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A CHARACTERIZATION ON BREAKDOWN OF SMOOTH SPHERICALLY SYMMETRIC SOLUTIONS OF THE ISENTROPIC SYSTEM OF COMPRESSIBLE NAVIER–STOKES EQUATIONS

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Abstract

We study an initial boundary value problem on a ball for the isentropic system of compressible Navier–Stokes equations, in particular, a criterion of breakdown of the classical solution. For smooth initial data away from vacuum, it is proved that the classical solution which is spherically symmetric loses its regularity in a finite time if and only if the *concentration* of *mass* forms around the center in *Lagrangian coordinate system*. In other words, *in Euler coordinate system*, either the *density concentrates* or *vanishes* around the center. For the latter case, one possible situation is that a vacuum ball appears around the center and the density may concentrate on the boundary of the vacuum ball simultaneously.

1. Introduction and main results

We are concerned with the isentropic system of compressible Navier–Stokes equations which reads as

$$\begin{cases} \rho_t + \operatorname{div}(\rho U) = 0, \\ (\rho U)_t + \operatorname{div}(\rho U \otimes U) + \nabla P = \mu \triangle U + (\mu + \lambda)\nabla(\operatorname{div} U), \end{cases} \tag{1.1}$$

where $t \geq 0$, $x \in \Omega \subset \mathbb{R}^N$ $(N = 2, 3)$, $\rho = \rho(t, x)$ and $U = U(t, x)$ are the density and fluid velocity respectively, and $P = P(\rho)$ is the pressure given by a state equation

$$P(\rho) = a\rho^\gamma \tag{1.2}$$

with the adiabatic constant $\gamma > 1$ and a positive constant a. The shear viscosity μ and the bulk one λ are constants satisfying the physical hypothesis

$$\mu > 0, \quad \mu + \frac{N}{2}\lambda \geq 0. \tag{1.3}$$

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The domain Ω is a bounded ball with a radius R, namely,

$$\Omega = B_R = \{x \in \mathbb{R}^N;\ |x| \leq R < \infty\}. \tag{1.4}$$

We study an initial boundary value problem for (1.1) with the initial condition

$$(\rho, U)(0, x) = (\rho_0, U_0)(x), \quad x \in \Omega, \tag{1.5}$$

and the boundary condition

$$U(t, x) = 0, \quad t \geq 0,\ x \in \partial\Omega, \tag{1.6}$$

and we are looking for the smooth spherically symmetric solution (ρ, U) of the problem (1.1), (1.5), (1.6) which enjoys the form

$$\rho(t, x) = \rho(t, |x|), \quad U(t, x) = u(t, |x|)\frac{x}{|x|}. \tag{1.7}$$

Then, for the initial data to be consistent with the form (1.7), we assume the initial data (ρ_0, U_0) also takes the form

$$\rho_0 = \rho_0(|x|), \quad U_0 = u_0(|x|)\frac{x}{|x|}. \tag{1.8}$$

In this paper, we further assume the initial density is uniformly positive, that is,

$$\rho_0 = \rho_0(|x|) \geq \underline{\rho} > 0, \quad x \in \Omega \tag{1.9}$$

for a positive constant $\underline{\rho}$. Then it is noted that as long as the classical solution of (1.1), (1.5), (1.6) exists the density ρ is positive, that is, the vacuum never occurs. It is also noted that since the assumption (1.7) implies

$$U(t, x) + U(t, -x) = 0, \quad x \in \Omega, \tag{1.10}$$

we necessarily have $U(t, 0) = 0$ (also $U_0(0) = 0$).

There are many results about the existence of local and global strong solutions in time of the isentropic system of compressible Navier–Stokes equations when the initial density is uniformly positive (refer to [1, 5, 6, 7, 13, 14, 15, 18, 19] and their generalization [10, 11, 12, 17] to the full system including the conservation law of energy). On the other hand, for the initial density allowing vacuum, the local well-posedness of strong solutions of the isentropic system was established by Kim [8]. For strong solutions with spatial symmetries, the authors in [9] proved the global existence of radially symmetric strong solutions of the isentropic system in an annular domain, even allowing vacuum initially. However, it still remains open whether there exist global strong solutions which are spherically symmetric in a ball. The main difficulties lie on the lack of estimates of the density and velocity near the center. In the case vacuum appears, it is worth noting that Xin [20] established a blow-up result which shows that if the initial density has a compact support, then any smooth solution to the Cauchy problem of the full system of compressible Navier–Stokes equations without heat conduction blows up in a finite time. The same blowup phenomenon occurs also for the isentropic system. Indeed, Zhang–Fang ([21], Theorem 1.8) showed that if $(\rho, U) \in C^1([0,T]; H^k)$ $(k > 3)$ is a spherically symmetric solution to the Cauchy problem with the compact supported initial density, then the upper limit of T must be finite. On the other hand, it's unclear whether the strong (classical) solutions lose their regularity in a finite time when the initial density is uniformly away from vacuum. Therefore, it is important to study the mechanism of possible blowup of smooth solutions, which is a main issue in this paper.

In the spherical coordinates, the original system (1.1) under the assumption (1.7) takes the form

$$\begin{cases} \rho_t + (\rho u)_r + (N-1)\dfrac{\rho u}{r} = 0, \\ (\rho u)_t + \big(\rho u^2 + P(\rho)\big)_r + (N-1)\dfrac{\rho u^2}{r} = \epsilon\Big(u_r + (N-1)\dfrac{u}{r}\Big)_r \end{cases} \tag{1.11}$$

where $\epsilon = 2\mu + \lambda$. Now, we consider the following Lagrangian transformation:

$$t = t, \quad y = \int_0^r \rho(t,s)s^{N-1}\,ds. \tag{1.12}$$

Then, it follows from (1.10) that

$$y_t = -\rho u r^{N-1}, \quad r_t = u, \quad r_y = (\rho r^{N-1})^{-1}, \tag{1.13}$$

and the system (1.11) can be further reduced to

$$\begin{cases} \rho_t + \rho^2(r^{N-1}u)_y = 0, \\ r^{1-N}u_t + p_y = \varepsilon(\rho(r^{N-1}u)_y)_y \end{cases} \tag{1.14}$$

where $t \geq 0$, $y \in [0, M_0]$ and M_0 is defined by

$$M_0 = \int_0^R \rho_0(r)r^{N-1}\,dr = \int_0^R \rho(t,r)r^{N-1}\,dr, \tag{1.15}$$

according to the conservation of mass. Note that

$$r(t,0) = 0, \quad r(t, M_0) = R. \tag{1.16}$$

We denote by E_0 the initial energy

$$E_0 = \int_0^R \Big(\rho_0\frac{u_0^2}{2} + \frac{a\rho_0^\gamma}{\gamma-1}\Big)r^{N-1}\,dr, \tag{1.17}$$

and define a cuboid $Q_{t,y}$, for $t \geq 0$ and $y \in [0, M_0]$, as

$$Q_{t,y} = [0, t] \times [y, M_0]. \tag{1.18}$$

Our main result is stated as follows.

Theorem 1.1. *Assume that the initial data* (ρ_0, U_0) *satisfy* (1.8), (1.9) *and*

$$(\rho_0, U_0) \in H^3(\Omega). \tag{1.19}$$

Let (ρ, U) *be a classical spherically symmetric solution to the initial boundary value problem* (1.1), (1.5), (1.6) *in* $[0, T] \times \Omega$, *and* T^* *be the upper limit of* T, *that is, the maximal time of existence of the classical solution. Then, if* $T^* < \infty$, *it holds the following.*

1. *In Euler coordinate system* (1.1), *for any* $r_0 \in (0, R)$,

$$\limsup_{t \to T^*-0} \left\{ \sup_{|x| \leq r_0} \left(\rho(t, x) + \frac{1}{\rho(t, x)} \right) \right\} = \infty. \tag{1.20}$$

2. *In Lagrangian coordinate system* (1.14), *for any* $y_0 \in (0, M_0)$,

$$\limsup_{t \to T^*-0} \left\{ \sup_{y \in [0, y_0]} \rho(t, y) \right\} = \infty. \tag{1.21}$$

Moreover, for system (1.14) *in Lagrangian coordinate, there exists a constant* C *depending only on* a, γ, N *such that for any given* $y_0 \in (0, M_0)$ *it holds*

$$\begin{aligned} \rho(t, y) \leq {} & C \left(\frac{\sup \rho_0}{\inf \rho_0} \right) \left(\frac{E_0}{M_0 - y_0} \right)^{1/(\gamma-1)} \exp(C\varepsilon^{-1} H) \\ & \cdot \exp\left\{ CT \left(\frac{\sup \rho_0}{\inf \rho_0} \right)^{\gamma} \left(\frac{E_0}{M_0 - y_0} \right)^{\gamma/(\gamma-1)} \exp(C\varepsilon^{-1} H) \right\}, \quad (t, y) \in Q_{T, y_0}, \end{aligned} \tag{1.22}$$

where

$$H = y_0^{-(N-1)\gamma/(N(\gamma-1))} M_0^{1/2} E_0^{(N\gamma+N-2)/(2N(\gamma-1))} + y_0^{-\gamma/(\gamma-1)} E_0^{\gamma/(\gamma-1)} T. \tag{1.23}$$

REMARK 1.1. The local existence of smooth solution with initial data as in Theorem 1.1 is classical and can be found, for example, in [8] and references therein. So the maximal time T^* is well defined.

REMARK 1.2. There are several results on the blowup criterion for classical solutions to the system (1.1) (refer to [2, 3, 4, 16] and references therein). Especially, the authors in [3] established the following Serrin-type blowup criterion:

- When $N = 3$,

$$\lim_{T\to T^*-0}(\|\rho\|_{L^\infty(0,T\,;\,L^\infty)} + \|u\|_{L^r(0,T\,;\,L^s)}) = \infty, \tag{1.24}$$

for any $r \in [2, \infty]$ and $s \in (3, \infty]$ satisfying

$$\frac{2}{r} + \frac{3}{s} \le 1. \tag{1.25}$$

- When $N = 2$,

$$\lim_{T\to T^*-0}\|\rho\|_{L^\infty(0,T\,;\,L^\infty)} = \infty. \tag{1.26}$$

REMARK 1.3. Theorem 1.1 asserts that the formation of singularity is only due to the *concentration* of the *mass* around the center in *Lagrangian coordinate system*. More precisely, the mass anywhere away from the center is bounded up to the maximal time.

On the hand, *in Euler coordinate system*, either the *density concentrates* or *vanishes* around the center. For the latter case, one possible situation is that a vacuum ball appears around the center and the density may concentrate on the boundary of the vacuum ball simultaneously.

2. Proof of Theorem 1.1

We only prove the case when $N = 3$ since the case $N = 2$ is even simpler. Throughout of this section, we assume that (ρ, U) is a classical spherically symmetric solution with the form (1.7) to the initial boundary value problem (1.1), (1.5), (1.6) in $[0, T]\times\Omega$, and the maximal time T^*, the upper limit of T, is finite, and we denote by C generic positive constants only depending on the initial data and the maximal time T^*.

We first have the following basic energy estimate. Since the proof is standard, we omit it.

Lemma 2.1. *It holds for any* $0 \le t \le T$,

$$\int_\Omega \left(\rho\frac{|U|^2}{2} + \frac{a\rho^\gamma}{\gamma-1}\right)dx + \varepsilon\int_0^t\int_\Omega |\nabla U|^2\,dx\,d\tau \le \int_\Omega\left(\rho_0\frac{|U_0|^2}{2} + \frac{a\rho_0^\gamma}{\gamma-1}\right)dx, \tag{2.1}$$

or equivalently,

$$\begin{aligned}&\int_0^R\left(\rho\frac{u^2}{2} + \frac{a\rho^\gamma}{\gamma-1}\right)r^{N-1}\,dr + \varepsilon\int_0^t\int_0^R\left(u_r^2 + \frac{u^2}{r^2}\right)r^{N-1}\,dr\,d\tau\\ &\le \int_0^R\left(\rho_0\frac{u_0^2}{2} + \frac{a\rho_0^\gamma}{\gamma-1}\right)r^{N-1}\,dr = E_0.\end{aligned} \tag{2.2}$$

Next, using the basic energy estimate above, we can refine the blowup criterion (1.24) in the present case of spherically symmetric solutions as follows.

Lemma 2.2.

$$\lim_{T \to T^*-0} \|\rho\|_{L^\infty(0,T\,;\,L^\infty)} = \infty. \tag{2.3}$$

Proof. Due to the Serrin type condition (1.24), it suffices to show that

$$\sup_{t \in [0,T^*)} \|\rho(t,\ \cdot)\|_{L^\infty} < \infty \tag{2.4}$$

implies

$$\lim_{T \to T^*-0} \|u\|_{L^4(0,T\,;\,L^{12})} < \infty. \tag{2.5}$$

To do that, making use of the identity

$$\triangle U = \nabla \operatorname{div} U = \left(u_r + (N-1)\frac{u}{r} \right)_r \frac{x}{r}, \tag{2.6}$$

we rewrite the equation of momentum as

$$(\rho U)_t + \operatorname{div}(\rho U \otimes U) + \nabla P = \varepsilon \triangle U, \quad \varepsilon = 2\mu + \lambda. \tag{2.7}$$

Multiplying the equation (2.7) by $4|U|^2 U$ and integrating it over Ω, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega \rho |U|^4\,dx + \varepsilon \int_\Omega (\nabla |U|^2)^2\,dx &\le C \int_\Omega P|U|^2|\nabla U|\,dx \\ &\le C \int_\Omega \rho|U|^4\,dx + C\int_\Omega |\nabla U|^2\,dx \end{aligned} \tag{2.8}$$

where we used the assumption (2.4). Then, it easily follows from the Gronwall's inequality and Lemma 2.1 that

$$\int_0^t \int_\Omega (\nabla |U|^2)^2\,dx\,d\tau \le C, \quad t \in [0, T^*), \tag{2.9}$$

which implies the desired estimate (2.5) by Sobolev's embedding theorem $H^1(\Omega) \subset L^6(\Omega)$. Thus, the proof of Lemma 2.2 is completed. □

Due to the refined criterion above, to prove Theorem 1.1, it remains to show that the density away from the center stays bounded up to the maximal time T^*. To do that, we prepare the next lemma which gives a relationship between r and y.

Lemma 2.3. *There exists a positive constant C depending only on a, γ, N such that*

$$r(t, y) \geq C y^{\gamma/(N(\gamma-1))} E_0^{-1/(N(\gamma-1))} \tag{2.10}$$

and

$$R^N - r(t, y)^N \geq C(M_0 - y)^{\gamma/(\gamma-1)} E_0^{-1/(\gamma-1)}, \quad (t, y) \in [0, T] \times [0, M_0]. \tag{2.11}$$

Proof. By the energy inequality (2.2), we have

$$\begin{aligned} y &= \int_0^r \rho s^{N-1}\, ds = \int_0^r \rho s^{(N-1)/\gamma} s^{N-1-(N-1)/\gamma}\, ds \\ &\leq \left(\int_0^r \rho^\gamma s^{N-1}\, ds \right)^{1/\gamma} \left(\int_0^r s^{N-1}\, ds \right)^{1-1/\gamma} \\ &\leq C r^{N(1-1/\gamma)} E_0^{1/\gamma}, \end{aligned} \tag{2.12}$$

which implies

$$r(t, y) \geq C y^{\gamma/(N(\gamma-1))} E_0^{-1/(N(\gamma-1))}. \tag{2.13}$$

Similarly, we have

$$\begin{aligned} M_0 - y &= \int_r^R \rho s^{N-1}\, ds = \int_r^R \rho s^{(N-1)/\gamma} s^{N-1-(N-1)/\gamma}\, ds \\ &\leq \left(\int_r^R \rho^\gamma s^{N-1}\, ds \right)^{1/\gamma} \left(\int_r^R s^{N-1}\, ds \right)^{1-1/\gamma} \\ &\leq C (R^N - r(t, y)^N)^{1-1/\gamma} E_0^{1/\gamma}, \end{aligned} \tag{2.14}$$

which implies

$$R^N - r(t, y)^N \geq C(M_0 - y)^{\gamma/(\gamma-1)} E_0^{-1/(\gamma-1)}. \tag{2.15}$$

Thus, the proof of Lemma 2.3 is completed. □

We are now in a position to establish the pointwise estimates of the density away from the center.

Lemma 2.4. *For any given $y_0 \in (0, M_0)$, there exists a constant C exactly as in* Theorem 1.1 *such that*

$$\rho(t, y) \leq C, \quad (t, y) \in [0, T] \times [y_0, M_0]. \tag{2.16}$$

Proof. In view of (1.14), it holds

$$\begin{aligned}\varepsilon(\log\rho)_{ty} &= \varepsilon\left(\frac{\rho_t}{\rho}\right)_y = -\varepsilon(\rho(r^{N-1}u)_y)_y = -r^{1-N}u_t - p_y \\ &= -(r^{1-N}u)_t - p_y - (N-1)\frac{u^2}{r^N}.\end{aligned} \tag{2.17}$$

Thus, for $y > y_0 > 0$, integrating (2.17) over $(0, t) \times (y_0, y)$, we deduce that

$$\begin{aligned}\varepsilon\log\frac{\rho(t, y)}{\rho(t, y_0)} &= \varepsilon\log\frac{\rho_0(y)}{\rho_0(y_0)} + \int_{y_0}^{y}((r^{1-N}u)(0, z) - (r^{1-N}u)(t, z))\,dz \\ &\quad + \int_0^t (p(s, y_0) - p(s, y))\,ds - \int_0^t\int_{y_0}^{y}(N-1)\frac{u^2(s, z)}{r^N}\,dz\,ds,\end{aligned} \tag{2.18}$$

which is equivalent to

$$\begin{aligned}\frac{\rho(t, y)}{\rho(t, y_0)} = \frac{\rho_0(y)}{\rho_0(y_0)} &\exp\left(\varepsilon^{-1}\int_{y_0}^{y}((r^{1-N}u)(0, z) - (r^{1-N}u)(t, z))\,dz\right) \\ &\cdot\exp\left(\varepsilon^{-1}\int_0^t (p(s, y_0) - p(s, y))\,ds\right) \\ &\cdot\exp\left(-\varepsilon^{-1}\int_0^t\int_{y_0}^{y}(N-1)\frac{u^2(s, z)}{r^N}\,dz\,ds\right).\end{aligned} \tag{2.19}$$

We can rewrite (2.19) as

$$\rho(t, y) = \mathcal{P}(t)\mathcal{U}(t, y)\exp\left(-\varepsilon^{-1}\int_0^t p(s, y)\,ds\right) \tag{2.20}$$

where

$$\mathcal{P}(t) = \frac{\rho(t, y_0)}{\rho_0(y_0)}\exp\left(\varepsilon^{-1}\int_0^t p(s, y_0)\,ds\right) \tag{2.21}$$

and

$$\begin{aligned}\mathcal{U}(t, y) = \rho_0(y)&\exp\left(\varepsilon^{-1}\int_{y_0}^{y}((r^{1-N}u)(0, z) - (r^{1-N}u)(t, z))\,dz\right) \\ &\cdot\exp\left(-\varepsilon^{-1}\int_0^t\int_{y_0}^{y}(N-1)\frac{u^2(s, z)}{r^N}\,dz\,ds\right).\end{aligned} \tag{2.22}$$

Hence, it follows from the equation of mass, the energy inequality (2.2) and Lemma 2.3 that

$$
\begin{aligned}
\int_{y_0}^{M_0} r^{1-N}|u|\,dy &= \int_{r(y_0)}^{R} \rho|u|\,dr \le Cr^{1-N}(y_0)\int_{r(y_0)}^{R}\rho|u|r^{N-1}\,dr \\
&\le Cr^{1-N}(y_0)\left(\int_0^R \rho r^{N-1}\,dr\right)^{1/2}\left(\int_0^R \rho u^2 r^{N-1}\,dr\right)^{1/2} \\
&\le Cr^{1-N}(y_0)M_0^{1/2}E_0^{1/2} \\
&\le Cy_0^{-(N-1)\gamma/(N(\gamma-1))}M_0^{1/2}E_0^{(N\gamma+N-2)/(2N(\gamma-1))}
\end{aligned} \tag{2.23}
$$

and

$$
\begin{aligned}
\int_0^T\int_{y_0}^{M_0} r^{1-N}\frac{|u|^2}{r}\,dy\,dt &= \int_0^T\int_{r(y_0)}^{R}\frac{\rho|u|^2}{r}\,dr\,dt \\
&\le Cr^{-N}(y_0)\int_0^T\int_0^R \rho|u|^2 r^{N-1}\,dr\,dt \\
&\le Cr^{-N}(y_0)E_0T \le Cy_0^{-\gamma/(\gamma-1)}E_0^{\gamma/(\gamma-1)}T.
\end{aligned} \tag{2.24}
$$

Consequently, we have

$$
\begin{aligned}
&\mathcal{U}(t,y) \\
&\le C\left(\left(\sup_{x\in\Omega}\rho_0(x)\right)\exp\left\{C\varepsilon^{-1}y_0^{-(N-1)\gamma/(N(\gamma-1))}M_0^{1/2}E_0^{(N\gamma+N-2)/(2N(\gamma-1))}\right\}\right)
\end{aligned} \tag{2.25}
$$

and

$$
\begin{aligned}
&\mathcal{U}(t,y)^{-1} \\
&\le C\left(\left(\inf_{x\in\Omega}\rho_0(x)\right)^{-1}\exp\left\{C\varepsilon^{-1}y_0^{-(N-1)\gamma/(N(\gamma-1))}M_0^{1/2}E_0^{(N\gamma+N-2)/(2N(\gamma-1))}\right.\right. \\
&\qquad\left.\left. + C\varepsilon^{-1}y_0^{-\gamma/(\gamma-1)}E_0^{\gamma/(\gamma-1)}T\right\}\right), \quad (t,y)\in Q_{T,y_0}.
\end{aligned} \tag{2.26}
$$

If we set

$$
H = y_0^{-(N-1)\gamma/(N(\gamma-1))}M_0^{1/2}E_0^{(N\gamma+N-2)/(2N(\gamma-1))} + y_0^{-\gamma/(\gamma-1)}E_0^{\gamma/(\gamma-1)}T, \tag{2.27}
$$

the above estimates for $\mathcal{U}(t,y)$ can be simply written as

$$
\begin{aligned}
\mathcal{U}(t,y) &\le C\left(\sup_{x\in\Omega}\rho_0(x)\right)\exp(C\varepsilon^{-1}H), \\
\mathcal{U}^{-1}(t,y) &\le C\left(\inf_{x\in\Omega}\rho_0(x)\right)^{-1}\exp(C\varepsilon^{-1}H).
\end{aligned} \tag{2.28}
$$

On the other hand, it follows from (2.20) that

$$\begin{aligned}\frac{d}{dt}\exp\Big(\frac{\gamma}{\varepsilon}\int_0^t p(s,y)\,ds\Big) &= \frac{a\gamma}{\varepsilon}\rho(t,y)^{\gamma}\exp\Big(\frac{\gamma}{\varepsilon}\int_0^t p(s,y)\,ds\Big)\\ &= \frac{a\gamma}{\varepsilon}(\mathcal{P}(t)\mathcal{U}(t,y))^{\gamma},\end{aligned}\tag{2.29}$$

which implies

$$\exp\Big(\frac{1}{\varepsilon}\int_0^t p(s,y)\,ds\Big) = \Big(1+\frac{a\gamma}{\varepsilon}\int_0^t(\mathcal{P}(s)\mathcal{U}(s,y))^{\gamma}\,ds\Big)^{1/\gamma}.\tag{2.30}$$

Next, we are in a position to estimate $\mathcal{P}(t)$. First, observe that

$$\begin{aligned}\int_{y_0}^{M_0}\frac{dy}{\rho(t,y)} = \int_{r(y_0)}^{R} r^{N-1}\,dr &= \frac{R^N - r(y_0)^N}{N}\\ &\geq C(M_0-y_0)^{\gamma/(\gamma-1)}E_0^{-1/(\gamma-1)}.\end{aligned}\tag{2.31}$$

In view of (2.20) and (2.30), we have

$$\rho(t,y) = \frac{\mathcal{P}(t)\mathcal{U}(t,y)}{\big(1+(a\gamma/\varepsilon)\int_0^t(\mathcal{P}(s)\mathcal{U}(s,y))^{\gamma}\,ds\big)^{1/\gamma}}.\tag{2.32}$$

Then, $\mathcal{P}(t)$ can be estimated as

$$\begin{aligned}&\frac{R^N - r^N(y_0)}{N}\mathcal{P}(t) = \int_{y_0}^{M_0}\frac{\mathcal{P}(t)}{\rho(t,y)}\,dy\\ &= \int_{y_0}^{M_0}\frac{\big(1+(a\gamma/\varepsilon)\int_0^t(\mathcal{P}(s)\mathcal{U}(s,y))^{\gamma}\,ds\big)^{1/\gamma}}{\mathcal{U}(t,y)}\,dy\\ &\leq C\int_{y_0}^{M_0}\frac{1}{\mathcal{U}(t,y)}\,dy\\ &\quad + C\Big(\frac{a\gamma}{\varepsilon}\Big)^{1/\gamma}\Big(\sup_{Q_{T,y_0}}\mathcal{U}(t,y)\Big)\Big(\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\Big)\int_{y_0}^{M_0}\Big(\int_0^t\mathcal{P}(s)^{\gamma}\,ds\Big)^{1/\gamma}\,dy\\ &\leq C(M_0-y_0)\Big(\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\Big)\\ &\quad + C(M_0-y_0)\Big(\sup_{Q_{T,y_0}}\mathcal{U}(t,y)\Big)\Big(\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\Big)\Big(\int_0^t\mathcal{P}(s)^{\gamma}\,ds\Big)^{1/\gamma}.\end{aligned}\tag{2.33}$$

Using (2.31) and taking γ-th power on both sides of (2.33), we have

$$\left(\frac{M_0-y_0}{E_0}\right)^{\gamma/(\gamma-1)}\mathcal{P}(t)^\gamma \le C\left(\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\right)^\gamma + C\left(\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\right)^\gamma\left(\sup_{Q_{T,y_0}}\mathcal{U}(t,y)\right)^\gamma\left(\int_0^t \mathcal{P}(s)^\gamma\,ds\right). \tag{2.34}$$

Therefore, by Gronwall's inequality, we deduce from (2.34) that

$$\begin{aligned}\mathcal{P}(t) \le C\left(\frac{E_0}{M_0-y_0}\right)^{1/(\gamma-1)}&\left(\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\right)\\ &\cdot\exp\left\{CT\left(\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\right)^\gamma\left(\sup_{Q_{T,y_0}}\mathcal{U}(t,y)\right)^\gamma\right\}.\end{aligned} \tag{2.35}$$

Finally, recalling (2.32), we have

$$\rho(t,y) \le \mathcal{P}(t)\left(\sup_{Q_{T,y_0}}\mathcal{U}(t,y)\right), \tag{2.36}$$

and plugging the estimates (2.25), (2.26) and (2.35) into (2.36), we can deduce the desired pointwise estimate (2.16). Thus the proof of Lemma 2.4 is completed. □

3. Proof of Theorem 1.1

Note that (1.21) and (1.22) are direct consequences of Lemma 2.2 and Lemma 2.4. Now we are in a position to prove (1.20).

We argue by contradiction. Suppose (1.20) fails to hold. In Euler coordinate system (1.11), there exist positive constants C, ϵ ($< T^*$) and r_1 ($< R$) such that

$$C^{-1} \le \rho(t,r) \le C, \quad \text{for all} \quad (t,r)\in[T^*-\varepsilon,T^*)\times[0,r_1]. \tag{3.1}$$

Then we claim that in Lagrangian coordinate system (1.14), there exist a positive constant y_1 ($< M_0$) such that

$$\rho(t,y) \le C \quad \text{for all} \quad (t,y)\in[T^*-\varepsilon,T^*)\times[0,y_1]. \tag{3.2}$$

In fact, by virtue of (3.1), it holds

$$y(t,r_1) = \int_0^{r_1}\rho(t,r)r^2\,dr \ge \int_0^{r_1} C^{-1}r^2\,dr \ge \frac{1}{3}C^{-1}{r_1}^3, \quad t\in[T^*-\varepsilon,T^*) \tag{3.3}$$

which immediately implies (3.2). Now, it follows from (3.2) and Lemma 2.4 that the density is bounded on $[0,T^*)\times\Omega$, which contradicts to the blowup criterion Lemma 2.2. Thus the proof of Theorem 1.1 is completed.

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