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CALIBRATED SUBMANIFOLDS AND REDUCTIONS OF G_2-MANIFOLDS

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Abstract

(Co)associative submanifolds in a G_2-manifold with a free S^1 or T^2 action are characterized by submanifolds in the quotient space. Using our method, we construct various examples of (co)associative submanifolds and fibrations on G_2-manifolds with the T^2-symmetry such as the cone of the Iwasawa manifold.

1. Introduction

In 1996, Strominger, Yau and Zaslow [23] presented a conjecture explaining mirror symmetry of compact Calabi–Yau 3-folds in terms of dual fibrations by special Lagrangian 3-tori, including singular fibers. In M-theory, fibrations of coassociative 4-folds in compact manifolds with G_2 holonomy are expected to play the same role as special Lagrangian fibrations in Calabi–Yau manifolds [1], [2] and [15].

In this paper, we focus on (co)associative submanifolds in a G_2-manifold Y with a free S^1 or T^2-action. Since many known examples of G_2-manifolds such as those constructed by Bryant and Salamon [10] admit T^2-actions which are free on the open dense subsets, it is natural to consider the case when S^1 or T^2 acts on Y. Then we consider (co)associative submanifolds which are invariant under the S^1 or T^2-action or perpendicular to S^1 or T^2-orbits and characterize them by submanifolds in the quotient space Y/S^1 and Y/T^2. These are described in Theorems 3.7 and 4.13, which are our main theorems. Then using our characterization, we construct several examples of (co)associative submanifolds and fibrations in many cases.

This paper is organized as follows. In Section 2, we review the fundamental facts of calibrated geometry and G_2 geometry.

In Section 3, we study the case when a G_2-manifold Y admits a free S^1-action. It is known that for any Calabi–Yau 3-fold M^3, $M^3 \times S^1$ admits a torsion-free G_2 structure and its (co)associative submanifolds can be constructed from holomorphic or special Lagrangian submanifolds in M^3 (Example 2.10). On the other hand, it is known ([4]) that for a torsion-free G_2-manifold Y with a free S^1-action the quotient space Y/S^1 admits an SU(3)-structure (a generalized notion of a Calabi–Yau structure). Note that the torsion-free property of Y is not needed to define an SU(3)-structure on Y/S^1. The

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G_2-structure on Y is recovered in terms of tensors on Y/S^1 (Remark 3.6) and similar to that in Example 2.10. As a generalization of Example 2.10, we can characterize (co)associative submanifolds in Y by submanifolds in Y/S^1 (Theorem 3.7). We apply the proof to the case when Y is a (sine) cone and obtain similar results. Bryant [7] characterized associative cones ($\mathbb{R}_{>0}$-invariant associative 3-folds) in $\mathbb{R}^7$ using pseudo-holomorphic curves in S^6 and studied them in detail via the theory of integrable systems. Theorem 3.7 is an analogue of this by considering the S^1-action instead of the $\mathbb{R}_{>0}$-action.

Section 4 is the main section in this paper. We study the case when an almost G_2-manifold Y admits a free T^2-action. As in Section 3, it is known that a torsion-free G_2-manifold and (co)associative submanifolds can be constructed from a Calabi–Yau 2-fold M^2 and its submanifolds (Example 2.11). Using the notion of multi-moment maps [20], we see the following: there exists a smooth map $\underline{\nu}\colon Y/T^2 \to \mathbb{R}$ whose fibers are almost hyperkähler 2-folds. In other words, Y/T^2 admits three almost CR-structures satisfying the quaternionic relation. A G_2-structure is recovered in terms of tensors on Y/T^2 (Remark 4.12) and similar to that in Example 2.11. As a generalization of Example 2.11, we can characterize (co)associative submanifolds in Y by submanifolds in Y/T^2.

In Section 5, we give examples of (co)associative submanifolds and fibrations in G_2-manifolds by using our method.

2. Preliminaries

2.1. Calibrated geometry. The notion of the calibration was introduced by Harvey and Lawson [16]. This is a generalization of the Wirtinger inequality to the effect that any compact complex submanifold in a Kähler manifold minimizes its volume in its homology class.

DEFINITION 2.1. Let (M, g) be an m-dimensional Riemannian manifold and φ be a closed k-form on M $(1 \leq k \leq m)$. Then φ is called a *calibration* on M if for every oriented k-dimensional subspace $V \subset T_pM$, $p \in M$, we have $\varphi|_V \leq \mathrm{vol}_V$.

Let $N \subset M$ be a k-dimensional oriented submanifold of M. Then N is called a *calibrated submanifold* (*φ-submanifold*) of M if we have $\varphi|_N = \mathrm{vol}_N$.

By definition, a calibrated submanifold has the homologically minimizing volume. Calibrations are meaningful when they have many calibrated submanifolds. Assuming that φ is invariant under the holonomy group $\mathrm{Hol}(g)$, we can produce various calibrations that have many calibrated submanifolds. For instance, we have the following calibrations and corresponding calibrated submanifolds.

Hol(g) ($\subset$)	U(m)	SU(m)	G_2
(M, g)	Kähler	Calabi–Yau	G_2
φ	$\omega^k/k!$ (ω: Kähler form)	$\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$ (Ω: hol. volume form)	$\varphi \in \Omega^3$ ($*\varphi \in \Omega^4$) (φ: G_2-structure)
φ-submanifolds	k-dim. complex submanifolds	special Lagrangian submanifolds	(co)associative submanifolds

2.2. The holonomy group G_2.

DEFINITION 2.2. Define a 3-form φ_0 on $\mathbb{R}^7$ by

$$\varphi_0 = e_{123} + e_1(e_{45} + e_{67}) + e_2(e_{46} - e_{57}) - e_3(e_{47} + e_{56}),$$

where $(e_1, \ldots, e_7)$ is the standard dual basis on $\mathbb{R}^7$ and wedge signs are omitted. The stabilizer of φ_0 is the exceptional Lie group G_2:

$$G_2 = \{g \in GL(7, \mathbb{R}) \mid g^*\varphi_0 = \varphi_0\}.$$

This is a 14-dimensional compact simply-connected semisimple Lie group.

The Lie group G_2 also fixes the standard metric $g_0 = \sum_{i=1}^{7} e_i^2$, the orientation on $\mathbb{R}^7$, and the 4-form

$$*\varphi_0 = e_{4567} + e_{23}(e_{67} + e_{45}) + e_{13}(e_{57} - e_{46}) - e_{12}(e_{56} + e_{47}).$$

Note that φ_0 and $*\varphi_0$ are related by the Hodge $*$-operator. These tensors are uniquely determined by φ_0 via the relation

$$6g_0(v_1, v_2)\,\mathrm{vol}_{g_0} = i(v_1)\varphi_0 \wedge i(v_2)\varphi_0 \wedge \varphi_0, \tag{2.1}$$

where vol_{g_0} is a volume form of g_0, $i(\,\cdot\,)$ is an interior product, and $v_i \in T(\mathbb{R}^7)$.

DEFINITION 2.3. Let Y be a 7-dimensional oriented manifold and φ a 3-form on Y. We call a 3-form $\varphi \in \Omega^3(Y)$ a *G_2-structure* on Y if for each point $y \in Y$, there exists an oriented isomorphism between T_yY and $\mathbb{R}^7$ identifying φ_y with φ_0. From (2.1), a G_2-structure φ induces the Riemannian metric g on Y, volume form on Y and $*\varphi \in \Omega^4(Y)$.

A triple (Y, φ, g) is called a *G_2-manifold* if Y is a 7-dimensional oriented manifold, $\varphi \in \Omega^3(Y)$ is a G_2-structure on Y and g is an associated metric. A G_2-manifold (Y,φ,g) is called an *almost G_2-manifold* if φ is closed: $d\varphi = 0$. A G_2-manifold (Y,φ,g) is called a *torsion-free G_2-manifold* if φ is closed and coclosed: $d\varphi = 0$, $d * \varphi = 0$.

Lemma 2.4 ([14]). *Let (Y,φ,g) be a G_2-manifold. Then* $\mathrm{Hol}(g) \subset G_2$ *if and only if $d\varphi = d*\varphi = 0$.*

Lemma 2.5 ([16]). *Let (Y, φ, g) be a G_2-manifold. Then for each point $p \in Y$ and every oriented k-dimensional subspace $V^k \subset T_pY$ $(k = 3,4)$, we have $\varphi|_{V^3} \leq \mathrm{vol}_{V^3}$, $*\varphi|_{V^4} \leq \mathrm{vol}_{V^4}$. If (Y, φ, g) is torsion-free, the G_2-structure φ and its Hodge dual $*\varphi$ define calibrations on Y.*

DEFINITION 2.6 ([16]). Let (Y,φ,g) be a G_2-manifold. An oriented 3-dimensional submanifold L^3 is called an *associative submanifold* of Y if $\varphi|_{L^3} = \mathrm{vol}_{L^3}$. An oriented 4-dimensional submanifold L^4 is called a *coassociative submanifold* of Y if $*\varphi|_{L^4} = \mathrm{vol}_{L^4}$.

REMARK 2.7. If $d\varphi \neq 0$ (resp. $d*\varphi \neq 0$), associative (resp. coassociative) submanifolds need not have the homologically minimizing volume.

Lemma 2.8 ([16]). *Let (Y, φ, g) be a G_2-manifold. A* 4*-dimensional submanifold L^4 is coassociative if and only if $\varphi|_{TL^4} = 0$.*

2.3. Relations to Calabi–Yau manifolds. The only connected Lie subgroups of G_2 which can be the holonomy group of a Riemannian metric on a 7-dimensional manifold are $\{1\}$, SU(2), SU(3) and G_2. The inclusions SU(2), SU(3) $\subset G_2$ imply that we can make a G_2-manifold from a Calabi–Yau 2- or 3-fold with holonomy SU(2) or SU(3). Showing how to do this, we learn how to construct (co)associative submanifolds in each case.

DEFINITION 2.9. A quintuple $(M, h, J, \omega, \Omega)$ is called a *Calabi–Yau m-fold* if

- A quadruple (M,h,J,ω) is an m-dimensional Kähler manifold with a Kähler metric h, a complex structure J, and an associated Kähler form ω.
- Ω is a nowhere vanishing holomorphic $(m, 0)$-form on M.
- $\omega^m/m! = (-1)^{m(m-1)/2}(\sqrt{-1}/2)^m\,\Omega \wedge \bar{\Omega}$.

Then for any $\theta \in \mathbb{R}$, $\mathrm{Re}(e^{-\sqrt{-1}\theta}\Omega)$ defines a calibration on M. A real oriented m-dimensional submanifold of M is called a *special Lagrangian submanifold* of M with phase $e^{\sqrt{-1}\theta}$ if it is a $\mathrm{Re}(e^{-\sqrt{-1}\theta}\Omega)$-submanifold.

By definition, the following examples appear immediately.

EXAMPLE 2.10. Let $(M, h, J, \omega, \Omega)$ be a Calabi–Yau 3-fold, $\mathcal{I}$ be a circle S^1 or $\mathbb{R}$ and x be a coordinate on $\mathcal{I}$. Then $(Y, \varphi, g) := (\mathcal{I} \times M, dx \wedge \omega + \mathrm{Re}\,\Omega, dx^2 + h)$ is a torsion-free G_2-manifold with $*\varphi = \omega^2/2 - dx \wedge \mathrm{Im}\,\Omega$. Suppose that

- Σ is a holomorphic curve in M (i.e. Σ is a ω-submanifold),
- $L_{e^{\sqrt{-1}\theta}}$ is a special Lagrangian submanifold of M with phase $e^{\sqrt{-1}\theta}$,
- S is a holomorphic surface in M (i.e. S is a $\omega^2/2$-submanifold).

Then with an appropriate orientation,

1. $\mathcal{I} \times \Sigma$ is an associative 3-fold in Y,
2. $\mathcal{I} \times L_{\pm\sqrt{-1}}$ is a coassociative 4-fold in Y,
3. $\{x\} \times L_1$ is an associative 3-fold in Y $(x \in \mathcal{I})$,
4. $\{x\} \times S$ is a coassociative 4-fold in Y $(x \in \mathcal{I})$.

Signs of the phases of special Lagrangian submanifolds depend on the orientation.

EXAMPLE 2.11. Let $(M, h, J, \omega, \Omega)$ be a Calabi–Yau 2-fold and (x_1, x_2, x_3) be a coordinate on $\mathcal{I}^3$. Then $Y = \mathcal{I}^3 \times M$ is a torsion-free G_2-manifold with a 3-form φ and a metric g defined by

$$\varphi = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \omega + dx_2 \wedge \operatorname{Re}\Omega - dx_3 \wedge \operatorname{Im}\Omega,$$

$$g = dx_1^2 + dx_2^2 + dx_3^2 + h,$$

$$*\varphi = \frac{\omega^2}{2} + dx_2 \wedge dx_3 \wedge \omega - dx_1 \wedge dx_3 \wedge \operatorname{Re}\Omega - dx_1 \wedge dx_2 \wedge \operatorname{Im}\Omega.$$

Since $\mathrm{SU}(2) = \mathrm{Sp}(1)$, a Calabi–Yau 2-fold M is hyperkähler. So we have complex structures J_0, J_1, J_2 on M satisfying $J_0 J_1 J_2 = -id_{TM}$ associated with h and $\operatorname{Im}\Omega$, ω, $\operatorname{Re}\Omega$, respectively. For $(x_1, x_2, x_3) \in \mathcal{I}^3$, $m \in M$, if

- $\mathcal{O} \subset \mathcal{I}$ is an open interval and $U_M \subset M$ is an open set (i.e. U_M is a $\omega^2/2$-submanifold),
- Σ_i is a J_i-holomorphic curve (i.e. Σ_0 is an $\operatorname{Im}\Omega$-submanifold, Σ_1 is a ω-submanifold and Σ_2 is a $\operatorname{Re}\Omega$-submanifold),

then with an appropriate orientation,

1. $\mathcal{I}^2 \times (\mathcal{O} \times \{m\})$ is an associative 3-fold in Y.
2. $\mathcal{I}^2 \times (\{x_3\} \times \Sigma_0)$ is a coassociative 4-fold in Y.
3. $\mathcal{I} \times \{x_2\} \times (\{x_3\} \times \Sigma_1)$ is an associative 3-fold in Y.
4. $\mathcal{I} \times \{x_2\} \times (\mathcal{O} \times \Sigma_2)$ is a coassociative 4-fold in Y.
5. $\{(x_1, x_2)\} \times (\mathcal{O} \times \Sigma_0)$ is an associative 3-fold in Y.
6. $\{(x_1, x_2)\} \times (\{x_3\} \times U_M)$ is a coassociative 4-fold in Y.

In next sections we generalize Examples 2.10, 2.11 to G_2-manifolds on which S^1 or T^2 acts freely.

3. S^1 reduction of G_2-manifolds

3.1. Calibrated submanifolds in the S^1 quotient spaces. Let $(Y, \tilde{\varphi}, \tilde{g})$ be a G_2-manifold and suppose that S^1 acts freely on Y preserving the G_2-structure. In this section, we discuss calibrated submanifolds in Y invariant under the S^1-action in terms of submanifolds in Y/S^1.

From [4], we know that the quotient space Y/S^1 admits an SU(3)-structure, a reduction of the total coframe bundle to an SU(3)-bundle. It is a generalization of the

Calabi–Yau structure (that is, the torsion-free SU(3)-structure). We define an SU(3)-structure in terms of tensors here.

DEFINITION 3.1 ([11], [12]). A quintuple $(g, J, \sigma, \psi^{\pm})$ on a real 6-dimensional manifold N is called an SU(3)-*structure* if

- A quadruple (N, g, J, σ) is an almost Hermitian manifold with a Hermitian metric g, an almost complex structure J and an associated Kähler form σ.
- $\psi^{\pm} \in \Omega^3(N)$ are 3-forms on N with norms $\|\psi^{\pm}\| = 2$ satisfying $\psi^- = \psi^+(J\,\cdot, J\,\cdot, J\,\cdot)$ and $\Psi := \psi^+ + \sqrt{-1}\psi^-$ is a (3, 0)-form w.r.t. J.

REMARK 3.2. The forms ψ^+ and ψ^- are $(3,0)$- and $(0,3)$-forms with respect to J so that $\psi^{\pm}(J\,\cdot, J\,\cdot, \,\cdot) = -\psi^{\pm}$. These forms are subject to the following compatibility relations:

$$\sigma \wedge \psi^{\pm} = 0, \quad \psi^+ \wedge \psi^- = \frac{2}{3}\sigma^3.$$

The former is equivalent to saying that σ is a $(1,1)$-form with respect to J. The latter is equivalent to $\sigma^3/3! = (-1)^{3(3-1)/2}(\sqrt{-1}/2)^3\Psi \wedge \bar{\Psi}$. Therefore if σ is closed, J is integrable, and Ψ is a holomorphic (3, 0)-form, then the SU(3)-structure is a Calabi–Yau structure.

REMARK 3.3. For any $\theta \in \mathbb{R}$, $p \in N$ and oriented 3-dimensional subspace $V \subset T_pN$, we have $\mathrm{Re}(e^{-\sqrt{-1}\theta}\Psi)|_V \leq \mathrm{vol}_V$. As in the Calabi–Yau case, an oriented 3-dimensional submanifold $L \subset N$ is called a *special Lagrangian submanifold* of N with phase $e^{\sqrt{-1}\theta}$ if $\mathrm{Re}(e^{-\sqrt{-1}\theta}\Psi)|_L = \mathrm{vol}_L$.

Proposition 3.4 ([4][1]). *Let $(Y, \tilde{\varphi}, \tilde{g})$ be a G_2-manifold and suppose that S^1 acts freely on Y preserving the G_2-structure. Then Y/S^1 admits an* SU(3)-*structure* $(g, J_1, \tau_1, \psi^{\pm})$.

REMARK 3.5. The torsion-free property of Y assumed in [4] is not needed to define an SU(3)-structure on Y/S^1. If Y is torsion-free, Y/S^1 admits symplectic structure. For a Kähler manifold N of real dimension 6, the conditions are given in [4] to be $N = Y/S^1$ for some torsion-free G_2-manifold Y with a free S^1-action.

The tensors defining an SU(3)-structure on Y/S^1 can be described as follows: Let $X_1^* \in \mathfrak{X}(Y)$ be a vector field generated by the S^1-action, and let $\pi_1 \colon Y \to Y/S^1$ be the

[1]An SU(3)-structure on Y/S^1 introduced in [4] seems to be different from ours. In fact, for any SU(3)-structure (g, J, σ, ψ^+) and any positive smooth function $r \colon Y/S^1 \to \mathbb{R}_{>0}$, $(g', J', \sigma', \psi^{+'}) := (r^2 g, J, r^2\sigma, r^3\psi^+)$ also defines an SU(3)-structure on Y/S^1. So we can define the SU(3)-structure on Y/S^1 as above.

projection. Define a function $\tilde{t}_1 = \tilde{g}(X_1^*, X_1^*)^{-1/2} \in C^\infty(Y)$, a 1-form $\tilde{\eta}_1 = \tilde{g}(\,\cdot\,, \tilde{t}_1^2 X_1^*) \in \Omega^1(Y)$, a 2-form $\tilde{\sigma}_1 = i(X_1^*)\tilde{\varphi} \in \Omega^2(Y)$ and a 3-form $\tilde{\Psi}^- = -i(X_1^*)(*\tilde{\varphi}) \in \Omega^3(Y)$.

The 1-form $\tilde{\eta}_1$ is a connection 1-form of $\pi_1 \colon Y \to Y/S^1$ since $\tilde{\eta}_1$ is S^1-invariant and satisfies $\tilde{\eta}_1(X_1^*) = 1$. The tensors $\tilde{t}_1, \tilde{\sigma}_1, \tilde{\Psi}^-$ induce a function $t_1 \in C^\infty(Y/S^1)$, a 2-form $\sigma_1 \in \Omega^2(Y/S^1)$ and a 3-form $\Psi^- \in \Omega^3(Y/S^1)$. Then

- $g = (\pi_1)_* \tilde{g}$, the pushforward of $\tilde{g}$,
- $\tau_1 = t_1 \sigma_1$,
- J_1: an almost complex structure satisfying $g(J_1 \,\cdot\,, \,\cdot\,) = \tau_1$,
- $\psi^- = t_1 \Psi^-$,
- $\psi^+ = -\psi^-(J_1 \,\cdot\,, J_1 \,\cdot\,, J_1 \,\cdot\,) = \psi^-(J_1 \,\cdot\,, \,\cdot\,, \,\cdot\,)$.

REMARK 3.6. We can recover the metric $\tilde{g}$ and the G_2-structure $\tilde{\varphi}$ on Y as follows:

$$\begin{aligned}
\tilde{g} &= \pi_1^* g + \tilde{t}_1^{-2} \tilde{\eta}_1 \otimes \tilde{\eta}_1, \\
\tilde{\varphi} &= \tilde{t}_1^{-1} \tilde{\eta}_1 \wedge \pi_1^* \tau_1 + \pi_1^* \psi^+, \\
*\tilde{\varphi} &= \frac{1}{2} \pi_1^* \tau_1^2 - \tilde{t}_1^{-1} \tilde{\eta}_1 \wedge \pi_1^* \psi^-.
\end{aligned}$$

These descriptions generalize Example 2.10. In fact, a similar statement holds for (co)associative submanifolds in Y invariant under the S^1-action.

Theorem 3.7. *Let $(Y, \tilde{\varphi}, \tilde{g})$ be a G_2-manifold with a free S^1-action preserving the G_2-structure. Let $\pi_1 \colon Y \to Y/S^1$ be the natural projection. By* Proposition 3.4, *Y/S^1 admits an* SU(3)*-structure $(g, J_1, \tau_1, \psi^\pm)$.*

If $L_{S^1}^k \subset Y$ is an oriented k-dimensional submanifold invariant under the S^1-action, then with respect to an appropriate orientation the following properties hold:

1. *$L_{S^1}^3 \subset Y$ is an associative* 3*-fold if and only if $\pi_1(L_{S^1}^3) \subset Y/S^1$ is a J_1-holomorphic curve (i.e. $T(\pi_1(L_{S^1}^3))$ is J_1-invariant).*
2. *$L_{S^1}^4 \subset Y$ is a coassociative* 4*-fold if and only if $\pi_1(L_{S^1}^4) \subset Y/S^1$ is a special Lagrangian submanifold with phase $\pm\sqrt{-1}$.*

If $L_p^k \subset Y$ is an oriented k-dimensional submanifold perpendicular to the S^1-orbits $(k = 3, 4)$*, then with respect to an appropriate orientation the following properties hold*:

3. *$L_p^3 \subset Y$ is an associative* 3*-fold if and only if $\pi_1(L_p^3) \subset Y/S^1$ is a special Lagrangian submanifold with phase* 1.
4. *$L_p^4 \subset Y$ is a coassociative* 4*-fold if and only if $\pi_1(L_p^4) \subset Y/S^1$ is a J_1-holomorphic surface.*

Corollary 3.8. *There is a one to one correspondence between S^1-invariant associative* 3*-folds* (*resp. S^1-invariant coassociative* 4*-folds*) *in Y and J_1-holomorphic curves* (*resp. special Lagrangian submanifolds with phase $\pm\sqrt{-1}$*) *in Y/S^1.*

REMARK 3.9. It is known that there is one to one correspondence between associative cones ($\mathbb{R}_{>0}$-invariant associative 3-folds) in $\mathbb{R}^7$ and pseudoholomorphic curves in S^6 (cf. [7]). The standard $\mathbb{R}_{>0}$-action on $\mathbb{R}^7$ preserves the G_2-structure on $\mathbb{R}^7$ up to constant. Considering the S^1-action instead of the $\mathbb{R}_{>0}$-action, we can regard Corollary 3.8 as an analogue of this fact.

It is known that 4-dimensional almost complex manifolds are fibrated locally by pseudoholomorphic discs (cf. [3]). From 1 of Theorem 3.7, we see the following.

Corollary 3.10. *A G_2-manifold with a free S^1-action which preserves the G_2-structure is locally fibrated by S^1-invariant associative 3-folds.*

REMARK 3.11 (Relations to evolution equations). Consider the case 1 of Theorem 3.7. Let $\gamma\colon \Sigma \to Y/S^1$ be a smooth immersion of a surface Σ. Let $(U, (s, t))$ be a local conformal coordinate of Σ. Then from the local triviality of $\pi_1\colon Y \to Y/S^1$, there exists a local lift $\tilde{\gamma}\colon U \to Y$ of γ on a small open set $U \subset \Sigma$ which is transverse to S^1-orbits.

The differential equation of J_1-holomorphic curve is $\partial\gamma/\partial s + J_1\,\partial\gamma/\partial t = 0$. By the definition of J_1, this equation can be described as

$$\left(\frac{\partial\tilde{\gamma}}{\partial s}\right)^d = -\tilde{\varphi}_{abc}\left(\frac{X_1^*}{\|X_1^*\|}\right)^a\left(\frac{\partial\tilde{\gamma}}{\partial t}\right)^b\tilde{g}^{cd}.$$

So the differential equation of J_1-holomorphic curve is considered as the special case of the evolution equations in [17], [18]. Lotay [17], [18] constructed examples of (co)associative submanifolds in $\mathbb{R}^7$ by evolution equations.

Proof of Theorem 3.7. The proof is implied from Example 2.10. Fix one of the orientations of submanifolds. The same proof is valid for another orientation.

Proof of 1. Take any $x \in L^3_{S^1}$ and choose an arbitrary oriented orthonormal basis $\{\tilde{t}_1(X_1^*)_x, \tilde{v}_1, \tilde{v}_2\}$ of $T_x L^3_{S^1}$. Then $\{v_1, v_2\} = \{\pi_{1*}\tilde{v}_1, \pi_{1*}\tilde{v}_2\}$ is an oriented orthonormal basis of $T_{\pi_1(x)}(\pi_1(L^3_{S^1}))$. Thus $L^3_{S^1}$ is associative if and only if $\tilde{\varphi}(\tilde{t}_1(X_1^*)_x, \tilde{v}_1, \tilde{v}_2) = 1$. By Remark 3.6, this condition is equivalent to $g(J_1 v_1, v_2) = 1$, and hence $v_2 = J_1 v_1$ follows from the Cauchy–Schwarz inequality. So $T(\pi_1(L^3_{S^1}))$ is J_1-invariant and $\pi_1(L^3_{S^1})$ is a J_1-holomorphic curve.

Proof of 2. Take any $x \in L^4_{S^1}$ and choose an arbitrary oriented orthonormal basis $\{\tilde{t}_1(X_1^*)_x, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ of $T_x L^4_{S^1}$ and $v_i = \pi_{1*}\tilde{v}_i \in T_{\pi_1(x)}(\pi_1(L^4_{S^1}))$. Then $L^4_{S^1}$ is coassociative if and only if $*\tilde{\varphi}(\tilde{t}_1(X_1^*)_x, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3) = 1$. This is equivalent to $-\psi^-(v_1, v_2, v_3) = 1$. Namely, $\pi_1(L^4_{S^1}) \subset Y/S^1$ is a special Lagrangian submanifold with phase $-\sqrt{-1}$. Note that for another orientation of $L^4_{S^1}$, we see that $\pi_1(L^4_{S^1}) \subset Y/S^1$ is a special Lagrangian submanifold with phase $\sqrt{-1}$.

Proof of 3. Take any $x \in L_p^3$ and choose an arbitrary oriented orthonormal basis $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ of $T_x L_p^3$ and $v_i = \pi_{1*}\tilde{v}_i \in T_{\pi_1(x)}(\pi_1(L_p^3))$. By definition, we have $\tilde{\eta}_1(\tilde{v}_i) = 0$. Then L_p^3 is associative if and only if $\tilde{\varphi}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) = 1$. This is equivalent to $\psi^+(v_1, v_2, v_3) = 1$.

Proof of 4. We follow the proof of the Wirtinger inequality in [5]. Similarly to the former proof, we see L_p^4 is coassociative if and only if $\tau_1^2/2|_{T_{\pi_1(x)}(\pi_1(L_p^4))} = \mathrm{vol}_{T_{\pi_1(x)}(\pi_1(L_p^4))}$ for any $x \in L_p^4$. By the spectral decomposition of the skew-symmetric 2-form $\tau_1|_{T_{\pi_1(x)}(\pi_1(L_p^4))}$, we know that there exists an oriented orthonormal basis $\{w_1, w_2, w_3, w_4\} \subset T_{\pi_1(x)}(\pi_1(L_p^4))$ and its dual basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset T^*_{\pi_1(x)}(\pi_1(L_p^4))$ satisfying

$$\tau_1|_{T_{\pi_1(x)}(\pi_1(L_p^4))} = \lambda_1\alpha_1 \wedge \alpha_2 + \lambda_2\alpha_3 \wedge \alpha_4$$

for some $\lambda_i \in \mathbb{R}$. Then $\tau_1^2/2|_{T_{\pi_1(x)}(\pi_1(L_p^4))} = \lambda_1\lambda_2\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 = \lambda_1\lambda_2 \,\mathrm{vol}_{T_{\pi_1(x)}(\pi_1(L_p^4))}$ follows. On the other hand, $\lambda_i = \tau_1(w_{2i-1}, w_{2i}) = g(J_1 w_{2i-1}, w_{2i}) \leq 1$ holds by the Cauchy–Schwarz inequality, where the equality holds if and only if $w_{2i} = J_1 w_{2i-1}$.

Since $\tau_1^2/2|_{T_{\pi_1(x)}(\pi_1(L_p^4))} = \mathrm{vol}_{T_{\pi_1(x)}(\pi_1(L_p^4))}$, we have $\lambda_1 = \lambda_2 = 1$. This implies that $T(\pi_1(L_p^4))$ is J_1-invariant and hence $\pi_1(L_p^4)$ is a J_1-holomorphic surface. □

3.2. Application to cones and sine cones. The similar statement holds when a G_2-manifold is a (sine) cone. First, we introduce the notion of nearly Kähler manifolds.

DEFINITION 3.12 ([11], [12], [24]). Let $(g, J, \sigma, \psi^\pm)$ be an SU(3)-structure on a 6-dimensional manifold N. An SU(3)-structure satisfying $d\sigma = 3\psi^+$ and $d\psi^- = -2\sigma^2$ is called *nearly Kähler*.

REMARK 3.13 ([24][1]). Let (N, g, J) be a 6-dimensional almost Hermitian manifold. Then the following are equivalent:

- N admits a nearly Kähler structure,
- $(\nabla_X J)X = 0$ for every vector field X on N and $\nabla_X J \neq 0$ for every $0 \neq X \in TN$, where ∇ is the Levi-Civita connection of g.

Lemma 3.14 ([12], [24]). *Let $(N, g, J, \sigma, \psi^\pm)$ be a nearly Kähler manifold. Then $C(N) = N \times \mathbb{R}_{>0}$ admits a torsion-free G_2-structure defined by*

$$\tilde{g} = dr^2 + r^2 g, \quad \tilde{\varphi} = r^2\, dr \wedge \sigma + r^3\psi^+ = \frac{1}{3}\, d(r^3\sigma),$$
$$*\tilde{\varphi} = r^3\psi^- \wedge dr + \frac{1}{2} r^4\sigma^2 = -\frac{1}{4}\, d(r^4\psi^-).$$

[1]Our definition of nearly Kähler manifolds corresponds to that of "strictly" nearly Kähler ones in [24].

The metric is just the cone metric on $C(N)$. Thus nearly Kähler manifolds are analogue of Sasakian manifolds whose cones are Kähler manifolds.

Lemma 3.15 ([6]). *Let* $(N, g, J, \sigma, \psi^{\pm})$ *be a nearly Kähler manifold. Then* $C_s(N) = N \times (0, \pi)$ (*a sine cone of* N) *admits a nearly parallel* G_2*-structure* $(\tilde{\varphi}, \tilde{g})$ *with*

$$\begin{aligned}\tilde{g} &= dt^2 + \sin^2 t g,\\ \tilde{\varphi} &= \sin^2 t\, dt \wedge \sigma + \cos t \sin^3 t \psi^+ - \sin^4 t \psi^-,\\ *\tilde{\varphi} &= \frac{1}{2} \sin^4 t \sigma^2 + \sin^3 t \cos t \psi^- \wedge dt - \sin^4 t\, dt \wedge \psi^+.\end{aligned}$$

Here, a G_2*-manifold* $(Y, \tilde{\varphi}, \tilde{g})$ *is said to be* nearly parallel *if* $d\tilde{\varphi} = 4 * \tilde{\varphi}$, $d * \tilde{\varphi} = 0$.

REMARK 3.16. Since $C(N)$ is a torsion-free G_2-manifold, $C(C_s(N)) \cong \mathbb{R} \times C(N)$ admits a torsion-free Spin(7)-structure. The nearly parallel G_2-structure on $C_s(N)$ is induced from the torsion-free Spin(7)-structure on $C(C_s(N))$.

REMARK 3.17 ([19]). There are no coassociative submanifolds of a nearly parallel G_2-manifold.

Proof. If L is a coassociative submanifold of a nearly parallel G_2-manifold $(Y, \tilde{\varphi}, \tilde{g})$, then $\tilde{\varphi}|_L = 0$, which implies that $4 * \tilde{\varphi} = d\tilde{\varphi}|_L = 0$. This contradicts the assumption that L is coassociative. $\square$

We can prove the results similar to Theorem 3.7 as follows.

Proposition 3.18. *Let* $(N, g, J, \sigma, \psi^{\pm})$ *be a nearly Kähler manifold. From* Lemma 3.14, *the cone* $C(N) = N \times \mathbb{R}_{>0}$ *admits a torsion-free* G_2 *structure. If* $L^k \subset N$ *is an oriented k-dimensional submanifold* $(k = 2, 3, 4)$ *and* $r \in \mathbb{R}_{>0}$, *then with respect to an appropriate orientation the following properties hold*:

1. $C(L^2) = L^2 \times \mathbb{R}_{>0} \subset C(N)$ *is an associative* 3*-fold if and only if* L^2 *is a* J*-holomorphic curve.*
2. $C(L^3) = L^3 \times \mathbb{R}_{>0} \subset C(N)$ *is a coassociative* 4*-fold if and only if* L^3 *is a special Lagrangian submanifold with phase* $\pm\sqrt{-1}$.
3. $L^3 \times \{r\} \subset Y$ *is an associative* 3*-fold if and only if* L^3 *is a special Lagrangian submanifold with phase* 1,
4. $L^4 \times \{r\} \subset Y$ *is a coassociative* 4*-fold if and only if* L^4 *is a* J*-holomorphic surface.*

Proposition 3.19. *Let* $(N, g, J, \sigma, \psi^{\pm})$ *be a nearly Kähler manifold. From* Lemma 3.15, *the sine cone* $C_s(N) = N \times (0, \pi)$ *admits a nearly parallel* G_2 *structure. If* $L^k \subset N$ *be an oriented k-dimensional submanifold* $(k = 2, 3)$ *and* $t \in (0, \pi)$, *then with respect to an appropriate orientation the following properties hold*:

1. $C_s(L^2) = L^2 \times (0, \pi) \subset C_s(N)$ *is an associative* 3*-fold if and only if* L^2 *is a* J*-holomorphic curve,*
2. $L^3 \times \{\pi/2\} \subset C_s(N)$ *is an associative* 3*-fold if and only if* L^3 *is a special Lagrangian submanifold with phase* $\pm\sqrt{-1}$.

REMARK 3.20. If $L^3 \times \{t\} \subset C_s(N)$ is associative for some $t \in (0, \pi)$, we have $t = \pi/2$.

Proof. Suppose that $L^3 \times \{t\} \subset C_s(N)$ is associative for some $t \in (0, \pi)$. Then we easily see that $\sigma|_{L^3} = 0$, $\mathrm{Im}(e^{\pm\sqrt{-1}t}(\psi^+ + \sqrt{-1}\psi^-))|_{L^3} = 0$, and $\mathrm{Re}(e^{\pm\sqrt{-1}t}(\psi^+ + \sqrt{-1}\psi^-))|_{L^3} = \mathrm{vol}_{L^3}$, which imply that $\psi^+|_{L^3} = (1/3)\, d\sigma|_{L^3} = 0$, $\cos t \cdot \psi^-|_{L^3} = 0$. Hence if $t \neq \pi/2$, we have $(\psi^+ + \sqrt{-1}\psi^-)|_{L^3} = 0$, which is a contradiction. □

4. T^2 reduction of almost G_2-manifolds

Let $(Y, \tilde{\varphi}, \tilde{g})$ be an almost G_2-manifold on which a 2-torus T^2 acts preserving the G_2-structure. As in the former section, we discuss the geometry of the quotient space "Y/T^2".

4.1. Multi-moment maps and reduced spaces. We use a notion of the multi-moment map introduced by Madsen and Swann [20], which is a generalization of the moment map in symplectic geometry.

DEFINITION 4.1. Let $(Y, \tilde{\varphi}, \tilde{g})$ be an almost G_2-manifold on which a 2-torus T^2 acts preserving the G_2-structure. Fix vector fields $X_1^*, X_2^* \in \mathfrak{X}(Y)$ generated by a basis $\{X_1, X_2\}$ of the Lie algebra $\mathfrak{t}^2$ of T^2. A T^2-invariant function $\tilde{\nu}\colon Y \to \mathbb{R}$ is called a *multi-moment map* for the T^2-action if we have

$$\tilde{\varphi}(X_1^*, X_2^*, \cdot) = d\tilde{\nu}.$$

The multi-moment map is defined for any Lie group G in [20]. We focus here on the case $G = T^2$. There are results on the existence for the multi-moment map, which correspond to those of the moment map in symplectic geometry.

Proposition 4.2 ([20]). *The multi-moment map for a* T^2*-action exists if either of the following conditions holds*:

- $b_1(Y) = 0$, *where* $b_1(Y)$ *is the first Betti number of* Y.
- $\tilde{\varphi} = d\tilde{\kappa}$ *with a* 2*-form* $\tilde{\kappa} \in \Omega^2(Y)$ *preserved by the* T^2*-action.*

Proposition 4.3. *For* $y \in Y$, *the following conditions are equivalent*:

- $(d\tilde{\nu})_y = 0$.
- $(X_1^*)_y$ *and* $(X_2^*)_y$ *are linearly dependent.*

- $\dim(T^2\text{-}orbit\ through\ y) < 2$.

Madsen and Swann [20] considered a "T^2-reduction" of a torsion-free G_2-manifold and show that the reduced space admits a "coherently tri-symplectic" structure, which consists of three symplectic structures satisfying some conditions, but not necessarily satisfy the hyperkähler relation as follows: We can show that reduced space admits three 2-forms which are not necessarily closed, but satisfy the hyperkähler relation.

Proposition 4.4. *Let $(Y, \tilde{\varphi}, \tilde{g})$ be an almost G_2-manifold on which a 2-torus T^2 acts preserving the G_2-structure. Suppose that there exists a multi-moment map $\tilde{\nu}\colon Y \to \mathbb{R}$, and that T^2 acts freely on $\tilde{\nu}^{-1}(\alpha)$ for a regular value α of $\tilde{\nu}$. Then $M_\alpha := \tilde{\nu}^{-1}(\alpha)/T^2$ (a T^2-*reduction *of Y at level α) is a smooth 4-manifold.*

On the reduced space M_α, there exists a metric $\underline{g_\alpha}$ induced from $\tilde{g}$. Define three nondegenerate 2-forms $\underline{\tau_{0,\alpha}}$, $\underline{\tau_{1,\alpha}}$, $\underline{\tau_{2,\alpha}} \in \Omega^2(M_\alpha)$ as those induced from 2-forms $\tilde{\tau}_0 = -i(X_2^{*'})i(X_1^{*'})*\tilde{\varphi}$, $\tilde{\tau}_1 = i(X_1^{*'})\tilde{\varphi}$, $\tilde{\tau}_2 = i(X_2^{*'})\tilde{\varphi} \in \Omega^2(Y)$, respectively. Here $\{X_1^{*'}, X_2^{*'}\}$ are orthonormal vector fields on Y obtained from $\{X_1^*, X_2^*\}$ via the Gram–Schmidt process. If $\underline{J_{i,\alpha}}$ is an almost complex structure associated with $\underline{g_\alpha}$ and $\underline{\tau_{i,\alpha}}$ $(0 \le i \le 2)$, then we have

- *A quintuple $(M_\alpha, \underline{J_{0,\alpha}}, \underline{J_{1,\alpha}}, \underline{J_{2,\alpha}}, \underline{g_\alpha})$ is an almost hyperkähler manifold, (i.e. $\underline{J_{0,\alpha} J_{1,\alpha} J_{2,\alpha}} = -id_{TM_\alpha}$ and $\underline{g_\alpha}$ is Hermitian w.r.t. each $\underline{J_{i,\alpha}}$.)*
- *$\underline{\tau_{i,\alpha}} = \underline{g_\alpha}(\underline{J_{i,\alpha}}\,\cdot\,,\ \cdot\,)$, for $0 \le i \le 2$.*

The proof is given by a local argument, which follows from the next lemma.

Lemma 4.5. *Choose any $y \in Y$, where $(X_1^*)_y$ and $(X_2^*)_y$ are linearly independent. Let $\{E_i\}_{1 \le i \le 7}$ be the standard basis of $\mathbb{R}^7$ and $\{e_i\}_{1 \le i \le 7}$ be its dual.*

Since G_2 acts transitively on the Grassmannian of oriented 2-planes in $\mathbb{R}^7$ [8] and by the definition of G_2-structure, there exists an oriented isomorphism between T_yY and $\mathbb{R}^7$ identifying $\tilde{\varphi}_y$, $(X_1^{*'})_y$, $(X_2^{*'})_y$ and φ_0, E_1, E_2, respectively. Via this identification, we see the following:

- $d\tilde{\nu} = (1/h)e_3$,
- $TM_\alpha \cong \mathrm{span}_{\mathbb{R}}\{E_4, E_5, E_6, E_7\}$,
- $\underline{g_\alpha} = \sum_{i=4}^7 (e_i)^2$,
- $\tilde{\tau}_0 = e_{56} + e_{47}$, $\tilde{\tau}_1 = e_{23} + e_{45} + e_{67}$, $\tilde{\tau}_2 = -e_{13} + e_{46} - e_{57}$,
- $\underline{\tau_{0,\alpha}} = e_{56} + e_{47}$, $\underline{\tau_{1,\alpha}} = e_{45} + e_{67}$, $\underline{\tau_{2,\alpha}} = e_{46} - e_{57}$,

where $1/h = \|X_1^* \wedge X_2^*\| = \sqrt{\|X_1^*\|^2\|X_2^*\|^2 - g(X_1^*, X_2^*)^2}$. With respect to $\{E_4, E_5,$

$E_6, E_7\}$, *we can express*

$$\underline{J_{0,\alpha}} = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \quad \underline{J_{1,\alpha}} = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix},$$

$$\underline{J_{2,\alpha}} = \begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}.$$

Proof. We can denote $X_1^{*'} = aX_1^*$, $X_2^{*'} = bX_1^* + cX_2^*$, where $1/a = \|X_1^*\|$, $b = -ahg(X_1^*, X_2^*)$, $c = h/a$. Then we may consider $X_1^* = (1/a)E_1$, $X_2^* = (a/h)E_2 - (b/h)E_1$. Hence $d\tilde{\nu} = \tilde{\varphi}(X_1^*, X_2^*, \cdot) = (1/h)\varphi_0(E_1, E_2, \cdot) = (1/h)e_3$. Other formulas follow similarly. □

4.2. Coassociative submanifolds in the reduced space. In this subsection, we consider coassociative submanifolds in almost G_2-manifolds using the multi-moment map and the reduced space.

Lemma 4.6. *Let $(Y, \tilde{\varphi}, \tilde{g})$ be an almost G_2-manifold with a T^2-action on Y preserving the G_2-structure. Suppose that there exists a multi-moment map $\tilde{\nu}\colon Y \to \mathbb{R}$ for the T^2-action. Then for every connected T^2-invariant coassociative* 4*-fold L, there exists $\alpha \in \mathbb{R}$ satisfying*

$$L \subset \tilde{\nu}^{-1}(\alpha).$$

Proof. Since L is T^2-invariant, for any $p \in L$, $(X_1^*)_p, (X_2^*)_p \in T_pL$. Moreover, L is a coassociative 4-fold if and only if $\tilde{\varphi}|_L = 0$. Then

$$d\tilde{\nu}|_{T_pL} = \tilde{\varphi}((X_1^*)_p, (X_2^*)_p, \cdot)|_{T_pL} = 0.$$

So $d\tilde{\nu}|_{TL} = 0$ and this implies the lemma because L is connected. □

Theorem 4.7. *Let $(Y, \tilde{\varphi})$ be an almost G_2-manifold with a T^2-action preserving the G_2-structure. Suppose that there exists a multi-moment map $\tilde{\nu}\colon Y \to \mathbb{R}$ for the T^2-action and that for a regular value α of $\tilde{\nu}$, T^2 acts freely on $\tilde{\nu}^{-1}(\alpha)$. Let $\pi_{2,\alpha}\colon \tilde{\nu}^{-1}(\alpha) \to \tilde{\nu}^{-1}(\alpha)/T^2 = M_\alpha$ be the projection. By* Proposition 4.4, *M_α admits an almost hyperkähler structure $(\underline{J_{0,\alpha}}, \underline{J_{1,\alpha}}, \underline{J_{2,\alpha}}, \underline{g_\alpha})$.*

Then for an oriented 2*-dimensional submanifold $\Sigma \subset M_\alpha$, the following are equivalent*:

1. $\pi_{2,\alpha}^{-1}(\Sigma)$ *is a T^2-invariant coassociative* 4*-fold of Y,*
2. $\underline{\tau_{1,\alpha}}|_\Sigma = \underline{\tau_{2,\alpha}}|_\Sigma = 0$,

3. Σ *is a* $\underline{J_{0,\alpha}}$*-holomorphic curve.*

Proof. First, we prove the equivalence of 1 and 2. Take any $x \in \pi_{2,\alpha}^{-1}(\Sigma)$ and choose an arbitrary basis $\{(X_1^{*'})_x, (X_2^{*'})_x, \tilde{v}_1, \tilde{v}_2\}$ of $T_x(\pi_{2,\alpha}^{-1}(\Sigma))$. Then $\{\underline{v_1}, \underline{v_2}\} = \{\pi_{2*}\tilde{v}_1, \pi_{2*}\tilde{v}_2\}$ is a basis of $T_{\pi_{2,\alpha}(x)}\Sigma$. So $\pi_{2,\alpha}^{-1}(\Sigma)$ is coassociative if and only if

$$d\tilde{\nu}(\tilde{v}_i) = \tilde{\varphi}(X_1^*, X_2^*, \tilde{v}_i) = 0 \quad (i = 1, 2), \quad \tilde{\varphi}(X_j^{*'}, \tilde{v}_1, \tilde{v}_2) = 0 \quad (j = 1, 2).$$

The first condition is always satisfied since $\pi_{2,\alpha}^{-1}(\Sigma) \subset \tilde{\nu}^{-1}(\alpha)$. The second condition is equivalent to $\underline{\tau_{j,\alpha}}(\underline{v_1}, \underline{v_2}) = 0$. This implies the equivalence of 1 and 2.

Next, we prove the equivalence of 2 and 3.

Take any $p \in \Sigma$ and choose any orthonormal basis $\{\underline{v_1}, \underline{v_2}\}$ of $T_p\Sigma$. We can take $\{\underline{v_1}, \underline{J_{0,\alpha}}\underline{v_1}, \underline{J_{1,\alpha}}\underline{v_1}, \underline{J_{2,\alpha}}\underline{v_1}\}$ as an orthonormal basis of T_pM_α. Then the statement 3 holds if and only if $\underline{\tau_{k,\alpha}}(\underline{v_1}, \underline{v_2}) = \underline{g_\alpha}(\underline{J_{k,\alpha}}\underline{v_1}, \underline{v_2}) = 0$ $(k = 1, 2)$, which is equivalent to $\underline{v_2} = \pm\underline{J_{0,\alpha}}\underline{v_1}$, namely Σ is a $\underline{J_{0,\alpha}}$-holomorphic curve. $\square$

Corollary 4.8. *As in* Corollary 3.10, *we see that* $\tilde{\nu}^{-1}(\alpha)$ $(\subset Y)$ *is locally fibrated by* T^2*-invariant coassociative* 4*-folds from* Theorem 4.7.

4.3. Calibrated submanifolds in the T^2 quotient spaces. Let $(Y, \tilde{\varphi}, \tilde{g})$ be an almost G_2-manifold on which T^2 acts freely preserving the G_2-structure. Consider the quotient space Y/T^2. As in the S^1 case (Theorem 3.7), we see the relation between submanifolds of Y and Y/T^2. First, we introduce the generalized notion of "pseudo-holomorphic curves" from [9].

DEFINITION 4.9. An *almost CR-structure* on a smooth manifold M is a sub-bundle $E \subset TM$ of even rank equipped with a bundle map $J\colon E \to E$ of $J^2 = -id_E$. A (real) submanifold $S \subset M$ is said to be *E-holomorphic* or *CR J-holomorphic* if $TS \subset E|_S$ and TS is J-invariant. An almost CR structure (E, J) is said to be a *CR structure* if the Nijenhuis tensor of J vanishes.

Proposition 4.10. *Let* $(Y, \tilde{\varphi}, \tilde{g})$ *be an almost* G_2*-manifold with a free* T^2*-action preserving the* G_2*-structure. Let* $\pi_2\colon Y \to Y/T^2$ *be the natural projection. Suppose that there exists a multi-moment map* $\tilde{\nu}\colon Y \to \mathbb{R}$ *for the* T^2*-action. Let* $\underline{\nu}\colon Y/T^2 \to \mathbb{R}$ *be a map induced from* $\tilde{\nu}$ *and put* $Q = \ker(d\underline{\nu}) \subset T(Y/T^2)$.

Then there exist bundle maps $\underline{J_i}\colon Q \to Q$ $(i = 0, 1, 2)$ *satisfying* $\underline{J_0 J_1 J_2} = -id_Q$ *and each* $(Q, \underline{J_i})$ *is an involutive almost CR-structure on* Y/T^2.

Proof. Since the T^2-action is free, $d\tilde{\nu} \neq 0$ holds everywhere and so $Q = \ker d\underline{\nu}$ is a rank 4 involutive subbundle. For an arbitrary point $q \in Y/T^2$, we see

$$Q_q = T_q(\underline{\nu}^{-1}(\underline{\nu}(q))) = T_q(M_{\underline{\nu}(q)}).$$

From Proposition 4.4, we can define an almost complex structure $\underline{J_i}$ on Q by $(\underline{J_i})_q = \underline{J_{i,\underline{\nu}(q)}}$. Thus we see that $(Q, \underline{J_i})$ is an almost CR-structure on Y/T^2. □

Decompose $T^2 = S^1_1 \times S^1_2$, and suppose the S^1_i-action generates the vector field X^*_i. Let $\pi_1 \colon Y \to Y/S^1_1$, $\pi_{2,1} \colon Y/S^1_1 \to Y/T^2$ and $\pi_2 \colon Y \to Y/T^2$ be the projections satisfying $\pi_2 = \pi_{2,1} \circ \pi_1$. If tensors $\tilde{\zeta}$ on Y induces tensors on Y/S^1_1 and Y/T^2, we denote these by ζ on Y/S^1_1 and $\underline{\zeta}$ on Y/T^2, respectively.

REMARK 4.11 (Relations between S^1 and T^2 reductions). By Propositions 3.4 and 4.10, we see that there exists an SU(3)-structure $(g, J_1, \tau_1, \psi^{\pm})$ on Y/S^1_1 and almost CR structures $(Q = \ker d\underline{\nu}, \underline{J_i})$ on Y/T^2 $(i = 0, 1, 2)$. Define 1-forms $\tilde{\theta}'_i = \tilde{g}(X^{*'}_i, \cdot) \in \Omega^2(Y)$ $(i = 1, 2)$ and 2-forms $\underline{\tau_i} = \underline{g}(\underline{J_i}\cdot, \cdot) \in \Omega^2(Y/T^2)$ $(i = 0, 1, 2)$. Then we have

$$
\begin{aligned}
&g = \pi^*_{2,1}\underline{g} + \theta'_2 \otimes \theta'_2,\\
&\tau_1 = \pi^*_{2,1}\underline{\tau_1} + h\theta'_2 \wedge \pi^*_{2,1} d\underline{\nu},\\
&(\pi_{2,1})_* \circ J_1 = \underline{J_1} \circ (\pi_{2,1})_* + h \cdot \mathrm{grad}(\underline{\nu}) \otimes \theta'_2,\\
&\psi^+ = \pi^*_{2,1}(h d\underline{\nu} \wedge \underline{\tau_0}),\\
&\psi^- = \pi^*_{2,1}(h d\underline{\nu} \wedge \underline{\tau_2}).
\end{aligned}
$$

REMARK 4.12. As in the S^1 case, we can recover the G_2-structure on Y. In the pointwise coordinate of Lemma 4.5, $\tilde{\theta}'_1 = e_1$, $\tilde{\theta}'_2 = e_2$. Then

$$
\begin{aligned}
\tilde{g} &= \tilde{\theta}'^2_1 + \tilde{\theta}'^2_2 + \pi^*_2 \underline{g},\\
\tilde{\varphi} &= \tilde{\theta}'_1 \wedge \tilde{\theta}'_2 \wedge \pi^*_2(h\, d\underline{\nu}) + \tilde{\theta}'_1 \wedge \pi^*_2 \underline{\tau_1} + \tilde{\theta}'_2 \wedge \pi^*_2 \underline{\tau_2} - \pi^*_2(h\, d\underline{\nu} \wedge \underline{\tau_0}),\\
*\tilde{\varphi} &= \frac{1}{2}\pi^*_2 \underline{\tau}^2_0 + \tilde{\theta}'_2 \wedge \pi^*_2(h\, d\underline{\nu}) \wedge \pi^*_2 \underline{\tau}_1 - \tilde{\theta}'_1 \wedge \pi^*_2(h\, d\underline{\nu}) \wedge \pi^*_2 \underline{\tau}_2 - \tilde{\theta}'_1 \wedge \tilde{\theta}'_2 \wedge \pi^*_2 \underline{\tau}_0.
\end{aligned}
$$

Proof of Remark 4.11. Choose an arbitrary point $y \in Y$ and a pointwise coordinate as in Lemma 4.5. Then $g = \pi^*_{2,1}\underline{g} + \theta'_2 \otimes \theta'_2 = \sum_{i=2}^7 e_i^2$, $\tau_1 = \pi^*_{2,1}\underline{\tau_1} + h\theta'_2 \wedge \pi^*_{2,1} d\underline{\nu} = e_{23} + e_{45} + e_{67}$, $\psi^+ = \pi^*_{2,1}(h\, d\underline{\nu} \wedge \underline{\sigma_0}) = e_3(e_{56} + e_{47})$, $\psi^- = \pi^*_{2,1}(h\, d\underline{\nu} \wedge \underline{\sigma_2}) = e_3(e_{46} - e_{57})$ follow. With respect to $\{E_2, E_3, E_4, E_5, E_6, E_7\}$, we have

$$
J_1 = \begin{pmatrix}
 & -1 & & & & \\
1 & & & & & \\
 & & & -1 & & \\
 & & 1 & & & \\
 & & & & & -1 \\
 & & & & 1 &
\end{pmatrix}.
$$

Comparing with Lemma 4.5, we obtain the equations desired. □

Theorem 4.13. *Let $(Y, \tilde{\varphi}, \tilde{g})$ be an almost G_2-manifold with a free T^2-action preserving the G_2-structure. Let $\pi_2\colon Y \to Y/T^2$ be the projection. Suppose that there exists a multi-moment map $\tilde{\nu}\colon Y \to \mathbb{R}$ for the T^2-action. Let $\underline{\nu}\colon Y/T^2 \to \mathbb{R}$ be the map induced from $\tilde{\nu}$ By* Proposition 4.10, *there exist almost CR-structures $(Q = \ker(d\underline{\nu}), \underline{J_i})$ $(i = 0, 1, 2)$ on Y/T^2 satisfying $\underline{J_0 J_1 J_2} = -id_Q$.*

Decomposing $T^2 = S^1_1 \times S^1_2$, let $\pi_1\colon Y \to Y/S^1_1$, $\pi_{2,1}\colon Y/S^1_1 \to Y/T^2$ and $\pi_2\colon Y \to Y/T^2$ be the projections satisfying $\pi_2 = \pi_{2,1} \circ \pi_1$. Then from Propositions 3.4 *and* 4.10, *we see that there exists an* SU(3)*-structure $(g, J_1, \tau_1, \psi^{\pm})$ on Y/S^1_1 and CR structures $(Q = \ker d\underline{\nu}, \underline{J_i})$ with induced* 2*-forms $\underline{\tau_i}$ on Y/T^2 $(i = 0, 1, 2)$.*

If $L^k_{T^2} \subset Y$ is an oriented k-dimensional submanifold invariant under the T^2-action $(k = 3, 4)$, then with respect to an appropriate orientation the following properties hold:

1. $L^3_{T^2} \subset Y$ *is an associative* 3*-fold if and only if $\pi_2(L^3_{T^2}) \subset Y/T^2$ is contained in the integral curve of* $\mathrm{grad}(\underline{\nu})$,
2. $L^4_{T^2} \subset Y$ *is a coassociative* 4*-fold if and only if $\pi_2(L^4_{T^2}) \subset Y/T^2$ is a CR $\underline{J_0}$-holomorphic curve.*

If $L^k_{S^1_1,p} \subset Y$ is an oriented k-dimensional submanifold invariant under the S^1_1-action with $\tilde{\theta}'_2\big(TL^k_{S^1_1,p}\big) = 0$ ($\tilde{\theta}'_2$ is defined in Remark 4.11) $(k = 3, 4)$, *then with respect to an appropriate orientation the following properties hold*:

3. $L^3_{S^1_1,p} \subset Y$ *is an associative* 3*-fold if and only if $\pi_2\big(L^3_{S^1_1,p}\big) \subset Y/T^2$ is a CR $\underline{J_1}$-holomorphic curve,*
4. $L^4_{S^1_1,p} \subset Y$ *is a coassociative* 4*-fold if and only if for each $\alpha \in \mathbb{R}$, $\pi_2\big(L^4_{S^1_1,p}\big) \cap \underline{\nu}^{-1}(\alpha) \subset Y/T^2$ is either an empty set or a CR $\underline{J_2}$-holomorphic curve and* $\mathrm{grad}(\underline{\nu})|_{\pi_2\left(L^4_{S^1_1,p}\right)}$ *is tangent to $\pi_2\big(L^4_{S^1_1,p}\big)$.*

If $L^k_{p,p} \subset Y$ is an oriented k-dimensional submanifold perpendicular to T^2-orbits $(k = 3, 4)$, then with respect to an appropriate orientation the following properties hold:

5. $L^3_{p,p} \subset Y$ *is an associative* 3*-fold if and only if for each $\alpha \in \mathbb{R}$, $\pi_2(L^3_{p,p}) \cap \underline{\nu}^{-1}(\alpha) \subset Y/T^2$ is either an empty set or a CR $\underline{J_0}$-holomorphic curve with* $\mathrm{grad}(\underline{\nu})|_{\pi_2(L^3_{p,p})}$ *tangent to $\pi_2(L^3_{p,p})$.*
6. $L^4_{p,p} \subset Y$ *is a coassociative* 4*-fold if and only if $\pi_2(L^4_{p,p}) \subset Y/T^2$ is a CR $\underline{J_0}$-holomorphic surface.*

REMARK 4.14. This theorem is a generalization of Example 2.11. T^2-orbits and the "$\underline{\nu}$-direction" correspond to $\mathcal{I}^2$ and x_3-direction in Example 2.11, respectively.

Corollary 4.15. *If $L^3_{p,p} \subset Y$ is an associative* 3*-fold, then for any $\alpha \in \mathbb{R}$ satisfying $\pi_2(L^3_{p,p}) \cap \underline{\nu}^{-1}(\alpha) \neq \phi$, $\pi_2^{-1}(\pi_2(L^3_{p,p}) \cap \underline{\nu}^{-1}(\alpha)) = \pi_2^{-1}(\pi_2(L^3_{p,p})) \cap \tilde{\nu}^{-1}(\alpha) = T^2 \cdot L^3_{p,p} \cap \tilde{\nu}^{-1}(\alpha)$ is a T^2-invariant coassociative* 4*-fold of Y.*

Corollary 4.16. *There is one to one correspondence between T^2-invariant associative* 3*-folds (resp. T^2-invariant coassociative* 4*-folds) in Y and* 1*-dimensional* *submanifolds of the integral curve of* $\mathrm{grad}(\underline{\nu})$ (*resp. CR* $\underline{J_0}$*-holomorphic curves*) *in* Y/T^2.

REMARK 4.17 (Relations to evolution equations). Consider the case 2 of Theorem 4.13. Let $\underline{\gamma}\colon \Sigma \to Y/T^2$ be a smooth immersion of a surface Σ and $(U,(s,t))$ be a local conformal coordinate of Σ. Then from the local triviality of $\pi_2\colon Y \to Y/T^2$, there exists a local lift $\tilde{\gamma}\colon U \to Y$ of $\underline{\gamma}$ on a small open set $U \subset \Sigma$ which is transverse to T^2-orbits.

The differential equation of $\underline{J_0}$-holomorphic curve is $\partial\underline{\gamma}/\partial s + \underline{J_0}\,\partial\underline{\gamma}/\partial t = 0$. By the definition of $\underline{J_0}$, this equation can be described as

$$\left(\frac{\partial\tilde{\gamma}}{\partial s}\right)^{e} = (*\tilde{\varphi})_{abcd}(X_1^{*'})^{a}(X_2^{*'})^{b}\left(\frac{\partial\tilde{\gamma}}{\partial t}\right)^{c}\tilde{g}^{de}.$$

Thus the differential equation of $\underline{J_0}$-holomorphic curve is considered as the special case of the evolution equations in [17], [18]. Lotay [17], [18] constructed examples of (co)associative submanifolds in $\mathbb{R}^7$ by evolution equations.

Proof of Theorem 4.13. We fix one of the orientations of submanifolds. The same proof is valid for another orientation.

Proof of 1. Take any $p \in L^3_{T^2}$ and choose an oriented orthonormal basis $\{(X_1^{*'})_p, (X_2^{*'})_p, \tilde{v}\}$ of $T_pL^3_{T^2}$. Then $\underline{v} = \pi_{2*}(\tilde{v})$ is a basis of $T_{\pi_2(p)}(\pi_2(L^3_{T^2}))$. Now, $L^3_{T^2}$ is associative if and only if $\tilde{\varphi}((X_1^{*'})_p, (X_2^{*'})_p, \tilde{v}) = 1$. By Remark 4.12, this condition is equivalent to $h\,d\underline{\nu}(\underline{v}) = 1$, which implies that $\pi_2(L^3_{T^2})$ is contained in the integral curve of $\mathrm{grad}(\underline{\nu})$.

The claim 2 follows from Theorem 4.7.

Proof of 3. By Theorem 3.7, $L^3_{S^1_1,p} \subset Y$ is associative if and only if $\pi_1\big(L^3_{S^1_1,p}\big)$ is a J_1-holomorphic curve. By Remark 4.11, we see that this is equivalent to saying that $\pi_2\big(L^3_{S^1_1,p}\big)$ is a CR $\underline{J_1}$-holomorphic curve. Actually, if $\pi_1\big(L^3_{S^1_1,p}\big)$ is a J_1-holomorphic curve, then it follows that

$$\underline{J_1}(\pi_2)_*\big(TL^3_{S^1_1,p}\big) = \{(\pi_{2,1})_*J_1 - h\cdot\mathrm{grad}(\underline{\nu})\otimes\theta_2'\}(\pi_1)_*\big(TL^3_{S^1_1,p}\big) = (\pi_2)_*\big(TL^3_{S^1_1,p}\big).$$

Thus $\pi_2\big(L^3_{S^1_1,p}\big)$ is a CR $\underline{J_1}$-holomorphic curve. Conversely, if $\pi_2\big(L^3_{S^1_1,p}\big)$ is a CR $\underline{J_1}$-holomorphic curve, we obtain $(\pi_{2,1})_*(J_1(\pi_1)_*\big(TL^3_{S^1_1,p}\big)) = (\pi_{2,1})_*\big((\pi_1)_*\big(TL^3_{S^1_1,p}\big)\big)$. On the other hand, since $\pi_2\big(L^3_{S^1_1,p}\big)$ is a CR $\underline{J_1}$-holomorphic curve, we see

$$d\underline{\nu}\big((\pi_2)_*\big(TL^3_{S^1_1,p}\big)\big) = 0.$$

This is equivalent to $g\big((\pi_1)_*\big(TL^3_{S^1_1,p}\big), \mathrm{grad}(\nu)\big) = 0$. Since g is Hermitian, we have $g\big(J_1(\pi_1)_*\big(TL^3_{S^1_1,p}\big), J_1(\mathrm{grad}(\nu))\big) = 0$.

Using a pointwise coordinate of Lemma 4.5, we see $g(\ \cdot\ , J_1(\mathrm{grad}(\nu))) = (-1/h)\theta'_2$. Hence we have $\theta'_2\big((\pi_1)_*\big(TL^3_{S^1_1,p}\big)\big) = \theta'_2\big(J_1(\pi_1)_*\big(TL^3_{S^1_1,p}\big)\big) = 0$, which means that $(\pi_1)_*\big(TL^3_{S^1_1,p}\big)$ and $J_1(\pi_1)_*\big(TL^3_{S^1_1,p}\big)$ are horizontal. Then $J_1(\pi_1)_*\big(TL^3_{S^1_1,p}\big) = (\pi_1)_*\big(TL^3_{S^1_1,p}\big)$ and so $\pi_1\big(L^3_{S^1_1,p}\big)$ is a J_1-holomorphic curve.

Proof of 4. Suppose that $L^4_{S^1_1,p} \subset Y$ is coassociative and $\pi_2\big(L^4_{S^1_1,p}\big) \cap \underline{\nu}^{-1}(\alpha) \neq \phi$. By Theorem 3.7, $L^4_{S^1_1,p}$ is coassociative if and only if $\pm\psi^-|_{\pi_1\left(L^4_{S^1_1,p}\right)} = \mathrm{vol}_{\pi_1\left(L^4_{S^1_1,p}\right)}$ which is equivalent to $\pm h d\underline{\nu} \wedge \underline{\tau_2}|_{\pi_1\left(L^4_{S^1_1,p}\right)} = \mathrm{vol}_{\pi_2\left(L^4_{S^1_1,p}\right)}$ by Remark 4.11.

Fix an arbitrary point $q \in \pi_2\big(L^4_{S^1_1,p}\big) \cap \underline{\nu}^{-1}(\alpha)$ and choose an oriented orthonormal basis $\{\underline{v_1}, \underline{v_2}\} \subset T_q\big(\pi_2\big(L^4_{S^1_1,p}\big) \cap \underline{\nu}^{-1}(\alpha)\big)$. There exists $\underline{v'_3} \in T_q\big(\pi_2\big(L^4_{S^1_1,p}\big)\big)$ with $d\underline{\nu}(\underline{v'_3}) \neq 0$. Via the Gram–Schmidt process, we have an orthonormal basis $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\} \subset T_q\big(\pi_2\big(L^4_{S^1_1,p}\big)\big)$. Then we have

$$\pm h d\underline{\nu}(\underline{v_3}) \cdot \underline{\tau_2}(\underline{v_1}, \underline{v_2}) = 1.$$

Since $|h d\underline{\nu}(\underline{v_3})| \leq 1$ and $|\underline{\tau_2}(\underline{v_1}, \underline{v_2})| \leq 1$ hold, we obtain $|h d\underline{\nu}(\underline{v_3})| = 1$ and $|\underline{\tau_2}(\underline{v_1}, \underline{v_2})| = 1$, respectively. The first equation implies that $\underline{v_3} = h \cdot \mathrm{grad}(\underline{\nu})$, and $\mathrm{grad}(\underline{\nu})|_{L^4_{S^1_1,p}}$ is tangent to $L^4_{S^1_1,p}$. In the same way as the proof of 1 in Theorem 3.7, the second equation implies that $\pi_2\big(L^4_{S^1_1,p}\big) \cap \underline{\nu}^{-1}(\alpha)$ is a CR $\underline{J_2}$-holomorphic curve.

Conversely, fixing $q \in \pi_1\big(L^4_{S^1_1,p}\big)$, take $\{\underline{v_1}, \underline{v_2} = \underline{J_2}\underline{v_1}, \underline{v_3} = h \cdot \mathrm{grad}(\underline{\nu})\}$ as an orthonormal basis of $T_{\pi_{2,1}(q)}(\pi_2\big(L^4_{S^1_1,p}\big))$, where $\underline{v_1}, \underline{v_2} \in T_{\pi_{2,1}(q)}(\pi_2\big(L^4_{S^1_1,p}\big) \cap \underline{\nu}^{-1}(\underline{\nu}(\pi_{2,1}(q))))$. Define $\{v_1, v_2, v_3\} \subset T_q(Y/S^1)$ as horizontal lifts of $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ by θ'_2.

Then $\{v_1, v_2, v_3\}$ is an orthonormal basis of $T_q\big(\pi_1\big(L^4_{S^1_1,p}\big)\big)$ satisfying $\psi^-(v_1, v_2, v_3) = \pm 1$. From Theorem 3.7, we see that $L^4_{S^1_1,p}$ is coassociative.

Proof of 5. Suppose that $L^3_{p,p}$ is associative and $\pi_2(L^3_{p,p}) \cap \underline{\nu}^{-1}(\alpha) \neq \phi$. Since $\tilde{\theta}_i|_{L^3_{p,p}} = 0$ $(i = 1, 2)$, we see from Remark 4.12, $-\pi_2^*(h\, d\underline{\nu} \wedge \underline{\tau_0})|_{L^3_{p,p}} = \mathrm{vol}_{L^3_{p,p}}$.

Fix an arbitrary point $y \in L^3_{p,p}$ and choose an oriented orthonormal basis $\{\tilde{v}_1, \tilde{v}_2\} \subset T_y(L^3_{p,p} \cap \tilde{\nu}^{-1}(\alpha))$. There exists $\tilde{v}'_3 \in T_y(L^3_{p,p})$ with $d\tilde{\nu}(\tilde{v}'_3) \neq 0$. Via the Gram–Schmidt process, we have an orthonormal basis $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} \subset T_y(L^3_{p,p})$. If we define $\underline{v_i} = \pi_{2*}(v_i)$, we have $-h\, d\underline{\nu}(\underline{v_3}) \cdot \underline{\tau_0}(\underline{v_1}, \underline{v_2}) = 1$.

Similarly to the proof of 4, we see that $\mathrm{grad}\,\underline{\nu}|_{\pi_2(L^3_{p,p})}$ is tangent to $\pi_2(L^3_{p,p})$. Thus $\pi_2(L^3_{p,p}) \cap \underline{\nu}^{-1}(\alpha)$ is a CR $\underline{J_0}$-holomorphic curve. The converse follows similarly to the proof of 4.

Proof of 6. Suppose that $L^4_{p,p}$ is associative. Since $\tilde{\theta}_i|_{L^4_{p,p}} = 0$ $(i = 1, 2)$, we see from Remark 4.12, $(1/2)\underline{\tau}_0|_{\pi_2(L^4_{p,p})} = \mathrm{vol}_{\pi_2(L^4_{p,p})}$. We can prove 6 as in the proof of 4 in Theorem 3.7. □

5. Examples

Basic examples of calibrated submanifolds are given in Examples 2.10, 2.11. We provide more examples on (sine) cones and T^2 bundles by using our method.

5.1. Examples of nearly Kähler manifolds.

EXAMPLE 5.1 ([21]). Let (N, g, J) be a real 6-dimensional Kähler manifold and (B, h) an even dimensional Riemannian manifold. Suppose that there exists a Riemannian submersion with totally geodesic fibers

$$\varpi : (N, g) \to (B, h).$$

Let $TN = \mathcal{V} \oplus \mathcal{H}$ be the corresponding splitting of TN, where $\mathcal{V}$ is a vertical subbundle and $\mathcal{H}$ is a horizontal subbundle such that J preserves $\mathcal{V}$ and $\mathcal{H}$.

If we define a Riemannian metric $\hat{g}$ and an almost complex structure $\hat{J}$ as

$$\hat{g}|_{\mathcal{V}} = \frac{1}{2} g|_{\mathcal{V}}, \quad \hat{g}|_{\mathcal{H}} = g|_{\mathcal{H}}, \quad \hat{J}|_{\mathcal{V}} = -J|_{\mathcal{V}}, \quad \hat{J}|_{\mathcal{H}} = J|_{\mathcal{H}},$$

then $(N, \hat{g}, \hat{J})$ is a nearly Kähler manifold.

Each fiber of $\varpi : (N, g) \to (B, h)$ is $\hat{J}$-holomorphic. Hence if $\dim_{\mathbb{R}} B = 4$, $N \times \mathbb{R}_{>0}$ (or $N \times (0, \pi)$) $\ni (x, r) \mapsto \varpi(x) \in B$ is an associative fibration. (i.e. each fiber is an associative 3-fold.) If $\dim_{\mathbb{R}} B = 2$, $N \times \mathbb{R}_{>0} \ni (x, r) \mapsto (\varpi(x), r) \in B \times \mathbb{R}_{>0}$ is a coassociative fibration.

Next, we give examples of homogeneous nearly Kähler manifolds which are classified by Butruille [11].

Lemma 5.2 ([11]). *Any 6-dimensional compact homogeneous nearly Kähler manifold is isomorphic to a finite quotient of a homogeneous space belonging to the following list*:

$$\mathrm{SU}(3)/T^2, \quad \mathbb{C}P^3, \quad S^3 \times S^3, \quad S^6.$$

The spaces $\mathrm{SU}(3)/T^2$, $\mathbb{C}P^3$ are the twistor spaces of $\mathbb{C}P^2$ and S^4, respectively. They satisfy the condition of Example 5.1 so those (sine) cones admit associative fibrations. Moreover, in Theorem 1.3.1 of [24], by using a real structure on $\mathbb{C}P^3$ it is shown that $\mathbb{R}P^3 \subset \mathbb{C}P^3$ is a special Lagrangian submanifold with phase 1. Hence $\mathbb{R}P^3 \times \{r\} \subset \mathbb{C}P^3 \times \mathbb{R}_{>0}$ is an associative 3-fold for any $r \in \mathbb{R}_{>0}$.

The cone of S^6 is $\mathbb{R}^7 - \{0\}$ so this case is well-studied. Pseudoholomorphic curves in S^6 and $\mathbb{C}P^3$ are investigated in [7] and [25], respectively.

In the case of $S^3 \times S^3$, define maps $pr_1\colon S^3 \ni (z_1, z_2) \mapsto [z_1 : \bar{z}_2] \in \mathbb{C}P^1$, $pr_2\colon S^3 \ni (z_1, z_2) \mapsto [z_1 - \sqrt{-1}z_2 : \bar{z}_1 - \sqrt{-1}\bar{z}_2] \in \mathbb{C}P^1$ and $pr_3\colon S^3 \ni (z_1, z_2) \mapsto [z_1 + z_2 : \bar{z}_1 - \bar{z}_2] \in \mathbb{C}P^1$, where we consider $S^3 \subset \mathbb{C}^2$. The map pr_1 is a slight modification of the Hopf fibration.

Proposition 5.3. *For each $i = 1, 2, 3$, the map $\varpi_i = pr_i \times pr_i\colon S^3 \times S^3 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ is a pseudoholomorphic fibration, and so induce associative fibrations $S^3 \times S^3 \times \mathbb{R}_{>0} \to S^2 \times S^2$ and $S^3 \times S^3 \times (0, \pi) \to S^2 \times S^2$.*

Proof. Note that each fiber of pr_1, pr_2 and pr_3 is of the form $\{(e^{\sqrt{-1}\theta}z_1, e^{-\sqrt{-1}\theta}z_2) \mid \theta \in \mathbb{R}\} \subset S^3$, $\{(z_1 \cos\theta + z_2 \sin\theta, -z_1 \sin\theta + z_2 \cos\theta) \mid \theta \in \mathbb{R}\} \subset S^3$ and $\{(z_1 \cos\theta + \sqrt{-1}z_2 \sin\theta, \sqrt{-1}z_1 \sin\theta + z_2 \cos\theta) \mid \theta \in \mathbb{R}\} \subset S^3$ for some $(z_1, z_2) \in S^3$.

By using the notation in [11], each fiber of ϖ_i is an integral submanifold of the distribution $\mathrm{span}_{\mathbb{R}}\{X_i^*, Y_i^*\} = \mathrm{span}_{\mathbb{R}}\{X_i^*, JX_i^*\}$ since we can take $X_1, Y_1 = (1/2)\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$, $X_2, Y_2 = (1/2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $X_3, Y_3 = (1/2)\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \in \mathfrak{su}(2)$ and the almost complex structure J on $S^3 \times S^3$ preserves $\mathrm{span}_{\mathbb{R}}\{X_i^*, Y_i^*\}$. Here X^* means the vector field on $S^3 \times S^3$ generated by $X \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Hence each fiber of ϖ_i is pseudoholomorphic. $\square$

REMARK 5.4. Define the inclusion $\iota\colon S^6 \times (0, \pi) \ni (\sigma, t) \mapsto (\cos t, \sigma \sin t) \in S^7$, where we consider $S^6 \subset \mathbb{R}^7$ and $S^7 \subset \mathbb{R} \times \mathbb{R}^7$. It is known that S^7 admits a nearly parallel G_2-structure induced from a Spin(7)-structure on $\mathbb{R}^8$. The inclusion ι preserves the G_2-structure from their constructions. Hence, if $L^k \subset S^6$ is an oriented k-dimensional submanifold ($k = 2, 3$) and $t \in (0, \pi)$, then

- $\iota(L^2 \times (0, \pi)) \subset S^7$ is an associative 3-fold iff L^2 is a J-holomorphic curve,
- $\iota(L^3 \times \{\pi/2\}) \subset S^7$ is an associative 3-fold iff L^3 is a special Lagrangian submanifold with phase $\pm\sqrt{-1}$.

This result is known by Lotay [19].

5.2. Cone of the Iwasawa manifold. For $x, y, z \in \mathbb{C}$, denote

$$A(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Define $G = \{A(x, y, z) \mid x, y, z \in \mathbb{C}\}$, $\Gamma = \{A(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in \mathbb{Z}[\sqrt{-1}]\}$. Let $N^6 = \Gamma \setminus G$ be the space of right cosets, which is called the Iwasawa manifold. It is a principal T^2-bundle over T^4 (the generic element is mapped to $(x, y) + \mathbb{Z}^2$). The Iwasawa manifold is a compact complex manifold which is not Kähler. (It is known that $h^{1,0}(N, \mathbb{C}) = 3$, $h^{0,1}(N, \mathbb{C}) = 2$, $b^1(N) = 4$).

First, we show that $Y = N \times \mathbb{R}_{>0}$ admits a torsion-free G_2-structure $(\tilde\varphi, \tilde g)$ with $\mathrm{Hol}(\tilde g) = G_2$. Define 1-forms $\tilde e_i \in \Omega^1(N)$ $(i = 1, 2, 4, 5, 6, 7)$ by

$$\begin{cases} dx = \tilde e_4 - \sqrt{-1}\tilde e_7, \\ dy = \tilde e_6 + \sqrt{-1}\tilde e_5, \\ -dz + x\,dy = \tilde e_1 - \sqrt{-1}\tilde e_2. \end{cases}$$

Left hand sides are Γ-invariant forms on G so they induce 1-forms on $N = \Gamma \setminus G$. Hence N is a nilmanifold with a global basis of 1-forms such that

$$d\tilde e_i = \begin{cases} \tilde e_{46} - \tilde e_{57} & (i = 1), \\ -\tilde e_{45} - \tilde e_{67} & (i = 2), \\ 0 & (i = 4, 5, 6, 7). \end{cases}$$

We also define vector fields $\{\tilde E_i\} \in \mathfrak{X}(N)$ dual to $\{\tilde e_i\}$. If we write $x = x_1 + \sqrt{-1}x_2$, $y = y_1 + \sqrt{-1}y_2$, $z = z_1 + \sqrt{-1}z_2$, we can describe $\{\tilde E_i\}$ explicitly as

$$\tilde E_1 = -\frac{\partial}{\partial z_1}, \quad \tilde E_2 = \frac{\partial}{\partial z_2}, \quad \tilde E_4 = \frac{\partial}{\partial x_1}, \quad \tilde E_7 = -\frac{\partial}{\partial x_2},$$
$$\tilde E_5 = \frac{\partial}{\partial y_2} - x_2\frac{\partial}{\partial z_1} + x_1\frac{\partial}{\partial z_2}, \quad \tilde E_6 = \frac{\partial}{\partial y_1} + x_1\frac{\partial}{\partial z_1} + x_2\frac{\partial}{\partial z_2}.$$

Extending $\tilde e_i$ and $\tilde E_i$ on Y, define 1-forms $\tilde e'_i \in \Omega^1(Y)$ by

$$(\tilde e'_1, \tilde e'_2, \tilde e'_3, \tilde e'_4, \tilde e'_5, \tilde e'_6, \tilde e'_7) = \left(\frac{1}{s}\tilde e_1, \frac{1}{s}\tilde e_2, s^2\,ds, s\tilde e_4, s\tilde e_5, s\tilde e_6, s\tilde e_7\right),$$

where $\mathbb{R}_{>0}$ is parametrized by s. We write $\tilde E_3 = \partial/\partial s$. Define a metric $\tilde g$ on Y, a 3-form $\tilde\varphi \in \Omega^3(Y)$ and its Hodge dual $*\tilde\varphi \in \Omega^4(Y)$ by

$$\tilde g = \sum_{i=1}^{7}(e'_i)^2,$$
$$\tilde\varphi = \tilde e'_{123} + \tilde e'_1(\tilde e'_{45} + \tilde e'_{67}) + \tilde e'_2(\tilde e'_{46} - \tilde e'_{57}) - \tilde e'_3(\tilde e'_{47} + \tilde e'_{56}),$$
$$*\tilde\varphi = \tilde e'_{4567} + \tilde e'_{23}(\tilde e'_{67} + \tilde e'_{45}) + \tilde e'_{13}(\tilde e'_{57} - \tilde e'_{46}) - \tilde e'_{12}(\tilde e'_{56} + \tilde e'_{47}).$$

Then $(Y, \tilde\varphi, \tilde g)$ is a torsion-free G_2-manifold with $\mathrm{Hol}(\tilde g) = G_2$. For more details, see [22][1].

***T^2*-action on *Y*.** Identifying $T^2 = \{A(0, 0, w) \mid w \in \mathbb{C}/\mathbb{Z}[\sqrt{-1}]\}$, T^2 acts on Y freely by right multiplication. Vector fields $\{X_1^*, X_2^*\} \subset \mathfrak{X}(Y)$ generated by the T^2-action are given by $X_1^* = \partial/\partial z_1 = -\tilde E_1$, $X_2^* = \partial/\partial z_2 = \tilde E_2$.

[1] Note that the notation in [22] differs from ours. The basis $(e_1,e_2,e_3,e_4,e_5,e_6,e_7)$ in [22] corresponds to $(\tilde e_4, \tilde e_5, \tilde e_6, -\tilde e_7, \tilde e_1, \tilde e_2, \tilde e_3)$ in our notation.

Since $\tilde{\varphi}(X_1^*, X_2^*, \cdot) = -ds$, there exists a multi-moment map $\tilde{\nu} = -s\colon Y \to \mathbb{R}_{<0}$ for the T^2-action.

Geometry of Y/T^2. Since N is a T^2-bundle over T^4 and since T^2 acts fiberwise, we have $Y/T^2 = T^4 \times \mathbb{R}_{>0}$, where we denote the projection by $\pi_2\colon Y \to T^4 \times \mathbb{R}_{>0}$. Define vector fields by $\underline{E_i} = (\pi_2)_*(\tilde{E}_i) \in \mathfrak{X}(Y/T^2)$ ($i = 4, 5, 6, 7$), namely,

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right) = (\underline{E_4}, -\underline{E_7}, \underline{E_6}, \underline{E_5}).$$

Then with respect to $\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial y_1, \partial/\partial y_2\}$ we have

$$\underline{J_0} = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, \quad \underline{J_1} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix},$$

$$\underline{J_2} = \begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}.$$

$(\underline{J_0}, \underline{J_1}, \underline{J_2})$ is a standard hyperkähler structure on T^4 induced by the left multiplication of $(i, -k, j)$ on the quaternion $\mathbb{H}$.

Calibrated submanifolds in Y.

T^2-invariant case. Since $(\underline{J_0}, \underline{J_1}, \underline{J_2})$ is a standard hyperkähler structure on T^4, there are many holomorphic curves. If $C \subset T^4$ is a $\underline{J_0}$-holomorphic curve and $s \in \mathbb{R}_{>0}$, then $\pi_2^{-1}(C \times \{s\})$ is a T^2-invariant coassociative 4-fold, which is compact if C is compact. T^4 is fibrated by $\underline{J_0}$-holomorphic curves so Y is fibrated by T^2-invariant coassociative 4-folds.

For the associative case, the integral curve of $\mathrm{grad}(\tilde{\nu})$ in Y is $\mathbb{R}_{>0} \subset Y$. If $x \in T^4$ and $\mathcal{O} \subset \mathbb{R}_{>0}$ is an open interval, $\pi_2^{-1}(\{x\} \times \mathcal{O})$ is a T^2-invariant associative 3-folds.

S_1^1-invariant and perpendicular to S_2^1-orbits case. Decomposing $T^2 = S_1^1 \times S_2^1$, let X_i^* be the vector field generated by the S_i^1-action. A submanifold $L \subset Y$ is perpendicular to S_2^1-orbits if and only if $\tilde{e}_2|_L = 0$. Then we have

$$\ker(\tilde{e}_2) = \mathrm{span}_{\mathbb{R}}\{\tilde{E}_1, \tilde{E}_3, \tilde{E}_4, \tilde{E}_5, \tilde{E}_6, \tilde{E}_7\}.$$

An S_1^1-invariant submanifold contains $\tilde{E}_1$ in its tangent space. If L_1 and L_2 are integral submanifolds of the involutive distributions $\mathrm{span}_{\mathbb{R}}\{\tilde{E}_1, \tilde{E}_3, \tilde{E}_4, \tilde{E}_6\}$ and $\mathrm{span}_{\mathbb{R}}\{\tilde{E}_1, \tilde{E}_3, \tilde{E}_5, \tilde{E}_7\}$, respectively, each L_i is a coassociative 4-fold perpendicular to the S_2^1-orbits. If L_i is maximal, L_i is an S_1^1-invariant coassociative 4-fold perpendicular to the S_2^1-orbits because $S_1^1 \cdot L_i$ is an integral submanifold of the same distribution containing L_i. We see that Y is foliated by these coassociative 4-folds. They can be described as follows.

$$L_1 = \{[(A(x_1 + \sqrt{-1}x_2^0, y_1 + \sqrt{-1}y_2^0, z_1 + \sqrt{-1}(x_2^0 y_1 + z_2^0)))] \mid x_1, y_1, z_1 \in \mathbb{R}_{>0}\} \times \mathbb{R}_{>0},$$
$$L_2 = \{[(A(x_1^0 + \sqrt{-1}x_2, y_1^0 + \sqrt{-1}y_2, z_1 + \sqrt{-1}(x_1^0 y_2 + z_2^0)))] \mid x_2, y_2, z_2 \in \mathbb{R}_{>0}\} \times \mathbb{R}_{>0},$$

where $x_i^0, y_i^0, z_i^0 \in \mathbb{R}$. These are S^1-bundle over T^2. Moreover, we have

$$(\pi_2)_*\{\tilde{E}_4, \tilde{E}_6\} = \{\underline{E_4}, \underline{J_2 E_4}\}, \quad (\pi_2)_*\{\tilde{E}_5, \tilde{E}_7\} = \{\underline{E_5}, -\underline{J_2 E_5}\},$$

and so $\pi_2(L) \cap \{\underline{\nu} = \text{const.}\} \subset T^4$ is a $\underline{J_2}$-holomorphic curve with $\text{grad}(\underline{\nu})|_{\pi_2(L)} = -(1/s^4)(\partial/\partial s)|_{\pi_2(L)} = -(1/s^4)\tilde{E}_3|_{\pi_2(L)} \in T(\pi_2(L))$, which corresponds to 4 of Theorem 4.13.

Perpendicular to T^2-orbits case. A submanifold $L \subset Y$ is perpendicular to T^2-orbits if and only if $\tilde{e}_1|_L = \tilde{e}_2|_L = 0$. Then we obtain

$$\ker(\tilde{e}_1) \cap \ker(\tilde{e}_2) = \text{span}_{\mathbb{R}}\{\tilde{E}_3, \tilde{E}_4, \tilde{E}_5, \tilde{E}_6, \tilde{E}_7\}.$$

If L is an integral submanifold of the involutive distribution $\text{span}_{\mathbb{R}}\{\tilde{E}_3, \tilde{E}_4, \tilde{E}_7\}$, L is an associative 3-fold perpendicular to the T^2-orbits. L is described as $L = \{[(A(x, y^0, z^0))] \mid x \in \mathbb{C}\} \times \mathbb{R}_{>0}$, where $y^0, z^0 \in \mathbb{C}$. We see that Y is foliated by these associative 3-folds. Moreover, we have $(\pi_2)_*\{\tilde{E}_4, \tilde{E}_7\} = \{\underline{E_4}, -\underline{J_0 E_4}\}$, and so $\pi_2(L) \cap \{\underline{\nu} = \text{const.}\} \subset T^4$ is a $\underline{J_0}$-holomorphic curve with $\text{grad}(\underline{\nu})|_{\pi_2(L)} = -(1/s^4)(\partial/\partial s)|_{\pi_2(L)} = -(1/s^4)\tilde{E}_3|_{\pi_2(L)} \in T(\pi_2(L))$, which corresponds to 5 of Theorem 4.13.

5.3. Further examples. In [20], it it shown that for a hyperkähler 2-fold M whose Kähler forms have integral periods, there exists a T^2-bundle $\mathcal{X}_0$ over M and an open interval $I \subset \mathbb{R}$ such that $\mathcal{X}_0 \times I$ admits a torsion-free G_2-structure.

Especially if M is a toric hyperkähler 2-fold, it is shown in [13] that M is fibrated by complex Lagrangian submanifolds (pseudoholomorphic curves in dimension 4) whose generic fibers are diffeomorphic to $T^1 \times \mathbb{R}$. Using this fibration, we see that a G_2-manifold $\mathcal{X}_0 \times I$ is fibrated by T^2-invariant coassociative 4-folds.

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