDIN CONSTRUCTION AND ASYMPTOTIC EXPANSION OF SZEGÖ KERNEL ON REINHARDT DOMAINS

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Abstract

We apply Forelli–Rudin construction and Nakazawa's hodograph transformation to prove a graph theoretic closed formula for invariant theoretic coefficients in the asymptotic expansion of the Szegö kernel on strictly pseudoconvex complete Reinhardt domains. The formula provides a structural analogy between the asymptotic expansion of the Bergman and Szegö kernels. It can be used to effectively compute the first terms of Fefferman's asymptotic expansion in CR invariants. Our method also works for the asymptotic expansion of the Sobolev–Bergman kernel introduced by Hirachi and Komatsu.

1. Introduction

Fefferman [14] proposed and initiated a program of expressing the asymptotic expansion of the boundary singularity of the Bergman kernel K^B (the Szegö kernel K^S) for smoothly bounded strictly pseudoconvex domains $\Omega \subset \mathbb{C}^n$ explicitly in terms of boundary invariants. In his groundbreaking work on C^∞ extensibility of biholomorphic maps, Fefferman [12] proved that

$$(1) \qquad K^B(z) = \frac{n!}{\pi^n}\left(\frac{\varphi^B(z)}{r(z)^{n+1}} + \psi^B(z)\log r(z)\right), \quad \varphi^B, \psi^B \in C^\infty(\overline{\Omega}),$$

where $r \in C^\infty(\overline{\Omega})$ is a defining function. See [1] and [20] for important refinements of Fefferman's program. Graham [18] and Hirachi–Komatsu–Nakazawa [24, 25] carried out computations of the first few terms of Fefferman's asymptotic expansion in terms of CR invariants. Fefferman's program has also been extended to conformal geometry (cf. [15, 16]).

There are many questions related to the asymptotic expansion of the Bergman kernel. We only mention Ramadanov's conjecture which asks whether Ω is biholomorphic to the ball whenever $\psi(z) = 0$ and Yau's question [41, p. 679] to classify pseudoconvex domains whose Bergman metrics are Kähler–Einstein.

In Question 3 of his book [36, p. 20], Stein posed the problem: what are the relations between K^B and K^S? In Problem 9 of [14, p. 259], Fefferman raised the question:

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how are the asymptotic expansion of the Bergman and Szegö kernels related? Inspired by these questions, we develop a uniform method for studying the asymptotic expansion of the Bergman and Szegö kernels on Hartogs domains using the Forelli–Rudin construction. In particular, we prove closed formulas for coefficients in their asymptotic expansions as summations over graphs. Our work shows in an explicit way the analogy of the asymptotic expansion of the Bergman and Szegö kernels, at least for strictly pseudoconvex complete Reinhardt domains.

Hirachi–Komatsu [23] (see also [22]) defined the Sobolev–Bergman kernel K^s of Ω for each $s \in \mathbb{R}$ with the transformation law of weight $n + 1 - s$ under biholomorphic maps and the asymptotic expansion of singularities, analogous to the Bergman kernel $(= K^0)$ and the Szegö kernel $(= K^1)$. We can use the general mechanism developed in §3 to find closed formulas for coefficients in the asymptotic expansion of K^s. We will discuss this in a separate paper.

Other recent works exploring the relations between the Bergman and Szegö kernels can be found in e.g. [6, 29, 42]. See [8] for the connection to the heat kernel.

Our work crucially relies on the existence of complete asymptotic expansion of weighted Bergman kernel (appearing in the Forelli–Rudin construction), which was established in [10] for bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with real analytic boundary, in the context of Berezin quantization.

The paper is organized as follows: In §2, we review the works of Graham [18] and Hirachi–Komatsu–Nakazawa [24] on the asymptotic expansion of the Szegö kernel. In §3, building on work of [9], we prove graph-theoretic closed formulas for the asymptotic expansion of weighted Bergman kernels. In §4, we prove a graph-theoretic closed formula for coefficients in the asymptotic expansion of the Szegö kernel on Hartogs domains. In the case of complete Reinhardt domains, our formula becomes quite explicit using Nakazawa's hodograph transformation.

The main technical part of this paper is §3. Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$ equipped with a strictly-plurisubharmonic function $\Phi(x)$. We study the asymptotic expansion of the Bergman kernel with respect to the measure $e^{-\alpha \Phi} g(x)^C \, dx$ as $\alpha \to \infty$, where $C \geq 0$ is a real number. By Forelli–Rudin construction, we will show that the Szegö kernel corresponds to $C = 1/(n + 1)$. We will prove a graph theoretic formula for the asymptotic coefficients generalizing the results of [9, 38, 40], where $C = 0$ and $C = 1$ were treated. For the proof, we will apply the asymptotic expansion of Laplace integrals on Kähler manifolds developed in [9] and the criterion of Weyl invariant polynomials proved in [39]. Some of the arguments are straightforward generalization of our previous work; for the sake of completeness, we include detailed proofs taking care of the extra weighted sum over linear subgraphs.

2. The asymptotic expansion of the Szegö kernel

Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Given a surface element σ on $\partial\Omega$, then the Sezgö kernel $K^S(z, \zeta)$ is defined as the

reproducing kernel associated with the Hardy space $H^2_\sigma(\Omega)$ consisting of holomorphic functions in Ω having L^2 boundary values with respect to σ. Namely,

(1) $K^S(z, \zeta) \in H^2_\sigma(\Omega)$ for every $\zeta \in \Omega$ fixed,

(2) $K^S(z, \zeta) = \overline{K^S(\zeta, z)}$,

(3) $f(z) = \int_{\partial\Omega} K^S(z, \zeta) f(\zeta) \sigma(\zeta)$ for any $f \in H^2_\sigma(\Omega)$ and $z \in \Omega$.

If $r \in C^\infty(\overline{\Omega})$ is a defining function in the sense $\Omega = \{r > 0\}$ with $dr \neq 0$ on $\partial\Omega$, then by the pioneering work of Fefferman [12] (see also [5]), the boundary singularity of the Szegö kernel has the form

$$(2) \qquad K^S(z, z) = \frac{(n-1)!}{2\pi^n}\left(\frac{\varphi^S(z)}{r(z)^n} + \psi^S(z)\log r(z)\right).$$

As pointed out in [24], in order to make the Szegö kernel invariant under biholomorphic change of coordinates, the surface element σ should satisfy

$$(3) \qquad \sigma \wedge dr = J[r]^{1/(n+1)}\,dV(z) \quad \text{on} \quad \partial\Omega,$$

where $dV(z) = (1/(-2\sqrt{-1}))^n\,dz_1 \wedge d\bar{z}_1 \cdots dz_n \wedge d\bar{z}_n$ and $J[r]$ is the complex Monge–Ampère operator

$$(4) \qquad J[r] = (-1)^n \det \begin{pmatrix} r & \partial r/\partial\bar{z}_j \\ \partial r/\partial z_i & \partial^2 r/\partial z_i \partial\bar{z}_j \end{pmatrix}_{1 \leq i,\, j \leq n}.$$

Starting from an arbitrary smooth defining function of Ω, Fefferman [13] devised a recursive algorithm to explicitly construct another defining function $r^F \in C^\infty(\overline{\Omega})$ which is an approximate solution to the Dirichlet problem of Monge–Ampère equation

$$(5) \qquad J[r^F] = 1 + O^{n+1}(r^F), \quad r^F > 0 \text{ in } \Omega, \quad r^F|_{\partial\Omega} = 0,$$

where $O^{n+1}(r^F)$ denotes a term of the form $(r^F)^{n+1}f$ with $f \in C^\infty(\overline{\Omega})$.

Let us recall the definition of CR invariants for strictly pseudoconvex hypersurfaces using Moser's normal form. Let $(z', z_n) = (z_1, \ldots, z_n) \in \mathbb{C}^n$. A hypersurface $0 \in \partial\Omega \subset \mathbb{C}^n$ with local equation

$$(6) \qquad 2u = |z'|^2 + \sum_{|\alpha|,|\beta|\geq 2, k\geq 0} A^k_{\alpha\bar{\beta}}(v) v^k z'_\alpha \bar{z}'_\beta, \quad z_n = u + iv$$

is said to be in Moser's normal form if the coefficients $A^k_{\alpha\bar{\beta}}$ satisfy:

(i) $A^k_{\alpha\bar{\beta}} = \overline{A^k_{\beta\bar{\alpha}}}$;

(ii) $\text{tr}(A_{2\bar{2}}) = 0$, i.e. $\sum_{p=1}^{n-1} A^k_{pi\bar{p}\bar{j}} = 0$ for all k, i, j;

(iii) $\text{tr}(A_{2\bar{3}}) = 0$, i.e. $\sum_{p,q=1}^{n-1} A^k_{pq\bar{p}\bar{q}\bar{j}} = 0$ for all k, j;

(iv) $\text{tr}(A_{3\bar{3}}) = 0$, i.e. $\sum_{p,q,r=1}^{n-1} A^k_{pqr\bar{p}\bar{q}\bar{r}} = 0$ for all k.

A classical result of Chern and Moser [7] says that any real analytic hypersurface may be placed in Moser's normal form through a biholomorphic map.

DEFINITION 2.1 ([14, 18, 24]).   Denote by $N(A^k_{\alpha\bar\beta})$ a real hypersurface in normal form (6). A polynomial P in variables $A^k_{\alpha\bar\beta}$ is said to be a CR invariant of weight $w \in \mathbb{N}_{\geq 0}$ if it satisfies the transformation law $P(A^k_{\alpha\bar\beta}) = |\det \Phi'(0)|^{2w/(n+1)} P(B^k_{\alpha\bar\beta})$ for any biholomorphic mapping $\Phi \colon N(A^k_{\alpha\bar\beta}) \to N(B^k_{\alpha\bar\beta})$ preserving the origin.

Let I_w denote the set of CR invariants of weight w. Then every $P \in I_w$ is a homogeneous polynomial of weight w if we define the weight of $A^k_{\alpha\bar\beta}$ to be $(|\alpha| + |\beta|)/2 + k - 1$. Graham [18] proved the following:

Theorem 2.2 ([18]).   (i)   *Let $n = 2$. Then $I_1 = I_2 = \{0\}$ and $\dim I_3 = \dim I_4 = 1$. Moreover, I_3 and I_4 are respectively spanned by $A^0_{4\bar4}$ and $|A^0_{2\bar4}|^2$.*
(ii) *Let $n \geq 3$. Then $I_1 = \{0\}$ and $\dim I_2 = 1$. Moreover, I_2 is spanned by $\|A^0_{2\bar2}\|^2 = \sum |A^0_{\alpha\bar\beta}|^2$, where the summation runs over $|\alpha| = |\beta| = 2$.*

When $n = 2$, a basis of the two dimensional I_5 has been determined in [18, 24] and a basis of the three dimensional I_6 has been determined by Hirachi [21].

Theorem 2.3 ([24]).   (i)   *Let $n = 2$ and $\eta_1 = 4A^0_{4\bar4}$. Then there exist constants k^S_1 and k^S_2 independent of Ω such that*

$$(7) \qquad \varphi^S = 1 + O(r^2), \quad \psi^S = k^S_1 \eta_1 r + k^S_2 |A^0_{2\bar4}|^2 r^2 + O(r^3).$$

(ii) *Let $n \geq 3$. There is a constant c^S_n depending only on n such that*

$$(8) \qquad \varphi^S = 1 + c^S_n \|A^0_{2\bar2}\|^2 r^2 + O(r^3).$$

Theorem 2.4 ([24]).   *The universal constants in (7) and (8) are given by $k^S_1 = -2$, $k^S_2 = 8/15$ and $(n-1)(n-2)c^S_n = 2/3$.*

The above theorems were proved by Hirachi, Komatsu and Nakazawa [24]. In [25], they extended the expansion of ψ^S in (7) to weight 5. They gave two different methods of identifying the universal constants. The first method is by using microlocal analysis of Kashiwara [27] and Boutet de Monvel [4]. Below we will outline their second method using explicit asymptotic expansion for Reinhardt domains.

Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex complete Reinhardt domain. Its logarithmic real representation domain is given by

$$-\log|\Omega| = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid (e^{-x_1}, \dots, e^{-x_{n-1}}, e^{-y}) \in \Omega\}.$$

First we assume $n = 2$. Let $f(x) := \inf\{y \in \mathbb{R} \mid (x, y) \in -\log|\Omega|\}$. Then $\lambda = y - f(x)(> 0)$ is a defining function of $\partial\Omega \cap \{z_1 z_2 \neq 0\}$. We make change of variables $(x, y) \rightarrow (\lambda, v)$ with $v = f'(x)$ and set $p(v) = f''(x)$, the hodograph transformation.

Theorem 2.5 ([24]).   *Let $n = 2$. Near $\partial\Omega \cap \{z_1 z_2 \neq 0\}$, we have*

$$\text{(9)} \qquad K^S(z) = \frac{1}{\pi^2} J[\lambda]^{2/3} \left(\frac{\tilde{\varphi}(v, \lambda)}{\lambda^3} + \tilde{\psi}(v, \lambda) \log \lambda \right),$$

where $J[\lambda] = p/|4z_1 z_2|^2$. Let $e_1 = p''$, $e_2 = (pp^{(3)})'$, $e_3 = (p^2 p^{(4)})''$, $e_{41} = e_1 e_3$, $e_{42} = (pe_3')'$ and $e_{43} = (pp^{(4)})^2$. Then

$$\text{(10)} \qquad \tilde{\varphi} = 1 + \frac{\lambda}{6} e_1, \quad \tilde{\psi} = -\frac{\lambda}{72} e_3 + \frac{\lambda^2}{4320}(12e_{42} + e_{43} - e_{41}) + O(\lambda^3).$$

Lemma 2.6 ([24]).   *Under the notation of the above theorem, we have $|A_{2\bar{4}}^0|^2 = J[\lambda]^{4/3} e_{43}/48^2$, $r^F = J[\lambda]^{-1/3}(\tilde{r} + O(\lambda^4))$ and $\eta_1 = J[\lambda](\tilde{\eta}_1 + O(\lambda^2))$, where*

$$\tilde{r} = \lambda - \frac{\lambda^2}{12} e_1 - \frac{\lambda^3}{36} \left(e_2 - \frac{e_1^2}{2} \right), \quad \tilde{\eta}_1 = \frac{e_3}{144} - \frac{\lambda}{720} \left(e_{42} - \frac{e_{41}}{2} \right).$$

Theorem 2.5 and Lemma 2.6 imply $k_1^S = -2$, $k_2^S = 8/15$ in (7).

Next we consider the higher dimensional case. Let $n \geq 3$ and $\Omega \subset \mathbb{C}^n$ a bounded strictly pseudoconvex complete Reinhardt domain satisfying $-\log|\Omega| = \{\lambda := y - (f_1(x_1) + \cdots + f_{n-1}(x_{n-1})) > 0\}$ with hodograph variables $v_j = f_j'(x_j)$ and $p_j(v_j) = f_j''(x_j)$. We introduce

$$e_1 = \sum_{j=1}^{n-1} p_j'', \quad e_{21} = \sum_{j=1}^{n-1} (p_j p_j''')', \quad e_{22} = \sum_{j=1}^{n-1} (p_j'')^2, \quad e_{23} = \sum_{j \neq k} p_j'' p_k''.$$

Theorem 2.7 ([24]).   *Under the above notation, we have*

$$\|A_{22}^0\|^2 = \frac{J[\lambda]^{2/(n+1)}}{16n(n+1)}((n-2)(n-1)e_{22} + 2e_{23}),$$

$$r^F = J[\lambda]^{-1/(n+1)} \left(\lambda - \frac{e_1 \lambda^2}{2n(n+1)} + \frac{-n(n+1)e_{21} + (n^2-1)e_{22} - e_{23}}{6(n-1)n^2(n+1)^2} \lambda^3 + O(\lambda^4) \right).$$

The Szegö kernel has the expansion

$$\text{(11)} \qquad K^S(z) = \frac{(n-1)!}{\pi^n} J[\lambda]^{n/(n+1)} \left(\frac{\tilde{\varphi}(v, \lambda)}{\lambda^n} + \tilde{\psi}(v, \lambda) \log \lambda \right),$$

where $J[\lambda] = p/(4^n |z_1 \cdots z_n|^2)$ and

$$(12) \qquad \tilde{\varphi} = 1 + \frac{\lambda}{2(n+1)} e_1 + \lambda^2 \left(\frac{1}{6(n^2-1)} e_{21} + \frac{n-1}{8(n+1)^2(n-2)} e_{23} \right) + O(\lambda^3).$$

Theorem 2.7 immediately implies $(n-1)(n-2)c_n^S = 2/3$ in (8).

(10) and (12) were obtained by computer-aided calculations in [24]. We will use our graph theoretic formulas to compute them in §4.

3. The asymptotic expansion of weighted Bergman kernels

Throughout this section, both the Bergman kernel $\mathcal{K}_\alpha(x, y)$ and the Berezin transform $\mathcal{I}_\alpha$ depend on a nonnegative real number $C \geq 0$. For simplicity, we suppress C in their notations. The following theorem was proved in [9] for $C = 0, 1$; the proof for general C is the same.

Theorem 3.1 (Engliš [9]). *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$ with real analytic boundary, $\Phi(x)$ a strictly-plurisubharmonic function on Ω, $g_{i\bar{j}}(x) = \partial^2 \Phi(x)/(\partial x_i \partial \bar{x}_j)$ the associated Kähler metric and $g = \det g_{i\bar{j}}$ its volume element. Let $x \in \Omega$ and assume $f \in C^\infty(\Omega)$ is supported in a small neighborhood of x. Then there is an asymptotic expansion for the Laplace integral as $\alpha \to \infty$,*

$$\int_\Omega f(y) e^{-\alpha(\Phi(x)+\Phi(y)-\Phi(x,y)-\Phi(y,x))} \frac{|g(x,y)|^{2-2C}}{g(x)^{1-C}} g(y)^C \, dy$$
$$\sim \pi^n \sum_{j \geq 0} \alpha^{-n-j} \mathcal{R}_j(f)(x),$$

where $\Phi(x, y)$ and $g(x, y)$ are the almost analytic extensions of the Kähler potential $\Phi(x)$ and $g(x)$ respectively, and $\mathcal{R}_j \colon C^\infty(\Omega) \to C^\infty(\Omega)$ are differential operators given by

$$(13) \quad \mathcal{R}_j f(x) = \frac{1}{g(x)^{2-C}} \sum_{k=j}^{3j} \frac{1}{k!(k-j)!} L^k[f(y)|g(x,y)|^{2(1-C)} g(y)^C S(x,y)^{k-j}]|_{y=x},$$

where L is the (constant-coefficient) differential operator

$$Lf(y) = g^{i\bar{j}}(x)\, \partial_i \partial_{\bar{j}} f(y)$$

and the function $S(x, y)$ satisfies

$$S = \partial_\alpha S = \partial_{\alpha\beta} S = \partial_{i_1 i_2 \cdots i_m} S = \partial_{\bar{i}_1 \bar{i}_2 \ldots \bar{i}_m} S = 0 \quad at \quad y = x,$$
$$\partial_{i\bar{j}\alpha_1\alpha_2\cdots\alpha_m} S|_{y=x} = -\partial_{\alpha_1\alpha_2\ldots\alpha_m} g_{i\bar{j}}(x), \quad m \geq 1.$$

Here the Greek indices α, β may represent either i or $\bar{i}$

Denote by $\mathcal{K}_\alpha(x, y)$ the reproducing kernel of the weighted Bergman space of all holomorphic functions on Ω square-integrable with respect to the measure $e^{-\alpha\Phi}g(x)^C\,dx$. It was shown in [10] that $\mathcal{K}_\alpha(x, y)$ has an asymptotic expansion in a small neighborhood of the diagonal as $\alpha \to \infty$,

$$(14) \qquad \mathcal{K}_\alpha(x, y) = \frac{1}{\pi^n}e^{\alpha\Phi(x,y)}g(x, y)^{1-C}\sum_{k=0}^{\infty}\mathcal{B}_k(x, y)\alpha^{n-k}.$$

The proof used Fefferman's expansion for the Bergman kernel in a certain Forelli–Rudin type domain over Ω.

The Berezin transform is given by

$$(15) \qquad \mathcal{I}_\alpha f(x) = \int_\Omega f(y)\frac{|\mathcal{K}_\alpha(x, y)|^2}{\mathcal{K}_\alpha(x, x)}e^{-\alpha\Phi(y)}g(y)^C\,dy,$$

which has an asymptotic expansion as $\alpha \to \infty$ (cf. [10]),

$$(16) \qquad \mathcal{I}_\alpha f(x) = \sum_{k=0}^{\infty}\mathcal{Q}_k f(x)\alpha^{-k}.$$

The Berezin transform was first introduced by Berezin [3] in the context of quantization of Kähler manifolds. The existence of the asymptotic expansion (14) on compact Kähler manifold was proved by Karabegov–Schlichenmaier [26].

The following lemma is the key result we will use, which slightly refines the formulae in [9].

Lemma 3.2. *We have $\mathcal{Q}_0 = id$ and $\mathcal{B}_0 = 1$. For $k \geq 1$,*

$$(17) \qquad \mathcal{Q}_k f(x) = \sum_{j=0}^{k}\sum_{i=0}^{k-j}\mathcal{R}_j(\mathcal{B}_i(x, y)\mathcal{B}_{k-j-i}(y, x)f(y))|_{y=x} - \sum_{m=1}^{k}\mathcal{B}_m(x)\mathcal{Q}_{k-m}f(x),$$

$$(18) \qquad \mathcal{B}_k(x) = -\sum_{\substack{i+j=k\\i,j\geq 1}}\mathcal{B}_i(x)\mathcal{B}_j(x) - \sum_{\substack{l+i+j=k\\1\leq l\leq k}}\mathcal{R}_l(\mathcal{B}_i(x, y)\mathcal{B}_j(y, x))|_{y=x}.$$

Proof. By multiplying $\mathcal{K}_\alpha(x, x)$ to both sides of (15) and using (14) and (16), we get

$$\sum_{m=0}^{\infty}\mathcal{B}_m(x, y)\sum_{i=0}^{\infty}\mathcal{Q}_i f(x)\alpha^{n-m-i}$$

$$(19) \qquad = \frac{1}{\pi^n}\int_\Omega f(y)e^{-\alpha(\Phi(x)+\Phi(y)-\Phi(x,y)-\Phi(y,x))}$$

$$\times \frac{|g(x, y)|^{2(1-C)}}{g(x)^{1-C}}g(y)^C\sum_{i,m=0}^{\infty}\mathcal{B}_i(x, y)\mathcal{B}_m(y, x)\alpha^{2n-m-i}\,dy.$$

By applying Theorem 3.1 to the right-hand side of the above equation and equating the coefficients of α^{n-k}, we get (17).

Since $Q_0 = id$ and $Q_k(f) = 0$ when $k \geq 1$ and f is either holomorphic or anti-holomorphic, by substituting $f = 1$ in (17), we get (18).   □

Before proceeding we need to introduce parallel notions for graphs and pointed graphs representing Weyl invariant polynomials in jets of metrics and functions.

A *digraph* or simply a *graph* $G = (V, E)$ is defined to be a finite directed multi-graph which may have multi-edges and loops. A vertex v of a digraph G is called *stable* if $\deg^-(v) \geq 2$, $\deg^+(v) \geq 2$, i.e. both the inward and outward degrees of v are no less than 2. A vertex v is called *semistable* if we have

$$\deg^-(v) \geq 1, \quad \deg^+(v) \geq 1, \quad \deg^-(v) + \deg^+(v) \geq 3.$$

The weight of a digraph G is defined to be the integer $w(G) = |E| - |V|$. A digraph G is *stable* (*semistable*) if each vertex of G is stable (semistable). The set of semi-stable and stable graphs of weight k will be denoted by $\mathcal{G}^{ss}(k)$ and $\mathcal{G}(k)$ respectively. A directed edge uv of a semistable digraph is called *contractible* if $u \neq v$ and at least one of the following two conditions holds:
(i) $\deg^+(u) = 1$;
(ii) $\deg^-(v) = 1$.
A semistable graph G is called *stabilizable* if after contractions of a finite number of contractible edges of G, the resulting graph becomes stable, which is called the *stabil-ization graph* of G and denoted by G^s.

A *pointed graph* $\Gamma = (V \cup \{\bullet\}, E)$ is defined to be a digraph with a distinguished vertex labeled by f. G or Γ is called *semistable* (*stable*) if each ordinary vertex $v \in V$ is semistable (stable). The weight of a pointed graph $\Gamma = (V \cup \{\bullet\}, E)$ is defined to be $w(\Gamma) = |E| - |V|$. By abuse of notation, we denote $V(\Gamma) = V \cup \{\bullet\}$. The set of semistable and stable pointed graphs of weight k will be denoted by $\mathcal{G}_1^{ss}(k)$ and $\mathcal{G}_1(k)$ respectively. Denote by $\text{Aut}(\Gamma)$ the set of all automorphisms of the pointed graph Γ fixing the distinguished vertex. A directed edge uv of a semistable pointed graph is called *contractible* if $u \neq v$ and at least one of the following two conditions holds:
(i) $u \in V$ and $\deg^+(u) = 1$;
(ii) $v \in V$ and $\deg^-(v) = 1$.
A semistable pointed graph Γ is called *stabilizable* if after contractions of a finite num-ber of contractible edges of Γ, the resulting graph becomes stable, which is called the *stabilization graph* of Γ and denoted by Γ^s.

We can canonically associate a polynomial in the variables $\{g_{i\bar{j}\alpha}\}_{|\alpha|\geq 1}$ or $\{f_\alpha\}_{|\alpha|\geq 0}$ to a semistable graph or pointed semistable graph, such that each ordinary vertex rep-resents a partial derivative of $g_{i\bar{j}}$, the distinguished vertex represents a partial derivative of f and each edge represents the contraction of a pair of indices. Abusing notation, we will denote this polynomial associated to a graph Γ also by Γ.

A *linear digraph* is a digraph in which $\deg^+(v) = \deg^-(v) = 1$ for each vertex v. We denote by $\mathscr{L}(G)$ the set of linear subgraphs of G. Note that we assume the empty graph $\emptyset \in \mathscr{L}(G)$.

A digraph G is called *strongly connected* or *strong* if there is a directed path from each vertex in G to every other vertex. We call a graph *quasi-strong* if all of its connected components are strong. A strongly connected component of a digraph G is called a *source* (*sink*) if it has only outward (inward) edges in G. A connected graph is strong if and only it has no proper source or sink.

A *Weyl invariant polynomial* is a polynomial of $\{g_{i\bar{j}\alpha}\}_{|\alpha|\geq 1}$ or $\{f_\alpha\}_{|\alpha|\geq 0}$ invariant under the transformation of coordinates. Recall the following criterion for Weyl invariant polynomials.

Theorem 3.3 ([39]). *Given two functions as summations over stabilizable semistable* (*pointed*) *graphs,*

$$(20) \qquad P_1 = \sum_{G \in \mathcal{G}^{ss}(k)}^{stabilizable} \frac{(-1)^{|V(G)|}c(G)}{|\mathrm{Aut}(G)|} G \quad and \quad P_2 = \sum_{\Gamma \in \mathcal{G}_1^{ss}(k)}^{stabilizable} \frac{(-1)^{|V(\Gamma)|}c(\Gamma)}{|\mathrm{Aut}(\Gamma)|} \Gamma,$$

then P_1 (or P_2) is a Weyl invariant polynomial if and only if $c(G_1) = c(G_2)$ whenever G_1, G_2 have the same stabilization graph.

DEFINITION 3.4. For convenience, a function $c(G)$ defined on the set of stabilizable semistable graphs is called a *Weyl function* if it satisfies $c(G_1) = c(G_2)$ whenever G_1, G_2 have the same stabilization graph.

The following lemma gives nontrivial examples of Weyl functions.

Lemma 3.5 ([39]). *For any constant C,*

$$(21) \qquad \tau_1(G) = \sum_{H \in \mathscr{L}(G)} C^{n(H)} \quad and \quad \tau_2(\Gamma) = \sum_{H \in \mathscr{L}(\Gamma_-)} C^{n(H)}$$

are Weyl functions. Here $n(H)$ is the number of connected components of H and Γ_- is the subgraph of Γ obtained by removing the distinguished vertex $\bullet$ and its adjacent edges from Γ. Note that when $n(H) = 0$, we adopt the convention $0^{n(H)} = 1$.

Following [9], we may show that $\mathcal{B}_k$, $\mathcal{R}_k$, $\mathcal{Q}_k$ all are Weyl invariant polynomials. We now prove closed formulas for the coefficients in the expansions

$$(22) \qquad \mathcal{B}_k = \sum_{G \in \mathcal{G}^{ss}(k)}^{stabilizable} \mathcal{B}_G G, \quad \mathcal{R}_k f = \sum_{\Gamma \in \mathcal{G}_1^{ss}(k)}^{stabilizable} \mathcal{R}_\Gamma \Gamma, \quad \mathcal{Q}_k f = \sum_{\Gamma \in \mathcal{G}_1^{ss}(k)}^{stabilizable} \mathcal{Q}_\Gamma \Gamma.$$

We only need to deal with stable (pointed) graphs and use Theorem 3.3 to recover the coefficients of stabilizable semistable (pointed) graphs.

We use the notations $\mathcal{G} = \bigcup_{k \geq 1} \mathcal{G}(k)$ and $\mathcal{G}_1 = \bigcup_{k \geq 1} \mathcal{G}_1(k)$.

Lemma 3.6. *Let $\Gamma = (V \cup \{\bullet\}, E) \in \mathcal{G}_1$ be a stable pointed graph. Then*

$$(23) \qquad \mathcal{R}_\Gamma = \frac{(-1)^{|V(\Gamma)|+1}}{|\mathrm{Aut}(\Gamma)|} \sum_{H \in \mathscr{L}(\Gamma_-)} (-C)^{n(H)}.$$

Proof. As noticed in [9, p. 34], in a normal coordinate system around x, the operators $\mathcal{R}_j$ in (13) simplify to

$$(24) \qquad \mathcal{R}_j f(x) = \sum_{k=j}^{2j} \frac{1}{k!\,(k-j)!} L^k (f(y)g(y)^C S^{k-j})|_{y=x}.$$

To connect it to the graph-theoretic picture, we regard L^k as k edges, S^{k-j} as $k-j$ vertices and $k!\,(k-j)!$ the symmetry factor.

We define an equivalence relation $\sim$ on $\mathscr{L}(\Gamma_-)$ as follows: $H_1 \sim H_2$ if there is an automorphism $h \in \mathrm{Aut}(\Gamma)$ such that $h(H_1) = H_2$.

Given $H \in \mathscr{L}$, denote by $\mathrm{Aut}(\Gamma)_H$ the isotropy subgroup of $\mathrm{Aut}(\Gamma)$ at H. Recall the following equation (cf. [38, Lemma 5.5])

$$(25) \qquad \frac{1}{g} \partial_{\alpha_1} \cdots \partial_{\alpha_r} g = \sum_{L \in \mathscr{L}(\alpha_1, \ldots, \alpha_r)} (-1)^{n(L)+|V(L)|} \cdot L,$$

where $\mathscr{L}(\alpha_1, \ldots, \alpha_r)$ is the set of all decorated linear digraphs with external legs $\alpha_1, \ldots, \alpha_r$ (i.e. attaching indices $\alpha_1, \ldots, \alpha_r$ to vertices of linear digraphs) such that each vertex is semistable. Two decorated linear digraphs are considered the same whenever they differ by a graph isomorphism preserving the labeling of external legs.

We have the natural action of $\mathrm{Aut}(\Gamma)$ on $\mathscr{L}(\Gamma_-)$. Then the orbits are in one-to-one correspondence with the equivalence classes $\mathscr{L}(\Gamma_-)/\sim$ and the isotropy group at H is $\mathrm{Aut}(\Gamma)_H$. See [38, 40] for more detailed discussions. By the graph-theoretic interpretations of (24) and (25), we have

$$R_\Gamma = \sum_{H \in \mathscr{L}(\Gamma_-)/\sim} \frac{(-1)^{n(H)+|V(\Gamma_-)|}}{|\mathrm{Aut}(\Gamma)_H|} C^{n(H)},$$

$$= \frac{(-1)^{|V(\Gamma)|+1}}{|\mathrm{Aut}(\Gamma)|} \sum_{H \in \mathscr{L}(\Gamma_-)/\sim} |\text{orbit of } H|(-C)^{n(H)},$$

which gives (23). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Corollary 3.7. *In any holomorphic coordinates, we have*

$$(26) \qquad \mathcal{R}_k f = \sum_{\Gamma \in \mathcal{G}_1^{ss}(k)}^{stabilizable} \frac{(-1)^{|V(\Gamma)|+1}}{|\mathrm{Aut}(\Gamma)|} \sum_{H \in \mathscr{L}(\Gamma_-)} (-C)^{n(H)} \Gamma.$$

Proof.   This follows from Theorem 3.3 and Lemma 3.5.   □

Theorem 3.8. *Let $G \in \mathcal{G}$ and $\Gamma \in \mathcal{G}_1$. Then*

$$(27) \qquad \mathcal{B}_G = \begin{cases} \dfrac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|} \displaystyle\sum_{H \in \mathscr{L}(G)} (-C)^{n(H)} & \textit{if } G \textit{ is quasi-strong,} \\ 0 & \textit{otherwise,} \end{cases}$$

$$(28) \qquad \mathcal{Q}_\Gamma = \begin{cases} \dfrac{(-1)^{|V(\Gamma)|+1}}{|\mathrm{Aut}(\Gamma)|} \displaystyle\sum_{H \in \mathscr{L}(\Gamma_-)} (-C)^{n(H)} & \textit{if } \Gamma \textit{ is strong,} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof.   First we assume that G is strong. Let us look at the right-hand side of (18). The first term contributes disconnected graphs. In the second term, the two factors $\mathcal{B}_i(x, y)$ and $\mathcal{B}_j(y, x)$ are sink and source respectively. Since G is strong, we must have $i = j = 0$. So it is not difficult to see from (18) and (23) that

$$(29) \qquad \mathcal{B}_G = -\mathcal{R}_{G \coprod \{\bullet\}} = \frac{(-1)^{|V(G)|+1}}{|\mathrm{Aut}(G)|} \sum_{H \in \mathscr{L}(G)} (-C)^{n(H)},$$

where $G \coprod \{\bullet\}$ is the disjoint union of G and the distinguished vertex $\bullet$.

If G is quasi-strong, we can prove (27) by induction on the weight of the graph and using [38, Lemma 3.9]. See [40, Theorem 3.6] for details.

If some connected component G_i of $G = G_1 \coprod \cdots \coprod G_n$ is not strongly connected, then G_i has a proper sink S. In order to prove $\mathcal{B}_G = 0$, we note that in $R_l(\mathcal{B}_i(x, y)\mathcal{B}_j(y, x))|_{y=x}$, the sink S may either belong to $B_i(x, y)$ or R_l, actually the contributions of these two cases to G exactly cancel out. We also need to note the fact that if $\tau(G) = \sum_{H \in \mathscr{L}(G)} C^{n(H)}$ and G has strongly connected components $H_1, \ldots, H_k$, then $\tau(G) = \prod_{i=1}^{k} \tau(H_i)$. The detailed argument is similar to the proof of [38, Proposition 3.3]. We omit the details. So we conclude the proof of the formula (27).

The formula for $\mathcal{Q}_\Gamma$ follows from (17), (23) and (27) by using the same argument as [38, Theorem 3.4].   □

Corollary 3.9. *In any holomorphic coordinates*, *we have*

$$\mathcal{B}_k = \sum_{G \in \mathcal{G}^{ss}(k)}^{\text{quasi-strong}} \frac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|} \sum_{H \in \mathscr{L}(G)} (-C)^{n(H)} G, \tag{30}$$

$$\mathcal{Q}_k f = \sum_{\Gamma \in \mathcal{G}_1^{ss}(k)}^{\text{strong}} \frac{(-1)^{|V(\Gamma)|+1}}{|\mathrm{Aut}(\Gamma)|} \sum_{H \in \mathscr{L}(\Gamma_-)} (-C)^{n(H)} \Gamma. \tag{31}$$

Proof. In [39, Lemma 3.4 and Lemma 4.5], it was proved that the stabilization graph of a semistable (pointed) graph G is strong if and only if G is strong. So the corollary follows from Theorem 3.3 and Lemma 3.5. $\qquad\square$

REMARK 3.10. Explicit computations of the first terms of $\mathcal{B}_k$ when $C = 0$ or 1 have been carried out in [9, 40] and [9, 32, 33, 37] respectively. In the next section, we will see that the Szegö kernel corresponds to $C = 1/(n+1)$. There has been much interest in the asymptotic expansion of the Szegö kernel (see e.g. [2, 19, 28, 30, 35]).

4. Forelli–Rudin construction

Let Ω be a domain in $\mathbb{C}^{n-1}$ and w a positive continuous weight function on Ω. Consider the domain

$$\tilde{\Omega} := \{(x, t) \in \Omega \times \mathbb{C}^m : |t|^2 < w(x)\}. \tag{32}$$

Similar construction was first used by Forelli and Rudin [17]. See also [10, 31]. For simplicity we take $m = 1$. Then $\tilde{\Omega}$ is a Hartogs domain in $\mathbb{C}^n$. If $d\sigma$ is the measure on $\partial\tilde{\Omega}$ defined by

$$\int_{\partial\tilde{\Omega}} f \, d\sigma := \int_\Omega \int_0^{2\pi} f(x, e^{i\theta} \sqrt{w(x)}) \frac{d\theta}{2\pi} \rho(x) \, dx \tag{33}$$

for some weight function ρ on Ω, then the Szegö kernel of the Hardy subspace in $L^2(\partial\tilde{\Omega}, d\sigma)$ is given by (cf. [31, 11])

$$K^S((x, t), (y, s)) = \sum_{k=0}^{\infty} \langle t, s \rangle^k K_{\Omega, 2\pi w^k \rho}(x, y), \tag{34}$$

where $K_{\Omega, w^k\rho}(x, y)$ is the weighted Bergman kernel on Ω with respect to the weight $w^k\rho$, i.e. the reproducing kernel of the subspace of holomorphic functions in $L^2(\Omega, w^k\rho)$. The formula (34) was generalized by Engliš and Zhang [11] to the situation when the fiber of the Hartogs domain is, instead of a ball, an arbitrary irreducible bounded symmetric domain.

Lemma 4.1. *The surface measure σ in (3) corresponds to the choice $\rho = (1/2)J[w]^{1/(n+1)}$.*

Proof.   As shown in (3), in order for the Szegö kernel to be invariant under biholomorphic change of coordinates, σ should be equal to

$$(35) \qquad \sigma = J[r]^{1/(n+1)}/\|\nabla r\| \, dS,$$

where dS is the ordinary surface measure on the boundary (i.e. the $(2n-1)$-dimensional Hausdorff measure) corresponding to the choice $\rho = \sqrt{w + \|\partial w\|^2}$. For the defining function of $\tilde{\Omega}$ one can take $r(x,t) = w(x) - |t|^2$, leading to $\|\nabla r\| = 2\sqrt{w + \|\partial w\|^2}$ and

$$J[r] = J[w] = (-1)^{n-1} \det \begin{pmatrix} r & \partial r/\partial \bar{x}_j \\ \partial r/\partial x_i & \partial^2 r/\partial x_i \partial \bar{x}_j \end{pmatrix}_{1 \leq i,j \leq n-1}$$

on the boundary. Thus σ in (35) corresponds to the choice $\rho = (1/2)J[w]^{1/(n+1)}$. $\square$

The following theorem is an analogue of [9, Theorem 10].

Theorem 4.2. *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^{n-1}$ with real-analytic boundary, Φ a strictly plurisubharmonic real-analytic defining function for Ω, $g_{i\bar{j}}$ the Kähler metric defined by the potential Φ, and $K^S(x,t)$ the on-diagonal Szegö kernel of the Hartogs domain*

$$(36) \qquad \tilde{\Omega} = \{(x,t) \in \Omega \times \mathbb{C} : |t|^2 < e^{-\Phi(x)}\} \subset \mathbb{C}^n.$$

Then
(i)   as (x,t) approaches a point of $\partial\tilde{\Omega} \setminus \{t = 0\}$, the reproducing kernel $K^S(x,t)$ admits an asymptotic expansion

$$(37) \qquad K^S(x,t) = \sum_{l=0}^{\infty} c_l(x) \cdot u_{n-1-l}(|t|^2 e^{\Phi(x)}),$$

in the sense that the partial sum of the first l terms of the right-hand side differs from the left-hand side by a function which is $O(u_{n-1-l}(|t|^2 e^{\Phi(x)}))$ if $l \leqq n$, and is in $C^{l-n-1}(\tilde{\Omega} \setminus \{t = 0\})$ if $l \geq qn + 1$. Here the function $u_l(w)$ is given by

$$(38) \qquad u_l(w) = \sum_{k=\max(0,-l)}^{\infty} \frac{(k+l)!}{k!} w^k$$

$$= \begin{cases} \dfrac{l!}{(1-w)^{l+1}}, & l \geq 0, \\[3mm] \dfrac{(-w)^{-l} + w(1-w)^{-l-1} - (w-1)^{-l-1}\log(1-w)}{(-l-1)!}, & l < 0. \end{cases}$$

(ii) *The coefficients $c_l(x)$ in (37) are given by the formula*

$$(39) \qquad c_l(x) = \frac{1}{\pi^n} g(x)^{n/(n+1)} e^{(n/(n+1))\Phi(x)} \sum_{j=0}^{l} a_{n-1-j,l-j} \mathcal{B}_j(x),$$

where $a_{m,l}$ ($m \in \mathbb{Z}$, $l \geq 0$) are functions of n with $a_{m0} = 1$, given below

$$\left(k + \frac{n}{n+1}\right)^m = \sum_{l=0}^{m} \frac{(k+m-l)!}{k!} a_{m,l}$$

and $\mathcal{B}_j(x)$ are the scalar invariants of $g_{i\bar{j}}$ from (14) with $C = 1/(n+1)$.

Proof.   First note that

$$J[e^{-\Phi(x)}]$$
$$= (-1)^{n-1} e^{-n\Phi(x)}$$
$$\times \det \begin{pmatrix} 1 & -\bar{\partial}_1\Phi & \cdots & -\bar{\partial}_{n-1}\Phi \\ -\partial_1\Phi & -\partial_1\Phi\bar{\partial}_1\Phi - \partial_1\bar{\partial}_1\Phi & \cdots & -\partial_1\Phi\bar{\partial}_{n-1}\Phi - \partial_1\bar{\partial}_{n-1}\Phi \\ \vdots & \vdots & \ddots & \vdots \\ -\partial_{n-1}\Phi & -\partial_{n-1}\Phi\bar{\partial}_1\Phi - \partial_{n-1}\bar{\partial}_1\Phi & \cdots & -\partial_{n-1}\Phi\bar{\partial}_{n-1}\Phi - \partial_{n-1}\bar{\partial}_{n-1}\Phi \end{pmatrix}$$
$$= e^{-n\Phi(x)} g(x).$$

The determinant of the matrix can be computed by adding $\bar{\partial}_j\Phi$ times the first column to the $(j+1)$-th column for each $1 \leq j \leq n-1$.

By (34) and Lemma 4.1, we have

$$(40) \qquad K^S(x,t) = \sum_{k=0}^{\infty} K_k(x)|t|^{2k} = \sum_{k=0}^{\infty} e^{-k\Phi(x)} K_k(x)(|t|^2 e^{\Phi(x)})^k,$$

where $K_k(x)$ is the on-diagonal weighted Bergman kernel on Ω with respect to the weight $\pi e^{-(k+n/(n+1))\Phi(x)} g(x)^{1/(n+1)}$. The convergence is uniform on compact subsets of Ω (cf. [10]).

On the other hand, by (14), as $k \to \infty$,

$$(41) \qquad K_k(x) = \frac{(n-1)!}{\pi^n} e^{(k+n/(n+1))\Phi(x)} g(x)^{n/(n+1)} \sum_{j=0}^{\infty} \mathcal{B}_j(x) \left(k + \frac{n}{n+1}\right)^{n-1-j},$$

where $\mathcal{B}_j(x)$ are the scalar invariants of $g_{i\bar{j}}$ from the last section with $C = 1/(n+1)$.

As noted in [9, p. 36], u_l is unbounded on the unit disk D for $l \geq -1$, and belongs to $C^{-l-2}(\bar{D})$ for $l \leq -2$. Let $f(w) = \sum_0^\infty f_k w^k$ be a holomorphic function on D for which

$$f_k = A_M \frac{(k+M)!}{k!} + A_{M-1} \frac{(k+M-1)!}{k!} + \cdots + A_{m+1} \frac{(k+m+1)!}{k!} + O(k^m)$$

as $k \to \infty$, where $M, m \in \mathbb{Z}$, $m < M$. Then we have

$$(42) \qquad f(w) = \sum_{l=m+1}^M A_l u_l(w) + h(w),$$

where $h(w) = O(u_m(w))$ if $m \geq -1$, and $h(w) \in C^{-m-2}(\bar{D})$ if $m \leq -2$.

Obviously $a_{m0} = 0$. Then the theorem follows from (40), (41) and (42). $\qquad \square$

Let $n = 2$ and $\tilde{\Omega} = \{(z_1, z_2) \in \Omega \times \mathbb{C} : |z_2|^2 < e^{-\Phi(z_1)}\}$ with $\Phi(z_1)$ depending only on $|z_1|$. By applying Theorem 4.2 to the complete Reinhardt domain $\tilde{\Omega} \subset \mathbb{C}^2$, we get (9) in Theorem 2.5,

$$(43) \qquad K^S = \frac{1}{4\pi^2} \left(\frac{p}{2|z_1 z_2|^2} \right)^{2/3} \left(\frac{L_0}{\lambda^2} + \frac{L_1}{\lambda} + \sum_{k=2}^\infty L_k \lambda^{k-2} \log \lambda \right).$$

The following lemma is an analogue of [34, Proposition 0]. It follows from the integral representation of L_k proved in [24, Proposition 3].

Lemma 4.3. *Each coefficient L_k is a linear combination of*

$$p^{(\eta_1)} \cdots p^{(\eta_{2k})}/p^k \quad with \quad \eta_1 + \cdots + \eta_{2k} = 2k.$$

Namely L_k is homogeneous of degree k and order $2k$.

Let $k \geq 0$ and C be any constant. Define the function $W_{C,k}(p)$ by

$$(44) \qquad W_{C,k}(p) = \frac{1}{p^k} \sum_{G \in \mathcal{G}^{ss}(k)}^{\text{quasi-strong}} \frac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|} \sum_{H \in \mathscr{L}(G)} (-C)^{n(H)} \prod_{v \in V(G)} h(\deg(v) - 2),$$

where G runs over all quasi-strong (i.e. all connected components are strongly connected) semistable graphs of weight k and $n(G)$ is the number of components of G; the function h is defined recursively by

$$h(1) = p', \quad h(k) = [p \cdot h(k-1)]', \quad k \geq 2.$$

We can now prove a closed formula for L_k by using (30).

Theorem 4.4. *The coefficients of* (43) *are given by*

$$
(45) \qquad L_k = \begin{cases} (1-k)!\, W_{1/3,k}(p), & 0 \le k \le 1, \\ \dfrac{(-1)^{k+1}}{(k-2)!} W_{1/3,k}(p), & k \ge 2. \end{cases}
$$

Proof.   In the notations of Theorem 2.5, for $(z_1, z_2) \in \mathbb{C}^2$, we have

$$
x = -\log|z_1| = -\frac{1}{2}(\log z_1 + \log \bar{z}_1), \quad y = -\log|z_2|, \quad f(x) = \frac{1}{2}\Phi,
$$

$$
e^{-2\lambda} = |z_2|^2 e^{\Phi(z_1)}, \quad \frac{\partial^2 \Phi}{\partial z_1 \bar{\partial} z_1} = \frac{1}{2|z_1|^2} \frac{\partial^2 f}{\partial x^2} = \frac{p}{2|z_1|^2}.
$$

By using these equations, (37) becomes

$$
(46) \qquad K^S = \frac{1}{\pi^2} \left(\frac{e^{-2\lambda} p}{2|z_1 z_2|^2} \right)^{2/3} \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_{1-j,k-j} \mathcal{B}_{1/3,j}(z_1) u_{1-k}(e^{-2\lambda}),
$$

where $\mathcal{B}_{1/3,j}$ denotes $\mathcal{B}_j$ for $C = 1/3$.

By (38), the singular part of $u_{1-k}(e^{-2\lambda})$ is given by

$$
(47) \qquad u_{1-k}(e^{-2\lambda}) = \begin{cases} \dfrac{(1-k)! + O(\lambda)}{2^{2-k}\lambda^{2-k}}, & 0 \le k \le 1, \\ \dfrac{[(-1)^{k+1} 2^{k-2}\lambda^{k-2} + O(\lambda^{k-1})]\log(\lambda)}{(k-2)!}, & k \ge 2. \end{cases}
$$

Note that the derivatives of p satisfy

$$
(48) \qquad \frac{\partial p}{\partial z_1} = -\frac{pp'}{2z_1}, \quad \frac{\partial p}{\partial \bar{z}_1} = -\frac{pp'}{2\bar{z}_1}.
$$

By (30), we express $\mathcal{B}_j(z_1)$ as a summation of rational differential functions of p,

$$
(49) \qquad \begin{aligned} \mathcal{B}_{1/3,j}(z_1) = &\sum_{G \in \mathcal{G}^{ss}(j)}^{\text{quasi-strong}} \frac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|} \sum_{H \in \mathscr{L}(G)} \left(-\frac{1}{3}\right)^{n(H)} \\ &\times \prod_{v \in V(G)} \frac{\partial^{\deg(v)-2}}{\partial z_1^{\deg^+(v)-1} \partial \bar{z}_1^{\deg^-(v)-1}} \left[\frac{p}{2|z_1|^2} \right] \bigg|_{|z_1|^2 = p/2}. \end{aligned}
$$

Note that $\mathcal{B}_{1/3,j}$ is of degree no more than j. The top degree is achieved only when

all derivatives are taken on the numerator p. It is not difficult to see from (49) that

$$
\mathcal{B}_{1/3,j}(z_1) = \sum_{G \in \mathcal{G}^{ss}(j)}^{\text{quasi-strong}} \frac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|} \sum_{H \in \mathcal{L}(G)} \left(-\frac{1}{3}\right)^{n(H)}
$$
$$
\times \frac{1}{2^j p^j} \prod_{v \in V(G)} \frac{1}{p} \cdot \frac{\partial^{\deg(v)-2} p}{\partial z_1^{\deg^+(v)-1} \partial \bar{z}_1^{\deg^-(v)-1}} + Low
$$
$$
= \sum_{G \in \mathcal{G}^{ss}(j)}^{\text{quasi-strong}} \frac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|} \sum_{H \in \mathcal{L}(G)} \left(-\frac{1}{3}\right)^{n(H)}
$$
$$
\times \frac{1}{2^j p^j} \prod_{v \in V(G)} h(\deg(v) - 2) + Low
$$
$$
= \frac{W_{1/3,j}(p)}{2^j} + Low,
$$

where Low denotes the terms of rational differential functions of p with degree strictly less than j, which may be discarded according to Lemma 4.3. It also implies that in the summation (46), we can discard all terms except when $j = k$, i.e. the term $a_{1-k,0}\mathcal{B}_{1/3,k}(z_1) = \mathcal{B}_{1/3,k}(z_1)$. In view of (47), Equation (45) follows immediately. □

EXAMPLE 4.5.   From

$$
h(1) = p', \quad h(2) = (p')^2 + pp'', \quad h(3) = (p')^3 + 4pp'p'' + p^2 p^{(3)},
$$
$$
h(4) = (p')^4 + 11p(p')^2 p'' + 7p^2 p' p^{(3)} + 4p^2(p'')^2 + p^3 p^{(4)},
$$
$$
h(5) = (p')^5 + 26p(p')^3 p'' + 32p^2(p')^2 p''' + 34p^2 p'(p'')^2 + 11p^3 p' p^{(4)}
$$
$$
+ 15p^3 p'' p''' + p^4 p^{(5)},
$$
$$
h(6) = (p')^6 + 57p(p')^4 p'' + 122p^2(p')^3 p''' + 180p^2(p')^2(p'')^2 + 76p^3(p')^2 p^{(4)}
$$
$$
+ 192p^3 p' p'' p''' + 16p^4 p' p^{(5)} + 34p^3(p'')^3 + 26p^4 p'' p^{(4)}
$$
$$
+ 15p^4(p''')^2 + p^5 p^{(6)}.
$$

We get the following formulas for $W_{C,k}$, $0 \le k \le 3$,

$$
W_{C,0}(p) = 1, \quad W_{C,1}(p) = \left(\frac{1}{2} - C\right) p'', \quad W_{C,2}(p) = \left(\frac{1}{6} - \frac{1}{2}C\right)(pp^{(3)})',
$$
$$
W_{C,3}(p) = \left(\frac{1}{24} - \frac{1}{6}C\right)(p^2 p^{(4)})'' + \left(\frac{1}{6}C - \frac{1}{2}C^2\right)(pp'' p^{(3)})',
$$

$$W_{C,4}(p) = \left(-\frac{1}{24}C + \frac{1}{120}\right)(p(p^2 p^{(4)})''')' + \left(-\frac{2}{3}C^2 + \frac{5}{24}C + \frac{1}{240}\right)(pp^{(4)})^2$$

$$+ \left(-\frac{1}{3}C^2 + \frac{1}{8}C - \frac{1}{240}\right)p''(p^2 p^{(4)})''$$

$$+ \left(-\frac{1}{2}C^3 - \frac{1}{12}C^2 + \frac{1}{12}C\right)(p'(p'')^2 p''' + p(p'')^2 p^{(4)} + 2pp''(p''')^2)$$

$$+ \left(-\frac{1}{4}C^2 + \frac{1}{12}C\right)(2(p')^2(p''')^2 - p'(p'')^2 p''' - p(p'')^2 p^{(4)}$$

$$+ 4p^2 p''' p^{(5)} + 12pp' p''' p^{(4)}).$$

The computation is routine. For example, there are two quasi-strong semistable graphs of weight 1 in $\mathcal{G}^{ss}(1)$,

So by (44), we have $W_{C,1}(p) = (1/p)[(1/2)(1 - 2C)h(2) - (1/2)(1 - 2C)h(1)^2] = ((1/2) - C)p''$.

The cardinality of $|\mathcal{G}^{ss}(k)|$ increases very rapidly with the growth of k. There are 19 quasi-strong graphs in $\mathcal{G}^{ss}(2)$, among which 4 are stable. There are 300 quasi-strong graphs in $\mathcal{G}^{ss}(3)$, among which 14 are stable. There are 8696 quasi-strong graphs in $\mathcal{G}^{ss}(4)$, among which 71 are stable.

The 19 quasi-strong graphs in $\mathcal{G}^{ss}(2)$ are depicted in Table 0. They are grouped according to their stabilization graphs. Also listed are the values of

$$\frac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|} \sum_{H \in \mathscr{L}(G)} (-C)^{n(H)}$$

for each graph G. Let $\tau(G) = \sum_{H \in \mathscr{L}(G)}(-C)^{n(H)}$. By [39, Lemma 3.8], if G_1 and G_2 have the same stabilization graph, then $\tau(G_1) = \tau(G_2)$.

We can now get L_k, $0 \leq k \leq 3$ by using Theorem 4.4.

$$L_0 = W_{1/3,0}(p) = 1, \quad L_1 = W_{1/3,1}(p) = \frac{1}{6}p'',$$

$$L_2 = -W_{1/3,2}(p) = 0, \quad L_3 = W_{1/3,3}(p) = -\frac{1}{72}(p^2 p^{(4)})'',$$

$$L_4 = -\frac{1}{2}W_{1/3,4} = \frac{1}{360}(p(p^2 p^{(4)})''')' + \frac{1}{4320}((pp^{(4)})^2 - p''(p^2 p^{(4)})'').$$

Table 0. The 19 quasi-strong semistable graphs of weight 2.

③	① ⇄ ∘ (2, 1)	∘ ⇄ ∘ (3, 1)	① ⇄ ∘ (1, 2)	triangle graph (1, 1, 1, 2)
$-\dfrac{1}{2}c + \dfrac{1}{6}$	$\dfrac{3}{2}c - \dfrac{1}{2}$	$\dfrac{1}{2}c - \dfrac{1}{6}$	$\dfrac{3}{2}c - \dfrac{1}{2}$	$-\dfrac{3}{2}c + \dfrac{1}{2}$
triangle graph	triangle graph	triangle graph	square graph	square graph
$-3c - 1$	$-\dfrac{3}{2}c + \dfrac{1}{2}$	$-\dfrac{3}{2}c + \dfrac{1}{2}$	$\dfrac{3}{2}c - \dfrac{1}{2}$	$3c - 1$

① ⇄ ① (1, 1)	triangle with ①	square graph
$-\dfrac{1}{2}c^2 + \dfrac{3}{2}c - \dfrac{1}{2}$	$c^2 - 3c + 1$	$-\dfrac{1}{2}c^2 + \dfrac{3}{2}c - \dfrac{1}{2}$

∘ ⇄ ∘ (2, 2)	triangle graph	square graph
$\dfrac{1}{2}c - \dfrac{1}{8}$	$-c + \dfrac{1}{4}$	$\dfrac{1}{2}c - \dfrac{1}{8}$

② | ②	∘ ⇄ ∘ (2, 1) | ②	∘ ⇄ ∘ (2, 1) | ∘ ⇄ ∘ (2, 1)
$\dfrac{1}{2}c^2 - \dfrac{1}{2}c + \dfrac{1}{8}$	$-c^2 + c - \dfrac{1}{4}$	$\dfrac{1}{2}c^2 - \dfrac{1}{2}c + \dfrac{1}{8}$

Now we consider the higher dimensional Reinhardt domains. Let $n \geq 2$. Under the notations of Theorem 2.7, denote

$$(50) \qquad K^S = \frac{(n-1)!}{\pi^n} \left(\frac{p_1 \cdots p_{n-1}}{4^n |z_1 \cdots z_n|^2} \right)^{n/n+1} \left(\sum_{k=0}^{n-1} \frac{L_k}{\lambda^{n-k}} + \sum_{k=n}^{\infty} L_k \lambda^{k-n} \log \lambda \right).$$

Theorem 4.6. *Let $k \geq 0$. Then the coefficients of* (50) *are given by*

$$(51) \qquad L_k = \begin{cases} \dfrac{(n-k-1)!}{(n-1)!} \displaystyle\sum_{\substack{k=m_1+\cdots+m_{n-1} \\ m_i \geq 0}} \prod_{i=1}^{n-1} W_{1/(n+1),m_i}(p_i), & 0 \leq k \leq n-1, \\[2em] \dfrac{(-1)^{n-k-1}}{(n-1)!\,(k-n)!} \displaystyle\sum_{\substack{k=m_1+\cdots+m_{n-1} \\ m_i \geq 0}} \prod_{i=1}^{n-1} W_{1/(n+1),m_i}(p_i), & k \geq n, \end{cases}$$

where $W_{1/(n+1),m_i}(p_i)$ is the function defined in (44) *with $p = p_i$ and $C = 1/(n+1)$.*

Proof.   Under the notations of Theorem 2.7, (37) becomes

$$K^S = \frac{1}{\pi^n} \left(\frac{e^{-2\lambda} p_1 \cdots p_{n-1}}{2^{n-1} |z_1 \cdots z_n|^2} \right)^{n/(n+1)}$$

$$\times \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_{n-1-j,k-j} \mathcal{B}_{1/(n+1),j}(z_1, \ldots, z_{n-1}) u_{n-k-1}(e^{-2\lambda}).$$

Note that we have the analogue of Lemma 4.3 for any $n \geq 2$. So we can use the same argument as Theorem 4.4. The singular part of $u_{n-k-1}(e^{-2\lambda})$ is given by

$$u_{n-k-1}(e^{-2\lambda}) = \begin{cases} \dfrac{(n-k-1)! + O(\lambda)}{2^{n-k} \lambda^{n-k}}, & 0 \leq k \leq n-1, \\[1.5em] \dfrac{[(-1)^{n-k-1} 2^{k-n} \lambda^{k-n} + O(\lambda^{k-n+1})] \log(\lambda)}{(k-n)!}, & k \geq n. \end{cases}$$

Finally, (51) is clear since the Bergman kernel of a product domain $D_1 \times D_2$ is the product of the Bergman kernels of D_1 and D_2. $\square$

EXAMPLE 4.7.   Assume $n \geq 3$, we can easily compute L_1, L_2 by using (51) and Example 4.5.

$$L_1 = \frac{1}{n-1} \sum_{i=1}^{n-1} W_{1/(n+1),1}(p_i) = \frac{1}{2(n+1)} \sum_{i=1}^{n-1} p_i''$$

and

$$L_2 = \frac{1}{(n-1)(n-2)} \left(\sum_{i=1}^{n-1} W_{1/(n+1),2}(p_i) + \frac{1}{2} \sum_{i \neq j} W_{1/(n+1),1}(p_i) W_{1/(n+1),1}(p_j) \right)$$

$$= \frac{1}{(n-1)(n-2)} \left(\frac{n-2}{6(n+1)} \sum_{i=1}^{n-1} (p_i p_i^{(3)})' + \frac{1}{2} \cdot \frac{(n-1)^2}{4(n+1)^2} \sum_{i \neq j} p_i'' p_j'' \right)$$

$$= \frac{1}{6(n^2-1)} \sum_{i=1}^{n-1} (p_i p_i^{(3)})' + \frac{n-1}{8(n+1)^2(n-2)} \sum_{i \neq j} p_i'' p_j'',$$

which agrees with (12).

Assume $n \geq 4$, we can compute L_3 by using (51) and Example 4.5.

$$L_3 = \frac{1}{(n-1)(n-2)(n-3)} \sum_{\substack{m_1+m_2+m_3=3 \\ m_i \geq 0}} \prod_{i=1}^{n-1} W_{1/(n+1),m_i}(p_i)$$

$$= \frac{1}{24(n+1)(n-1)(n-2)} \sum_{i=1}^{n-1} (p_i^2 p_i^{(4)})'' + \frac{1}{6(n+1)^2(n-1)(n-3)} \sum_{i=1}^{n-1} (p_i p_i'' p_i^{(3)})'$$

$$= \frac{1}{12(n+1)^2(n-3)} \sum_{i \neq j} (p_i p_i^{(3)})' p_j'' + \frac{(n-1)^2}{48(n+1)^3(n-2)(n-3)} \sum_{i \neq j \neq k} p_i'' p_j'' p_k''.$$

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