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THE CONVERGENCE OF THE EXPLORATION PROCESS FOR CRITICAL PERCOLATION ON THE k-OUT GRAPH

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Abstract

We consider the percolation on the k-out graph $G_{\mathrm{out}}(n,k)$. The critical probability of it is $p_c = 1/(k + \sqrt{k^2 - k})$. Similarly to the random graph $G(n, p)$, in a scaling window $p_c(1 + O(n^{-1/3}))$, the sequence of sizes of large components rescaled by $n^{-2/3}$ converges to the excursion lengths of a Brownian motion with some drift. Also, the size of the largest component is $O(\log n)$ in the subcritical phase, and $O(n)$ in the supercritical phase. The proof is based on the analysis of the exploration process.

1. Introduction

The random graph $G(n, p)$ has n vertices $\{1, \dots, n\}$ and each edge $\langle i, j\rangle$ is realized independently with probability p. The random graph is interpreted as the percolation on the complete graph having n vertices with percolation probability p. It is well known that the random graph has "double jump". Namely, for $G(n,c/n)$, the size of the largest component $\mathcal{C}_1$ is of order $\log n$ for $c < 1$, of order $n^{2/3}$ for $c = 1$, and of order n for $c > 1$, we refer the reader to [5], [6]. Furthermore, it became clear that the structure of the random graph rapidly changes in a "scaling window" $(1/n)(1+O(n^{-1/3}))$, we refer the reader to [7]. In the scaling window, the rescaled size of the largest component $|\mathcal{C}_1|\cdot n^{-2/3}$ converges in distribution to a non-trivial random variable, we refer the reader to [8]. Aldous [1] showed that the sequence of sizes of components $(|\mathcal{C}_1|, |\mathcal{C}_2|, \dots)$ arranged in decreasing order converges to the sequence of excursion lengths of Brownian motion with some drift when scaled by $n^{2/3}$. This fact is provided by the analysis of the exploration process. In this process, we explore one vertex w_t at each time t, i.e., we count vertices connected to w_t by an edge. So, the exploration process reveals the structure of connected components of $G(n, p)$.

The convergence to the Brownian excursions appears in another graph model too. In [9], Nachmias and Peres investigated it in the random d-regular graph $G_{\mathrm{reg}}(n, d)$. It is known that the d-regular graph is almost surely connected for $d \geq 3$. But when we consider the percolation on $G_{\mathrm{reg}}(n, d)$, i.e., each edge remains with probability p and is removed with probability $1 - p$ independently, we can consider a similar problem for

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percolation clusters. For this model, in the scaling window $p = (1 + O(n^{-1/3}))/(d-1)$, the sequence of the rescaled sizes of the connected components converges to the sequence of the excursion lengths of a Brownian motion with some drift, both arranged in decreasing order, we refer the reader to [2].

We prove this convergence for the k-out graph $G_{\text{out}}(n, k)$, where $k \geq 2$ is a given integer and we fix it throughout this paper. The k-out graph model is explained in [9]. First we construct the graph with directed edges $\vec{G}_{\text{out}}(n,k)$. The vertex set of $\vec{G}_{\text{out}}(n,k)$ is $\mathcal{V} = \{1, \dots, n\}$. For each $v \in \mathcal{V}$, choose k distinct edges $(v, v_1), \dots, (v, v_k)$ uniformly from $\{(v, 1), (v, 2), \dots, (v, n)\} \setminus \{(v, v)\}$. This is a construction of the $\vec{G}_{\text{out}}(n, k)$. The vertex set of $G_{\text{out}}(n, k)$ is $\mathcal{V}$ too and a non-directed edge $\langle v, w\rangle$ is in the edge set of $G_{\text{out}}(n, k)$ if either (v, w) or (w, v) is an edge of $\vec{G}_{\text{out}}(n, k)$. However, it is convenient to keep the information of the direction of each edge until we construct the exploration process. We also consider percolation on $G_{\text{out}}(n,k)$. Each edge is open with probability p and closed with probability $1 - p$ independently. Our exploration process explores open clusters of $G_{\text{out}}(n, k)$.

The k-out graph model is related to the Watts–Strogatz model, which was introduced in [11] to explain the small world property of real networks (for more details, see [11] and references therein.). The Watts–Strogatz model is constructed in the following way. We start the one-dimensional lattice ring. For each edge, we rewire each edge at random with probability p. When the rewiring probability is 1, the Watts–Strogatz model resembles the k-out graph. Further, the critical point of percolation on a graph can be considered as an indicator of the robustness of the graph model. We hope that our analysis of the k-out graph model will lead to knowing the critical point of percolation on the Watts–Strogatz model, hence the robustness of it.

Bollobás and Riordan [4] proved the phase transition of the growing k-out graph. It is constructed in the following way. At time 0, we prepare k vertices having no edges. For each time $t \geq 1$, we add one vertex, and reach undirected edges from new vertex to k vertices chosen uniformly at random. We continue this until the number of vertices become n. Similar to the k-out graph, without first k vertices, each vertex reach k edges from itself. But this model has inhomogeneous degree of vertices, and it belongs to the group of the uniformly grown graph. The growing k-out graph has more robustness than the k-out graph. In fact, it is interesting to see that the critical point of the growing k-out graph is half of the critical point of the k-out graph.

In Section 2 we state main results. There are some theorems in each phase of the percolation for the k-out graph. Section 3 introduces the construction of the exploration process for the k-out graph. Proofs of Theorems 1, 3 and 4 are in Section 4. In Section 5 the proof of Theorem 2 is provided by the convergence of the exploration process.

For the construction and the calculation of the exploration process of the k-out graph, we follow the argument in [2]. However, we have to overcome some difficulties peculiar to the k-out graph.

2. Main results

Let $\mathcal{C}_l$ be the l-th largest open cluster of $G_{\text{out}}(n, k)$.

Theorem 1 (critical phase). *Let $\lambda \in \mathbb{R}$ be fixed and let*

$$p = p(\lambda) = \frac{1 + \lambda n^{-1/3}}{k + \sqrt{k^2 - k}}.$$

Then there exist positive constants $c(\lambda, k)$, $C(\lambda, k)$ such that for a large enough n and a large enough A,

$$\mathbb{P}(|\mathcal{C}_1| \geq An^{2/3}) \leq \frac{C(\lambda, k)e^{-c(\lambda,k)A^3}}{A}.$$

Even if A is not a constant, the above inequality is correct when $A = O(n^{1/10})$.

Let $\{\mathcal{B}(s): s \in [0, \infty)\}$ be a standard Brownian motion. For $\lambda \in \mathbb{R}$, we define the process $\mathcal{B}^\lambda$ by

$$\mathcal{B}^\lambda(s) = \mathcal{B}(2(k - \sqrt{k^2 - k})s) + 2(k - \sqrt{k^2 - k})\lambda s - (\sqrt{k^2 - k} - k + 1)s^2 \tag{1}$$

for $s \in [0, \infty)$. Next we define the reflected process of $\mathcal{B}^\lambda$ by

$$W^\lambda(s) = \mathcal{B}^\lambda(s) - \min_{0 \leq s' \leq s} \mathcal{B}^\lambda(s'). \tag{2}$$

Let $(|\gamma_j|)_{j\geq 1}$ be the sequence of all excursion lengths of W^λ arranged in a decreasing order. Also let $(|\mathcal{C}_j|)_{j\geq 1}$ be the sequence of sizes of open clusters in a decreasing order, concatenated by zeros to form a vector in l^2.

Theorem 2 (scaling window). *Let $\lambda \in \mathbb{R}$ be fixed and let*

$$p = p(\lambda) = \frac{1 + \lambda n^{-1/3}}{k + \sqrt{k^2 - k}}.$$

Then

$$n^{-2/3} \cdot (|\mathcal{C}_j|)_{j\geq 1} \overset{f.d.}{\Longrightarrow} (|\gamma_j|)_{j\geq 1}$$

as $n \to \infty$.

Next, we state the results of the largest open cluster size in subcritical and supercritical phases. Let $p = c/(k + \sqrt{k^2 - k})$.

Theorem 3 (subcritical phase). *If $c < 1$ and $A > 0$ is sufficiently large, then*

$$\mathbb{P}(|\mathcal{C}_1| > A \log n) \to 0, \quad \textit{as} \quad n \to \infty.$$

Theorem 4 (supercritical phase). *If $c > 1$ and $\delta > 0$ is sufficiently small, then*

$$\mathbb{P}(|\mathcal{C}_1| < \delta n) \to 0, \quad as \quad n \to \infty.$$

3. Exploration process of the *k*-out graph

We consider the k-out graph with directed edges $\vec{G}_{\text{out}}(n, k)$. Each vertex has state *neutral* or *active* or *explored*, and each edge has state *checked* or *unchecked*. We explain the algorithm of the exploration of $\vec{G}_{\text{out}}(n, k)$.

3.1. The algorithm of the exploration process. For each time, we explore one vertex. Let w_t be the vertex that we explore at time t. Let $\mathcal{N}_t$ be the set of neutral vertices at time t, $\mathcal{A}_t$ be the set of active vertices at time t, and $\mathcal{E}_t$ be the set of explored vertices at time t. Let $\mathcal{N}_t^{(i)}$ be the set of vertices at time t such that, for $v \in \mathcal{N}_t^{(i)}$,
(i) v is neutral, and
(ii) among k edges starting from v, exactly i edges are checked.
Therefore $\mathcal{N}_t$ equals the union of $\mathcal{N}_t^{(0)}, \mathcal{N}_t^{(1)}, \dots, \mathcal{N}_t^{(k)}$. Similarly, let $\mathcal{A}_t^{(i)}$ be the set of active vertices at time t with exactly i edges being checked. So $\mathcal{A}_t$ equals the union of $\mathcal{A}_t^{(0)}$, $\mathcal{A}_t^{(1)}$, ..., $\mathcal{A}_t^{(k)}$. Further we define the set $\mathcal{N}_t^{(\geq i)}$ as $\bigcup_{j=i}^{k} \mathcal{N}_t^{(j)}$. Similarly we define $\mathcal{N}_t^{(\leq i)}$, $\mathcal{A}_t^{(\geq i)}$, $\mathcal{A}_t^{(\leq i)}$.

Let N_t and $N_t^{(i)}$ denote the cardinalities of $\mathcal{N}_t$ and $\mathcal{N}_t^{(i)}$, respectively. Also let A_t and $A_t^{(i)}$ denote the cardinalities of $\mathcal{A}_t$ and $\mathcal{A}_t^{(i)}$, respectively.

At time 0, all vertices are in $\mathcal{N}_0^{(0)}$, and all edges are unchecked. The exploration at each time consists of $k + 2$ steps. Let $(t - 1, l)$ denote the l-th step at time t for $0 \leq l \leq k + 1$. We write $\mathcal{N}_{t-1,l}$ for $\mathcal{N}$ at the step $(t - 1, l)$. Similarly we define $\mathcal{A}_{t-1,l}$, $\mathcal{N}_{t-1,l}^{(i)}$ and $\mathcal{A}_{t-1,l}^{(i)}$.

At the step $(t-1, 0)$, we choose w_t. At the step $(t-1, l)$ for $1 \leq l \leq k$, we explore each directed edge from w_t. At the step $(t - 1, k + 1)$, we explore one directed edge to w_t. The number of edges from w_t depends on the state of w_t at time $t - 1$. So, it is necessary to divide the exploration of directed edges from w_t. From now on, we will explain these steps in more detail.

The step $(t-1, 0)$; choosing w_t. If $A_{t-1}^{(i)} > 0$ and $A_{t-1}^{(\geq i+1)} = 0$, then w_t is chosen from $\mathcal{A}_{t-1}^{(i)}$ uniformly. If $A_{t-1} = 0$, w_t is chosen from $\mathcal{N}_{t-1}$ uniformly. We put $\mathcal{N}_{t-1,0}^{(i)} = \mathcal{N}_{t-1}^{(i)} \setminus \{w_t\}$ and $\mathcal{A}_{t-1,0}^{(i)} = \mathcal{A}_{t-1}^{(i)} \setminus \{w_t\}$ for $0 \leq i \leq k$.

The step $(t - 1, l)$, $1 \leq l \leq k$; directed edge from w_t. At this step, we execute the arm stretch process (AS process) $\rho_{t-1,l}$. Let T_l be the total time required for AS processes $\rho_{t-1,1}, \dots, \rho_{t-1,l}$. $\rho_{t-1,l}$ and T_l will be described later. If $w_t \in \mathcal{N}_{t-1}^{(\leq l-1)} \cup \mathcal{A}_{t-1}^{(\leq l-1)}$, we execute the AS process $\rho_{t-1,l}$. If $w_t \in \mathcal{N}_{t-1}^{(\geq l)} \cup \mathcal{A}_{t-1}^{(\geq l)}$, do nothing and put $\mathcal{N}_{t-1,l}^{(i)} = \mathcal{N}_{t-1,l-1}^{(i)}$, $\mathcal{A}_{t-1,l}^{(i)} = \mathcal{A}_{t-1,l-1}^{(i)}$ for $0 \leq i \leq k$, and $T_l = T_{l-1}$ at this step.

The construction of the AS process. The AS process is composed of the following algorithm. We start with $T_0 = 0$.

First, we check one directed edge from w_t, and write $\eta_{t-1,T_{l-1}+1}$ for the head of this edge. We declare that the edge $(w_t, \eta_{t-1,T_{l-1}+1})$ is checked. If $\eta_{t-1,T_{l-1}+1}$ is chosen from $\mathcal{N}^{(0)}_{t-1,l-1}$ and the edge $\langle w_t, \eta_{t-1,T_{l-1}+1}\rangle$ is open, we say that $\eta_{t-1,T_{l-1}+1}$ *is good*, and let $\eta_{t-1,T_{l-1}+1}$ change to active. Otherwise we say that $\eta_{t-1,T_{l-1}+1}$ *is bad*, and stop the process here.

Next, for $x \geq 1$, suppose that we have distinct x good vertices $\{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_{l-1}+x}\}$. Then we check one directed edge from $\eta_{t-1,T_{l-1}+x}$, and write $\eta_{t-1,T_{l-1}+x+1}$ for the head of this edge. We declare that the edge $(\eta_{t-1,T_{l-1}+x}, \eta_{t-1,T_{l-1}+x+1})$ is checked. If $\eta_{t-1,T_{l-1}+x+1}$ is chosen from $\mathcal{N}^{(0)}_{t-1,l-1}$ and the edge $\langle \eta_{t-1,T_{l-1}+x}, \eta_{t-1,T_{l-1}+x+1}\rangle$ is open, we say that $\eta_{t-1,T_{l-1}+x+1}$ is good, and let $\eta_{t-1,T_{l-1}+x+1}$ change to active. Otherwise we say that $\eta_{t-1,T_{l-1}+x+1}$ is bad, and stop the process here. We continue it until we get to a bad vertex, where we stop this process.

By this construction, new good vertices $\{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_l-1}\}$ at the step $(t-1,l)$ are in $\mathcal{A}^{(1)}_{t-1,l}$.

We consider the state of the bad vertex and renew the states of vertices. Let $\{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_{l-1}+x}\}$ be vertices obtained by the above procedure. Then $T_l = T_{l-1} + x$, $\eta_{t-1,T_{l-1}+y}$ is good for $y < x$ and $\eta_{t-1,T_{l-1}+x}$ is bad.

a) If η_{t-1,T_l} is chosen from $\mathcal{A}_{t-1,l-1} \cup \{w_t\} \cup \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_l-1}\}$, or η_{t-1,T_l} is chosen from $\mathcal{N}_{t-1,l-1}$ and the edge $\langle \eta_{t-1,T_l-1}, \eta_{t-1,T_l}\rangle$ is closed, then we put

$$\begin{aligned}
\mathcal{N}^{(0)}_{t-1,l} &= \mathcal{N}^{(0)}_{t-1,l-1} \setminus \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_l-1}\},\\
\mathcal{N}^{(i)}_{t-1,l} &= \mathcal{N}^{(i)}_{t-1,l-1} \quad \text{for} \quad 1 \leq i \leq k,\\
\mathcal{A}^{(1)}_{t-1,l} &= \mathcal{A}^{(1)}_{t-1,l-1} \cup \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_l-1}\},\\
\mathcal{A}^{(i)}_{t-1,l} &= \mathcal{A}^{(i)}_{t-1,l-1} \quad \text{for} \quad i \neq 1.
\end{aligned}$$

b) If η_{t-1,T_l} is chosen from $\mathcal{N}^{(1)}_{t-1,l-1}$ and the edge $\langle \eta_{t-1,T_l-1}, \eta_{t-1,T_l}\rangle$ is open, then we do the same thing as in the case a) except that we put

$$\begin{aligned}
\mathcal{N}^{(1)}_{t-1,l} &= \mathcal{N}^{(1)}_{t-1,l-1} \setminus \{\eta_{t-1,T_l}\},\\
\mathcal{A}^{(1)}_{t-1,l} &= \mathcal{A}^{(1)}_{t-1,l-1} \cup \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_l}\}.
\end{aligned}$$

c) If η_{t-1,T_l} is chosen from $\mathcal{N}^{(i)}_{t-1,l-1}$ for $2 \leq i \leq k$ and the edge $\langle \eta_{t-1,T_l-1}, \eta_{t-1,T_l}\rangle$ is open, then we do the same thing as a) except that we put

$$\begin{aligned}
\mathcal{N}^{(i)}_{t-1,l} &= \mathcal{N}^{(i)}_{t-1,l-1} \setminus \{\eta_{t-1,T_l}\},\\
\mathcal{A}^{(i)}_{t-1,l} &= \mathcal{A}^{(i)}_{t-1,l-1} \cup \{\eta_{t-1,T_l}\}.
\end{aligned}$$

When all the above operations are over, the AS process $\rho_{t-1,l}$ is finished.

The step $(t-1, k+1)$; directed edge to w_t. For $v \in \mathcal{N}^{(i)}_{t-1,k}$, if there is an unchecked edge (v, w_t) in $\vec{G}_{\text{out}}(n, k)$, then we declare that the edge (v, w_t) is checked. Further, if the non-directed edge $\langle v, w_t \rangle$ is open, then $v \in \mathcal{A}^{(i+1)}_{t-1,k+1}$, if this edge is closed, then $v \in \mathcal{N}^{(i+1)}_{t-1,k+1}$. If there is not such an edge, then $v \in \mathcal{N}^{(i)}_{t-1,k+1}$. For $v \in \mathcal{A}^{(i)}_{t-1,k}$, if there is an unchecked edge (v, w_t) in $\vec{G}_{\text{out}}(n, k)$, then we declare (v, w_t) checked and $v \in \mathcal{A}^{(i+1)}_{t-1,k+1}$. If there is not such an edge, then $v \in \mathcal{A}^{(i)}_{t-1,k+1}$.

REMARK 1. For the directed edge (v, w_t) checked at the step $(t-1, k+1)$, there are cases when the revers edge (w_t, v) is checked before the step $(t-1, k)$. In this case, the non-directed edge $\langle w_t, v \rangle$ has already decided open or closed. So, the state of this edge must not change. This fact influences the calculation of the exploration process. We solve this problem in Remark 3.

When all the above steps are over, we put $\mathcal{E}_t = \mathcal{E}_{t-1} \cup \{w_t\}$, $\mathcal{N}^{(i)}_t = \mathcal{N}^{(i)}_{t-1,k+1}$, $\mathcal{A}^{(i)}_t = \mathcal{A}^{(i)}_{t-1,k+1}$ for $0 \le i \le k$, and the exploration of w_t is finished.

3.2. The stochastic process characterizing the exploration process. We define ξ_t by

$$\xi_t = A_t - A_{t-1,1} - 1. \tag{3}$$

Namely, $\xi_t + 1$ is the number of vertices changing their state from neutral to active, from step $(t-1, 2)$ to step $(t-1, k+1)$.

Next we define $N(w_t)$ by $N(w_t) = A_{t-1,1} - A_{t-1,0} + \mathbf{1}_{\{w_t \in \mathcal{N}_{t-1}\}}$. Roughly $N(w_t)$ is the number of vertices changing their state from neutral to active, during the step $(t-1, 1)$ (including w_t). Thus $A_t = A_{t-1} + N(w_t) + \xi_t$.

The process $N(w_t)$ is positive exactly when $A_{t-1} = 0$, and A_t can be written as $A_t = \sum_{s=1}^t N(w_s) + \sum_{s=1}^t \xi_s$. We will investigate the behavior of

$$X_t = \sum_{s=1}^t \xi_s.$$

REMARK 2. By the above construction, $\mathcal{A}^{(0)}_t = \mathcal{A}^{(0)}_{t,l} = \emptyset$ for any t and l.

By definition, we have

$$X_t = A_t - Z_t, \tag{4}$$

where Z_t is a non-decreasing process given by

$$Z_t = \sum_{s=1}^t N(w_s).$$

Let $0 = t_0 < t_1 < \cdots$ be the times at which A_t vanishes. Then $Z_t = Z_{t_j+1}$ for all $t \in \{t_j + 1, \dots, t_{j+1}\}$. Since $X_{t_j} = -Z_{t_j}$, we have $X_{t_{j+1}} = -Z_{t_{j+1}} = -Z_t < X_t$ for all $t \in \{t_j + 1, \dots, t_{j+1} - 1\}$. Therefore each t_j is one of renewal times of the process $\min_{0 \le s \le t} X_s$.

4. Proofs of Theorems 1, 3, and 4

For vertices u and v, we mean by $u \leftarrow v$ that there is a directed edge (v, u) in $\vec{G}_{\text{out}}(n, k)$, and we mean by $\{\eta_{t-1,T_l} \in \mathcal{N}^{(i)}_{t-1,l-1}, \text{ open}\}$ that $\eta_{t-1,T_l} \in \mathcal{N}^{(i)}_{t-1,l-1}$ and the edge $\langle \eta_{t-1,T_l-1}, \eta_{t-1,T_l} \rangle$ is open (if $\eta_{t-1,T_l-1} = \eta_{t-1,T_{l-1}}$, then $\langle w_t, \eta_{t-1,T_l} \rangle$ is open). Also, $\{w_t \leftarrow v, \text{ closed}\}$ means that $w_t \leftarrow v$ and the edge $\langle w_t, v \rangle$ is closed. $r_{t-1,l}$ denotes the number of good vertices in the AS process $\rho_{t-1,l}$. Let

$$r'_{t-1,l} = r_{t-1,l} + \mathbf{1}_{\{\eta_{t-1,T_l} \in \mathcal{N}^{(\ge 1)}_{t-1,l-1}, \text{ open}\}},$$

i.e., it is the number of new active vertices in $\rho_{t-1,l}$.

Assume that $w_t \in \mathcal{N}^{(i)}_{t-1} \cup \mathcal{A}^{(i)}_{t-1}$ for some $1 \le i \le k$. Then we introduce fictitious AS processes $\hat{\rho}_{t-1,l}$ for $l = 1, \dots, i$, which are independent copies of an AS process with data $\{\mathcal{N}^{(j)}_{t-1,l-1} \cup \mathcal{A}^{(j)}_{t-1,l-1}\}$, $0 \le j \le k$, which always executes, and $\mathcal{N}^{(j)}_{t-1,l}$, $\mathcal{A}^{(j)}_{t-1,l}$ are unchanged from $\mathcal{N}^{(j)}_{t-1,l-1}$, $\mathcal{A}^{(j)}_{t-1,l-1}$. Let $\hat{r}_{t-1,l}$ and $\hat{r}'_{t-1,l}$ be the number of good vertices in $\hat{\rho}_{t-1,l}$ and the number of new active vertices in $\hat{\rho}_{t-1,l}$. For $l \ge i+1$, we put $\hat{\rho}_{t-1,l}$ as the l-th AS process as before. Therefore $\hat{r}_{t-1,l} = r_{t-1,l}$ and $\hat{r}'_{t-1,l} = r'_{t-1,l}$ for $l \ge i + 1$. By (3), ξ_t also has the following expression.

$$\begin{aligned}
\xi_t &= \sum_{l=2}^{k} \mathbf{1}_{\{w_t \in \mathcal{N}^{(\le l-1)}_{t-1} \cup \mathcal{A}^{(\le l-1)}_{t-1}\}} \hat{r}'_{t-1,l} + \sum_{v \in \mathcal{N}_{t-1,k}} \mathbf{1}_{\{w_t \leftarrow v, \text{ open}\}} - 1 \\
&= \sum_{l=2}^{k} \hat{r}'_{t-1,l} + \sum_{v \in \mathcal{N}_{t-1,k}} \mathbf{1}_{\{w_t \leftarrow v, \text{ open}\}} - 1 \\
&\quad - \sum_{j=2}^{k} \big\{ \mathbf{1}_{\{A^{(j)}_{t-1} > 0, A^{(\ge j+1)}_{t-1} = 0\}} + \mathbf{1}_{\{A_{t-1} = 0, w_t \in \mathcal{N}^{(j)}_{t-1}\}} \big\} \sum_{l=2}^{j} \hat{r}'_{t-1,l}.
\end{aligned}$$

Let $\mathcal{F}_t$ be the σ-algebra generated by all the information up to time t, and let $\mathcal{F}_{t-1,l} \supset \mathcal{F}_{t-1}$ be the σ-algebra generated by all the information up to the end of the step $(t-1, l)$.

Hereafter, we use the following notation for $f_n = f_n(\omega)$ and $g_n = g_n(\omega)$. $f_n = o(g_n)$ if there is Ω_0 such that $\mathbb{P}(\Omega_0) = 1$ and $\lim_{n\to\infty} \sup_{\omega \in \Omega_0} f_n/g_n = 0$. $f_n = O(g_n)$ if there exists $C > 0$ such that $|f_n| \le C|g_n|$ for any n and almost every ω.

We use following notations for some distributions. Let $Be(p)$, $Ge(p)$, $Bin(n; p)$ and $NB(k; p)$ denote distributions of the Bernoulli distribution with the success probability p, the geometric distribution with the success probability p starting 1, the binomial distribution with the population n and the success probability p, and the negative binomial distribution with the number of failures k and the success probability p, respectively.

Next, we introduce an abbreviated notation,

$$X \sim NB(n;\ p) + Bin(m;\ q) + c,$$

to mean that $X = X_1 + X_2 + c$ for independent random variables X_1 and X_2 and a constant c such that $X_1 \sim NB(n;\ p)$ and $X_2 \sim Bin(m;\ q)$.

4.1. Fundamental lemmas.

Lemma 1. *Let $\{\alpha_{t-1,l}\}$ be independent random variables distributed as*

$$\alpha_{t-1,l} \sim Ge(1-p) \quad (1 \le l \le k), \tag{5}$$

$$\alpha_{t-1,k+1} \sim \begin{cases} Bin\left(n-t;\ \dfrac{kp}{n-t}\right) & (t \le n-1), \\ Be(kp) & (t = n) \end{cases} \tag{6}$$

for each $1 \le t \le n$ and $1 \le l \le k+1$. Then we can couple $\{\hat{r}'_{t-1,l}\}$ and $\{\alpha_{t-1,l}\}$ such that almost surely $\hat{r}'_{t-1,l} \le \alpha_{t-1,l}$ and $\alpha_{t-1,l}$ is independent of $\mathcal{F}_{t-1,l-1}$ for each $1 \le t \le n$ and $1 \le l \le k$. Also we can couple $\left\{\sum_{v \in \mathcal{N}_{t-1,k}} \mathbf{1}_{\{w_t \leftarrow v,\ open\}}\right\}$ and $\{\alpha_{t-1,k+1}\}$ such that almost surely $\sum_{v \in \mathcal{N}_{t-1,k}} \mathbf{1}_{\{w_t \leftarrow v,\ open\}} \le \alpha_{t-1,k+1}$ and $\alpha_{t-1,k+1}$ is independent of $\mathcal{F}_{t-1,k}$ for each $1 \le t \le n$. Further, let $\alpha_t = \sum_{l=2}^{k} \alpha_{t-1,l} + \alpha_{t-1,k+1} - 1$. Then α_t is distributed as

$$\alpha_s \sim \begin{cases} NB(k-1;\ 1-p) + Bin\left(n-t;\ \dfrac{kp}{n-t}\right) - 1 & (t \le n-1), \\ NB(k-1;\ 1-p) + Be(kp) - 1 & (t = n) \end{cases} \tag{7}$$

for each $1 \le t \le n$, and we can couple $\{\xi_t\}$ and $\{\alpha_t\}$ such that almost surely $\xi_t \le \alpha_t$ for each t and α_t is independent of $\mathcal{F}_{t-1}$.

Proof. We have for each $d \ge 1$,

$$\begin{aligned} \mathbb{P}(\hat{r}'_{t-1,l} \ge d \mid \mathcal{F}_{t-1,l-1}) &\le \mathbb{P}(\{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_{l-1}+d-1} \text{ are good}, \\ &\qquad \langle \eta_{t-1,T_{l-1}+d-1}, \eta_{t-1,T_{l-1}+d} \rangle \text{ is open} \mid \mathcal{F}_{t-1,l-1}) \\ &\le p^d. \end{aligned}$$

So we can couple $\hat{r}'_{t-1,l}$ and $\alpha_{t-1,l}$ such that almost surely $\hat{r}'_{t-1,l} \le \alpha_{t-1,l}$.

Other facts are trivial. □

Lemma 2. *Let $\delta \in [0,1)$. For any $0 \le t \le \delta n$, there exist positive constants $C_1 = C_1(k)$ and $C_2 = C_2(k)$,*

$$\mathbb{E}\left[N_t^{(\ge 1)} + A_t\right] \le C_1 t, \quad \mathbb{E}\left[\left(N_t^{(\ge 1)} + A_t\right)^2\right] \le C_2 t^2.$$

Proof. Let $\zeta_t = \big(N_t^{(\geq 1)} + A_t\big) - \big(N_{t-1}^{(\geq 1)} + A_{t-1}\big)$. Also, let $\bar{\zeta}_t$ be a random variable distributed as $NB(k; 1-p) + Bin(n-t; k/(n-t))$, where $\bar{\zeta}_t$ is independent of $\mathcal{F}_{t-1}$. By Lemma 1, we can couple ζ_t and $\bar{\zeta}_t$, such that almost surely $\zeta_t \leq \bar{\zeta}_t$ for all $t \leq \delta n$. Hence we get some constant $C_1 = C_1(k)$ depending on k,

$$\mathbb{E}[N_t^{(\geq 1)} + A_t] = \sum_{s=1}^{t} \mathbb{E}\zeta_s \leq \sum_{s=1}^{t} \mathbb{E}\bar{\zeta}_s \leq \sum_{s=1}^{t} \left\{ \frac{kp}{1-p} + k \right\} \leq C_1 t.$$

Also there exists $C_2 = C_2(k)$ depending on k,

$$\begin{aligned} \mathbb{E}\big[\big(N_t^{(\geq 1)} + A_t\big)^2\big] &= \mathbb{E}\big[\big(N_{t-1}^{(\geq 1)} + A_{t-1} + \zeta_t\big)^2\big] \\ &\leq \mathbb{E}\big[\big(N_{t-1}^{(\geq 1)} + A_{t-1}\big)^2\big] + 2C_1^2 t + C_3 \leq C_2 t^2, \end{aligned}$$

where $C_3 = C_3(k)$ depends on k. □

Lemma 3. *Let $0 \leq p < 1$ and $m = m(n) \leq n$ be a sequence such that $m(n) \to \infty$ as $n \to \infty$. Then*

$$\begin{aligned} \mathbb{E}[\xi_t \mid \mathcal{F}_{t-1}] = {} & (k-1)\frac{p}{1-p} + kp - 1 - \sum_{j=2}^{k} F_j \frac{N_{t-1}^{(j)}}{n-t} - G\frac{A_{t-1}}{n-t} \\ & - \sum_{j=2}^{k} \big\{ \mathbf{1}_{\{A_{t-1}^{(j)} > 0, A_{t-1}^{(\geq j+1)} = 0\}} + \mathbf{1}_{\{A_{t-1} = 0, w_t \in \mathcal{N}_{t-1}^{(j)}\}} \big\} \\ & \times \left\{ (j-1)\frac{p}{1-p} - \frac{(j-1)p^2}{(1-p)^2}\frac{N_{t-1}^{(\geq 1)}}{n-t} - \frac{(j-1)p}{(1-p)^2}\frac{A_{t-1}}{n-t} \right\} \\ & + \frac{\big(N_{t-1}^{(\geq 1)} + A_{t-1}\big)^2}{(n-t)^2} O(1) + O\left(\frac{m^{1/3}}{n-t} + m^{1/3} p^{m^{1/3}} \right) \\ & + \mathbb{E}\big[\mathbf{1}_{\{N_t^{(0)} < m^{1/3}\}} O(1) \,\big|\, \mathcal{F}_{t-1}\big], \end{aligned} \tag{8}$$

where $F_j = (k-1)p^2/(1-p)^2 + jp$ and $G = (k-1)p/(1-p)^2 + kp$. Furthermore, let

$$D_k = 2(k - \sqrt{k^2 - k}).$$

If $p = (1+\epsilon)/(k + \sqrt{k^2 - k})$ for some $\epsilon = o(1)$, then

$$(k-1)\frac{p}{1-p} + kp - 1 = D_k \epsilon + O(\epsilon^2). \tag{9}$$

REMARK 3. (i) Suppose that w_t is a good vertex of the AS process $\rho_{s-1,l}$ for some $s < t$ and some $1 \leq l \leq k$. Namely, there exist $s < t$, $1 \leq l \leq k$ and $y \in \mathbb{N}$

such that $w_t = \eta_{s-1,T_{l-1}+y}$ and $T_{l-1} + y < T_l$. Then we have checked the directed edge $(\eta_{s-1,T_{l-1}+y}, \eta_{s-1,T_{l-1}+y+1})$. Hence we have to exclude $\eta_{s-1,T_{l-1}+y+1}$ when we check other directed edges from w_t $(= \eta_{s-1,T_{l-1}+y})$. Furthermore, when we execute the l-th AS process $\rho_{t-1,l}$, $\eta_{t-1,T_{l-1}+1}$ is chosen from $\mathcal{N}_{t-1,l-1} \cup \mathcal{A}_{t-1,l-1}$ outside $\{\eta_{t-1,T_0+1}, \eta_{t-1,T_1+1}, \dots, \eta_{t-1,T_{l-2}+1}\}$. However, if we take conditional expectation without considering these things, then the error is $O(1/(n-t))$.
(ii) Suppose that w_t appears in the AS process $\rho_{s-1,l}$ for some $s < t$ and some $1 \le l \le k$. Namely, there exist $s < t$, $1 \le l \le k$ and $y \in \mathbb{N}$ such that $w_t = \eta_{s-1,T_{l-1}+y}$ and $T_{l-1} + y \le T_l$. When $y = 1$, we understand that $\eta_{s-1,T_{l-1}+y-1} = w_s$. Then we have already decided the edge $\langle \eta_{s-1,T_{l-1}+y}, \eta_{s-1,T_{l-1}+y-1} \rangle$ to be open or closed at time s. However, since $\eta_{s-1,T_{l-1}+y-1}$ is not neutral, so we don't have to exclude $\eta_{s-1,T_{l-1}+y-1}$ when we execute AS processes at time t.
(iii) Suppose that there exist $s \le t$, $1 \le l \le k$ and $y \in \mathbb{N} \cup \{0\}$ such that $w_t = \eta_{s-1,T_{l-1}+y}$ and $T_{l-1}+y < T_l$, where we understand $t = s$ if $y = 0$. We consider the step $(t-1,k+1)$. If $\{w_t \leftarrow \eta_{s-1,T_{l-1}+y+1}\}$ occurs, then we have already decided the edge $\langle w_t, \eta_{s-1,T_l} \rangle$ is open or closed at time s. However, similarly to (i), if we take conditional expectation without considering these things, then the error is $O(1/(n-t))$.

Proof of Lemma 3. First we consider steps $(t-1, l)$ for $2 \le l \le k$. Note that

$$\begin{aligned}\mathbf{1}_{\{\hat{r}'_{t-1,l}=x\}} &= \mathbf{1}_{\{\hat{r}_{t-1,l}=x-1\}} \mathbf{1}_{\{\eta_{t-1,T_l} \in \mathcal{N}^{(\ge 1)}_{t-1,l-1},\ \text{open}\}} \\ &\quad + \mathbf{1}_{\{\hat{r}_{t-1,l}=x\}} \big[\mathbf{1}_{\{\eta_{t-1,T_l} \in \mathcal{N}_{t-1,l-1} \setminus \{\eta_{t-1,T_{l-1}+1},\dots,\eta_{t-1,T_l-1}\},\ \text{closed}\}} \\ &\qquad + \mathbf{1}_{\{\eta_{t-1,T_l} \in \mathcal{A}_{t-1,l-1} \cup \{w_t\} \cup \{\eta_{t-1,T_{l-1}+1},\dots,\eta_{t-1,T_l-1}\}\}} \big].\end{aligned} \tag{10}$$

By (i) and (ii) of Remark 3,

$$\mathbb{P}(\eta_{t-1,T_{l-1}+1} \text{ is good} \mid \mathcal{F}_{t-1,l-1}) = \frac{N^{(0)}_{t-1,l-1}}{n-t} p + O\left(\frac{1}{n-t}\right),$$

$$\mathbb{P}(\eta_{t-1,T_{l-1}+1} \text{ is bad and changes to active from neutral} \mid \mathcal{F}_{t-1,l-1}) = \frac{N^{(\ge 1)}_{t-1,l-1}}{n-t} p + O\left(\frac{1}{n-t}\right),$$

$$\mathbb{P}(\eta_{t-1,T_{l-1}+1} \text{ is bad and doesn't change its state} \mid \mathcal{F}_{t-1,l-1}) = \frac{N_{t-1,l-1}}{n-t}(1-p) + \frac{A_{t-1,l-1}}{n-t} + O\left(\frac{1}{n-t}\right).$$

Further for $y \ge 2$,

$$\mathbb{P}(\eta_{t-1,T_{l-1}+y} \text{ is good} \mid \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_{l-1}+y-1}\} \text{ are good}) = \frac{N^{(0)}_{t-1,l-1} - (y-1)}{n-t} p,$$

$$\begin{aligned}
&\mathbb{P}(\eta_{t-1,T_{l-1}+y} \text{ is bad and changes to active from neutral} \mid \\
&\quad \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_{l-1}+y-1}\} \text{ are good}) \\
&= \frac{N^{(\geq 1)}_{t-1,l-1}}{n-t} p, \\
&\mathbb{P}(\eta_{t-1,T_{l-1}+y} \text{ is bad and doesn't change its state} \mid \\
&\quad \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_{l-1}+y-1}\} \text{ are good}) \\
&= \frac{N_{t-1,l-1} - (y-1)}{n-t}(1-p) + \frac{A_{t-1,l-1} + (y-1) + 1}{n-t}.
\end{aligned}$$

Therefore, from (10),

$$\begin{aligned}
&\mathbb{E}[\hat{r}'_{t-1,l} \mid \mathcal{F}_{t-1,l-1}] = \mathbb{E}\left[\sum_{x=0}^{n} x \mathbf{1}_{\{\hat{r}'_{t-1,l}=x\}} \,\middle|\, \mathcal{F}_{t-1,l-1}\right] \\
&= \sum_{x=1}^{N^{(0)}_{t-1,l-1}} x \cdot \left[\prod_{y=0}^{x-2}\left\{\frac{N^{(0)}_{t-1,l-1} - y}{n-t} p + \mathbf{1}_{\{y=0\}} \cdot O\left(\frac{1}{n-t}\right)\right\}\left\{\frac{N^{(\geq 1)}_{t-1,l-1}}{n-t} p + O\left(\frac{1}{n-t}\right)\right\}\right. \\
&\qquad + \prod_{y=0}^{x-1}\left\{\frac{N^{(0)}_{t-1,l-1} - y}{n-t} p + \mathbf{1}_{\{y=0\}} \cdot O\left(\frac{1}{n-t}\right)\right\} \\
&\qquad \left.\times \left\{\frac{N_{t-1,l-1} - x}{n-t}(1-p) + \frac{A_{t-1,l-1} + x}{n-t} + O\left(\frac{1}{n-t}\right)\right\}\right] \\
&= \frac{N_{t-1,l-1}}{n-t} p \left(1 - \frac{N^{(0)}_{t-1,l-1}}{n-t} p\right) \sum_{x=1}^{N^{(0)}_{t-1,l-1}} x \prod_{y=0}^{x-2}\left\{\frac{N^{(0)}_{t-1,l-1} - y}{n-t} p\right\} + O\left(\frac{1}{n-t}\right),
\end{aligned}$$

where we understand that $\prod_{x=0}^{-1}\{\,\cdot\,\} = 1$. Here

$$\begin{aligned}
&\sum_{x=1}^{N^{(0)}_{t-1,l-1}} x \prod_{y=0}^{x-2}\left\{\frac{N^{(0)}_{t-1,l-1} - y}{n-t} p\right\} \\
&= \mathbf{1}_{\{N^{(0)}_{t-1,l-1} \geq m^{1/3}\}} \left[\sum_{x=1}^{\lfloor m^{1/3}\rfloor} x \prod_{y=0}^{x-2}\left\{\frac{N^{(0)}_{t-1,l-1} - y}{n-t} p\right\} + \sum_{x=\lfloor m^{1/3}\rfloor+1}^{N^{(0)}_{t-1,l-1}} x \prod_{y=0}^{x-2}\left\{\frac{N^{(0)}_{t-1,l-1} - y}{n-t} p\right\}\right] \\
&\qquad + \mathbf{1}_{\{N^{(0)}_{t-1,l-1} < m^{1/3}\}} \sum_{x=1}^{N^{(0)}_{t-1,l-1}} x \prod_{y=0}^{x-2}\left\{\frac{N^{(0)}_{t-1,l-1} - y}{n-t} p\right\} \\
&= \frac{1}{\left(1 - \left(N^{(0)}_{t-1,l-1}/(n-t)\right)p\right)^2} + O\left(\frac{m^{1/3}}{n-t} + m^{1/3} p^{m^{1/3}}\right) + \mathbf{1}_{\{N^{(0)}_{t-1,l-1} < m^{1/3}\}} O(1).
\end{aligned}$$

Thus

$$\mathbb{E}[\hat{r}'_{t-1,l} \mid \mathcal{F}_{t-1,l-1}] = \frac{N_{t-1,l-1}}{n-t} p \frac{1}{1-\big(N^{(0)}_{t-1,l-1}/(n-t)\big)p} + O\bigg(\frac{m^{1/3}}{n-t} + m^{1/3} p^{m^{1/3}}\bigg) + \mathbf{1}_{\{N^{(0)}_{t-1,l-1} < m^{1/3}\}} O(1).$$

By the Taylor expansion in $\big(N^{(\geq 1)}_{t-1,l-1} + A_{t-1,l-1}\big)/(n-t)$, the right hand side of the above equality is equal to

$$\frac{p}{1-p} - \frac{p^2}{(1-p)^2} \frac{N^{(\geq 1)}_{t-1,l-1}}{n-t} - \frac{p}{(1-p)^2} \frac{A_{t-1,l-1}}{n-t} + \frac{\big(N^{(\geq 1)}_{t-1,l-1} + A_{t-1,l-1}\big)^2}{(n-t)^2} O(1) + O\bigg(\frac{m^{1/3}}{n-t} + m^{1/3} p^{m^{1/3}}\bigg) + \mathbf{1}_{\{N^{(0)}_{t-1,l-1} < m^{1/3}\}} O(1).$$

By the proof of Lemma 2, we have

$$\mathbb{E}\bigg[\frac{N^{(\geq 1)}_{t-1,l-1} + A_{t-1,l-1}}{n-t} \;\bigg|\; \mathcal{F}_{t-1}\bigg] = \frac{N^{(\geq 1)}_{t-1} + A_{t-1}}{n-t} + O\bigg(\frac{1}{n-t}\bigg),$$

$$\mathbb{E}\bigg[\frac{\big(N^{(\geq 1)}_{t-1,l-1} + A_{t-1,l-1}\big)^2}{(n-t)^2} \;\bigg|\; \mathcal{F}_{t-1}\bigg] = \frac{\big(N^{(\geq 1)}_{t-1} + A_{t-1}\big)^2}{(n-t)^2} + O\bigg(\frac{1}{n-t}\bigg).$$

So we obtain that $\mathbb{E}\big[\sum_{l=2}^{k} \hat{r}'_{t-1,l} \;\big|\; \mathcal{F}_{t-1}\big]$ is equal to

$$\begin{aligned}(11)\qquad & (k-1)\frac{p}{1-p} - (k-1)\frac{p^2}{(1-p)^2} \frac{N^{(\geq 1)}_{t-1}}{n-t} - (k-1)\frac{p}{(1-p)^2} \frac{A_{t-1}}{n-t} \\ & + \frac{\big(N^{(\geq 1)}_{t-1} + A_{t-1}\big)^2}{(n-t)^2} O(1) + O\bigg(\frac{m^{1/3}}{n-t} + m^{1/3} p^{m^{1/3}}\bigg) \\ & + \mathbb{E}\big[\mathbf{1}_{\{N^{(0)}_t < m^{1/3}\}} O(1) \mid \mathcal{F}_{t-1}\big],\end{aligned}$$

where we used that $N^{(0)}_{t-1,l-1} \geq N^{(0)}_t$.

Next, we consider the step $(t-1, k+1)$. By (iii) of Remark 3,

$$\begin{aligned}\mathbb{E}\bigg[\sum_{v \in \mathcal{N}_{t-1,k}} \mathbf{1}_{\{w_t \leftarrow v,\ \text{open}\}} \;\bigg|\; \mathcal{F}_{t-1,k}\bigg] &= \sum_{j=0}^{k} N^{(j)}_{t-1,k} \frac{(k-j)p}{n-t} + O\bigg(\frac{1}{n-t}\bigg) \\ &= kp - \sum_{j=1}^{k} jp \frac{N^{(j)}_{t-1,k}}{n-t} - kp \frac{A_{t-1,k}}{n-t} + O\bigg(\frac{1}{n-t}\bigg),\end{aligned}$$

hence we have

$$\mathbb{E}\left[\sum_{v\in\mathcal{N}_{t-1,k}} \mathbf{1}_{\{w_t\leftarrow v,\ \text{open}\}} \,\middle|\, \mathcal{F}_{t-1}\right] = kp - \sum_{j=1}^{k} jp\frac{N_{t-1}^{(j)}}{n-t} - kp\frac{A_{t-1}}{n-t} + O\left(\frac{1}{n-t}\right). \tag{12}$$

Combining (11) and (12), we have (8).

(9) is trivial. □

Lemma 4. *Let* $\delta \in (0, 1)$ *and* $p < 1$.

If $a > k(p/(1-p) + 1/(1-\delta))$, *then there exists some constant* $\bar{I}(a) > 0$ *such that*

$$\mathbb{P}(N_t^{(0)} < n - t - at) \le e^{-\bar{I}(a)t} \tag{13}$$

for any $t \le \delta n$. *In particular, if* $p < 1/3$ *and* $\delta < (1-2p)/(2-3p)$, *then we can take* $a = 2k$ *in* (13).

If $b < (k-1-k\delta)/(1+k\delta)$ *for some* $\delta > 0$, *then there exists some constant* $\underline{I}(b) > 0$ *such that*

$$\mathbb{P}(N_t^{(0)} > n - t - bt) \le e^{-\underline{I}(b)t} \tag{14}$$

for any $t \le \delta n$. *In particular, if* $\delta < (2k-3)/(k(k+3))$, *then we can take* $b = k/3$ *in* (14).

Proof. Take an $s \le t$. Recall ζ_s in the proof of Lemma 2 and let $\{\bar{X}_s\}$ be i.i.d. random variables distributed as $NB(k;\, 1-p) + Bin(n;\, k/((1-\delta)n))$ for each $s \le t$. We couple ζ_s and $\bar{X}_s$ for each $s \le t$ such that almost surely $\zeta_s \le \bar{X}_s$. Therefore (13) follows from usual large deviation estimates for $\bar{X}$ (for usual large deviation estimates, we refer the reader to the Section 1.9 of [10].).

Next, let $\underline{X}_s$ be i.i.d. random variables distributed as $Bin(n-(1+b)\delta n;\, k/n) - 1$. We can couple ζ_s and $\underline{X}_s$ such that $\zeta_s \ge \underline{X}_s$ on $\{N_{s-1,k}^{(0)} > n-(1+b)\delta n\}$ for each $s \le t$, and $\underline{X}_s$ is independent of $\mathcal{F}_{s-1,k}$. Since $N_{s,l}^{(0)}$ is decreasing for each time and step, and

$$\mathbb{P}(N_t^{(0)} > n - t - bt) \le \mathbb{E}\left[\mathbf{1}_{\{N_t^{(\ge 1)} + A_t < bt\}} \mathbf{1}_{\{N_t^{(0)} > n-(1+b)\delta n\}}\right],$$

we obtain (14) from usual large deviation estimates for $\underline{X}$. □

4.2. Proof of Theorem 1. Let $\{\alpha_s\}$ be defined by (7), where we extend the definition for $s > n$ to be independent copies of α_n. Let $W_t = d + \sum_{s=1}^{t} \alpha_s$, where d is a fixed positive integer with $d < n^{1/3}$. Let $h = n^{1/3}$ and

$$\gamma_h = \min\{t \colon W_t = 0 \text{ or } W_t \ge h\}.$$

When $s \le n-1$, expanding in c around $c=0$, we obtain that

$$\log \mathbb{E}e^{-c\alpha_s} = c\left\{-(k-1)\frac{p}{1-p} - kp + 1\right\} + \frac{c^2}{2}\left\{(k-1)\frac{p}{1-p} + (k-1)\left(\frac{p}{1-p}\right)^2 + kp - \frac{k^2p^2}{n-s}\right\} + O(c^3).$$

Writing $\epsilon = \lambda n^{-1/3}$, from (9) we can see that the right hand side of the above equality is equal to

$$-D_k c\epsilon + \frac{c^2}{2}\left\{(k-1)\frac{p}{1-p} + (k-1)\left(\frac{p}{1-p}\right)^2 + kp - \frac{k^2p^2}{n-s}\right\} + O(c^3 + c\epsilon^2). \tag{15}$$

When $s \ge n$, we get the same equality, except that $k^2p^2/(n-s)$ in the second term of the right hand side of (15) is replaced with k^2p^2.

Using these facts, we first prove that for a large enough n,

$$\mathbb{P}(W_{\gamma_h} > 0) \le \begin{cases} \dfrac{4d\lambda}{1-e^{-4\lambda}} n^{-1/3} & \lambda > 0, \\ \dfrac{-2d\lambda}{e^{-\lambda}-1} n^{-1/3} & \lambda < 0, \\ dn^{-1/3} & \lambda = 0. \end{cases} \tag{16}$$

When $\lambda > 0$, we put $c = 4\epsilon$ and we have $-D_k c\epsilon = -8(k - \sqrt{k^2-k})\epsilon^2$. Since $p = 1/(k + \sqrt{k^2-k}) + O(\epsilon)$, by (15) and the remark after (15), we can see that

$$\log \mathbb{E}e^{-c\alpha_s} \ge 8(-kp + kp^2 - k^2p^2 + 1)\epsilon^2 + O(\epsilon^3).$$

Noting that $p \le 1/(2k-1)$, the right hand side of the above inequality is positive for a small ϵ and this implies that e^{-cW_s} is a submartingale. By the optional stopping theorem we have

$$e^{-cd} = \mathbb{E}e^{-cW_0} \le \mathbb{E}e^{-cW_{\gamma_h}} \le e^{-cn^{1/3}}\mathbb{P}(W_{\gamma_h} > 0) + \mathbb{P}(W_{\gamma_h} = 0).$$

This proves the first part of (16).

When $\lambda < 0$, we put $c = -\epsilon$. By a similar argument we can see that e^{cW_s} is a supermartingale. Also when $\lambda = 0$, W_s is a martingale. By the optional stopping theorem we obtain the second and the third part of (16).

Lemma 5. *For a large enough n,*

$$\mathbb{E}W_{\gamma_h} \le (3kn^{1/3} + 1)\mathbb{P}(W_{\gamma_h} > 0), \tag{17}$$

$$\mathbb{E}W_{\gamma_h}^2 \le (9k^2n^{2/3} + 1)\mathbb{P}(W_{\gamma_h} > 0). \tag{18}$$

Proof. Since $\mathbb{E}W_{\gamma_h} = \mathbb{E}[W_{\gamma_h}; W_{\gamma_h} > 0]$, noting that $W_{\gamma_h} = W_{\gamma_{h-1}} + \alpha_{\gamma_h}$, and that $W_{\gamma_{h-1}} \le n^{1/3}$, we can see that $\mathbb{E}W_{\gamma_h}$ is not larger than

$$n^{1/3}\mathbb{P}(W_{\gamma_h} > 0) + (3k-1)n^{1/3}\mathbb{P}(W_{\gamma_h} > 0) + \mathbb{E}[\alpha_{\gamma_h}; \alpha_{\gamma_h} > (3k-1)n^{1/3}].$$

To prove (17), we will show that the third term is not larger than $\mathbb{P}(W_{\gamma_h} > 0)$ for a large enough n. In fact,

$$\begin{aligned}
&\mathbb{E}[\alpha_{\gamma_h}; \alpha_{\gamma_h} > (3k-1)n^{1/3}] \\
&\le \sum_{x=1}^{n^{1/3}-1} \sum_{s=1}^{\infty} \mathbb{E}[\alpha_s \mathbf{1}_{\{\alpha_s > (3k-1)n^{1/3}\}} \mid 0 < W_1, \ldots, W_{s-2} < n^{1/3}, \\
&\qquad\qquad W_{s-1} = x,\ \alpha_s \ge n^{1/3} - x] \\
&\qquad \times \mathbb{P}(0 < W_1, \ldots, W_{s-2} < n^{1/3},\ W_{s-1} = x,\ \alpha_s \ge n^{1/3} - x) \qquad (19)\\
&\le \sum_{s=1}^{\infty} \frac{\mathbb{E}[\alpha_s \mathbf{1}_{\{\alpha_s > (3k-1)n^{1/3}\}}]}{\mathbb{P}(\alpha_s \ge n^{1/3})} \mathbb{P}(\gamma_h = s,\ W_s > 0) \\
&\le (3k-1)n^{1/3} \sum_{s=1}^{\infty} \sum_{j=1}^{\infty} (j+1) \frac{\mathbb{P}(\alpha_s \ge (3k-1)jn^{1/3})}{\mathbb{P}(\alpha_s \ge n^{1/3})} \mathbb{P}(\gamma_h = s,\ W_s > 0).
\end{aligned}$$

By (5)–(7), since $\alpha_s \ge \alpha_{s-1,2}$, the denominator in the right hand side of the above inequality is not less than $\mathbb{P}(\alpha_{s-1,2} \ge n^{1/3}) = p^{n^{1/3}}$.

If $\alpha_s \ge (3k-1)jn^{1/3}$, then either $\alpha_{s-1,l} \ge 2jn^{1/3}$ for some $2 \le l \le k$, or $\alpha_{s-1,k+1} \ge (k+1)jn^{1/3}$. Therefore we have

$$\mathbb{P}(\alpha_s \ge (3k-1)jn^{1/3}) \le (k-1)p^{2jn^{1/3}} + C_1(kp)^{(k+1)jn^{1/3}} \qquad (20)$$

for some constants $C_1 = C_1(k) > 0$. Since $k(kp)^k \le k(1/(2-1/k))^k \le 8/9$, we have

$$\frac{\mathbb{P}(\alpha_s \ge (3k-1)jn^{1/3})}{\mathbb{P}(\alpha_s \ge n^{1/3})} \le (k-1)p^{jn^{1/3}} + C_1\left(\frac{8}{9}\right)^{jn^{1/3}}. \qquad (21)$$

From (19) and (21), there is a constant $C_2 = C_2(k) > 0$ such that

$$\mathbb{E}[\alpha_{\gamma_h}; \alpha_{\gamma_h} > (3k-1)n^{1/3}] \le C_2 n^{1/3}\left\{p^{n^{1/3}} + \left(\frac{8}{9}\right)^{n^{1/3}}\right\}\mathbb{P}(W_{\gamma_h} > 0). \qquad (22)$$

To prove (18), we proceed as above. By $W_{\gamma_h} = W_{\gamma_h - 1} + \alpha_{\gamma_h}$ and (22),

$$\begin{aligned}
\mathbb{E}W_{\gamma_h}^2 &\le 9k^2 n^{2/3}\mathbb{P}(W_{\gamma_h} > 0) + 2n^{1/3} \cdot C_2 n^{1/3}\left\{p^{n^{1/3}} + \left(\frac{8}{9}\right)^{n^{1/3}}\right\}\mathbb{P}(W_{\gamma_h} > 0) \\
&\quad + \mathbb{E}[\alpha_{\gamma_h}^2; \alpha_{\gamma_h} > (3k-1)n^{1/3}].
\end{aligned}$$

By (21), it is easy to see that there is a constant $C_3 = C_3(k) > 0$ such that

$$\mathbb{E}[\alpha_{\gamma_h}^2; \alpha_{\gamma_h} > 3(k-1)n^{1/3}] \le C_3 n^{2/3} \left\{ p^{n^{1/3}} + \left(\frac{8}{9}\right)^{n^{1/3}} \right\} \mathbb{P}(W_{\gamma_h} > 0),$$

which proves (18) for a large enough n. □

Lemma 6. *For a large enough n,*

$$\mathbb{E}\gamma_h \le 100k^2 d n^{1/3}. \tag{23}$$

Proof. Assume first that $\lambda > 1/4$. By $\mathbb{E}\alpha_s = D_k \lambda n^{-1/3} + O(n^{-2/3})$ and $D_k > 1$, if n is large, then $\{W_t - t\lambda n^{-1/3}\}$ is a submartingale. By the optional stopping theorem and (17),

$$d \le \mathbb{E}W_{\gamma_h} - \lambda n^{-1/3}\mathbb{E}\gamma_h \le (3kn^{1/3} + 1)\mathbb{P}(W_{\gamma_h} > 0) - \lambda n^{-1/3}\mathbb{E}\gamma_h.$$

From (16), this shows that (23) is correct when $\lambda > 1/4$.

When n is large, it is easy to see that $\mathbb{P}(\alpha_s = 0) < 3/4$, and that $(\mathbb{E}\alpha_t)^2 = O(\epsilon^2)$. Hence $\{(W_t - \mathbb{E}W_t)^2 - (1/5)t\}$ is a submartingale, and by the optional stopping theorem we get

$$0 \le \mathbb{E}\left[(W_{\gamma_h} - \mathbb{E}W_{\gamma_h})^2 - \frac{1}{5}\gamma_h\right] = \mathbb{E}W_{\gamma_h}^2 - \frac{1}{5}\mathbb{E}\gamma_h.$$

Combining this with (16) and (18), we see that (23) is correct for $\lambda \le 1/4$, as well. □

Let v be a vertex in the component explored first, and let $\mathcal{C}(v)$ be the component including v. By definition t_1 is the first time X_t hits $-N(w_1)$, and $|\mathcal{C}(v)| = t_1$. Therefore $|\mathcal{C}(v)| \ge An^{2/3}$ implies that X_t never hits $-N(w_1)$ until $t = An^{2/3}$. Setting $W_0 = N(w_1)$, by our coupling we have $W_t \ge N(w_1) + X_t$ almost surely.

Following [2], we introduce $\gamma_h^* = \gamma_h \wedge n^{2/3}$, and note that if $|\mathcal{C}(v)| \ge An^{2/3}$ for some $A > 1$, then $W_{\gamma_h^*} > 0$ as well as $X_{\gamma_h^* + (A-1)n^{2/3}} > -N(w_1)$ almost surely. We shall estimate this probability.

By Lemma 1, $N(w_1)$ is stochastically dominated by a random variable distributed as $Ge(1-p)+1$. So the probability that $N(w_1) \ge n^{1/3}$ is bounded by $p^{n^{1/3}-1}$ from above.

Let $\mathcal{D}$ and $\mathcal{D}_t$ for $n^{1/3} - 1 \le t < \delta n$ be events given by

$$\mathcal{D}_t = \left\{ N_t^{(0)} \le n - t - \frac{k}{3}t \right\}, \quad \text{and} \quad \mathcal{D} = \bigcap_{n^{1/3}-1 \le t < \delta n} \mathcal{D}_t,$$

where $\delta > 0$ is a small constant. By Lemma 4, we have

$$\mathbb{P}(\mathcal{D}^C) \le \sum_{s=n^{1/3}-1}^{\delta n - 1} \mathbb{P}\left(N_s^{(0)} \ge n - s - \frac{k}{3}s \right) \le n e^{-\underline{I}(k/3)n^{1/3}}.$$

Therefore we can concentrate on the event $\mathcal{D} \cap \{N(w_1) < n^{1/3}\}$. For $c > 0$, we have

$$
\begin{aligned}
&\mathbb{P}(\mathcal{D} \cap \{N(w_1) < n^{1/3},\ W_{\gamma_h^*} > 0,\ X_{\gamma_h^*+(A-1)n^{2/3}} \geq -N(w_1)\}) \\
&\leq \sum_{d=1}^{n^{1/3}-1} \sum_{s=1}^{n^{2/3}} \mathbb{E}\big[e^{c \sum_{u=s+1}^{s+(A-1)n^{2/3}} \xi_u + cW_s + cd} \mathbf{1}_{\{\mathcal{D}\}};\ N(w_1) = d,\ \gamma_h^* = s,\ W_s > 0\big].
\end{aligned} \tag{24}
$$

To estimate $\mathbb{E}[e^{c\xi_u} \mid \mathcal{F}_{u-1}]$, we introduce another coupling. Let $\{w_t \overset{(l)}{\longleftarrow} v\}$ be the event that we choose uniformly l directed edges from $\{(v, v')\colon v' \notin \mathcal{E}_{t-1} \cup \{v\}\}$, and (v, w_t) is among these chosen edges. If $t \geq k - l$, then we understand that $\{w_t \overset{(l)}{\longleftarrow} v\}$ is equal to the total set Ω. Further we mean by $\{w_t \overset{(l)}{\longleftarrow} v,\ \text{open}\}$ the event that $\{w_t \overset{(l)}{\longleftarrow} v\}$ occurs and the edge $\langle v, w_t\rangle$ is open. Let $G_t(v)$ denote the family of events $\{\{w_t \overset{(l)}{\longleftarrow} v,\ \text{open}\}\colon 0 \leq l \leq k\}$. Then we can couple $\{\{w_t \leftarrow v,\ \text{open}\}\colon v \in \mathcal{V} \setminus \{\mathcal{E}_{t-1} \cup \{w_t\}\}\}$ with $\{G_t(v)\colon v \in \mathcal{V} \setminus \{\mathcal{E}_{t-1} \cup \{w_t\}\}\}$ such that

(i) $G_t(v)$ is independent of $\{G_t(v')\colon v' \in \mathcal{V} \setminus \{\mathcal{E}_{t-1} \cup \{w_t\}\},\ v' \neq v\}$,

(ii) $\{w_t \overset{(l)}{\longleftarrow} v,\ \text{open}\} \subset \{w_t \overset{(m)}{\longleftarrow} v,\ \text{open}\}$ almost surely if $m > l$, and

(iii) if $v \in \mathcal{N}_{t-1,k}^{(k-l)}$, then $\{w_t \overset{(l)}{\longleftarrow} v,\ \text{open}\} = \{w_t \leftarrow v,\ \text{open}\}$.

Then since $N_t^{(0)} \geq N_{t-1,k}^{(0)}$, we have

$$
\begin{aligned}
&\sum_{v \in \mathcal{N}_{t-1,k}} \mathbf{1}_{\{w_t \leftarrow v,\ \text{open}\}} \\
&\leq \sum_{v \in \mathcal{N}_{t-1}^{(0)}} \mathbf{1}_{\{w_t \overset{(k)}{\leftarrow} v,\ \text{open}\}} + \sum_{v \in \mathcal{N}_{t-1}^{(\geq 1)} \cup \mathcal{A}_{t-1}} \mathbf{1}_{\{w_t \overset{(k-1)}{\leftarrow} v,\ \text{open}\}}.
\end{aligned}
$$

Therefore, since $D_k < 2$, we obtain that for $t < n$ and for small $c > 0$,

$$
\begin{aligned}
&\mathbb{E}[e^{c\xi_t} \mid \mathcal{F}_{t-1}] \\
&\leq \exp\Bigg[(k-1)\Bigg\{\frac{p}{1-p}(c + c^2) + \bigg(\frac{p}{1-p}\bigg)^2 (\sqrt{2}c)^2\Bigg\} \\
&\qquad + N_{t-1}^{(0)}\Bigg\{\frac{kp}{n-t}(c + c^2)\Bigg\} + (n - t - N_{t-1}^{(0)})\Bigg\{\frac{(k-1)p}{n-t}(c + c^2)\Bigg\} - c\Bigg] \\
&\leq \exp\Bigg[2(c + c^2)|\epsilon| + \Bigg\{1 + \frac{2(k-1)p^2}{(1-p)^2}\Bigg\}c^2 - \frac{n - t - N_{t-1}^{(0)}}{n-t} p(c + c^2)\Bigg]
\end{aligned} \tag{25}
$$

for a large enough n. Since $\epsilon = \lambda n^{-1/3}$, when $n^{1/3} \leq t \leq \delta n$, (25) implies that

$$
\mathbb{E}[e^{c\xi_t} \mid \mathcal{F}_{t-1}]\mathbf{1}_{\{D_{t-1}\}} \leq \exp\bigg[3c|\epsilon| - \frac{kpct}{3n} + 2c^2\bigg]
$$

for a large enough n. Also by (25), we have $\mathbb{E}[e^{c\xi_t} \mid \mathcal{F}_{t-1}] \leq e^{2c|\epsilon|+2c^2}$ for every $1 \leq$

$t < n$. Since $\mathcal{D}_u \supset \mathcal{D}$ if $u \in [n^{1/3} - 1, \delta n)$, we have for $s \geq 0$ and $0 \leq s + t \leq \delta n$,

$$\begin{aligned}
&\mathbb{E}\big[e^{c\sum_{u=s+1}^{s+t}\xi_u}\mathbf{1}_{\{\mathcal{D}\}} \bigm| \mathcal{F}_s\big] \\
&\leq \exp\left[\sum_{u=s+1}^{s+t}\left(3c|\epsilon| - \frac{kpcu}{3n} + 2c^2\right) + 2c|\epsilon|n^{1/3} + 2c^2n^{1/3}\right] \\
&\leq \exp\left[3c|\epsilon|t - \frac{kpct^2}{6n} + 2c^2t + 2c|\epsilon|n^{1/3} + 2c^2n^{1/3}\right],
\end{aligned}$$

if $c > (kp/6)n^{-2/3} - (3/2)|\epsilon|$. But since $\epsilon = \lambda n^{-1/3}$, this is true as long as $c > 0$ and n is large enough. Now we put $t = (A-1)n^{2/3}$. If A is large, then the minimizer of the quadratic form in the exponent is

$$c = \frac{kpt^2/(6n) - 3|\epsilon|t - 2|\epsilon|n^{1/3}}{4(t + n^{1/3})},$$

and this is positive. Also, note that this c is of order $n^{-1/3}$. So, hereafter we use this c. Then we have

$$\mathbb{E}\big[e^{c\sum_{u=s+1}^{s+t}\xi_u}\mathbf{1}_{\{\mathcal{D}\}} \bigm| \mathcal{F}_s\big] \leq \exp\left[-\frac{\{kpt^2/(6n) - 3|\epsilon|t - 2|\epsilon|n^{1/3}\}^2}{8(t + n^{1/3})}\right]. \tag{26}$$

Next, we estimate $\mathbb{E}[e^{cW_s};\ N(w_1) = d,\ \gamma_h^* = s,\ W_s > 0]$. If $\gamma_h^* = s$, then $W_s = W_{s-1} + \alpha_s \leq n^{1/3} + \alpha_s$, and we have

$$\begin{aligned}
&\mathbb{E}[e^{cW_s};\ N(w_1) = d,\ \gamma_h^* = s,\ W_s > 0] \\
&\leq e^{3kcn^{1/3}}\mathbb{P}(N(w_1) = d,\ \gamma_h^* = s,\ W_s > 0) + e^{cn^{1/3}}\mathbb{E}[e^{c\alpha_s};\ \alpha_s > (3k-1)n^{1/3}].
\end{aligned}$$

Using (20), we obtain that

$$\begin{aligned}
&\mathbb{E}[e^{c\alpha_s};\ \alpha_s > (3k-1)n^{1/3}] \\
&\leq \sum_{j=1}^{\infty} e^{c(3k-1)(j+1)n^{1/3}}\{(k-1)p^{2jn^{1/3}} + C_1(kp)^{(k+1)jn^{1/3}}\}.
\end{aligned}$$

The right hand side converges and is bounded by

$$C_4(p^{n^{1/3}} + (kp)^{(k+1)n^{1/3}})$$

for some constant $C_4 = C_4(k, \lambda) > 0$, since c is of order $n^{-1/3}$. Therefore we have

$$\begin{aligned}
&\sum_{d=1}^{n^{1/3}}\sum_{s=1}^{n^{2/3}}\mathbb{E}[e^{cW_s};\ N(w_1) = d,\ \gamma_h^* = s,\ W_s > 0] \\
&\leq C_5\left\{\sum_{d=1}^{n^{1/3}}\mathbb{P}(N(w_1) = d)\mathbb{P}(W_{\gamma_h^*} > 0 \mid N(w_1) = d) + n(p^{n^{1/3}} + (kp)^{(k+1)n^{1/3}})\right\}
\end{aligned} \tag{27}$$

for some constant $C_5 = C_5(k, \lambda) > 0$. Finally by definition,

$$\begin{aligned}&\mathbb{P}(W_{\gamma_h^*} > 0 \mid N(w_1) = d)\\ &\le \mathbb{P}(W_{\gamma_h} \ge n^{1/3} \mid N(w_1) = d) + \mathbb{P}(\gamma_h \ge n^{2/3} \mid N(w_1) = d).\end{aligned}$$

By (16) and (23), we obtain that

$$\mathbb{P}(W_{\gamma_h^*} > 0 \mid N(w_1) = d) \le C_6 d n^{-1/3} \tag{28}$$

for some constant $C_6 = C_6(k, \lambda) > 0$. Substituting (26), (27) and (28) into (24), we obtain that

$$\begin{aligned}&\mathbb{P}(\mathcal{D} \cap \{N(w_1) < n^{1/3},\ W_{\gamma_h^*} > 0,\ X_{\gamma_h^* + (A-1)n^{2/3}} \ge -N(w_1)\})\\ &\le C_7 n^{-1/3} e^{-r(A-1)^3}\end{aligned}$$

for some constants $C_7 = C_7(k, \lambda) > 0$, and $r = r(k, \lambda) > 0$ if A is large but $A = O(n^{1/10})$. From this we have

$$\mathbb{P}(|\mathcal{C}(v)| \ge A n^{2/3}) \le C n^{-1/3} e^{-r(A-1)^3}$$

for some constant $C = C(k, \lambda) > 0$.

A similar argument applies to components explored after the first one.

Denote by S_T the number of vertices contained in components larger than T. Then $|\mathcal{C}_1| \ge T$ implies that $S_T \ge T$. So, taking $T = An^{2/3}$, we have

$$\mathbb{P}(|\mathcal{C}_1| \ge T) \le \mathbb{P}(S_T \ge T) \le \frac{\mathbb{E}S_T}{T} \le \frac{n\mathbb{P}(|\mathcal{C}(v)| \ge T)}{T} \le \frac{C}{A} e^{-r(A-1)^3},$$

completing the proof of Theorem 1.

4.3. Proofs of Theorems 3, 4. In cases of above or below the critical point, we also use the calculation of the X_t.

Proof of Theorem 3. Let $\{\alpha_s\}$ be the random variables given by (7).

By a calculation similar to (25), we have for a small $\theta > 0$,

$$\mathbb{E}e^{\theta\alpha_s} \le \exp\left[\left\{(k-1)\frac{p}{1-p} + kp - 1\right\}(\theta + \theta^2) + \left\{1 + \frac{2(k-1)p^2}{(1-p)^2}\right\}\theta^2\right].$$

The right hand side of the above inequality is not larger than $e^{-r\theta}$ for some $r > 0$ and a small enough $\theta > 0$. We have the same estimate for $s = n$. Let v be a vertex v such that it is included in the component explored at first. Lemma 1 and $|\mathcal{C}(v)| > A \log n$

imply that $\sum_{s=1}^{A\log n} \alpha_s \geq X_{A\log n} > -N(w_1)$ almost surely. Also, $N(w_1)$ is stochastically dominated by a random variable distributed as $Ge(1-p)+1$. Therefore we have

$$\begin{aligned}\mathbb{P}(|\mathcal{C}(v)| > A\log n) &\leq \sum_{d=1}^{\infty} \mathbb{P}(N(w_1)=d)\mathbb{P}(X_{A\log n} > -d)\\ &\leq \sum_{d=1}^{\infty} \mathbb{P}(N(w_1)=d)e^{-r\theta A\log n+\theta d}\\ &\leq Cn^{-r\theta A}\end{aligned}$$

for some constant $C>0$. The argument is similar for the components explored after first time. Similarly to the proof of Theorem 1,

$$\mathbb{P}(|\mathcal{C}_1| > A\log n) \leq \frac{n\mathbb{P}(|\mathcal{C}(v)| > A\log n)}{A\log n} \leq \frac{Cn^{1-r\theta A}}{A\log n}.$$

If A is large enough, $\mathbb{P}(|\mathcal{C}_1| > A\log n) \to 0$ as $n\to\infty$. □

Proof of Theorem 4. When $c = k+\sqrt{k^2-k}$ (i.e., $p=1$), since the k-out graph is almost surely connected, $|\mathcal{C}_1| = n$. We consider the case where $1 < c < k+\sqrt{k^2-k}$ (i.e., $p<1$). Let $r = (k-1)(p/(1-p)) + kp - 1$. Then r is positive in this case. Let

$$\begin{aligned}\xi_t' &= \sum_{l=2}^{k} \hat{r}_{t-1,l}' + \sum_{v\in\mathcal{N}_{t-1,k}} \mathbf{1}_{\{w_t\leftarrow v,\ \text{open}\}} - 1,\\ \xi_t'' &= \sum_{j=2}^{k} \bigl\{\mathbf{1}_{\{A_{t-1}^{(j)}>0, A_{t-1}^{(\geq j+1)}=0\}} + \mathbf{1}_{\{A_{t-1}=0, w_t\in\mathcal{N}_{t-1}^{(j)}\}}\bigr\} \sum_{l=2}^{j} \hat{r}_{t-1,l}'.\end{aligned} \tag{29}$$

Then $\xi_t = \xi_t' - \xi_t''$. Further, let $X_t' = \sum_{s=1}^{t} \xi_s'$ and $X_t'' = \sum_{s=1}^{t} \xi_s''$. So $X_t = X_t' - X_t''$. Therefore, we can see that

$$\{|\mathcal{C}_1| < \delta n\} \subset \bigcup_{t=\delta n+1}^{2\delta n} \{X_t < 0\} \subset \bigcup_{t=\delta n+1}^{2\delta n} \left\{X_t' < \frac{rt}{2} \text{ or } X_t'' \geq \frac{rt}{2}\right\}.$$

Take $a > k(p/(1-p) + 1/(1-2\delta))$ with a small $\delta > 0$. For $0 \leq s \leq 2\delta n$ and $1 \leq l \leq k$, we define the events $\tilde{\mathcal{D}}_s$ and $\tilde{\mathcal{D}}_{s,l}$ by

$$\begin{aligned}\tilde{\mathcal{D}}_s &= \{N_s^{(0)} > n - 2(1+a)\delta n\},\\ \tilde{\mathcal{D}}_{s,l} &= \{N_{s,l}^{(0)} > n - 2(1+a)\delta n - 1 - 2k\delta n\}.\end{aligned}$$

For $s \le t$ and $2 \le l \le k$, let $\beta_{s-1,l}$ be following independent non-negative random variables;

$$\mathbb{P}(\beta_{s-1,l} = x) = \prod_{y=0}^{x-1} \left\{ \frac{n - 2(1+a)\delta n - 1 - 2k\delta n - y}{n-s} p \right\} (1-p),$$

for an integer $x \ge 1$, and it puts on 0 the remaining probability. We can couple $\hat{r}'_{s-1,l}$ and $\beta_{s-1,l}$ such that $\hat{r}'_{s-1,l} \ge \beta_{s-1,l}$ on $\tilde{\mathcal{D}}_{s-1,l-1}$. We also let $\beta_{s-1,k+1}$ be independent random variables distributed as $Bin(n-2(1+a)\delta n-1-2k\delta n;\ kp/n)$ for each s. We can couple $\sum_{v \in \mathcal{N}_{s-1,k}} \mathbf{1}_{\{w_s \leftarrow v,\ \text{open}\}}$ and $\beta_{s-1,k+1}$ such that $\sum_{v \in \mathcal{N}_{s-1,k}} \mathbf{1}_{\{w_s \leftarrow v,\ \text{open}\}} \ge \beta_{s-1,k+1}$ on $\tilde{\mathcal{D}}_{s-1,k}$. We get for $\theta > 0$,

$$\mathbb{E}e^{-\theta X'_t} \le \mathbb{E}\left[\prod_{s=0}^{t-1} \prod_{l=2}^{k+1} \{ e^{-\theta \beta_{s,l}} \mathbf{1}_{\{\tilde{\mathcal{D}}_{s,l-1}\}} + \mathbf{1}_{\{\tilde{\mathcal{D}}^C_{s,l-1}\}} \} e^{\theta} \right].$$

Since $\tilde{\mathcal{D}}_{s,l}$ is decreasing in both s and l, and $0 < e^{-\theta\beta_{s,l}} < 1$, the right hand side of the above inequality is not larger than

$$\begin{aligned} &\mathbb{E}\left[\prod_{s=0}^{t-1} \left(\prod_{l=2}^{k+1} e^{-\theta\beta_{s,l}} \mathbf{1}_{\{\tilde{\mathcal{D}}_{2\delta n,k}\}} + \mathbf{1}_{\{\tilde{\mathcal{D}}^C_{2\delta n,k}\}} \right) e^{\theta} \right] \\ &\quad \le \mathbb{E}\left[\prod_{s=0}^{t-1} \left\{ \prod_{l=2}^{k+1} e^{-\theta\beta_{s,l}} \right\} e^{\theta} \left(1 + \prod_{s=0}^{t-1} \prod_{l=2}^{k+1} e^{\theta\beta_{s,l}} \mathbf{1}_{\{\tilde{\mathcal{D}}^C_{2\delta n,k}\}} \right) \right]. \end{aligned}$$

By Schwarz's inequality and Minkowski's inequality, the right hand side is not larger than

$$\sqrt{\mathbb{E}\left[\prod_{s=0}^{t-1} \left\{ \prod_{l=2}^{k+1} e^{-2\theta\beta_{s,l}} \right\} e^{2\theta} \right]} \times \left[1 + \sqrt{\mathbb{E}\left[\prod_{s=0}^{t-1} \left\{ \prod_{l=2}^{k+1} e^{2\theta\beta_{s,l}} \right\} \mathbf{1}_{\{\tilde{\mathcal{D}}^C_{2\delta n,k}\}} \right]} \right].$$

Again by Schwarz's inequality,

$$\mathbb{E}\left[\prod_{s=0}^{t-1} \left\{ \prod_{l=2}^{k+1} e^{2\theta\beta_{s,l}} \right\} \mathbf{1}_{\{\tilde{\mathcal{D}}^C_{2\delta n,k}\}} \right] \le \prod_{s=0}^{t-1} \prod_{l=2}^{k+1} \sqrt{\mathbb{E}e^{4\theta\beta_{s,l}}} \times \sqrt{\mathbb{P}(\tilde{\mathcal{D}}^C_{2\delta n,k})}.$$

By definition, we can find positive constants $\{C_l\}_{l=2}^{k+1}$ and $\theta_0 > 0$ such that we have

$$\mathbb{E}e^{\theta\beta_{s,l}} \le e^{\theta\mathbb{E}\beta_{s,l} + (\theta^2/2)C_l} \tag{30}$$

for every $2 \le l \le k+1$, $0 \le s \le 2\delta n$, and $|\theta| \le \theta_0$. Now we have for $0 \le s \le 2\delta n$,

$$\mathbb{E}\beta_{s,l} = \frac{p}{1-p} + O(\delta), \quad \text{and} \quad \mathbb{E}\beta_{s,l} \le \frac{p}{1-p}, \quad 2 \le l \le k, \tag{31}$$

$$\mathbb{E}\beta_{s,k+1} = kp + O(\delta), \quad \text{and} \quad \mathbb{E}\beta_{s,k+1} \le kp. \tag{32}$$

From (30)–(32), and from the fact that

$$\begin{aligned}\mathbb{P}\big(\tilde{\mathcal{D}}^C_{2\delta n,k}\big) &\le \mathbb{P}\big(\tilde{\mathcal{D}}^C_{2\delta n}\big) + \mathbb{P}\big(N^{(0)}_{2\delta n} - N^{(0)}_{2\delta n,k} \ge 1 + 2k\delta n\big) \\ &\le e^{-\bar{I}(a)\cdot 2\delta n} + kp^{2\delta n},\end{aligned}$$

choosing $\theta > 0$ small enough, we get

$$\mathbb{E}\left[\prod_{s=0}^{t-1}\left\{\prod_{l=2}^{k+1} e^{4\theta\beta_{s,l}}\right\}\right] P\big(\tilde{\mathcal{D}}^C_{2\delta n,k}\big) = o(1)$$

as $n \to \infty$. Also from (30)–(32), and by the definition of r,

$$\sqrt{\mathbb{E}\prod_{s=0}^{t-1}\left\{\prod_{l=2}^{k+1} e^{-2\theta\beta_{s,l}}\right\} e^{2\theta}} \le \exp\left[-\theta\{r + O(\delta)\}t + \theta^2 \sum_{l=2}^{k+1} C_l t\right].$$

Choosing $\delta > 0$ and $\theta > 0$ small enough, we can find $r_1 > 0$ such that

$$\mathbb{P}\left(X'_t < \frac{rt}{2}\right) \le e^{-r_1 t}. \tag{33}$$

To estimate $\mathbb{E}e^{\theta X''_t}$, take a vertex v which appears as $w_s \in \mathcal{N}^{(\ge 2)}_{s-1} \cup \mathcal{A}^{(\ge 2)}_{s-1}$. Then there is some time $u \le s-1$ such that $v \in \mathcal{N}^{(1)}_{u-1,k} \cup \mathcal{A}^{(1)}_{u-1,k}$ and that $\{w_u \leftarrow v\}$ occurs. Therefore $\sum_{s=1}^t \xi''_s \le \sum_{s=1}^t \mathbf{1}_{\{w_s \in \mathcal{N}^{(\ge 2)}_{s-1} \cup \mathcal{A}^{(\ge 2)}_{s-1}\}} \sum_{l=2}^k \hat{r}'_{s-1,l}$ and the right hand side is stochastically dominated by

$$\sum_{u=1}^t \sum_{v \in \mathcal{N}^{(1)}_{u-1,k} \cup \mathcal{A}^{(1)}_{u-1,k}} \mathbf{1}_{\{w_u \overset{(k-1)}{\leftarrow} v\}} \sum_{l=2}^k \alpha_{u-1,l}(v),$$

where $\{\alpha_{u-1,l}(v)\}$ for $1 \le u \le t$, $2 \le l \le k$, $v \in \mathcal{V}$ are i.i.d. random variables distributed as $Ge(1-p)$. Here we define random variables $f(x)$, g_s and h_s by

$$\begin{aligned} f(x) &\sim NB((k-1)x;\, 1-p), \\ g_s &\sim Bin\left(2(1+a)\delta n;\, \frac{k-1}{(1-2\delta)n}\right), \\ h_s &\sim Bin\left(n;\, \frac{k-1}{(1-2\delta)n}\right). \end{aligned}$$

We can couple $f(g_s)$ and $f(h_s)$ such that almost surely $f(g_s) \le f(h_s)$. Thus, using Lemma 4, we obtain that

$$\begin{aligned}\mathbb{E}e^{\theta X_t''} &\le \mathbb{E}\left[\prod_{s=1}^{t}\left\{e^{\theta f(g_s)}\mathbf{1}_{\{\tilde{\mathcal{D}}_{s-1}\}} + e^{\theta f(h_s)}\mathbf{1}_{\{\tilde{\mathcal{D}}_{s-1}^C\}}\right\}\right] \\ &\le \mathbb{E}\left[\prod_{s=1}^{t} e^{\theta f(g_s)} + \prod_{s=1}^{t} e^{\theta f(h_s)}\mathbf{1}_{\{\tilde{\mathcal{D}}_{2\delta n}^C\}}\right] \\ &\le \prod_{s=1}^{t}\mathbb{E}e^{\theta f(g_s)} + \sqrt{\prod_{s=1}^{t}\mathbb{E}e^{2\theta f(h_s)}}\sqrt{\mathbb{P}\big(\tilde{\mathcal{D}}_{2\delta n}^C\big)} \\ &\le e^{O(\delta)t\theta}.\end{aligned}$$

Therefore

$$\mathbb{P}\left(X_t'' \ge \frac{rt}{2}\right) \le e^{-r_2 t} \tag{34}$$

for some $r_2 > 0$.

Combining (33) and (34), we get the proof of the theorem. □

5. Proof of Theorem 2

5.1. Detailed calculation. From now on, we assume that $\epsilon = \epsilon(n)$ is a sequence such that $\epsilon(n) \to 0$ as $n \to \infty$ and we write $p = p(n) = (1 + \epsilon(n))/(k + \sqrt{k^2 - k})$. Also we assume that $s_0 \in [0, \infty)$ and $m = \lfloor s_0 n^{2/3} \rfloor$.

Lemma 7. *If n is large enough, then for all $t \le s_0 n^{2/3}$,*

$$\mathbb{E}A_t = O(\epsilon n^{2/3} + n^{1/3}), \quad \mathbb{E}Z_t = O(\epsilon n^{2/3} + n^{1/3}).$$

Proof. By Lemma 1, we can couple ξ_t and α_t such that almost surely $\xi_t \le \alpha_t$ and α_t is independent of $\mathcal{F}_{t-1}$ for $1 \le t \le s_0 n^{2/3}$. From (9), we have

$$\mathbb{E}[\xi_t \mid \mathcal{F}_{t-1}] \le \mathbb{E}\alpha_t = D_k\epsilon + O(\epsilon^2) \quad \text{a.s.}$$

Recall that $\{t_0, t_1, \dots\}$ are times at which A_t equals zero. By definition, we have $N(w_t) = 0$ for any $t \ne t_j + 1$. Let $\{\kappa_t\}_{t=1}^n$ be i.i.d. random variables distributed as $Ge(1 - p) + 1$. By Lemma 1, we can couple $N(w_{t_j+1})$ and a random variable κ_{t_j+1} such that almost surely $N(w_{t_j+1}) \le \kappa_{t_j+1}$ and κ_{t_j+1} is independent of $\mathcal{F}_{t_j}$. Also from (4), $Z_{t_j+1} = -X_{t_j+1} + N(w_{t_j+1}) + \xi_{t_j+1}$. Thus by coupling, almost surely

$$\begin{aligned} Z_t &= \sum_{j=0}^{\infty}\mathbf{1}_{\{t_j < t \le t_{j+1}\}}\{-X_{t_j+1} + N(w_{t_j+1}) + \xi_{t_j+1}\} \\ &\le -\min_{s\le t} X_s + \max_{s\le t}\{\kappa_s + \alpha_s\}.\end{aligned} \tag{35}$$

Here $X_t - \sum_{s=1}^{t} \mathbb{E}[\xi_s \mid \mathcal{F}_{s-1}]$ is an $(\mathcal{F}_t)$-martingale. So, using Doob's L^2 maximal inequality and Lemmas 2, 3 and 4, we have

$$\begin{aligned}
-\mathbb{E}\Big[\min_{s\le t} X_s\Big] &\le \mathbb{E}\Bigg[\max_{s\le t}\Bigg|X_s - \sum_{u=1}^{s}\mathbb{E}[\xi_u \mid \mathcal{F}_{u-1}]\Bigg| + \max_{s\le t}\Bigg|\sum_{u=1}^{s}\mathbb{E}[\xi_u \mid \mathcal{F}_{u-1}]\Bigg|\Bigg] \\
&\le \sqrt{4\mathbb{E}\Bigg[\Bigg|X_t - \sum_{s=1}^{t}\mathbb{E}[\xi_s \mid \mathcal{F}_{s-1}]\Bigg|^2\Bigg]} + \mathbb{E}\Bigg[\sum_{s=1}^{t}\big|\mathbb{E}[\xi_s \mid \mathcal{F}_{s-1}]\big|\Bigg] \qquad (36)\\
&\le \sum_{s=1}^{t}\mathbb{E}\big[\mathbf{1}_{\{A_{s-1}^{(\ge 2)}>0\}} + \mathbf{1}_{\{A_{s-1}=0,\, w_t\in\mathcal{N}_{s-1}^{(\ge 2)}\}}\big]O(1) \\
&\quad + O(\epsilon n^{2/3} + n^{1/3}).
\end{aligned}$$

For $1 \le i \le k$, let

$$a_s^{(i)} = A_s^{(i)} - A_{s-1,0}^{(i)}. \qquad (37)$$

Further we define $a_s^{(\ge i)} = A_s^{(\ge i)} - A_{s-1,0}^{(\ge i)}$. By definition, $a_s^{(i)} \in \mathcal{F}_s$ and

$$\mathbf{1}_{\{A_{s-1}^{(\ge 2)}>0\}} \le \mathbf{1}_{\{a_{s-1}^{(\ge 2)}\ge 1\}} + \mathbf{1}_{\{A_{s-2}^{(\ge 2)}\ge 2\}}. \qquad (38)$$

Now we use the following lemma. We prove this later.

Lemma 8. *For $s \le t \le s_0 n^{2/3}$ and $d \in \mathbb{N} \cup \{0\}$,*

$$\mathbb{P}(\{a_s^{(\ge 2)} = d\} \cap \{N_t^{(0)} \ge n - t - 2kt\} \mid \mathcal{F}_{s-1}) \le 2^{k\mathbf{1}_{\{d\ne 0\}}}\left\{\frac{k(t+2kt)}{n-t}\right\}^d,$$

$$\mathbb{P}(\{A_s^{(\ge 2)} \ge 2\} \cap \{N_t^{(0)} \ge n - t - 2kt\}) = O(n^{-2/3}).$$

Using (38) and Lemmas 4, 8, we have

$$\sum_{s=1}^{t}\mathbb{E}\mathbf{1}_{\{A_{s-1}^{(\ge 2)}>0\}}O(1) = O(n^{1/3}). \qquad (39)$$

Next, we have

$$\mathbb{E}\big[\mathbf{1}_{\{A_{s-1}=0,\, w_t\in\mathcal{N}_{s-1}^{(\ge 2)}\}} \mid \mathcal{F}_{s-1}\big] \le \mathbf{1}_{\{A_{s-1}=0\}}\frac{N_{s-1}^{(\ge 2)}}{N_{s-1}} \le \frac{N_{s-1}^{(\ge 2)}}{n-s}.$$

By Lemma 2,

$$\sum_{s=1}^{t}\mathbb{E}\mathbf{1}_{\{A_{s-1}=0,\, w_t\in\mathcal{N}_{s-1}^{(\ge 2)}\}}O(1) = O(n^{1/3}). \qquad (40)$$

Combining (36), (39) and (40), we have

$$-\mathbb{E}\Big[\min_{s\le t} X_s\Big] = O(\epsilon n^{2/3} + n^{1/3}). \tag{41}$$

Therefore, using (35) and (41), we obtain that

$$\mathbb{E}Z_t \le -\mathbb{E}\Big[\min_{s\le t} X_s\Big] + O(\log n) = O(\epsilon n^{2/3} + n^{1/3}).$$

Also, by Lemmas 2, 3, 4 and (39), (40), we have

$$\mathbb{E}X_t = O(\epsilon n^{2/3} + n^{1/3}).$$

From this and (4), we obtain that $\mathbb{E}A_t = \mathbb{E}X_t + \mathbb{E}Z_t = O(\epsilon n^{2/3} + n^{1/3})$. □

Proof of Lemma 8. When $d = 0$, the first part of this lemma is trivially true. So we assume that d is a natural number. We set $(\mathrm{I})_{s,c} = \{A^{(\ge 2)}_{s-1,k} - A^{(\ge 2)}_{s-1,0} = c\}$ and $(\mathrm{II})_{s,c} = \{A^{(\ge 2)}_{s-1,k+1} - A^{(\ge 2)}_{s-1,k} = c\}$. Then we have

$$\mathbf{1}_{\{a_s^{(\ge 2)} = d\}} = \sum_{c=0}^{d\wedge k} \mathbf{1}_{\{(\mathrm{I})_{s,c}\}} \mathbf{1}_{\{(\mathrm{II})_{s,d-c}\}}.$$

For each c with $0 \le c \le d \wedge k$, we have

$$\mathbf{1}_{\{(\mathrm{I})_{s,c}\}} = \sum_{1\le l_1<l_2<\cdots<l_c\le k} \prod_{i=1}^{c} \mathbf{1}_{\{(\mathrm{II})_{s,i}\}},$$

where we set $(\mathrm{III})_{s,i} = \{A^{(\ge 2)}_{s-1,l_i} - A^{(\ge 2)}_{s-1,l_i-1} = 1\}$. Since $(\mathrm{III})_{s,i}$ is a subset of $\{\eta_{s-1,T_{l_i}} \in \mathcal{N}^{(\ge 2)}_{s-1,l_i-1}\}$, we have

$$\mathbb{E}[\mathbf{1}_{\{(\mathrm{III})_{s,i}\}} \mid \mathcal{F}_{s-1,l_i-1}] \le \frac{N^{(\ge 2)}_{s-1,l_i-1}}{n-s} \le \frac{N^{(\ge 1)}_{s-1,l_i-1} + A_{s-1,l_i-1}}{n-s}.$$

Note that $N^{(\ge 1)}_{s-1,l_i-1} + A_{s-1,l_i-1} \le t - s + 2kt$ if $N^{(0)}_{s-1,l_i-1} \ge n - t - 2kt$ occurs. Therefore

$$\mathbb{E}\big[\mathbf{1}_{\{(\mathrm{III})_{s,i}\}} \mathbf{1}_{\{N^{(0)}_{s-1,l_i-1} \ge n-t-2kt\}} \mid \mathcal{F}_{s-1,l_i-1}\big] \le \frac{t-s+2kt}{n-s} \le \frac{k(t+2kt)}{n-t}.$$

Next, we estimate $(\mathrm{II})_{s,d-c}$. If $v \in \mathcal{A}^{(\ge 2)}_{t-1,k+1} \setminus \mathcal{A}^{(\ge 2)}_{t-1,k}$, then either

(i) $v \in \mathcal{N}^{(\geq 1)}_{t-1,k}$ and $\{w_t \leftarrow v, \text{open}\}$ occurs, or

(ii) $v \in \mathcal{A}^{(1)}_{t-1,k}$ and $\{w_t \leftarrow v\}$ occurs. This means also that $\{w_t \leftarrow v\} \subset \{w_t \overset{(k-1)}{\leftarrow} v\}$ because we have already checked at least one edge starting from v.

Therefore

$$\mathbb{E}[\mathbf{1}_{\{(\mathrm{II})_{s,d-c}\}} \mid \mathcal{F}_{s-1,k}] \leq \sum_{\{v_1,\dots,v_{d-c}\} \in \mathcal{N}^{(\geq 1)}_{s-1,k} \cup \mathcal{A}^{(1)}_{s-1,k}} \prod_{j=1}^{d-c} \mathbb{P}(w_s \overset{(k-1)}{\leftarrow} v_j)$$
$$\leq \binom{N^{(\geq 1)}_{s-1,k} + A_{s-1,k}}{d-c} \left(\frac{k-1}{n-s}\right)^{d-c}$$
$$\leq \left\{(N^{(\geq 1)}_{s-1,k} + A_{s-1,k}) \frac{k}{n-s}\right\}^{d-c},$$

so on the event $\{N^{(0)}_{s-1,k} \geq n-t-2kt\}$, we have

$$\mathbb{E}[\mathbf{1}_{\{(\mathrm{II})_{s,d-c}\}} \mid \mathcal{F}_{s-1,k}] \leq \left\{\frac{k(t-s+2kt)}{n-s}\right\}^{d-c} \leq \left\{\frac{k(t+2kt)}{n-t}\right\}^{d-c}.$$

Now, since $N^{(0)}_{t,l}$ is decreasing in both t and l, we have

$$\mathbf{1}_{\{N^{(0)}_t \geq n-t-2kt\}} \leq \prod_{s=1}^{t} \prod_{l=0}^{k} \mathbf{1}_{\{N^{(0)}_{s-1,l} \geq n-t-2kt\}}$$

for $s \leq t$. Thus

$$\mathbb{E}\big[\mathbf{1}_{\{a^{(\geq 2)}_s = d\}} \mathbf{1}_{\{N^{(0)}_t \geq n-t-2kt\}} \bigm| \mathcal{F}_{s-1,k}\big]$$
$$\leq \sum_{c=0}^{d \wedge k} \sum_{1 \leq l_1 < l_2 < \cdots < l_c \leq k} \prod_{i=1}^{c} \mathbf{1}_{\{(\mathrm{III})_{s,i}\}} \mathbf{1}_{\{N^{(0)}_{s-1,l_i-1} \geq n-t-2kt\}} \left\{\frac{k(t+2kt)}{n-t}\right\}^{d-c},$$

and therefore

$$\mathbb{E}\big[\mathbf{1}_{\{a^{(\geq 2)}_s = d\}} \mathbf{1}_{\{N^{(0)}_t \geq n-t-2kt\}} \bigm| \mathcal{F}_{s-1}\big]$$
$$\leq \sum_{c=0}^{d \wedge k} \binom{k}{c} \left\{\frac{k(t+2kt)}{n-t}\right\}^{c} \left\{\frac{k(t+2kt)}{n-t}\right\}^{d-c} \leq 2^k \left\{\frac{k(t+2kt)}{n-t}\right\}^{d}.$$

This proves the first part of this lemma.

To prove the second part of this lemma, let $\{d(q)\}_{q=r}^{s}$ be non-negative integers for $0 \le r \le s < t$. Using notations (I)–(III) again, we obtain that

$$\prod_{q=r}^{s} \mathbf{1}_{\{a_q^{(\ge 2)}=d(q)\}} \mathbf{1}_{\{N_t^{(0)} \ge n-t-2kt\}}$$
$$\le \prod_{q=r}^{s} \sum_{c(q)=0}^{d(q)\wedge k} \sum_{1\le l_1<l_2<\cdots<l_{c(q)}\le k} \prod_{i=1}^{c(q)} \mathbf{1}_{\{(\mathrm{III})_{q,i}\}} \mathbf{1}_{\{N_{q-1,l_i-1}^{(0)} \ge n-t-2kt\}} \mathbf{1}_{\{(\mathrm{II})_{q,d(q)-c(q)}\}} \mathbf{1}_{\{N_{q-1,k}^{(0)} \ge n-t-2kt\}}.$$

Thus, by the first part of this lemma,

$$\begin{aligned} &\mathbb{E}\left[\prod_{q=r}^{s} \mathbf{1}_{\{a_q^{(\ge 2)}=d(q)\}} \mathbf{1}_{\{N_t^{(0)} \ge n-t-2kt\}} \,\middle|\, \mathcal{F}_{r-1}\right] \\ &\quad \le \prod_{q=r}^{s} 2^{k\mathbf{1}_{\{d(q)\neq 0\}}} \left\{\frac{k(t+2kt)}{n-t}\right\}^{d(q)} \le \left\{\frac{2^k k(t+2kt)}{n-t}\right\}^{\sum_{q=r}^{s} d(q)}. \end{aligned} \tag{42}$$

Now

$$\begin{aligned} \mathbf{1}_{\{A_s^{(\ge 2)}\ge 2\}} \le \mathbf{1}_{\{A_s^{(\ge 2)}\ge 2\}} \Big[&\mathbf{1}_{\{\forall u\le s,\, a_u^{(\ge 2)}\le 2\}} + \mathbf{1}_{\{\exists u\le s,\, a_u^{(\ge 2)}=3, \forall u'\neq u, a_u^{(\ge 2)}\le 2\}} \\ &+ \mathbf{1}_{\{\exists u,\, u'\le s,\, a_u^{(\ge 2)}=a_{u'}^{(\ge 2)}=3\}} + \mathbf{1}_{\{\exists u\le s,\, a_u^{(\ge 2)}\ge 4\}} \Big]. \end{aligned}$$

When $A_{u-1}^{(\ge 2)} = 0$, if $a_u^{(\ge 2)} = x$ then $A_u^{(\ge 2)} = x$. When $A_{u-1}^{(\ge 2)} \ge 1$, since w_u is chosen from $\mathcal{A}_{u-1}^{(\ge 2)}$, if $a_u^{(\ge 2)} = x$ then $A_u^{(\ge 2)} = A_{u-1}^{(\ge 2)} + x - 1$. On the event $\{A_s^{(\ge 2)} \ge 2,\ \forall u \le s,\ a_u^{(\ge 2)} \le 2\}$, if

$$\#\{u \in [s-r, s];\ a_u^{(\ge 2)} = 2\} - \#\{u \in [s-r, s];\ a_u^{(\ge 2)} = 0\} \le 0$$

for all $r \le s$, then $A_0^{(\ge 2)} > 0$, which is a contradiction. So there exists a time $r \in [0, s]$ such that

$$\#\{u \in [s-r, s];\ a_u^{(\ge 2)} = 2\} - \#\{u \in [s-r, s];\ a_u^{(\ge 2)} = 0\} = 1.$$

Therefore

$$\begin{aligned} &\mathbf{1}_{\{A_s^{(\ge 2)}\ge 2\}} \mathbf{1}_{\{\forall u\le s, a_u^{(\ge 2)}\le 2\}} \\ &\quad \le \sum_{r=0}^{s} \mathbf{1}_{\{\#\{u\in[s-r,s];\ a_u^{(\ge 2)}=2\}-\#\{u\in[s-r,s];\ a_u^{(\ge 2)}=0\}=1\}} \mathbf{1}_{\{\forall u\le s, a_u^{(\ge 2)}\le 2\}} \\ &\quad = \sum_{r=0}^{s} \sum_{x=0}^{\lfloor r/2\rfloor} \mathbf{1}_{\{\#\{u\in[s-r,s];\ a_u^{(\ge 2)}=0\}=x\}} \mathbf{1}_{\#\{u\in[s-r,s];\ a_u^{(\ge 2)}=2\}=x+1\}} \mathbf{1}_{\{\#\{u\in[s-r,s];\ a_u^{(\ge 2)}=1\}=r-2x\}}. \end{aligned} \tag{43}$$

Thus from (42),

$$\begin{aligned}
&\mathbb{E}\big[\mathbf{1}_{\{A_s^{(\geq 2)}\geq 2\}}\mathbf{1}_{\{\forall u\leq s, a_u^{(\geq 2)}\leq 2\}}\mathbf{1}_{\{N_t^{(0)}\geq n-t-2kt\}}\big]\\
&\leq \sum_{r=0}^{s}\sum_{x=0}^{\lfloor r/2\rfloor}\frac{(r+1)!}{x!(x+1)!(r-2x)!}\left\{\frac{2^k k(t+2kt)}{n-t}\right\}^{2(x+1)+r-2x}\\
&\leq \sum_{r=0}^{s}\left\{\frac{2^k k(t+2kt)}{n-t}\right\}^{r+2}3^{r+1} = O\left(\frac{t^2}{n^2}\right).
\end{aligned}$$

Next, we have

$$\begin{aligned}
&\mathbf{1}_{\{A_s^{(\geq 2)}\geq 2\}}\mathbf{1}_{\{\exists u\leq s, a_u^{(\geq 2)}=3, \forall u'\neq u, a_u^{(\geq 2)}\leq 2\}}\\
&\leq \mathbf{1}_{\{a_s^{(\geq 2)}\geq 2\}}+\mathbf{1}_{\{a_{s-1}^{(\geq 2)}\geq 2\}}+\{\mathbf{1}_{\{a_s^{(\geq 2)}=1\}}+\mathbf{1}_{\{a_{s-1}^{(\geq 2)}=1\}}\}\mathbf{1}_{\{\exists u\leq s-2, a_u^{(\geq 2)}=3\}}\\
&\quad+\mathbf{1}_{\{\exists u\leq s-2, a_u^{(\geq 2)}=3, \forall u'\neq u, a_u^{(\geq 2)}\leq 2\}}\mathbf{1}_{\{a_s^{(\geq 2)}=a_{s-1}^{(\geq 2)}=0\}}\mathbf{1}_{\{A_{s-2}^{(\geq 2)}\geq 4\}}.
\end{aligned}\tag{44}$$

From the first part of this lemma, we know that

$$\mathbb{E}\big[\{\text{the first and the second term of (44)}\}\mathbf{1}_{\{N_t^{(0)}\geq n-t-2kt\}}\big] = O\left(\frac{t^2}{n^2}\right),$$

$$\mathbb{E}\big[\{\text{the third term of (44)}\}\mathbf{1}_{\{N_t^{(0)}\geq n-t-2kt\}}\big] = O\left(\frac{t}{n}t\frac{t^3}{n^3}\right)= O\left(\frac{t^5}{n^4}\right).$$

We consider the fourth term of (44). Using a similar argument to get (43) for the event $\{A_s^{(\geq 2)}\geq 2\}\cap\{\forall u\leq s,\ a_u^{(\geq 2)}\leq 2\}$, there is a time $r\in[0,s]$ such that

$$\#\{u\in[s-r,s];\ a_u^{(\geq 2)}=2\}-\#\{u\in[s-r,s];\ a_u^{(\geq 2)}=0\}=1$$

in this case too. Therefore

$$\begin{aligned}
&\mathbb{E}\big[\mathbf{1}_{\{A_s^{(\geq 2)}\geq 2\}}\mathbf{1}_{\{\exists u\leq s, a_u^{(\geq 2)}=3, \forall u'\neq u, a_u^{(\geq 2)}\leq 2\}}\mathbf{1}_{\{N_t^{(0)}\geq n-t-2kt\}}\big]\\
&= O\left(\frac{t^2}{n^2}+\frac{t^5}{n^4}\right).
\end{aligned}$$

Finally, by a straightforward calculation

$$\mathbb{E}\big[\mathbf{1}_{\{\exists u,u'\leq s-2, a_u^{(\geq 2)}=a_{u'}^{(\geq 2)}=3\}}\mathbf{1}_{\{N_t^{(0)}\geq n-t-2kt\}}\big] = O\left(\frac{t^2}{n^2}\right),$$

$$\mathbb{E}\big[\mathbf{1}_{\{\exists u\leq s-2,\ a_u^{(\geq 2)}\geq 4\}}\mathbf{1}_{\{N_t^{(0)}\geq n-t-2kt\}}\big] = O\left(\frac{t^2}{n^2}\right).$$

Thus we have proved the second part of this lemma. □

Lemma 9. *If n is large enough, then for all $t \le s_0 n^{2/3}$,*

$$\mathbb{E}N_t^{(0)} = n - \left\{(k-1)\frac{p}{1-p} + k\right\}t + O(\epsilon n^{2/3} + n^{1/3}),$$
$$\mathbb{E}N_t^{(1)} = k(1-p)t + O(\epsilon n^{2/3} + n^{1/3}),$$
$$\mathbb{E}N_t^{(i)} = O(\epsilon n^{2/3} + n^{1/3}), \quad 2 \le i \le k.$$

Proof. $N_t^{(0)} - N_{t-1}^{(0)}$ is equal to

$$-\sum_{l=2}^{k} \hat{r}_{t-1,l} - \sum_{v \in \mathcal{N}_{t-1,k}^{(0)}} \mathbf{1}_{\{w_t \leftarrow v\}} - \mathbf{1}_{\{A_{t-1}=0,\ w_t \in \mathcal{N}_{t-1}^{(0)}\}}\{1 + \hat{r}_{t-1,1}\}$$
$$+ \sum_{j=2}^{k}\left\{\mathbf{1}_{\{A_{t-1}^{(j)}>0,\ A_{t-1}^{(\ge j+1)}=0\}} + \mathbf{1}_{\{A_{t-1}=0,\ w_t \in \mathcal{N}_{t-1}^{(j)}\}}\right\}\sum_{l=2}^{j} \hat{r}_{t-1,l}.$$

Recall $m = \lfloor s_0 n^{2/3} \rfloor$ for some $s_0 \in [0, \infty)$. Similarly to (11), we have

$$\mathbb{E}[\hat{r}_{t-1,l} \mid \mathcal{F}_{t-1,l-1}] = \mathbb{E}\left[\sum_{x=0}^{n} x \mathbf{1}_{\{\hat{r}_{t-1,l}=x\}} \,\middle|\, \mathcal{F}_{t-1,l-1}\right]$$
$$= \frac{p}{1-p} + \frac{N_{t-1,l-1}^{(\ge 1)} + A_{t-1,l-1}}{n} O(1) + O(n^{-2/3}) + \mathbf{1}_{\{N_{t-1,l-1}^{(0)} < m^{1/3}\}} O(1).$$

Therefore

$$\mathbb{E}[N_t^{(0)} \mid \mathcal{F}_{t-1}] = N_{t-1}^{(0)} - (k-1)\frac{p}{1-p} - k + \mathbf{1}_{\{A_{t-1}^{(\ge 2)}>0\}} O(1)$$
$$+ \mathbf{1}_{\{A_{t-1}=0\}} O(1) + \frac{N_{t-1,l-1}^{(\ge 1)} + A_{t-1,l-1}}{n} O(1)$$
$$+ O(n^{-2/3}) + \mathbf{1}_{\{N_{t-1,l-1}^{(0)} < m^{1/3}\}} O(1).$$

Since $\sum_{s=1}^{t} \mathbf{1}_{\{A_{s-1}=0\}} \le Z_t$, by Lemma 7, $\mathbb{E}\big[\sum_{s=1}^{t} \mathbf{1}_{\{A_{s-1}=0\}}\big] = O(\epsilon n^{2/3} + n^{1/3})$. By (38) and Lemmas 4, 8, we have $\mathbb{E}\big[\sum_{s=1}^{t} \mathbf{1}_{\{A_{s-1}^{(\ge 2)}>0\}}\big] = O(n^{1/3})$. Combining these with Lemma 4, we obtain the first statement of the lemma.

Next, for $i \geq 1$, $N_t^{(i)} - N_{t-1}^{(i)}$ is equal to

$$
\begin{aligned}
&- \sum_{l=2}^{k} \mathbf{1}_{\{\eta_{t,T_l} \in \mathcal{N}_{t-1,l-1}^{(i)},\text{ open}\}} + \sum_{v \in \mathcal{N}_{t-1,k}^{(i-1)}} \mathbf{1}_{\{w_t \leftarrow v,\text{ closed}\}} - \sum_{v \in \mathcal{N}_{t-1,k}^{(i)}} \mathbf{1}_{\{w_t \leftarrow v\}} \\
&+ \sum_{j=2}^{k} \left\{ \mathbf{1}_{\{A_{t-1}^{(j)} > 0,\, A_{t-1}^{(\geq j+1)} = 0\}} + \mathbf{1}_{\{A_{t-1} = 0,\, w_t \in \mathcal{N}_{t-1}^{(j)}\}} \right\} \\
&\times \sum_{l=2}^{j} \mathbf{1}_{\{\eta_{t,T_l} \in \mathcal{N}_{t-1,l-1}^{(i)},\text{ open}\}} \\
&- \mathbf{1}_{\{w_t \in \mathcal{N}_{t-1}^{(i)}\}} - \mathbf{1}_{\{w_t \in \mathcal{N}_{t-1}^{(0)}\}} \mathbf{1}_{\{\eta_{t,T_1} \in \mathcal{N}_{t-1,0}^{(i)},\text{ open}\}}.
\end{aligned} \tag{45}
$$

So, we have

$$
\begin{aligned}
\mathbb{E}[N_t^{(i)} \mid \mathcal{F}_{t-1}] = N_{t-1}^{(i)} &+ N_{t-1}^{(i-1)} \frac{(k-i+1)(1-p)}{n-t} + \mathbf{1}_{\{A_{t-1}=0\}} O(1) \\
&+ \mathbf{1}_{\{A_{t-1}^{(\geq 2)} > 0\}} O(1) + \frac{N_{t-1}^{(\geq 1)} + A_{t-1}}{n} O(1) \\
&+ O(n^{-2/3}) + \mathbb{E}\left[\mathbf{1}_{\{N_{t-1,l-1}^{(0)} < m^{1/3}\}} \mid \mathcal{F}_{t-1}\right] O(1).
\end{aligned}
$$

Using (38) and Lemmas 2, 4 and 8, we complete the proof this lemma. □

Lemma 10. *If n is large enough, then for all $t \leq s_0 n^{2/3}$ and $0 \leq i \leq k$,*

$$
\mathbb{E}|N_t^{(i)} - \mathbb{E}N_t^{(i)}| = O(\epsilon n^{2/3} + n^{1/3}).
$$

Proof. By Lemma 9, for $i \geq 2$, $\mathbb{E}|N_t^{(i)} - \mathbb{E}N_t^{(i)}| \leq 2\mathbb{E}N_t^{(i)} = O(\epsilon n^{2/3} + n^{1/3})$, also by Lemma 7, $\mathbb{E}|A_t - \mathbb{E}A_t| \leq O(\epsilon n^{2/3} + n^{1/3})$. So we first assume that $i = 1$. Recall (45). By (38) and Lemmas 2, 4 and 8, we have

$$
\begin{aligned}
\mathbb{E}|N_t^{(1)} - \mathbb{E}N_t^{(1)}| \leq \mathbb{E}\Bigg| &\sum_{s=1}^{t} \sum_{v \in \mathcal{N}_{s-1,k}^{(0)}} \mathbf{1}_{\{w_s \leftarrow v,\text{ closed}\}} \\
&- \mathbb{E} \sum_{s=1}^{t} \sum_{v \in \mathcal{N}_{s-1,k}^{(0)}} \mathbf{1}_{\{w_s \leftarrow v,\text{ closed}\}} \Bigg| + O(\epsilon n^{2/3} + n^{1/3}).
\end{aligned}
$$

Here recall the definition of $\{w_s \overset{(l)}{\leftarrow} v\}$ in the proof of Theorem 1,

$$\begin{aligned}
&\mathbb{E}\left|\sum_{s=1}^{t}\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ closed}\}}-\mathbb{E}\sum_{s=1}^{t}\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ closed}\}}\right|\\
&=\mathbb{E}\left|\sum_{s=1}^{t}\sum_{v\notin\mathcal{E}_s}\mathbf{1}_{\{w_s\overset{(k)}{\leftarrow}v,\text{ closed}\}}-\mathbb{E}\sum_{s=1}^{t}\sum_{v\notin\mathcal{E}_s}\mathbf{1}_{\{w_s\overset{(k)}{\leftarrow}v,\text{ closed}\}}\right.\\
&\qquad+\sum_{s=1}^{t}\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ closed}\}}-\mathbb{E}\sum_{s=1}^{t}\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ closed}\}}\\
&\qquad\left.-\sum_{s=1}^{t}\sum_{v\notin\mathcal{E}_s}\mathbf{1}_{\{w_s\overset{(k)}{\leftarrow}v,\text{ closed}\}}+\mathbb{E}\sum_{s=1}^{t}\sum_{v\notin\mathcal{E}_s}\mathbf{1}_{\{w_s\overset{(k)}{\leftarrow}v,\text{ closed}\}}\right|.
\end{aligned}$$

By Schwarz's inequality,

$$\mathbb{E}\left|\sum_{s=1}^{t}\sum_{v\notin\mathcal{E}_s}\mathbf{1}_{\{w_s\overset{(k)}{\leftarrow}v,\text{ closed}\}}-\mathbb{E}\sum_{s=1}^{t}\sum_{v\notin\mathcal{E}_s}\mathbf{1}_{\{w_s\overset{(k)}{\leftarrow}v,\text{ closed}\}}\right|=O(\sqrt{t})=O(n^{1/3}).$$

Also, using Lemma 2,

$$\begin{aligned}
&\mathbb{E}\left|\sum_{s=1}^{t}\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ closed}\}}-\mathbb{E}\sum_{s=1}^{t}\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ closed}\}}\right.\\
&\qquad\left.-\sum_{s=1}^{t}\sum_{v\notin\mathcal{E}_s}\mathbf{1}_{\{w_s\overset{(k)}{\leftarrow}v,\text{ closed}\}}+\mathbb{E}\sum_{s=1}^{t}\sum_{v\notin\mathcal{E}_s}\mathbf{1}_{\{w_s\overset{(k)}{\leftarrow}v,\text{ closed}\}}\right|\\
&\le 2\sum_{s=1}^{t}\mathbb{E}\left|\sum_{v\in\mathcal{N}_{s-1,k}^{(\ge1)}\cup\mathcal{A}_{s-1,k}}\mathbf{1}_{\{w_s\overset{(k)}{\leftarrow}v,\text{ closed}\}}\right|=O(n^{1/3}).
\end{aligned}$$

Therefore $\mathbb{E}\left|N_t^{(1)}-\mathbb{E}N_t^{(1)}\right|=O(\epsilon n^{2/3}+n^{1/3})$.

Finally by $N_t^{(0)}=n-\sum_{i=1}^{k}N_t^{(i)}-A_t-t$ and the above facts, we obtain that $\mathbb{E}\left|N_t^{(0)}-\mathbb{E}N_t^{(0)}\right|=O(\epsilon n^{2/3}+n^{1/3})$. □

5.2. Convergence of the exploration process. Recall $\mathcal{B}^\lambda$ and W^λ in (1) and (2), respectively. Hereafter for a process $\{S_t\}$ indexed by positive integers we write S_t for $t\in\mathbb{R}$ to denote the continuous linear interpolation of S_t. Recall that $\{t_j\}$ are times at which A_t equals zero. We define the process $\hat{X}_t$ by $\hat{X}_0 = X_0 = 0$ and for any $t \in [t_j, t_{j+1})$,

$$\hat{X}_t = \begin{cases} X_t & \text{if} \quad X_t \geq X_{t_j}, \\ X_{t_j} & \text{otherwise}, \end{cases} \tag{46}$$

and $\hat{X}_t = X_n$ for any $t \geq n$. By this definition, the times when the minima of $\hat{X}_t$ updates are only $\{t_j\}$. Recording minima for the process $\hat{X}_t$ occurs only when $A_t = 0$. Since we explore one vertex at each time, the size of the j-th explored open cluster is $t_{j+1} - t_j$. Therefore the analysis of the process $\hat{X}_t$ characterizes open clusters of the k-out graph.

Theorem 5. *Let λ be in $\mathbb{R}$ and $\epsilon = \lambda n^{-1/3}$. Then*

$$n^{-1/3} \hat{X}_{n^{2/3}\cdot} \overset{d}{\Rightarrow} \mathcal{B}^{\lambda}(\,\cdot\,), \quad as \quad n \to \infty,$$

where this convergence is on finite intervals.

Recall that $s_0 \in [0, \infty)$ and $m = \lfloor s_0 n^{2/3} \rfloor$. Let $t \in [0, 1]$. We define a function $\psi_m(f)$ by $\psi_m(f) = f \mathbf{1}_{\{f \leq m^{1/3}\}}$. Let $\chi_{s-1,l} = \psi_m(\hat{r}'_{s-1,l})$ for $1 \leq l \leq k$, $\chi_{s-1,k+1} = \sum_{j=0}^{k} \psi_m \big(\sum_{v \in \mathcal{N}^{(j)}_{s-1,k}} \mathbf{1}_{\{w_s \leftarrow v, \text{ open}\}} \big)$,

$$\xi_s^{(m)} = \sum_{l=2}^{k+1} \chi_{s-1,l} - 1,$$

and $X_s^{(m)} = \sum_{u=1}^{s} \xi_u^{(m)}$.

To prove the convergence of the process $X^{(m)}$, we use a central limit theorem for martingales (cf. [10] Chapter 7, Theorem 7.2). Namely, if

1. $|m^{-1/2}(\xi_s^{(m)} - \mathbb{E}[\xi_s^{(m)} \mid \mathcal{F}_{s-1}])| \leq \epsilon_m$ for all $s \leq m$ with $\epsilon_m \to 0$, and

(47) 2. $$m^{-1} \sum_{s=1}^{\lfloor mt \rfloor} \mathbb{E}[(\xi_s^{(m)} - \mathbb{E}[\xi_s^{(m)} \mid \mathcal{F}_{s-1}])^2 \mid \mathcal{F}_{s-1}] \to Ct \text{ i.p.}$$

for any $t \in [0, 1]$ and some $C > 0$,

then $m^{-1/2} \sum_{s=1}^{\lfloor mt \rfloor} (\xi_s^{(m)} - \mathbb{E}[\xi_s^{(m)} \mid \mathcal{F}_{s-1}]) \overset{d}{\Rightarrow} \mathcal{B}(Ct)$, where $\mathcal{B}(t)$ is a standard Brownian motion.

We start the calculation of $\mathbb{E}[\xi_s^{(m)} \mid \mathcal{F}_{s-1}]$. Similarly to the calculation of $\mathbb{E}[\hat{r}'_{s-1,l} \mid \mathcal{F}_{s-1,l-1}]$ in the proof of Lemma 3,

$$\mathbb{E}[\chi_{s-1,l} \mid \mathcal{F}_{s-1,l-1}] = \mathbb{E}\left[\sum_{x=0}^{\lfloor m^{1/3} \rfloor \wedge N_{s-1,l-1}^{(0)}} x \mathbf{1}_{\{\hat{r}'_{t-1,l}=x\}} \,\middle|\, \mathcal{F}_{t-1,l-1} \right]$$

$$= \frac{p}{1-p} - \frac{p^2}{(1-p)^2} \frac{N_{s-1,l-1}^{(1)}}{n-s} + O\left(\frac{m^{1/3}}{n} + m^{1/3} p^{m^{1/3}} \right) + \frac{N_{s-1,l-1}^{(\geq 2)} + A_{s-1,l-1}}{n} O(1) + \frac{(N_{s-1,l-1}^{(\geq 1)} + A_{s-1,l-1})^2}{n^2} O(1) + \mathbf{1}_{\{N_{s-1,l-1}^{(0)} < m^{1/3}\}} O(1).$$

Also, similarly to the calculation of $\mathbb{E}\left[\sum_{v \in \mathcal{N}_{s-1,k}} \mathbf{1}_{\{w_s \leftarrow v, \text{ open}\}} \,\middle|\, \mathcal{F}_{s-1,k}\right]$ in the proof of Lemma 3, we have

$$\mathbb{E}[\chi_{s-1,k+1} \mid \mathcal{F}_{s-1,k}] = kp - \frac{N_{s-1,k}^{(1)}}{n-s} p + O(m^{1/3}(kp)^{m^{1/3}}) + \frac{N_{s-1,k}^{(\geq 2)} + A_{s-1,k}}{n} O(1) + \mathbf{1}_{\{N_{s-1,k}^{(0)} < m^{1/3}\}} O(1).$$

Thus $\mathbb{E}[\xi_s^{(m)} \mid \mathcal{F}_{s-1}]$ is equal to

$$(48) \quad D_k \epsilon - p \left\{ (k-1) \frac{p}{(1-p)^2} + 1 \right\} \frac{N_{s-1}^{(1)}}{n-s} + O\left(\frac{m^{1/3}}{n} + m^{1/3} (kp)^{m^{1/3}} + \epsilon^2 \right) + \frac{N_{s-1}^{(\geq 2)} + A_{s-1}}{n} O(1) + \frac{(N_{s-1}^{(\geq 1)} + A_{s-1})^2}{n^2} O(1) + \mathbb{E}\left[\mathbf{1}_{\{N_s^{(0)} < m^{1/3}\}} \,\middle|\, \mathcal{F}_{s-1}\right] O(1).$$

In the last line we used the fact that $N_{s-1,l}^{(0)} > N_s^{(0)}$. Furthermore using Lemmas 4, 7 and 9, we get for a large enough n,

$$\mathbb{E}\xi_s^{(m)} = D_k \epsilon - kp(1-p) \left\{ (k-1) \frac{p}{(1-p)^2} + 1 \right\} \frac{s}{n-s} + O(\epsilon^2 + \epsilon n^{-1/3} + n^{-2/3}).$$

Lemma 11. *If n is large enough, then for all $s \leq s_0 n^{2/3}$*

(i) $\mathbb{E}\xi_s^{(m)} - D_k \epsilon + kp(1-p)\{(k-1)(p/(1-p)^2) + 1\} s/(n-s) = O(\epsilon^2 + \epsilon n^{-1/3} + n^{-2/3})$,

(ii) $\mathbb{E}|\mathbb{E}[\xi_s^{(m)} \mid \mathcal{F}_{s-1}] - \mathbb{E}\xi_s^{(m)}| = O(\epsilon^2 + \epsilon n^{-1/3} + n^{-2/3})$,

(iii) $\mathbb{E}[(\xi_s^{(m)})^2 \mid \mathcal{F}_{s-1}] = 2(k - \sqrt{k^2 - k}) + O(\epsilon + n^{-2/3}) + (N_{s-1}^{(\geq 1)} + A_{s-1}) O(1/n) + \mathbb{E}\left[\mathbf{1}_{\{N_s^{(0)} < m^{1/3}\}} \,\middle|\, \mathcal{F}_{s-1}\right] O(1)$.

Proof. (i) has already been confirmed. By (48) and Lemmas 2, 7, 9 and 10,

$$\begin{aligned}\mathbb{E}|\mathbb{E}[\xi_s^{(m)} \mid \mathcal{F}_{s-1}] - \mathbb{E}\xi_s^{(m)}| &= p\left\{(k-1)\frac{p}{(1-p)^2}+1\right\}\frac{1}{n-s}\mathbb{E}\big|N_{s-1}^{(1)} - \mathbb{E}N_{s-1}^{(1)}\big| \\ &\quad + O(\epsilon^2 + \epsilon n^{-1/3} + n^{-2/3}) \\ &= O(\epsilon^2 + \epsilon n^{-1/3} + n^{-2/3}),\end{aligned}$$

which proves (ii).

To prove (iii), we consider $(\xi_s^{(m)}+1)^2$. We have

$$\mathbb{E}[(\xi_s^{(m)}+1)^2 \mid \mathcal{F}_{s-1}] = \mathbb{E}\left[\left(\sum_{l=2}^{k}\chi_{s-1,l}\right)^2 + \chi_{s-1,k+1}^2 + 2\sum_{l=2}^{k}\chi_{s-1,l}\chi_{s-1,k+1} \;\middle|\; \mathcal{F}_{s-1}\right].$$

Similarly to the calculation of $\mathbb{E}[\xi_s \mid \mathcal{F}_{s-1}]$ in the proof of Lemma 3, we have for $l < l'$,

$$\begin{aligned}&\mathbb{E}[\chi_{s-1,l}\chi_{s-1,l'} \mid \mathcal{F}_{s-1}] \\ &= \frac{p^2}{(1-p)^2} + O(n^{-2/3}) + \frac{N_{s-1}^{(\geq 1)} + A_{s-1}}{n}O(1) \\ &\quad + \mathbb{E}\big[\mathbf{1}_{\{N_{s-1,l-1}^{(0)} < m^{1/3}\}} \mid \mathcal{F}_{s-1}\big]O(1) + \mathbb{E}\big[\mathbf{1}_{\{N_{s-1,l'-1}^{(0)} < m^{1/3}\}} \mid \mathcal{F}_{s-1}\big]O(1).\end{aligned}$$

Also we have

$$\begin{aligned}\mathbb{E}[(\hat{r}'_{s,l})^2\mathbf{1}_{\{\hat{r}'_{s,l} \leq m^{1/3}\}} \mid \mathcal{F}_{s-1}] &= \mathbb{E}\left[\sum_{x=0}^{N_{s-1,l-1}^{(0)} \wedge m^{1/3}} x^2\mathbf{1}_{\{\hat{r}'_{s,l}=x\}} \;\middle|\; \mathcal{F}_{s-1}\right] \\ &= \frac{p(1+p)}{(1-p)^2} + O(n^{-2/3}) + \frac{N_{s-1}^{(\geq 1)} + A_{s-1}}{n}O(1) \\ &\quad + \mathbb{E}\big[\mathbf{1}_{\{N_{s-1,l-1}^{(0)} < m^{1/3}\}} \mid \mathcal{F}_{s-1}\big]O(1).\end{aligned}$$

By $N_{s-1,l-1}^{(0)} \geq N_s^{(0)}$, we obtain that $\mathbb{E}\big[\big(\sum_{l=2}^{k}\chi_{s-1,l}\big)^2 \mid \mathcal{F}_{s-1}\big]$ is equal to

$$\begin{aligned}&(k-1)(k-2)\frac{p^2}{(1-p)^2} + (k-1)\frac{p(1+p)}{(1-p)^2} + O(n^{-2/3}) \\ &+ \frac{N_{s-1}^{(\geq 1)} + A_{s-1}}{n}O(1) + \mathbb{E}\big[\mathbf{1}_{\{N_s^{(0)} < m^{1/3}\}} \mid \mathcal{F}_{s-1}\big]O(1).\end{aligned}$$

On the other hand, for a large enough n,

$$\begin{aligned}\mathbb{E}[\chi_{s-1,k+1}^2 \mid \mathcal{F}_{s-1}] &= k^2p^2 + kp + O(n^{-2/3}) + \frac{N_{s-1}^{(\geq 1)} + A_{s-1}}{n}O(1) \\ &\quad + \mathbb{E}\big[\mathbf{1}_{\{N_s^{(0)} < m^{1/3}\}} \mid \mathcal{F}_{s-1}\big]O(1),\end{aligned}$$

also we obtain that $\mathbb{E}\big[\sum_{l=2}^{k}\chi_{s-1,l}\chi_{s-1,k+1}\;\big|\;\mathcal{F}_{s-1}\big]$ is equal to

$$(k-1)\frac{p}{1-p}kp+O(n^{-2/3})+\frac{N_{s-1}^{(\geq 1)}+A_{s-1}}{n}O(1)+\mathbb{E}\big[\mathbf{1}_{\{N_s^{(0)}<m^{1/3}\}}\;\big|\;\mathcal{F}_{s-1}\big]O(1),$$

where we used $N_{s-1,l}^{(0)}>N_s^{(0)}$ again. Using

$$\begin{aligned}&(k-1)(k-2)\frac{p^2}{(1-p)^2}+(k-1)\frac{p(1+p)}{(1-p)^2}+k^2p^2+kp+2(k-1)\frac{p}{1-p}kp\\&=2(k-\sqrt{k^2-k})+1+O(\epsilon),\end{aligned}$$

we obtain that

$$\begin{aligned}\mathbb{E}[(\xi_s^{(m)}+1)^2\mid\mathcal{F}_{s-1}]&=2(k-\sqrt{k^2-k})+1+O(\epsilon+n^{-2/3})\\&\quad+\frac{N_{s-1}^{(\geq 1)}+A_{s-1}}{n}O(1)+\mathbb{E}\big[\mathbf{1}_{\{N_s^{(0)}<m^{1/3}\}}\;\big|\;\mathcal{F}_{s-1}\big]O(1).\end{aligned}$$

Also we have

$$\mathbb{E}[(\xi_s^{(m)}+1)^2\mid\mathcal{F}_{s-1}]=\mathbb{E}[(\xi_s^{(m)})^2\mid\mathcal{F}_{s-1}]+2\mathbb{E}[\xi_s^{(m)}\mid\mathcal{F}_{s-1}]+1,$$

so, combining Lemma 3, we obtain (iii). □

Lemma 12. *Let λ be in $\mathbb{R}$ and $\epsilon=\lambda n^{-1/3}$. Then for $t\in[0,1]$,*

$$\begin{aligned}m^{-1/2}X_{\lfloor mt\rfloor}^{(m)}\overset{d}{\Rightarrow}\;&\mathcal{B}(2(k-\sqrt{k^2-k})t)\\&+2(k-\sqrt{k^2-k})\lambda\sqrt{s_0}t-(1-(k-\sqrt{k^2-k})^2)\frac{s_0^{3/2}t^2}{2}.\end{aligned}$$

Proof. By definition, the condition 1 of (47) is satisfied. Also by Lemmas 2, 4 and the part (iii) of Lemma 11,

$$\mathbb{E}[(\xi_s^{(m)})^2\mid\mathcal{F}_{s-1}]\to 2(k-\sqrt{k^2-k})\quad\text{i.p.}$$

for $1\leq s\leq\lfloor mt\rfloor$, thus by Lemma 3,

$$m^{-1}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}[(\xi_s^{(m)}-\mathbb{E}[\xi_s^{(m)}\mid\mathcal{F}_{s-1}])^2\mid\mathcal{F}_{s-1}]\to 2(k-\sqrt{k^2-k})t\quad\text{i.p.},$$

and it means that the condition 2 of (47) is satisfied. Therefore

$$m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\{\xi_s^{(m)}-\mathbb{E}[\xi_s^{(m)}\mid\mathcal{F}_{s-1}]\}\overset{d}{\Rightarrow}\mathcal{B}(2(k-\sqrt{k^2-k})t).$$

On the other hand,

$$m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}[\xi_s^{(m)}\mid\mathcal{F}_{s-1}]=m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}\xi_s^{(m)}+m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\{\mathbb{E}[\xi_s^{(m)}\mid\mathcal{F}_{s-1}]-\mathbb{E}\xi_s^{(m)}\}. \tag{49}$$

By the part (i) of Lemma 11,

$$m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}\xi_s^{(m)}\to 2(k-\sqrt{k^2-k})\lambda\sqrt{s_0}t-(1-(k-\sqrt{k^2-k})^2)\frac{s_0^{3/2}t^2}{2}$$

as $n\to\infty$. Also, by the part (ii) of Lemma 11, expectation of absolute value of the second term of the right hand side of (49) converges to 0 as $n\to\infty$. Therefore, we obtain the statement of this lemma. □

Lemma 13. *Let λ be in $\mathbb{R}$ and $\epsilon=\lambda n^{-1/3}$. Then for $t\in[0,1]$,*

$$m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbf{1}_{\{A_{s-1}^{(2)}>0,A_{s-1}^{(\geq3)}=0\}}\hat{r}'_{s,2}\to\frac{k-1}{k}(k-\sqrt{k^2-k})^2\frac{s_0^{3/2}t^2}{2},\quad i.p.$$

as $n\to\infty$.

Proof. Let $h_s=\mathbf{1}_{\{A_{s-1}^{(2)}>0,A_{s-1}^{(\geq3)}=0\}}\hat{r}'_{s,2}$ for $1\leq s\leq\lfloor mt\rfloor$. We have

$$\begin{aligned}\mathbb{E}\left|m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}h_s-m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}h_s\right|&\leq\mathbb{E}\left|m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}h_s-m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}[h_s\mid\mathcal{F}_{s-2}]\right|\\&\quad+m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}|\mathbb{E}[h_s\mid\mathcal{F}_{s-2}]-\mathbb{E}h_s|.\end{aligned}$$

First, (38) and Lemmas 4, 8 imply that $\mathbb{E}h_s^2=O(n^{-1/3})$ for $(1/\bar{I}(2k))\log n\leq s\leq\lfloor mt\rfloor$. Also, $\mathbb{E}h_s^2$ is bounded from above by the second moment of $Ge(1-p)$. Therefore

$$\begin{aligned}&\mathbb{E}\left|m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}h_s-m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}[h_s\mid\mathcal{F}_{s-2}]\right|^2\\&\leq 2m^{-1}\sum_{s=3}^{\lfloor mt\rfloor}\mathbb{E}[\{h_{s-1}-\mathbb{E}[h_{s-1}\mid\mathcal{F}_{s-3}]\}\{h_s-\mathbb{E}[h_s\mid\mathcal{F}_{s-2}]\}]\\&\quad+m^{-1}\sum_{s=2}^{\lfloor mt\rfloor}\mathbb{E}[\{h_s-\mathbb{E}[h_s\mid\mathcal{F}_{s-2}]\}^2]\\&=O(n^{-1/3}+n^{-2/3}\log n)=O(n^{-1/3}).\end{aligned}\tag{50}$$

Next, from (11) in the proof of Lemma 3,

$$\mathbb{E}[h_s \mid \mathcal{F}_{s-1}] = \mathbf{1}_{\{A_{s-1}^{(2)}>0, A_{s-1}^{(\geq 3)}=0\}} \Bigg\{ \frac{p}{1-p} + \frac{N_{s-1}^{(\geq 1)} + A_{s-1}}{n} O(1) + \frac{(N_{s-1}^{(\geq 1)} + A_{s-1})^2}{n^2} O(1) + O\bigg(\frac{m^{1/3}}{n} + m^{1/3} p^{m^{1/3}}\bigg) + \mathbb{E}[\mathbf{1}_{\{N_s^{(0)}<m^{1/3}\}} O(1) \mid \mathcal{F}_{s-1}] \Bigg\}.$$

So, $\mathbb{E}[h_s \mid \mathcal{F}_{s-2}]$ is equal to

$$\mathbb{E}[\mathbf{1}_{\{A_{s-1}^{(2)}>0, A_{s-1}^{(\geq 3)}=0\}} \mid \mathcal{F}_{s-2}] \times \Bigg\{ \frac{p}{1-p} + \frac{N_{s-2}^{(\geq 1)} + A_{s-2}}{n} O(1) + \frac{\big(N_{s-2}^{(\geq 1)} + A_{s-2}\big)^2}{n^2} O(1) + O\bigg(\frac{m^{1/3}}{n} + m^{1/3} p^{m^{1/3}}\bigg) \Bigg\} + \mathbb{E}\big[\mathbf{1}_{\{N_s^{(0)}<m^{1/3}\}} \bigm| \mathcal{F}_{s-2}\big] O(1) + O\bigg(\frac{1}{n}\bigg).$$

Recall $a_s^{(i)}$ in (37), and let s be $(1/\bar{I}(2k)) \log n \leq s \leq \lfloor mt \rfloor$. We have

$$\begin{aligned}\mathbf{1}_{\{A_{s-1}^{(2)}>0, A_{s-1}^{(\geq 3)}=0\}} &= \mathbf{1}_{\{A_{s-1}^{(2)}>0,\ A_{s-1}^{(\geq 3)}=0,\ a_{s-1}^{(2)}\geq 1\}} + \mathbf{1}_{\{A_{s-1}^{(2)}>0,\ A_{s-1}^{(\geq 3)}=0,\ a_{s-1}^{(2)}\leq 0\}} \\ &= \mathbf{1}_{\{a_{s-1}^{(2)}\geq 1\}} - \mathbf{1}_{\{A_{s-1}^{(\geq 3)}>0,\ a_{s-1}^{(2)}\geq 1\}} \\ &\quad + \mathbf{1}_{\{A_{s-1}^{(2)}>0,\ A_{s-1}^{(\geq 3)}=0, a_{s-1}^{(2)}\leq 0\}}.\end{aligned}$$

Using Lemmas 4 and 7–9,

$$\begin{aligned}\mathbb{E}\mathbf{1}_{\{A_{s-1}^{(\geq 3)}>0,\ a_{s-1}^{(2)}\geq 1\}} &\leq \mathbb{E}\mathbf{1}_{\{A_{s-1}^{(\geq 3)}>0\}} = \mathbb{E}\mathbf{1}_{\{a_{s-1}^{(\geq 3)}\geq 1\}} + O(n^{-2/3}), \\ \mathbb{E}\mathbf{1}_{\{A_{s-1}^{(2)}>0,\ A_{s-1}^{(\geq 3)}=0,\ a_{s-1}^{(2)}\leq 0\}} &\leq \mathbb{E}\mathbf{1}_{\{A_{s-2}^{(\geq 2)}\geq 2\}} = O(n^{-2/3}).\end{aligned}$$

Furthermore, $a_{s-1}^{(\geq 3)} \geq 1$ implies that some bad vertex of the AS process is chosen from $\mathcal{N}_{s-2}^{(\geq 3)}$ or there is a directed edge from $\mathcal{N}_{s-2,k}^{(\geq 2)} \cup \mathcal{A}_{s-2,k}^{(2)}$ to w_{s-1}. So, by Lemmas 7 and 9,

$$\mathbb{E}\mathbf{1}_{\{a_{s-1}^{(\geq 3)}\geq 1\}} = O(n^{-2/3}). \tag{51}$$

We again use Lemmas 2, 4, 8 and the above facts to obtain that

$$\begin{aligned}\mathbb{E}h_s &= \mathbb{E}\bigg[\bigg\{\frac{p}{1-p} + \frac{N_{s-1}^{(1)} + A_{s-1}}{n} O(1)\bigg\} \mathbf{1}_{\{a_{s-1}^{(2)}\geq 1\}}\bigg] + O(n^{-2/3}) \\ &= \frac{p}{1-p} \mathbb{E}\mathbf{1}_{\{a_{s-1}^{(2)}\geq 1\}} + O(n^{-2/3}).\end{aligned} \tag{52}$$

Now

$$\begin{aligned}\mathbf{1}_{\{a_{s-1}^{(2)}\geq 1\}} &= \mathbf{1}_{\{a_{s-1}^{(2)}=1\}} + \mathbf{1}_{\{a_{s-1}^{(2)}\geq 2\}} \\ &= \mathbf{1}_{\{A_{s-2,k+1}^{(2)}-A_{s-2,k}^{(2)}=1\}} - \mathbf{1}_{\{A_{s-2,k}^{(2)}-A_{s-2,0}^{(2)}\geq 1,\ A_{s-2,k+1}^{(2)}-A_{s-2,k}^{(2)}=1\}} \\ &\quad + \mathbf{1}_{\{A_{s-2,k}^{(2)}-A_{s-2,0}^{(2)}\geq 1,\ a_{s-1}^{(2)}=1\}} + \mathbf{1}_{\{a_{s-1}^{(2)}\geq 2\}}.\end{aligned}$$

$A_{s-2,k}^{(2)} - A_{s-2,0}^{(2)} \geq 1$ implies that some bad vertex of the AS process is chosen from $\mathcal{N}_{s-2}^{(2)}$. So, by Lemmas 4, 8 and 9,

$$\mathbb{E}\mathbf{1}_{\{a_{s-1}^{(2)}\geq 1\}} = \mathbb{E}\mathbf{1}_{\{A_{s-2,k+1}^{(2)}-A_{s-2,k}^{(2)}=1\}} + O(n^{-2/3}).$$

Furthermore,

$$\begin{aligned}\mathbf{1}_{\{A_{s-2,k+1}^{(2)}-A_{s-2,k}^{(2)}=1\}} &= \mathbf{1}_{\{\forall v\in\mathcal{A}_{s-2,k}^{(2)},\ w_{s-1}\not\leftarrow v,\ A_{s-2,k+1}^{(2)}-A_{s-2,k}^{(2)}=1\}} \\ &\quad \times \left\{\mathbf{1}_{\{N_{s-2,k}^{(0)}\geq n-t-2kt\}} + \mathbf{1}_{\{N_{s-2,k}^{(0)}<n-t-2kt\}}\right\} \\ &\quad + \mathbf{1}_{\{\exists v\in\mathcal{A}_{s-2,k}^{(2)},\ w_{s-1}\leftarrow v,\ A_{s-2,k+1}^{(2)}-A_{s-2,k}^{(2)}=1\}}.\end{aligned}$$

So,

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}_{\{A_{s-2,k+1}^{(2)}-A_{s-2,k}^{(2)}=1\}} \;\middle|\; \mathcal{F}_{s-2,k}\right] \\ &= N_{s-2,k}^{(1)}\frac{(k-1)p}{n-s+1}(1+O(n^{-1/3}))\mathbf{1}_{\{N_{s-2,k}^{(0)}\geq n-t-2kt\}}r \\ &\quad + \mathbf{1}_{\{N_{s-2,k}^{(0)}<n-t-2kt\}}O(1) + \frac{A_{s-2,k}^{(2)}}{n}O(1).\end{aligned} \tag{53}$$

By Lemmas 2, 4 and (52), (53), we have

$$\mathbb{E}h_s = \mathbb{E}\left[\frac{p}{1-p}\frac{(k-1)p}{n-s+1}N_{s-2}^{(1)}\right] + O(n^{-2/3}). \tag{54}$$

Therefore, by Lemma 10, and facts that $h_s \geq 0$ and that $\mathbb{E}h_s$ is bounded from above by the expectation of $Ge(1-p)$,

$$\begin{aligned}&m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}|\mathbb{E}[h_s \mid \mathcal{F}_{s-2}] - \mathbb{E}h_s| \\ &\leq m^{-1/2}\sum_{s=(1/\bar{I}(2k))\log n}^{\lfloor mt\rfloor}\frac{p}{1-p}\frac{(k-1)p}{n-s+1}\mathbb{E}\left|N_{s-2}^{(1)} - \mathbb{E}N_{s-2}^{(1)}\right| + O(n^{-1/3}\log n) \\ &= O(n^{-1/3}\log n).\end{aligned} \tag{55}$$

Using (50) and (55), we get

$$\mathbb{E}\left|m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor} h_s - m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}h_s\right| \to 0,$$

as $n \to \infty$. So we consider $m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}h_s$. However, by (54) and Lemma 9,

$$\begin{aligned}\mathbb{E}h_s &= \frac{p}{1-p}\frac{(k-1)p}{n-s+1}k(1-p)s + O(n^{-2/3})\\ &= k(k-1)p^2\frac{s}{n} + O(n^{-2/3}),\end{aligned}$$

for $(2/\bar{I}(2k))\log n \le s \le \lfloor mt\rfloor$. Therefore, by this and the fact that $\mathbb{E}h_s \le (k-1)p^2/(1-p)$,

$$m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}h_s = k(k-1)p^2\frac{s_0^{3/2}t^2}{2} + O(n^{-1/3}\log n).$$

Taking the limit, we complete the proof of this lemma. □

Proof of Theorem 5. Recall the definition (46) of $\hat{X}_t$, which means that

$$|\hat{X}_t - X_t| \le N(w_{t_j+1})$$

for each $t \in [t_j, t_{j+1})$. By Lemma 1, we can couple $N(w_t)$ and a random variable κ_t distributed as $Ge(1+p)+1$, hence we have

$$m^{-1/2}\mathbb{E}|\hat{X}_t - X_t| \le m^{-1/2}\mathbb{E}\left|\max_{1\le s\le m}\kappa_s\right| \to 0 \tag{56}$$

as $n \to \infty$ for $t \le m$. Therefore it suffices to prove the convergence of X_t.

Recall ξ_s' in (29). Let

$$\begin{aligned}Y_t = \sum_{s=1}^{t}\Bigg[&(\xi_s'+1) - \sum_{l=2}^{k+1}\chi_{s-1,l}\\ &- \sum_{j=3}^{k}\mathbf{1}_{\{A_{s-1}^{(j)}>0,A_{s-1}^{(\ge j+1)}=0\}}\sum_{l=2}^{j}\hat{r}_{s-1,l}' - \sum_{j=2}^{k}\mathbf{1}_{\{A_{s-1}=0,w_s\in\mathcal{N}_{s-1}^{(j)}\}}\sum_{l=2}^{j}\hat{r}_{s-1,l}'\Bigg],\end{aligned}$$

and recall h_s in the proof of Lemma 13. We have

$$m^{-1/2}X_{\lfloor mt\rfloor} = m^{-1/2}X_{\lfloor mt\rfloor}^{(m)} - m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}h_s + m^{-1/2}Y_{\lfloor mt\rfloor}.$$

Using the coupling in Lemma 1, we have

$$\mathbb{E}\left[(\xi_s' + 1) - \sum_{l=2}^{k+1} \chi_{s-1,l}\right] \le (k-1)m^{1/3} p^{m^{1/3}} + \frac{(kp)^{m^{1/3}}}{1-kp}.$$

By Lemmas 4, 8 and (51), we have

$$\begin{aligned}\mathbb{E}\left[\sum_{j=3}^{k} \mathbf{1}_{\{A_{s-1}^{(j)}>0, A_{s-1}^{(\ge j+1)}=0\}}\right] &\le \mathbb{E}\mathbf{1}_{\{A_{s-1}^{(\ge 3)}>0\}} \\ &\le \mathbb{E}\mathbf{1}_{\{a_{s-1}^{(\ge 3)}\ge 1\}} + \mathbb{E}\mathbf{1}_{\{A_{s-2}^{(\ge 3)}\ge 2\}} = O(n^{-2/3}),\end{aligned}$$

for $(1/\bar{I}(2k))\log n \le s \le \lfloor mt \rfloor$. Also Lemma 9 implies that

$$\mathbb{E}\left[\sum_{j=2}^{k} \mathbf{1}_{\{A_{s-1}=0, w_s \in \mathcal{N}_{s-1}^{(j)}\}}\right] \le \mathbb{E}\mathbf{1}_{\{w_s \in \mathcal{N}_{s-1}^{(\ge 2)}\}} = O(n^{-2/3}).$$

Thus $m^{-1/2} Y_{\lfloor mt \rfloor} \to 0$ in probability as $n \to 0$.

Therefore, combining Lemma 12 and Lemma 13, we obtain that

$$\begin{aligned}m^{-1/2} X_{\lfloor mt \rfloor} &\overset{d}{\Rightarrow} \mathcal{B}(2(k - \sqrt{k^2-k})t) \\ &\quad + 2(k - \sqrt{k^2-k})\lambda\sqrt{s_0}t - (\sqrt{k^2-k} - k + 1)s_0^{3/2}t^2.\end{aligned}$$

This, together with (56) implies the statement of Theorem 5. □

We have proved the convergence of the exploration process, but to state the convergence of the sequence of component sizes, we need a bit more work.

Lemma 14. *Let λ be in $\mathbb{R}$ and $\epsilon = \lambda n^{-1/3}$. We define $\mathcal{C}_1^{(s_0 n^{2/3})}$ by the largest component explored after time $s_0 n^{2/3}$. Then for any $\alpha > 0$ we have*

$$\lim_{s_0 \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\left|\mathcal{C}_1^{(s_0 n^{2/3})}\right| \ge \alpha n^{2/3}\right) = 0.$$

Proof. Recall (9) and the fact that $D_k < 2$. Similarly to (25),

$$\begin{aligned}\mathbb{E}[\xi_t \mid \mathcal{F}_{t-1}] &\le (k-1)\frac{p}{1-p} + N_{t-1}^{(0)}\frac{kp}{n-t} + \left(n - t - N_{t-1}^{(0)}\right)\frac{(k-1)p}{n-t} - 1 \\ &= D_k\epsilon - \frac{n-t-N_{t-1}^{(0)}}{n-t}p + O(\epsilon^2) \\ &\le 2\epsilon - \frac{n-t-N_{t-1}^{(0)}}{n-t}p\end{aligned}$$

for a large enough n. Let some $\delta > 0$ be fixed. We define

$$\mathcal{D} = \left\{N_t^{(0)} \le n - t - \frac{k}{3}t, \text{ for every } t \text{ with } s_0 n^{2/3} \le t \le \delta n\right\},$$
$$\mathcal{D}_t = \left\{N_t^{(0)} \le n - t - \frac{k}{3}t\right\}, \quad \text{for each } t \text{ with } \quad s_0 n^{2/3} \le t \le \delta n.$$

Then we have

$$\mathbb{E}[\xi_t \mathbf{1}_{\{\mathcal{D}_{t-1}\}} \mid \mathcal{F}_{t-1}] \le 2\lambda n^{-1/3} - \frac{kt}{3(n-t)} p$$

for all $t \in (s_0 n^{2/3}, \delta n]$ and a large enough n. Now if $s_0 = s_0(\delta, \lambda) > 0$ is large enough, then for all $t \in (s_0 n^{2/3}, \delta n]$,

$$\mathbb{E}[\xi_t \mathbf{1}_{\{\mathcal{D}_{t-1}\}} \mid \mathcal{F}_{t-1}] \le -\delta^{-1} n^{-1/3}. \tag{57}$$

Let $\hat{t}_0 > s_0 n^{2/3}$ be the first time after $s_0 n^{2/3}$ such that $A_t = 0$. We define the stopping time γ such that

$$\gamma = \min\{t > 0 \colon X_{\hat{t}_0 + t} = X_{\hat{t}_0} - N(w_{\hat{t}_0+1})\}.$$

By Lemma 1, we can couple $N(w_{\hat{t}_0+1})$ and a random variable κ distributed as $Ge(1 - p) + 1$. Let $N(w_{\hat{t}_0+1}) = d$ be fixed. We define an $(\mathcal{F}_t)$-supermartingale Q_t by $Q_t = \sum_{s=1}^{t} \xi_{\hat{t}_0+s} \mathbf{1}_{\{\mathcal{D}_{\hat{t}_0+s-1}\}} + \delta^{-1} n^{-1/3} t$. By (57) and the optional stopping theorem,

$$\begin{aligned}
0 &\ge \mathbb{E}[Q_{\gamma \wedge \delta n} \mid N(w_{\hat{t}_0+1}) = d] \\
&\ge \mathbb{E}\left[\sum_{s=1}^{\gamma\wedge\delta n} \xi_{\hat{t}_0+s} - \sum_{s=1}^{\gamma\wedge\delta n} \xi_{\hat{t}_0+s} \mathbf{1}_{\{\mathcal{D}^C_{\hat{t}_0+s-1}\}} + \delta^{-1} n^{-1/3}\{\gamma \wedge \delta n\} \,\middle|\, N(w_{\hat{t}_0+1}) = d\right] \\
&\ge -d + \mathbb{P}\big(\mathcal{D}^C_{\hat{t}_0}\big) O(n) + \delta^{-1} n^{-1/3} \mathbb{E}[\gamma \wedge \delta n \mid N(w_{\hat{t}_0+1}) = d].
\end{aligned}$$

By Lemma 4, $\mathbb{P}(\mathcal{D}^C_{\hat{t}_0}) \le n^{-2}$ for a large enough n. Thus we have

$$\mathbb{E}[\gamma \wedge \delta n \mid N(w_{\hat{t}_0+1}) = d] \le 2\delta d n^{1/3},$$

and it provides

$$\begin{aligned}
\mathbb{E}[\gamma \wedge \delta n] &\le \sum_{d=1}^{\infty} \mathbb{P}(N(w_{\hat{t}_0+1}) = d) \mathbb{E}[\gamma \wedge \delta n \mid N(w_{\hat{t}_0+1}) = d] \\
&\le \sum_{d=1}^{\infty} \mathbb{P}(N(w_{\hat{t}_0+1}) = d) \cdot 2\delta d n^{1/3} \le \frac{2}{1-p} \delta n^{1/3}.
\end{aligned}$$

Also by $\gamma = |\mathcal{C}(w_{\hat{t}_0+1})|$ and Theorem 1, $\mathbb{P}(\gamma > \delta n) \leq n^{-1}$ for a large enough n. Thus

$$\mathbb{E}\gamma \leq n\mathbb{P}(\gamma > \delta n) + \mathbb{E}[\gamma \mathbf{1}_{\{\gamma \leq \delta n\}}] \leq 1 + \mathbb{E}[\gamma \wedge \delta n] \leq \frac{3}{1-p}\delta n^{1/3}$$

for a large enough s_0 and a large enough n. Similar argument applies to components explored after first time. Hence $\mathbb{E}|\mathcal{C}(v)| = O(\delta)n^{1/3}$ for any $v \in \mathcal{V} \setminus \mathcal{E}_{s_0 n^{2/3}}$. Therefore for any fixed $\alpha > 0$, we get

$$\mathbb{P}(|\mathcal{C}(v)| > \alpha n^{2/3}) = O(\delta)n^{-1/3}.$$

Let S be the number of vertices $v \in \mathcal{V} \setminus \mathcal{E}_{s_0 n^{2/3}}$ such that $|\mathcal{C}(v)| > \alpha n^{2/3}$. We have checked that $\mathbb{E}S \leq n\mathbb{P}(|\mathcal{C}(v)| > \alpha n^{2/3}) = O(\delta)n^{2/3}$. Also $|\mathcal{C}_1^{(s_0 n^{2/3})}| > \alpha n^{2/3}$ implies that $S > \alpha n^{2/3}$. Hence

$$\mathbb{P}(|\mathcal{C}_1^{(s_0 n^{2/3})}| > \alpha n^{2/3}) \leq \mathbb{P}(S > \alpha n^{2/3}) = \frac{O(\delta)n^{2/3}}{\alpha n^{2/3}} = O(\delta).$$

Since $\delta > 0$ was arbitrary and s_0 was large enough depending only on δ and λ, this completes the proof. □

Finally, we prove Theorem 2. This can be done parallel to the proof of Theorem 5 in [2].

Proof of Theorem 2. We define for $f \in C[0, s]$,

$$\Xi = \Big\{(r, l) \subset [0, s] \colon f(r) = f(l) = \min_{u \leq l} f(u), \quad \text{and } f(x) > f(r) \text{ for every } x \text{ with } r < x < l\Big\}$$

and

$$(\mathcal{L}_1, \mathcal{L}_2, \dots) = (l_1 - r_1, l_2 - r_2, \dots) \quad \text{with} \quad l_i - r_i \geq l_{i+1} - r_{i+1} \quad \text{for all} \quad i.$$

$(\mathcal{L}_1, \mathcal{L}_2, \dots)$ is the sequence of lengths of members of Ξ arranged in decreasing order, i.e., it is the decreasing sequence of excursion lengths of $f(r) - \min_{q \leq r} f(q)$. Since the sum of excursion lengths is at most s, we can get such sequence. We call l an *ending point* if $(r, l) \in \Xi$ for some $0 \leq r < l$. If for almost every $x \in [0, s]$, there exists $(r, l) \in \Xi$ such that $r < x < l$, we say *the function* $f \in C[0, s]$ *is good*. For $m \in \mathbb{N}$, we define the function $\phi_m \colon C[0, s] \to \mathbb{R}^m$ by

$$\phi_m(f) = (\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_m).$$

Now we use the following proposition in [2].

Proposition 1 (Nachmias and Peres [2]). *If $f \in C[0,s]$ is good, then $\phi_m(f)$ is continuous at f with respect to the $\|\cdot\|_\infty$ norm.*

A sample path of a Brownian motion is good with probability 1. By the Cameron Martin theorem, the process $\mathcal{B}^\lambda(\cdot)$ is good with probability 1 too, see [3]. Thus $\phi_m(f)$ is continuous on almost every sample point of $\mathcal{B}^\lambda$. Furthermore, using Theorem 5 and Theorem 2.3 in Durrett [10], Chapter 2, we have

$$n^{-2/3}\phi_m(\hat{X}) \stackrel{d}{\Rightarrow} \phi_m(\mathcal{B}^\lambda).$$

Now we rescale by $n^{-2/3}$ and order the sequence of the sizes of components explored before time $sn^{2/3}$. Then this sequence converges in distribution to the ordered sequence of excursion lengths of $W^\lambda[0,s]$. By Lemma 14, we get the proof of Theorem 2. □

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