Fujimori, S., Kim, Y.W., Koh, S.-E., Rossman, W., Shin, H., Umehara, M., Yamada, K. and Yang, S.-D.
Osaka J. Math.
52 (2015), 285–297

ZERO MEAN CURVATURE SURFACES IN LORENTZ–MINKOWSKI 3-SPACE WHICH CHANGE TYPE ACROSS A LIGHT-LIKE LINE

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(Received October 2, 2013)

Abstract

It is well-known that space-like maximal surfaces and time-like minimal surfaces in Lorentz–Minkowski 3-space $\boldsymbol{R}^3_1$ have singularities in general. They are both characterized as zero mean curvature surfaces. We are interested in the case where the singular set consists of a light-like line, since this case has not been analyzed before. As a continuation of a previous work by the authors, we give the first example of a family of such surfaces which change type across a light-like line. As a corollary, we also obtain a family of zero mean curvature hypersurfaces in $\boldsymbol{R}^{n+1}_1$ that change type across an $(n-1)$-dimensional light-like plane.

Introduction

Many examples of space-like maximal surfaces containing singular curves in the Lorentz–Minkowski 3-space $(\boldsymbol{R}^3_1; t, x, y)$ of signature $(-++)$ have been constructed in [11], [1], [12], [8], [4] and [5].

In this paper, we are interested in the zero mean curvature surfaces in $\boldsymbol{R}^3_1$ changing their causal type: Klyachin [10] showed under a sufficiently weak regularity assumption that a zero mean curvature surface in $\boldsymbol{R}^3_1$ changes its causal type only on the following two subsets:

- null curves (i.e., regular curves whose velocity vector fields are light-like) which are non-degenerate (i.e., their projections into the xy-plane are locally convex plane curves), or
- light-like lines, which are degenerate everywhere.

Given a non-degenerate null curve γ in $\boldsymbol{R}^3_1$, there exists a zero mean curvature surface which changes its causal type across this curve from a space-like maximal surface to

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2010 Mathematics Subject Classification. Primary 53A10; Secondary 53B30, 35M10.

Kim was supported by NRF 2009-0086794, Koh by NRF 2009-0086794 and NRF 2011-0001565, Shin by NRF-2014R1A2A2A01007324 and Yang by NRF 2012R1A1A2042530. Fujimori was partially supported by the Grant-in-Aid for Young Scientists (B) No. 21740052, Rossman was supported by Grant-in-Aid for Scientific Research (B) No. 20340012, Umehara by (A) No. 22244006 and Yamada by (B) No. 21340016 from Japan Society for the Promotion of Science.

a time-like minimal surface (cf. [6], [10], [9] and [7]). This construction can be accomplished using the Björling formula for the Weierstrass-type representation formula of maximal surfaces. (The reference [3] is an expository article on this subject.) However, if γ is a light-like line, the aforementioned construction fails, since the isothermal coordinates break down at the light-like singular points. Locally, such a surface is the graph of a function $t = f(x, y)$ satisfying

$$(1 - f_y^2) f_{xx} + 2 f_x f_y f_{xy} + (1 - f_x^2) f_{yy} = 0, \tag{$*$}$$

where $f_x = \partial f/\partial x$, $f_{xy} = \partial^2 f/\partial y \partial x$, etc. We call this and its graph the *zero mean curvature equation* and a *zero mean curvature surface*, respectively. Until now, zero mean curvature surfaces which actually change type across a light-like line were unknown. As announced in [2], the main purpose of this paper is to construct such an example. In Section 1, we give a formal power series solution of the zero mean curvature equation describing all zero mean curvature surfaces which contain a light-like line. Using this, we give the precise statement of our main result and show how the statement can be reduced to a proposition (cf. Proposition 1.3). In Section 2, we then prove it. As a consequence, we obtain the first example of (a family of) zero mean curvature surfaces which change type across a light-like line.

1. The main theorem

We discuss solutions of the zero mean curvature equation ($*$) which have the following form

$$f(x, y) = b_0(y) + \sum_{k=1}^{\infty} \frac{b_k(y)}{k} x^k, \tag{1.1}$$

where $b_k(y)$ $(k = 1, 2, \dots)$ are C^∞-functions. When f contains a singular light-like line, we may assume without loss of generality that (cf. [2])

$$b_0(y) = y, \quad b_1(y) = 0. \tag{1.2}$$

As was pointed out in [2], there exists a real constant μ called the *characteristic* of f such that $b_2(y)$ satisfies the following equation

$$b_2'(y) + b_2(y)^2 + \mu = 0 \quad ({}' = d/dy). \tag{1.3}$$

Now we derive the differential equations satisfied by $b_k(y)$ for $k \geq 3$ assuming (1.2). If we set

$$Y := f_y - 1 = \sum_{k=2}^{\infty} \frac{b_k'(y)}{k} x^k$$

and

$$P := 2(Y f_{xx} - f_x f_{xy}), \quad Q := Y^2 f_{xx} - 2 f_x f_{xy} Y, \quad R := f_x^2 f_{yy},$$

then, by straightforward calculations, we see that

$$P = -b_2 b_2' x^2 - \frac{4}{3} b_2 b_3' x^3 - \sum_{k=4}^{\infty} \left(P_k + \frac{2(k-1)}{k} b_2 b_k' + (3-k) b_2' b_k \right) x^k,$$
$$Q = -\sum_{k=4}^{\infty} Q_k x^k, \quad R = \sum_{k=4}^{\infty} R_k x^k,$$

where

$$\begin{aligned} P_k &:= \sum_{m=3}^{k-1} \frac{2(k-2m+3)}{k-m+2} b_m b_{k-m+2}', \\ Q_k &:= \sum_{m=2}^{k-2} \sum_{n=2}^{k-m} \frac{3n-k+m-1}{mn} b_m' b_n' b_{k-m-n+2}, \\ R_k &:= \sum_{m=2}^{k-2} \sum_{n=2}^{k-m} \frac{b_m b_n b_{k-m-n+2}''}{k-m-n+2} \end{aligned} \tag{1.4}$$

for $k \geq 4$, and that the zero mean curvature equation $(*)$ reduces to

$$\sum_{k=2}^{\infty} \frac{b_k''}{k} x^k = f_{yy} = P + Q + R.$$

It is now immediate, by comparing the coefficients of x^k from both sides, to see that each b_k $(k \geq 3)$ satisfies the following ordinary differential equation

$$b_k''(y) + 2(k-1) b_2(y) b_k'(y) + k(3-k) b_2'(y) b_k(y) = -k(P_k + Q_k - R_k), \tag{1.5}$$

where $P_3 = Q_3 = R_3 = 0$ and P_k, Q_k and R_k are as in (1.4) for $k \geq 4$. Note that P_k, Q_k and R_k are written in terms of b_j $(j = 1, \dots, k-1)$ and their derivatives.

Now, we consider the case that $1 - f_x^2 - f_y^2$ changes sign across the light-like line $\{t = y,\ x = 0\}$. This case occurs only when the characteristic μ as in (1.3) of f vanishes [2]. If we set

$$b_2(y) = 0 \quad (y \in \boldsymbol{R}),$$

then (1.3) holds for $\mu = 0$. So we assume

$$b_0(y) = y, \quad b_1(y) = 0, \quad b_2(y) = 0, \quad b_3(y) = 3cy, \tag{1.6}$$

where c is a non-zero constant. Then $f(x, y)$ in (1.1) can be rewritten as

$$f(x, y) = y + cyx^3 + \sum_{k=4}^{\infty} \frac{b_k(y)}{k} x^k. \tag{1.7}$$

In this situation, we will find a solution satisfying

$$b_k(0) = b'_k(0) = 0 \quad (k \geq 4). \tag{1.8}$$

Then (1.5) reduces to

$$b''_k(y) = -k(P_k + Q_k - R_k), \quad b_k(0) = b'_k(0) = 0, \quad (k \geq 4), \tag{1.9}$$

$$P_k = \sum_{m=3}^{k-1} \frac{2(k-2m+3)}{k-m+2} b_m(y) b'_{k-m+2}(y) \quad (k \geq 4), \tag{1.10}$$

$$Q_k = \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{3n-k+m-1}{mn} b'_m(y) b'_n(y) b_{k-m-n+2}(y) \quad (k \geq 7), \tag{1.11}$$

$$R_k = \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{b_m(y) b_n(y) b''_{k-m-n+2}(y)}{k-m-n+2} \quad (k \geq 7), \tag{1.12}$$

and $Q_k = R_k = 0$ for $4 \leq k \leq 6$, where the fact that $b_2(y) = 0$ has been extensively used. For example,

$$b_0 = y, \quad b_1 = b_2 = 0, \quad b_3 = 3cy, \quad b_4 = -4c^2y^3, \quad b_5 = 9c^3y^5,$$
$$b_6 = -24c^2y^7, \quad b_7 = 70c^5y^9 - 14c^3y^3, \ldots.$$

In this article, we show the following assertion:

Theorem 1.1. *For each positive number c, the formal power series solution $f(x,y)$ uniquely determined by* (1.9), (1.10), (1.11) *and* (1.12) *gives a real analytic zero mean curvature surface on a neighborhood of* $(x, y) = (0, 0)$. *In particular, there exists a non-trivial* 1*-parameter family of real analytic zero mean curvature surfaces each of which changes type across a light-like line* (*see* Fig. 1).

As a consequence, we get the following:

Corollary 1.2. *There exists a family of zero-mean curvature hypersurfaces in Lorentz–Minkowski space $\boldsymbol{R}_1^{n+1}$ each of which changes type across an* $(n-1)$*-dimensional light-like plane.*

Proof. Let f be as in the theorem. The graph of the function defined by

$$\boldsymbol{R}^n \ni (x_1, \ldots, x_n) \mapsto f(x_1, x_2) \in \boldsymbol{R}$$

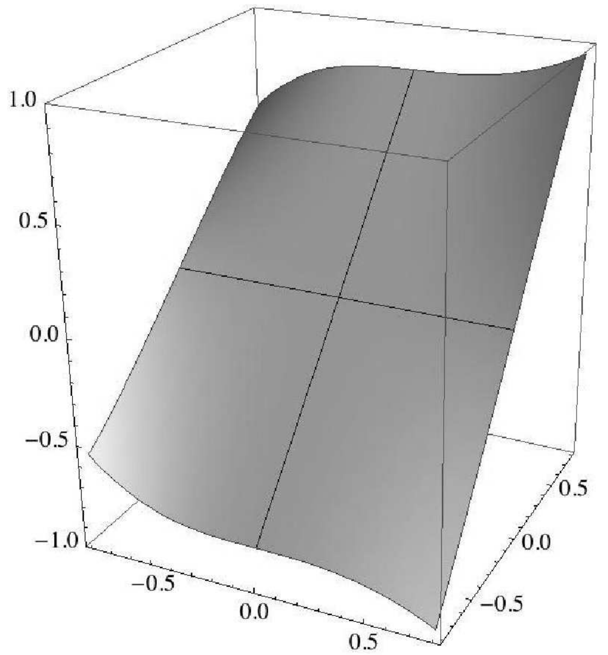

Fig. 1. The graph of $t = f(x, y)$ for $c = 1/2$ and $|x|, |y| < 0.8$ (The range of the graph is wider than the range used in our mathematical estimation. However, this figure still has a sufficiently small numerical error term in the Taylor expansion.)

gives the desired hypersurface. In this case, the zero mean curvature equation

$$\left(1 - \sum_{j=1}^{n} f_{x_j}^2\right) \sum_{i=1}^{n} f_{x_i x_i} + \sum_{i,j=1}^{n} f_{x_i x_j} f_{x_i} f_{x_j} = 0 \quad (f_{x_i} := \partial f/\partial x_i,\ f_{x_i x_j} := \partial^2 f/\partial x_j \partial x_i)$$

reduces to $(*)$ in the introduction. □

To prove Theorem 1.1, it is sufficient to show that for arbitrary positive constants $c > 0$ and $\delta > 0$ there exist positive constants n_0, θ_0, and C such that

$$|b_k(y)| \leq \theta_0 C^k \quad (|y| \leq \delta) \tag{1.13}$$

holds for $k \geq n_0$. In fact, if (1.13) holds, then the series (1.7) converges uniformly over the rectangle $[-C^{-1}, C^{-1}] \times [-\delta, \delta]$.

The key assertion to prove (1.13) is the following

Proposition 1.3. *For each $c > 0$ and $\delta > 0$, we set*

$$M := 3\max\{144c\tau|\delta|^{3/2}, \sqrt[4]{192c^2\tau}\}, \tag{1.14}$$

where τ is the positive constant given by (A.3) *in the appendix, such that*

$$t\int_t^{1-t} \frac{du}{u^2(1-u)^2} \le \tau \quad (0 < t < 1/2). \tag{1.15}$$

Then the function $\{b_l(y)\}_{l\ge 3}$ formally determined by the recursive formulas (1.9)–(1.12) *satisfies the inequalities*

$$|b_l''(y)| \le c|y|^{l^*}M^{l-3}, \tag{1.16}$$

$$|b_l'(y)| \le \frac{3c|y|^{l^*+1}}{l^*+2}M^{l-3}, \tag{1.17}$$

$$|b_l(y)| \le \frac{3c|y|^{l^*+2}}{(l^*+2)^2}M^{l-3} \tag{1.18}$$

for any

$$y \in [-\delta, \delta], \tag{1.19}$$

where

$$l^* := \frac{1}{2}(l-1) - 2 \quad (l \ge 3). \tag{1.20}$$

Once this proposition is proven, (1.13) follows immediately. In fact, if we set

$$\theta_0 = \frac{3}{c}(\delta M)^3, \quad C := \delta M$$

and $n_0 \ge 7$, then $1 \le l^* + 2 < l - 3$ and (1.13) follows from

$$\frac{3c|y|^{l^*+2}}{(l^*+2)^2}M^{l-3} \le \theta_0 C^l.$$

2. Proof of Proposition 1.3

We prove the proposition using induction on the number $l \ge 3$. If $l = 3$, then

$$|b_3''(y)| = 0 \le \frac{c}{|y|} = c|y|^{3^*}M^0,$$

$$|b_3'(y)| = 3c = \frac{3c|y|^{3^*+1}}{3^*+2}M^0,$$

$$|b_3(y)| = 3c|y| = \frac{3c|y|^{3^*+2}}{(3^*+2)^2}M^0$$

hold, using that $b_3(y) = 3cy$, $M^0 = 1$ and $3^* = -1$. So we prove the assertion for $l \geq 4$. Since (1.17), (1.18) follow from (1.16) by integration, it is sufficient to show that (1.16) holds for each $l \geq 4$. (In fact, the most delicate case is $l = 4$. In this case $l^* = -1/2$ and we can use the fact that $\int_0^{y_0} 1/\sqrt{y}\, dy$ for $y_0 > 0$ converges.)

The inequality (1.16) follows if one shows that, for each $k \geq 4$

$$|kP_k(y)|,\ |kQ_k(y)|,\ |kR_k(y)| \leq \frac{c}{3}|y|^{k^*} M^{k-3} \quad (|y| \leq \delta) \tag{2.1}$$

under the assumption that (1.16), (1.17) and (1.18) hold for all $3 \leq l \leq k-1$. In fact, if (2.1) holds, (1.16) for $l = k$ follows immediately. Then by the initial condition (1.9) (cf. (1.8)), we have (1.17) and (1.18) for $l = k$ by integration.

The estimation of $|kP_k|$ for $k \geq 4$. By (1.10) and using the fact that (1.17), (1.18) hold for $l \leq k-1$, we have for each $|y| < \delta$ that

$$\begin{aligned}
|kP_k| &\leq \sum_{m=3}^{k-1} \frac{2k|k-2m+3|}{k-m+2} |b_m(y)|\, |b'_{k-m+2}(y)| \\
&\leq \sum_{m=3}^{k-1} \frac{2k|k-2m+3|}{k-m+2} \left(\frac{3cM^{m-3}|y|^{m^*+2}}{(m^*+2)^2} \right) \left(\frac{3cM^{k-m+2-3}|y|^{(k-m+2)^*+1}}{(k-m+2)^*+2} \right) \\
&= cM^{k-3}|y|^{k^*} \frac{144c|y|^{3/2}}{M} \sum_{m=3}^{k-1} \frac{k|k-2m+3|}{(m-1)^2(k-m+1)(k-m+2)} \\
&\leq cM^{k-3}|y|^{k^*} \frac{144c|\delta|^{3/2}}{M} \sum_{m=3}^{k-1} \frac{k|k-2m+3|}{(m-1)^2(k-m+1)(k-m+2)} \\
&\leq \frac{c}{3\tau} M^{k-3}|y|^{k^*} \sum_{m=3}^{k-1} \frac{k|k-2m+3|}{(m-1)^2(k-m+1)^2}.
\end{aligned}$$

Here, we used (1.14). Since

$$\max_{m=3,\dots,k-1} |k-2m+3| = \max_{m=3,k-1} |k-2m+3| = \max\{|k-3|, |-k+5|\},$$

by setting $q = m-1$, we have that

$$\begin{aligned}
|kP_k| &\leq \frac{c}{3\tau} M^{k-3}|y|^{k^*} \sum_{m=3}^{k-1} \frac{k^2}{(m-1)^2(k-m+1)^2} = \frac{c}{3\tau} M^{k-3}|y|^{k^*} \frac{1}{k} \sum_{q=2}^{k-2} \frac{k^3}{q^2(k-q)^2} \\
&\leq \frac{c}{3\tau} M^{k-3}|y|^{k^*} \frac{1}{k} \int_{1/k}^{1-1/k} \frac{du}{u^2(1-u)^2} \leq \frac{c}{3} M^{k-3}|y|^{k^*},
\end{aligned}$$

where we applied Lemma A.1 and (1.15) at the last step of the estimations. Hence, we get (2.1) for kP_k.

The estimation of $|kQ_k|$ for $k \geq 7$. By (1.11) and the induction assumption, we have that

$$\begin{aligned}|kQ_k| &\leq \sum_{m=3}^{k-4}\sum_{n=3}^{k-m-1}\frac{k|3n-k+m-1|}{mn}|b'_m(y)|\,|b'_n(y)|\,|b_{k-m-n+2}(y)|\\ &\leq \sum_{m=3}^{k-4}\sum_{n=3}^{k-m-1}\frac{k|3n-k+m-1|}{mn}\left(\frac{3cM^{m-3}|y|^{m^*+1}}{m^*+2}\right)\\ &\qquad\times\left(\frac{3cM^{n-3}|y|^{n^*+1}}{n^*+2}\right)\left(\frac{3cM^{k-m-n+2-3}|y|^{(k-m-n+2)^*+2}}{((k-m-n+2)^*+2)^2}\right)\\ &= cM^{k-3}|y|^{k^*}\frac{432c^2}{M^4}\sum_{m=3}^{k-4}\sum_{n=3}^{k-m-1}\frac{k|3n-k+m-1|}{(m-1)^2(n-1)^2(k-m-n+2)^2}.\end{aligned}$$

Now we apply the inequalities

$$\begin{aligned}\max_{\substack{3\leq m\leq k-4\\ 3\leq n\leq k-m-1}}|3n-k+m-1| &= \max_{(m,n)=(3,3),(3,k-4),(k-4,3)}|3n-k+m-1|\\ &= \max\{|-k+11|,4,|2k-10|\}\leq 2k,\end{aligned}$$

and also

$$\frac{432c^2}{M^4}\leq\frac{1}{36\tau},$$

which follows from (1.14). Setting $p := m-1$, $q = n-1$, we have that

$$\begin{aligned}|kQ_k| &\leq \frac{c}{36\tau}M^{k-3}|y|^{k^*}\sum_{m=3}^{k-4}\sum_{n=3}^{k-m-1}\frac{2k^2}{(m-1)^2(n-1)^2(k-m-n+2)^2}\\ &= \frac{c}{18\tau}M^{k-3}|y|^{k^*}\sum_{p=2}^{k-5}\sum_{q=2}^{k-p-2}\frac{k^2}{p^2q^2(k-p-q)^2}.\end{aligned}$$

Now applying Lemma A.2, we have that

$$|kQ_k|\leq\frac{c}{18\tau}M^{k-3}|y|^{k^*}\times 6\tau\leq\frac{c}{3}M^{k-3}|y|^{k^*},$$

which proves (2.1) for kQ_k.

The estimation of $|kR_k|$ for $k \geq 7$. As in the case of $|kQ_k|$, we have that

$$\begin{aligned}
|kR_k| &\leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|b_m(y)|\,|b_n(y)|\,|b''_{k-m-n+2}(y)|}{k-m-n+2} \\
&\leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k}{k-m-n+2} \left(\frac{3cM^{m-3}|y|^{m^*+2}}{(m^*+2)^2} \right) \\
&\qquad \times \left(\frac{3cM^{n-3}|y|^{n^*+2}}{(n^*+2)^2} \right) (cM^{k-m-n+2-3}|y|^{(k-m-n+2)^*}) \\
&= 144c^3 M^{k-7} |y|^{k^*} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k}{(k-m-n+2)(m-1)^2(n-1)^2} \\
&= cM^{k-3}|y|^{k^*} \frac{144c^2}{M^4} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k^2}{(k-m-n+2)^2(m-1)^2(n-1)^2}.
\end{aligned}$$

Now we set $p = m-1$, $q = n-1$, and using the inequality

$$3^4 \times 144c^2\tau \leq 3^4 \times 192c^2\tau < M^4,$$

we have that

$$|kR_k| \leq \frac{c}{3^4\tau} M^{k-3}|y|^{k^*} \sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2q^2(k-p-q)^2}.$$

By applying Lemma A.2, we have that

$$|kR_k| \leq \frac{c}{3^4\tau} M^{k-3}|y|^{k^*} \times 6\tau < \frac{c}{3} M^{k-3}|y|^{k^*},$$

which proves (2.1) for kR_k. This completes the proof of Proposition 1.3.

Appendix A. Inequalities used in the proof of Theorem 1.1

For $a > 0$, it holds that

$$\text{(A.1)} \qquad \frac{1}{u^2(a-u)^2} = \frac{1}{a^3}\left(\frac{a}{u^2} + \frac{2}{u} + \frac{a}{(a-u)^2} + \frac{2}{a-u} \right).$$

Therefore,

$$\text{(A.2)} \qquad \int_t^{a-t} \frac{du}{u^2(a-u)^2} = \frac{2}{a^3}\left(\frac{a(a-2t)}{t(a-t)} + 2\log\frac{a-t}{t} \right) \quad (0 < t < a/2).$$

In particular, one can show that there exists a positive constant τ such that

$$t \int_t^{1-t} \frac{du}{u^2(1-u)^2} \le \tau \quad (0 < t < 1/2). \tag{A.3}$$

The following assertion is needed to prove (2.1) for $kP_k(y)$:

Lemma A.1. *Let p be a non-negative integer and k an integer satisfying $k \ge p+4$. Then the inequality*

$$\sum_{q=2}^{k-p-2} \frac{k^3}{q^2(k-p-q)^2} \le \int_{1/k}^{a-1/k} \frac{du}{u^2(a-u)^2} \quad (a := 1 - p/k)$$

holds.

Proof. In fact, if we set $a := 1 - p/k$, then (A.1) yields that

$$\begin{aligned}\frac{k^3}{q^2(k-p-q)^2} &= \frac{1}{k}\frac{1}{(q/k)^2(a-q/k)^2}\\ &= \frac{1}{a^3}\left[\frac{1}{k}\left(\frac{a}{(q/k)^2} + \frac{2}{q/k}\right) + \frac{1}{k}\left(\frac{a}{(a-q/k)^2} + \frac{2}{a-q/k}\right)\right].\end{aligned}$$

Since $x \mapsto (a+2x)/x^2$ is a monotone decreasing function and the function $x \mapsto (a + 2(a-x))/(a-x)^2$ is monotone increasing on the interval $(0, a/2)$, we have that

$$\frac{k^3}{q^2(k-p-q)^2} \le \frac{1}{a^3}\left[\int_{(q-1)/k}^{q/k}\left(\frac{a}{u^2} + \frac{2}{u}\right) du + \int_{q/k}^{(q+1)/k}\left(\frac{a}{(a-u)^2} + \frac{2}{a-u}\right) du\right],$$

which yields that

$$\begin{aligned}\sum_{q=2}^{k-p-2} \frac{a^3k^3}{q^2(k-p-q)^2} &\le \int_{1/k}^{a-2/k}\left(\frac{a}{u^2} + \frac{2}{u}\right) du + \int_{2/k}^{a-1/k}\left(\frac{a}{(a-u)^2} + \frac{2}{a-u}\right) du\\ &\le \int_{1/k}^{a-1/k}\left(\frac{a}{u^2} + \frac{2}{u}\right) du + \int_{1/k}^{a-1/k}\left(\frac{a}{(a-u)^2} + \frac{2}{a-u}\right) du\\ &\le \int_{1/k}^{a-1/k}\left(\frac{a}{u^2} + \frac{2}{u} + \frac{a}{(a-u)^2} + \frac{2}{a-u}\right) du\\ &= \int_{1/k}^{a-1/k} \frac{du}{u^2(a-u)^2}.\end{aligned}$$

This proves the assertion. □

The following assertion is needed to prove (2.1) for $kQ_k(y)$ and $kR_k(y)$:

Lemma A.2. *For any integer $k \geq 7$, the following inequalities holds*:

$$\sum_{p=2}^{k-5}\sum_{q=2}^{k-p-2}\frac{k^2}{p^2q^2(k-p-q)^2} \leq \frac{6}{k}\int_{1/k}^{1-1/k}\frac{du}{u^2(1-u)^2} \leq 6\tau,$$

where τ is a constant satisfying (A.3).

Proof. We set $a = a(p) := 1-(p/k)$. Applying Lemma A.1 and the identity (A.2), we have that

$$\begin{aligned}
\sum_{p=2}^{k-5}\sum_{q=2}^{k-p-2}\frac{k^2}{p^2q^2(k-p-q)^2} &= \sum_{p=2}^{k-5}\left[\frac{1}{kp^2}\sum_{q=2}^{k-p-2}\frac{k^3}{q^2(k-p-q)^2}\right] \\
&\leq \sum_{p=2}^{k-5}\left[\frac{1}{kp^2}\int_{1/k}^{a-1/k}\frac{du}{u^2(a-u)^2}\right] = \sum_{p=2}^{k-5}\left[\frac{1}{p^2}\frac{2}{a^2}\left(\frac{a-2/k}{a-1/k}+2\frac{\log(ka-1)}{ka}\right)\right] \\
&\leq \sum_{p=2}^{k-5}\left[\frac{2}{p^2a^2}\left(1+2\frac{\log ka}{ka}\right)\right] \leq \sum_{p=2}^{k-5}\frac{6}{p^2a^2},
\end{aligned}$$

where we used the fact that $(\log ka)/(ka) < 1$. By applying Lemma A.1 and by using the property (A.3) of the constant τ, it holds that

$$\begin{aligned}
\sum_{p=2}^{k-5}\sum_{q=2}^{k-p-2}\frac{k^2}{p^2q^2(k-p-q)^2} &\leq 6\sum_{p=2}^{k-5}\frac{1}{p^2(1-p/k)^2} = \frac{6}{k}\sum_{p=2}^{k-5}\frac{k^3}{p^2(k-p)^2} \\
&\leq \frac{6}{k}\sum_{p=2}^{k-2}\frac{k^3}{p^2(k-p)^2} \leq \frac{6}{k}\int_{1/k}^{1-1/k}\frac{du}{u^2(1-u)^2} < 6\tau,
\end{aligned}$$

which proves the assertion. □

References

[1] F.J.M. Estudillo and A. Romero: *Generalized maximal surfaces in Lorentz–Minkowski space* L^3, Math. Proc. Cambridge Philos. Soc. **111** (1992), 515–524.

[2] S. Fujimori, Y.W. Kim, S.-E. Koh, W. Rossman, H. Shin, H. Takahashi, M. Umehara, K. Yamada and S.-D. Yang: *Zero mean curvature surfaces in* $\mathbf{L}^3$ *containing a light-like line*, C.R. Math. Acad. Sci. Paris **350** (2012), 975–978.

[3] S. Fujimori, Y.W. Kim, S.-E. Koh, W. Rossman, M. Umehara, K. Yamada and S.-D. Yang: *Zero mean curvature surfaces in Lorentz–Minkowski* 3*-space and* 2*-dimensional fluid mechanics*, Math. J. Okayama Univ. **57** (2015), 173–200.

[4] S. Fujimori, W. Rossman, M. Umehara, K. Yamada and S.-D. Yang: *New maximal surfaces in Minkowski* 3*-space with arbitrary genus and their cousins in de Sitter* 3*-space*, Results Math. **56** (2009), 41–82.

[5] S. Fujimori, W. Rossman, M. Umehara, K. Yamada and S.-D. Yang: *Embedded triply periodic zero mean curvature surfaces of mixed type in Lorentz–Minkowski* 3*-space*, Michigan Math. J. **63** (2014), 189–207.

[6] C.H. Gu: *The extremal surfaces in the* 3*-dimensional Minkowski space*, Acta Math. Sinica (N.S.) **1** (1985), 173–180.

[7] Y.W. Kim, S.-E. Koh, H. Shin and S.-D. Yang: *Spacelike maximal surfaces, timelike minimal surfaces, and Björling representation formulae*, J. Korean Math. Soc. **48** (2011), 1083–1100.

[8] Y.W. Kim and S.-D. Yang: *A family of maximal surfaces in Lorentz–Minkowski three-space*, Proc. Amer. Math. Soc. **134** (2006), 3379–3390.

[9] Y.W. Kim and S.-D. Yang: *Prescribing singularities of maximal surfaces via a singular Björling representation formula*, J. Geom. Phys. **57** (2007), 2167–2177.

[10] V.A. Klyachin: *Zero mean curvature surfaces of mixed type in Minkowski space*, Izv. Math. **67** (2003), 209–224.

[11] O. Kobayashi: *Maximal surfaces in the* 3*-dimensional Minkowski space* L^3, Tokyo J. Math. **6** (1983), 297–309.

[12] M. Umehara and K. Yamada: *Maximal surfaces with singularities in Minkowski space*, Hokkaido Math. J. **35** (2006), 13–40.

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